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A Geometric Preferential Attachment Model of Networks II

Abraham D. Flaxman, Alan M. Frieze, and Juan Vera

Abstract. We study a random graph G_n that combines certain aspects of geometric random graphs and preferential attachment graphs. This model yields a graph with power law degree distribution where the expansion property depends on a tunable parameter of the model.

The vertices of G_n are n sequentially generated points, $x_1, x_2, \ldots, x_n$, chosen uniformly at random from the unit sphere in R^3 After generating x_t, we randomly connect it to m points from those points $x_1, x_2, \ldots, x_{t-1}$.

1. Introduction

During the last decade, a large body of research has centered on understanding and modeling the structure of large-scale networks like the Internet and the World Wide Web. Several recent books provide a general introduction to this topic [Newman et al. 06, **?**]. One important feature identified in early experimental studies (including [Aiello et al. 99, Broder el al. 00, Faloutsos et al. 99]) is that the vertex degree distribution of many real-world networks has a heavy-tailed property, which may follow a power law (i.e., the proportion of vertices of degree at least k is proportional to $k^{-\alpha}$ for some constant α). This has driven the investigation of random graph models that generate heavy-tailed degree distributions, including the fixed degree sequence model, the copying model, and the preferential attachment model.

The preferential attachment model and its derivatives have been particularly popular for theoretical analysis. Preferential attachment was proposed as a

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model for real-world complex networks by Barabási and Albert [Barabass and Albert 99]. The distribution was formalized by Bollobás and Riordan [Bollobás and Riordan 04b], and in [Bollobás et al. 01] it was proved rigorously that with high probability (whp) a graph chosen according to this distribution has a power-law degree distribution with complementary cumulative distribution function (ccdf) $\Pr[\deg(v) \geq k] = \Theta(k^{-2})$. By changing the initial attactiveness or extending allowable graph operations, the power of the ccdf power law can be tuned to take any value in the interval $(1, \infty)$[Buckley and Osthus 04, Cooper at al. 04].

However, there are some significant differences between graphs generated by preferential attachment and those found in the real world. One major difference is found in their expansion properties. Mihail, Papadimitriou, and Saberi showed that whp the preferential attachment model has conductance bounded below by a constant [Mihail et al. 03]. On the other hand, Blandford, Blelloch, and Kash found that some World Wide Web–related graphs have smaller separators than the preferential attachment model predicts [Blandford et al. 03]. This observation is consistent with observations due to Estrada, who found that half of the real-world networks he looked at were good expanders and the other half were not so good [Estrada 06]. The perturbed random graph framework provides one approach to understanding expansion in real-world networks [Flaxman xx], but it does not give a generative procedure. This paper investigates a generative procedure, based on a geometric modification of the preferential attachment model, that yields a graph that might or might not be a good expander, depending on a tunable parameter of the geometry. This is a strict generalization of the geometric preferential attachment graph developed in [Flaxman et al. 06], which was designed specifically to avoid being a good expander.

The primary contribution of this paper is to provide a parameterised model that exhibits a sharp transition between low and high conductance. Choosing this parameter appropriately provides a unified approach to generating preferential attachment graphs with and without good expansion processes.

1.1. The Random Process

In [Flaxman et al. 06] we studied a process that generates a sequence of graphs $G_t, t = 1, 2, \ldots, n$. The graph $G_t = (V_t, E_t)$ has t vertices and mt edges. Here, V_t is a subset of t random points on S, the surface of the sphere in $\mathbf{R}^3$ of radius $\frac{1}{2\sqrt{\pi}}$ (so that area$(S) = 1$). After randomly choosing $x_{t+1} \in S$, it is connected, by preferential attachment (i.e., proportional to degree), to m vertices in V_t among those of distance at most r from x_{t+1}. We showed that this graph has a power law degree distribution, small separators, and a moderate diameter. In this paper we provide a "smoothed" version of this model. Instead of being chosen proportional to degree among those vertices within distance r of x_{t+1}, the m neighbors of x_t are chosen proportional to degree and some function of the distance to x_{t+1}.

Let $F : \mathbf{R}_+ \to \mathbf{R}_+$. Define

$$I = \int_S F(|u-u_0|)du = \frac{1}{2}\int_{x=0}^{\pi} F(x)\sin x dx,$$

where u_0 is any point in S and $0 \le |u-u_0| \le \pi$ is the angular distance from u to u_0 along a great circle. Other parameters of the process are $m > 0$, which denotes the number of edges added in every step, and $\alpha \ge 0$, a measure of the bias towards self loops.

- Time step 0: To initialize the process, we start with G_0 being the Empty Graph.
- Time step t+1: We choose vertex x_{t+1} uniformly at random in S and add it to G_t. Let

$$T_t(x_{t+1}) = \sum_{v \in V_t} F(|x_{t+1}-v|)\deg_t(v).$$

  We add m random edges (x_{t+1}, y_i), $i = 1, 2, \dots, m$, incident with x_{t+1}. Here, each y_i is chosen independently from $V_{t+1} = V_t \cup \{x_{t+1}\}$ (parallel edges and loops are permitted), such that for each $i = 1, \dots, m$ and for all $v \in V_t$,

$$\Pr(y_i = v) = \frac{\deg_t(v)F(|x_{t+1}-v|)}{\max(T_t(x_{t+1}), \alpha m I t)}$$

  and

$$\Pr(y_i = x_{t+1}) = 1 - \frac{T_t(x_{t+1})}{\max(T_t(x_{t+1}), \alpha m I t)}.$$

  (When $t = 0$ we have $\Pr(y_i = x_1) = 1$.)

For $z > 0$, we define

$$I_z = \frac{1}{2}\int_{x=0}^{z} F(x)\sin x\,dx \text{ and } J_z = I - I_z.$$

We will first prove the following result about the degree distribution and the existence of small separators:

Let $d_k(t)$ denote the number of vertices of degree k at time t, and let $\overline{d}_k(t)$ denote the expectation of $d_k(t)$.

Theorem 1.1.

(a) *Suppose that $\alpha > 2$ and in addition that*

$$\int_{x=0}^{\pi} F(x)^2 \sin x dx = O(n^{\theta} I^2), \tag{1.1}$$

where $\theta < 1$ is a constant.

Then, there exists a constant $\gamma_1 > 0$ such that for all $k = k(n) \geq m$,

$$\overline{d}_k(n) = e^{\varphi_k(m,\alpha)} \left(\frac{m}{k}\right)^{1+\alpha} n + O(n^{1-\gamma_1}), \tag{1.2}$$

where $\varphi_k(m,\alpha) = O(1)$ tends to a constant $\varphi_\infty(m,\alpha)$ as $k \to \infty$.

Furthermore, for n sufficiently large, the random variable $d_k(n)$ satisfies the following concentration inequality:

$$\Pr(|d_k(n) - \overline{d}_k(n)| \geq I^2 n^{\max\{1/2, 2/\alpha\}+\delta}) \leq e^{-n^{\delta}}. \tag{1.3}$$

(b) *Suppose that $\alpha > 0$, $m \geq m_0$, where m_0 is a sufficiently large constant, and $\varphi, \eta = o(1)$ such that $\eta n \to \infty$ and $J_\eta \leq \varphi I$. Then whp, V_n can be partitioned into $T, \bar{T}$ such that $|T|, |\bar{T}| \sim n/2$, and there are $\tilde{O}((\eta + \varphi)mn)$ edges between T and $\bar{T}$.*

Remark 1.2. Note that the exponent in Theorem1.1(a) does not depend on the particular function F and that F manifests itself only through the error terms.

We now consider the connectivity and diameter of G_n. For this, we will place some more restrictions on F.

Define the parameter $\rho(\mu)$ by

$$I_\rho = \mu I. \tag{1.4}$$

We will say that F is *smooth* (for some value of μ) if

(S1) F is monotone non-increasing,

(S2) $\rho^2 n \geq L \ln n$ for some sufficiently large constant L,

(S3) $\rho^2 F(2\rho) \geq c_3 I$ for some c_3 that is bounded below.

Property (S3) ensures that if x_t is close (within distance ρ) to an existing vertex v, then it has a large enough probability of becoming a neighbor of v. This helps to ensure the connectivity of the subgraph induced by the vertices in $B_\rho = \{v \in S : |v - u| \leq \rho\}$. Property (S2) ensures that B_ρ contains enough vertices.

Theorem 1.3. *Suppose that $\alpha > 2$ and F is smooth for some constant $\mu > 0$ and $m \geq K \ln n$ for K sufficiently large. Then whp*

(a) G_n is connected.

(b) G_n has diameter $O(\ln n/\rho)$.

We now consider conditons under which G_n is an expander.

Let F be *tame* if there exist absolute constants C_1, C_2 such that

(T1) $F(x) \geq C_1$ for $0 < x \leq \pi$,

(T2) $I \leq C_2$.

The *conductance* Φ of G_n is defined by

$$\Phi = \min_{\deg_n(K) \leq mn} \Phi(K) = \min_{\deg_n(K) \leq mn} \frac{|E(K : \bar{K})|}{\deg_n(K)}.$$

Theorem 1.4. *If $\alpha > 2$ and F is tame and $m \geq K \ln n$ for K sufficiently large, then whp*

(a) G_n has conductance bounded from below by a constant,

(b) G_n is connected,

(c) G_n has diameter $O(\log_m n)$.

1.2. Canonical Functions

Where possible, we will illustrate our theorems using the canonical functions:

$$\begin{aligned} F_0(u) &= 1_{|u| \leq r}, & r \geq n^{\epsilon - 1/2}; \\ F_1(u) &= \frac{1}{\max\{n^{-\delta}, u\}^\beta}, & \text{where } \delta < 1/2; \\ F_2(u) &= e^{-\beta u}, & \beta = \beta(n) \geq 0. \end{aligned}$$

Function F_0 forces the neighbors of a vertex to be within r of that vertex. Function F_1 tries to smooth this out in a "polynomial" fashion, discouraging long-range attachments. As we will see, there is a qualitative difference between the cases $\beta < 2$ and $\beta > 2$, the latter behaving more like F_0 with respect to being an expander. Function F_2 tries to discourage long-range attachments using an "exponential" smoothing.

Notice also that F_0 corresponds to the model presented in [Flaxman et al. 06]. Also notice that without the $n^{-\delta}$ term in the definition of F_1 for $\beta \geq 2$, we would have $I = \infty$. One can justify its inclusion (for some value of δ) from the fact that whp the minimum distance between the points in V_n is greater than $1/n \ln n$.

Observe that

$$\begin{aligned} I_z(F_0) &= \frac{1}{2}(1 - \cos(\min\{z, r\})), \\ I_z(F_1) &= \begin{cases} \frac{\beta n^{\delta(\beta-2)}}{4(\beta-2)} + O(n^{(\beta-4)\delta} + z^{2-\beta}) & z \geq n^{-\delta},\ \beta > 2 \\ \Theta(z^{2-\beta}) + O(n^{(\beta-2)\delta}) & z \geq n^{-\delta},\ \beta < 2 \\ \ln(n^{\delta} z) + O(1) & z \geq n^{-\delta},\ \beta = 2, \end{cases} \\ I_z(F_2) &= \frac{1}{2(1+\beta^2)}(1 - e^{-\beta z}(\cos z + \beta \sin z)). \end{aligned}$$

For part (a) of Theorem 1.1,

$$F = F_0\text{:}\ \theta = 1 - 2\epsilon.$$

$$F = F_1,\ \beta > 2\text{:}\ \theta = 2\delta.$$

$$F = F_1,\ \beta < 2\text{:}\ \theta = 0.$$

$$F = F_1,\ \beta = 2\text{:}\ \theta = 2\delta.$$

$$F = F_2\text{:}\ \theta = 0.$$

For part (b) of Theorem 1.1,

$$F = F_0\text{:}\ \eta = r,\ \varphi = 0.$$

$$F = F_1,\ \beta > 2\text{:}\ \eta = n^{-\delta/2},\ \varphi = O(n^{-(\beta-2)\delta/2}).$$

$$F = F_1,\ \beta = 2\text{:}\ \eta = \tfrac{\ln \ln n}{\ln n},\ \varphi = O(\eta).$$

As we will see, in Theorem 1.4, $F = F_1, \beta < 2$ does not fit the hypotheses of part (b) of this theorem.

For Theorem 1.4,

$$F = F_0\text{:}\ I \sim r^2/4, \text{ so we can take } \mu \sim 1/4,\ \rho = r/2,\ c_3 \sim 1.$$

$$F = F_1,\ \beta > 2\text{:}\ I \sim \tfrac{n^{\delta(\beta-2)}}{2(\beta-2)}, \text{ so we can take } \mu \sim 1/4,\ \rho = n^{-\delta}/2,\ c_3 \sim (\beta-2)/2.$$

$$F = F_1,\ \beta < 2\text{:}\ I = \Theta(1), \text{ and we can take } \rho = 1,\ \mu = \Omega(1),\ c_3 = \Omega(1).$$

$$F = F_2\text{:}\ I = \Theta(1), \text{ and we can take } \rho = 1,\ \mu = \Omega(1),\ c_3 = \Omega(1).$$

We have a problem fitting the case of F_1 with $\beta = 2$ into the theorem.

Mihail described empirical results on the conductance of G_n in the case where $F = F_1$ [Mihail 06]. He observed poor conductance when $\beta < 2$ and good conductance when $\beta > 2$. This fits nicely with the results of Theorems 1.3 and 1.4.

We note that F_1 with $\beta < 2$ is tame since $F_1(x) \geq \pi^{-\beta}$ for $0 \leq \pi$ and

$$I = \frac{1}{2}\int_{x=0}^{\pi} \sin x x^{-\beta} dx \leq \frac{\pi^{2-\beta}}{2(2-\beta)}.$$

1.3. Open Question: The Role of α

This parameter was introduced in [Flaxman et al. 06] as a means of overcoming a difficult technical problem. When $\alpha > 2$, it facilitates a proof of Lemma 3.2. On the positive side, it does give a parameter that effects the power law. On the negative side, when $\alpha > 2$, there will whp be isolated vertices, unless we make m grow at least as fast as $\ln n$. It is for us, an interesting open question, as to how to prove our results with $\alpha = 0$.

2. Outline of the Paper

We prove a likely power law for the degree sequence in Section 3. We follow a standard practise and prove a recurrence for the expected number of vertices of degree k at time step t. Unfortunately, this involves the estimation of the expectation of the reciprocal of a random variable and to handle this, we show that this random variable is concentrated. This is quite technical and is done in Section 3.2. In Section 4 we show that under the assumptions of Theorem 1.1(b), there are small separators. This is relatively easy, since any given great circle can whp be used to define a small separator.

Section 5 proves connectivity when m grows logarithmically with n. The idea is to show that whp the subgraph $G_n(B)$, induced by a ball $B_\rho(u)$ of radius ρ

centered in $u \in S$, is connected and has small diameter. We then show that the union of the $G_n(B)$'s for $u = x_1, x_2, \ldots, x_n$ is connected and has small diameter.

Section 6 deals with the case of tame functions and shows that whp they give rise to graphs with good expansion properties.

3. Proving a Power Law

3.1. Establishing a Recurrence for $\overline{\mathbf{d}}_{\mathbf{k}}(\mathbf{t})$

Our approach to proving Theorem 1.1(a) is to find a recurrence for $\overline{d}_k(t)$, the expected number of vertices of degree k at time t. For $k \in \mathbf{N}$, define $D_k(t) = \{v \in V_t : \deg_t(v) = k\}$. Thus, $d_k(t) = |D_k(t)|$. Also, define $d_{m-1}(t) = 0$ and $\overline{d}_{m-1}(t) = 0$ for all integers t with $t > 0$. Let $\eta_k(G_t, x_{t+1})$ denote the (conditional) probability that a parallel edge from x_{t+1} to a vertex of degree no more than k is created at time $t+1$. Then,

$$\eta_k(G_t, x_{t+1}) = O\left(\min\left\{\sum_{i=m}^{k}\sum_{v\in D_i(t)} \frac{F(|x_{t+1}-v|)^2\, i^2}{\max\{\alpha m I t, T_t(x_{t+1})\}^2}, 1\right\}\right). \tag{3.1}$$

We remark that the assumption in Equation (1.1) shows that η_k is a small quantity; see Equation (3.7).

Then for $k \geq m$,

$$\begin{aligned} E[d_k(t+1) \mid G_t, x_{t+1}] = d_k(t) & \\ &+ m \sum_{v\in D_{k-1}(t)} \frac{(k-1)F(|x_{t+1}-v|)}{\max\{\alpha m I t, T_t(x_{t+1})\}} \\ &- m \sum_{v\in D_k(t)} \frac{kF(|x_{t+1}-v|)}{\max\{\alpha m I t, T_t(x_{t+1})\}} \\ &+ \Pr[deg_{t+1}(x_{t+1} = k) \mid G_t, x_{t+1}] \\ &+ O(m\eta_k(G_t, x_{t+1})). \end{aligned} \tag{3.2}$$

Let $\mathcal{A}_t$ be the event

$$\{|T_t(x_{t+1}) - 2mIt| \leq C_1 I m t^{\gamma} \ln n\},$$

where

$$\max\{2/\alpha, \theta\} < \gamma < 1$$

and C_1 is some sufficiently large constant.

Note that if

$$t \geq t_0 = (\ln n)^{2/(1-\gamma)}, \tag{3.3}$$

then

$$\mathcal{A}_t \text{ implies } T_t(x_{t+1}) \leq \alpha m I t.$$

Then, for $t \geq t_0$,

$$\begin{aligned}
&E\left[\sum_{v\in D_k(t)} \frac{kF(|x_{t+1}-v|)}{\max\{\alpha m I t, T_t(x_{t+1})\}}\right] \\
&= E\left[\sum_{v\in D_k(t)} \frac{kF(|x_{t+1}-v|)}{\max\{\alpha m I t, T_t(x_{t+1})\}} \;\middle|\; \mathcal{A}_t\right] \Pr[\mathcal{A}_t] \\
&\quad + E\left[\sum_{v\in D_k(t)} \frac{kF(|x_{t+1}-v|)}{\max\{\alpha m I t, T_t(x_{t+1})\}} \;\middle|\; \neg\mathcal{A}_t\right] \Pr[\neg\mathcal{A}_t] \\
&= \frac{k}{\alpha m t} E[d_k(t)|\mathcal{A}_t]\Pr[\mathcal{A}_t] + O(1)\Pr[\neg\mathcal{A}_t] \\
&= \frac{k\overline{d}_k(t)}{\alpha m t} - \frac{k}{\alpha m t} E[d_k(t)|\neg\mathcal{A}_t]\Pr[\neg\mathcal{A}_t] + O(1)\Pr[\neg\mathcal{A}_t] \\
&= \frac{k\overline{d}_k(t)}{\alpha m t} + O(k)\Pr[\neg\mathcal{A}_t].
\end{aligned}$$

In Lemma 3.2 below we prove that

$$\Pr[\neg\mathcal{A}_t] = O\left(n^{-2}\right). \tag{3.4}$$

Thus, if $t \geq t_0$ then

$$E\left[\sum_{v\in D_k(t)} \frac{kF(|x_{t+1}-v|)}{\max\{\alpha m I t, T_t(x_{t+1})\}}\right] = \frac{k\overline{d}_k(t)}{\alpha m t} + O\left(k/n^2\right). \tag{3.5}$$

In a similar way,

$$E\left[\sum_{v\in D_{k-1}(t)} \frac{(k-1)F(|x_{t+1}-v|)}{\max\{\alpha m I t, T_t(x_{t+1})\}}\right] = \frac{(k-1)\overline{d}_{k-1}(t)}{\alpha m t} + O\left(k/n^2\right). \tag{3.6}$$

On the other hand, given G_t, x_{t+1}, if

$$p = 1 - \frac{T_t(x_{t+1})}{\max\left(T_t(x_{t+1}), \alpha m I t\right)},$$

then

$$\Pr[\deg_{t+1}(x_{t+1}) = k \mid G_t, x_{t+1}] = \Pr[\mathrm{Bi}(m,p) = k-m].$$

So, if $t \geq t_0$,

$$\begin{aligned}\Pr[\deg_{t+1}(x_{t+1} = k)] &= \binom{m}{k-m} E\left[p^{k-m}(1-p)^{2m-k} \,\middle|\, \mathcal{A}_t\right] \\ &\quad \times \Pr[\mathcal{A}_t] + O(\Pr[\neg\mathcal{A}_t)] \\ &= \binom{m}{k-m}\left(1-\frac{2}{\alpha}\right)^{k-m}\left(\frac{2}{\alpha}\right)^{2m-k} \\ &\quad \times (1+O(mt^{\gamma-1}\ln n))\Pr[\mathcal{A}_t] + O(n^{-2}) \\ &= \binom{m}{k-m}\left(1-\frac{2}{\alpha}\right)^{k-m}\left(\frac{2}{\alpha}\right)^{2m-k} + O(mt^{\gamma-1}\ln n).\end{aligned}$$

Now note that from Equations (3.1) and (3.4) that if

$$t \geq t_1 = n^{(\gamma+\theta)/2\gamma}$$

and

$$k \leq k_0(t) = n^{(\gamma-\theta)/4},$$

then, from (1.1), we see that

$$E(m\eta_k(G_t, x_{t+1})) = O\left(\frac{k^2 n^\theta}{mt}\right) = O(t^{\gamma-1}). \tag{3.7}$$

Taking expectations on both sides of (3.2) and using (3.5), (3.6), and (3.7), we see that if $t \geq t_0$ and $k \leq k_0(t)$, then

$$\begin{aligned}\overline{d}_k(t+1) &= \overline{d}_k(t) + \frac{k-1}{\alpha t}\overline{d}_{k-1}(t) - \frac{k}{\alpha t}\overline{d}_k(t) \\ &\quad + \binom{m}{k-m}\left(1-\frac{2}{\alpha}\right)^{k-m}\left(\frac{2}{\alpha}\right)^{2m-k} + O\left(mt^{\gamma-1}\ln n\right)\end{aligned} \tag{3.8}$$

We consider the recurrence given by $f_{m-1} = 0$ and for $k \geq m$,

$$f_k = \frac{k-1}{\alpha} f_{k-1} - \frac{k}{\alpha} f_k + \binom{m}{k-m}\left(1-\frac{2}{\alpha}\right)^{k-m}\left(\frac{2}{\alpha}\right)^{2m-k}, \tag{3.9}$$

which, for $k > 2m$, has solution

$$\begin{aligned} f_k &= f_{2m} \prod_{i=2m+1}^{k} \frac{i-1}{i+\alpha} \\ &= f_{2m} e^{\varphi_k(m,\alpha)} \left(\frac{m}{k}\right)^{\alpha+1}. \end{aligned} \tag{3.10}$$

Here, $\varphi_k(m,\alpha) = O(1)$ tends to a limit $\varphi_\infty(m,\alpha)$ depending only on m, α as $k \to \infty$. Furthermore, $\lim_{m\to\infty} \varphi_\infty(\alpha, m) = 0$. We also have

$$f_{m+i} = f_{2m} \prod_{j=i+1}^{m} \left(1 + \frac{\alpha+1}{m+j-1}\right) \le e^{2\alpha+3} f_{2m}.$$

It follows that (3.10) is also valid for $m \le k \le 2m$ with $\varphi_k(m,\alpha) = O(1)$.

We finish the proof of (1.2) by showing that there exists a constant $M > 0$ such that

$$|\overline{d}_k(t) - f_k t| \le M(t_1 + m t^\gamma \ln n) \tag{3.11}$$

for all $0 \le t \le n$ and $m \le k \le k_0(t)$.

We have that (3.11) is trivially true for $t < t_1$, and for $t \ge t_1$ and $k > k_0(t)$ it follows from $\overline{d}_k(t) \le 2mt/k$.

Now, let $\Psi_k(t) = \overline{d}_k(t) - f_k t$. Then, for $t \ge t_1$ and $m \le k \le k_0(t)$,

$$\Psi_k(t+1) = \frac{k-1}{\alpha t} \Psi_{k-1}(t) - \frac{k}{\alpha t} \Psi_k(t) + O(m t^{\gamma-1} \ln n). \tag{3.12}$$

Let L denote the hidden constant in $O(m t^{\gamma-1} \ln n)$ of (3.12). Our inductive hypothesis $\mathcal{H}_t$ is that

$$|\Psi_k(t)| \le M(t_1 + m t^\gamma \ln n)$$

for every $m \le k \le k_0(t)$ and M sufficiently large. Assume that $t \ge t_1$. Then, $k \ll t$ in the current range of interest, and so from (3.12),

$$\begin{aligned} |\Psi_k(t+1)| &\le M(t_1 + m t^\gamma \ln n) + L m t^{\gamma-1} \ln n \\ &\le M(t_1 + m(t+1)^\gamma \ln n). \end{aligned}$$

This verifies $\mathcal{H}_{t+1}$ and completes the proof by induction.

3.2. Concentration of $\mathbf{T_t(u)}$

Now we turn our attention to prove that $T_t(u)$ is concentrated around its mean.

Lemma 3.1. *Let $u \in S$ and $t > 0$. Then, $E[T_t(u)] = 2Imt$.*

Proof.

$$ET_t(u) = E\left[\sum_{v\in V_t} \deg_t(v)F(|u-v|)\right] = I\sum_{v\in V_t}\deg_t(v) = 2Imt.$$

□

Lemma 3.2. *If $t > 0$ and u is chosen randomly from S, then*

$$\Pr[|T_t(u) - 2Imt| \geq mI(t^{2/\alpha} + t^{1/2}\ln t)\ln n] = O\left(n^{-2}\right).$$

Proof. We use the Azuma-Hoeffding inequality (see, for example, [Alon and Spencer 00]). One may be a little concerned here that our probability space is not discrete. Although it is not really necessary, one could replace S by 2^{2^n} randomly chosen points X and sample uniformly from these. Then, whp the change in distribution would be negligible. With this re-assurance, fix τ, with $1 \leq \tau < t$. Fix G_τ, and let $G_t = G_t(G_\tau, x_{\tau+1}, y_1, \ldots, y_m)$ and $\hat{G}_t = G_t(G_\tau, \hat{x}_{\tau+1}, \hat{y}_1, \ldots, \hat{y}_m)$, where $x_{\tau+1}, \hat{x}_{\tau+1} \in S$ and $y_1, \ldots, y_m, \hat{y}_1, \ldots, \hat{y}_m \in V_\tau$. We couple the construction of G_t and $\hat{G}_t$, starting at time step $\tau + 1$ with the graphs G_τ and $\hat{G}_\tau$, respectively. Then, for every step $\sigma > \tau + 1$, we choose the same point $x_\sigma \in S$ in both, and for every $i = 1, \ldots, m$, we choose $u_i, \hat{u}_i \in V_\sigma$ such that each marginal is the correct marginal and such that the probability of choosing the same vertex is maximized.

Notice that we have

$$\Pr[u_i = v = \hat{u}_i] = \min\left(\frac{\deg_{G_{\sigma-1}}(v)F(|v - x_\sigma|)}{\max\left(T_{\sigma-1}(x_\sigma), \alpha mI(\sigma-1)\right)}, \frac{\deg_{\hat{G}_{\sigma-1}}(v)F(|v - x_\sigma|)}{\max\left(\hat{T}_{\sigma-1}(x_\sigma), \alpha mI(\sigma-1)\right)}\right)$$

for every $v \in V_{\sigma-1}$. Also,

$$\Pr[u_i = x_\sigma = \hat{u}_i] = 1 - \max\left(\frac{T_{\sigma-1}(x_\sigma)}{\max\left(T_{\sigma-1}(x_\sigma), \alpha mI(\sigma-1)\right)}, \frac{\hat{T}_{\sigma-1}(x_\sigma)}{\max\left(\hat{T}_{\sigma-1}(x_\sigma), \alpha mI(\sigma-1)\right)}\right)$$

Now, for $u \in S$ let

$$\Delta_\sigma(u) := \Delta_{\sigma,\tau}(u) = \sum_{\rho=\tau}^{\sigma}\sum_{i=1}^{m} |F(|u - u_i^\rho|) - F(|u - \hat{u}_i^\rho|)|.$$

Before applying the Azuma-Hoeffding inequality, we prove the following lemma.

Lemma 3.3. *Let $t \geq 1$ and let u be a random point in S. Then, for some constant $C > 0$,*

$$E[\Delta_t(u)] \leq CmI \left(\frac{t}{\tau}\right)^{2/\alpha}.$$

Proof of Lemma 3.3. We begin with

$$E[|F(|w-u_i^\rho|)-F(|w-\hat{u}_i^\rho|)|\big|u_i^j,\hat{u}_i^j : i=1,\dots,m,\ j=1,\dots,\sigma] \leq 2I1_{u_i^\rho \neq \hat{u}_i^\rho}.$$

Therefore, if we define for every $\tau < \sigma \leq t$

$$\Delta_\sigma = \sum_{\rho=\tau}^{\sigma}\sum_{i=1}^{m} 1_{u_i^\sigma \neq \hat{u}_i^\sigma},$$

we have

$$E[\Delta_\sigma(u)] \leq 2IE[\Delta_\sigma].$$

Fix $\tau < \sigma \leq t$. We then have

$$\Delta_\sigma = \Delta_{\sigma-1} + \sum_{i=1}^{m} 1_{u_i^\sigma \neq \hat{u}_i^\sigma}. \tag{3.13}$$

Now fix $1 \leq i \leq m$. Taking expectations with respect to our coupling,

$$E[1_{u_i^\sigma \neq \hat{u}_i^\sigma}|G_{\sigma-1},\hat{G}_{\sigma-1},x_\sigma] = 1 - \Pr[u_i^\sigma = \hat{u}_i^\sigma|G_{\sigma-1},\hat{G}_{\sigma-1},x_\sigma]$$

$$= \max\left(\frac{T_{\sigma-1}(x_\sigma)}{\max\left(T_{\sigma-1}(x_\sigma),\alpha mI(\sigma-1)\right)}, \frac{\hat{T}_{\sigma-1}(x_\sigma)}{\max\left(\hat{T}_{\sigma-1}(x_\sigma),\alpha mI(\sigma-1)\right)}\right)$$

$$- \sum_{v\in V_{\sigma-1}} \min\left(\frac{\deg_{G_{\sigma-1}}(v)F(|v-x_\sigma|)}{\max\left(T_{\sigma-1}(x_\sigma),\alpha mI(\sigma-1)\right)}, \frac{\deg_{\hat{G}_{\sigma-1}}(v)F(|v-x_\sigma|)}{\max\left(\hat{T}_{\sigma-1}(x_\sigma),\alpha mI(\sigma-1)\right)}\right)$$

$$\leq \frac{\max\left(T_{\sigma-1}(x_\sigma),\hat{T}_{\sigma-1}(x_\sigma)\right) - \sum_{v\in V_{\sigma-1}}\min\left(\deg_{G_{\sigma-1}}(v),\deg_{\hat{G}_{\sigma-1}}(v)\right)F(|v-x_\sigma|)}{\max\left(T_{\sigma-1}(x_\sigma),\hat{T}_{\sigma-1}(x_\sigma),\alpha mI(\sigma-1)\right)} \tag{3.14}$$

$$\leq \frac{\sum_{v\in V_{\sigma-1}}|\deg_{G_{\sigma-1}}(v)-\deg_{\hat{G}_{\sigma-1}}(v)|F(|v-x_\sigma|)}{\max\left(T_{\sigma-1}(x_\sigma),\hat{T}_{\sigma-1}(x_\sigma),\alpha mI(\sigma-1)\right)} \tag{3.15}$$

$$\leq \frac{\sum_{v\in V_{\sigma-1}}|\deg_{G_{\sigma-1}}(v)-\deg_{\hat{G}_{\sigma-1}}(v)|F(|v-x_\sigma|)}{\alpha mI(\sigma-1)}.$$

Inequality (3.14) follows from

$$\max\left(\frac{a}{\max(a,c)}, \frac{b}{\max(b,c)}\right) = \frac{\max(a,b)}{\max(a,b,c)}$$

and

$$\min\left(\frac{a}{b}, \frac{c}{d}\right) \geq \frac{\min(a,c)}{\max(b,d)}.$$

Inequality (3.15) is a consequence of

$$\max\{\sum_i a_i, \sum_i b_i\} - \sum_i \min\{a_i, b_i\} \leq \sum_i |a_i - b_i|.$$

Therefore,

$$E\left[\Delta_\sigma \;\middle|\; G_{\sigma-1}, \hat{G}_{\sigma-1}\right] \leq \Delta_{\sigma-1} + \frac{\sum_{v\in V_{\sigma-1}} |\deg_{G_{\sigma-1}}(v) - \deg_{\hat{G}_{\sigma-1}}(v)|}{\alpha(\sigma-1)}. \quad (3.16)$$

But, for each $v \in V_{\sigma-1}$, we have

$$|\deg_{G_{\sigma-1}}(v) - \deg_{\hat{G}_{\sigma-1}}(v)| \leq \sum_{j=\tau}^{\sigma-1}\sum_{i=1}^{m}(1_{u_i^j=v,\, \hat{u}_i^j\neq v} + 1_{u_i^j\neq v,\, \hat{u}_i^j=v})$$

and thus

$$\sum_{v\in V_{\sigma-1}} |\deg_{G_{\sigma-1}}(v) - \deg_{\hat{G}_{\sigma-1}}(v)| \leq$$

$$\sum_{j=\tau}^{\sigma-1}\sum_{i=1}^{m}\sum_{v\in V_{\sigma-1}}\left(1_{u_i^j=v,\, \hat{u}_i^j\neq v} + 1_{u_i^j\neq v,\, \hat{u}_i^j=v}\right) \leq 2\Delta_{\sigma-1}.$$

Going back to (3.16) we have

$$E[\Delta_\sigma] \leq E[\Delta_{\sigma-1}]\left(1 + \frac{2}{\alpha(\sigma-1)}\right),$$

so, $E[\Delta_t] \leq e^{O(1)}\left(\frac{t}{\tau}\right)^{2/\alpha} E[\Delta_\tau]$. Now, $\Delta_\tau \leq m$, because the graphs G_τ and $\hat{G}_\tau$ differ at most in the last m edges. Therefore, $E[\Delta_t] \leq e^{O(1)} m \left(\frac{t}{\tau}\right)^{2/\alpha}$. □

To finally apply the Azuma-Hoeffding inequality, we note first that

$$\left|E_{G_t}[T_t(u)] - Ex_{\hat{G}_t}[T_t(u)]\right| = \left|E\left[\sum_{\rho=\tau}^{t}\sum_{i=1}^{m}(F(|u-u_i^\rho|) - F(|u-\hat{u}_i^\rho|))\right]\right|$$

$$\leq E[\Delta_t(u)], \quad (3.17)$$

and from Lemma 3.3

$$\sum_{\tau=1}^{t} E[\Delta_t(u)]^2 \le (e^{O(1)}mI)^2 t^{4/\alpha} \sum_{\tau=1}^{t} \tau^{-4/\alpha} = O\left(I^2m^2(t\ln t + t^{4/\alpha})\right).$$

Therefore, there is C_1 such that

$$\Pr\left[|T_t(u) - E[T_t(u)]| \ge C_1 Im(t^{2/\alpha} + t^{1/2}\ln t)(\ln n)^{1/2}\right] \le e^{-2\ln n} = n^{-2}.$$

□

3.3. Concentration of $d_k(t)$

We follow the proof of Lemma 3, replacing $T_t(u)$ by $d_k(t)$ and using the same coupling. When we reach (3.17) we find that $\left|E_{G_t}[d_k(t)] - E_{\hat{G}_t}[d_k(t)]\right| \le 2\mathbf{E}[\Delta_t]$; the rest is the same.

This proves (1.1) and completes the proof of Theorem 1.1(a) .

4. Small Separators

In this section we prove Theorem 1.1(b). For this, we assume $\alpha > 0$ and $m_0 \le m$, where m_0 is a sufficiently large constant and $\varphi, \eta = o(1)$ such that $\eta n \to \infty$ and $J_\eta \le \varphi I$.

We use the geometry of the instance to obtain a sparse cut. Consider partitioning the vertices in V_n using a great circle of S. This will divide V_n into sets T and $\bar{T}$, which each contain about $n/2$ vertices. More precisely, we have

$$\Pr\left[|T| < (1-\xi)n/2\right] = \Pr\left[|\bar{T}| < (1-\xi)n/2\right] \le e^{-\xi^2 n/4}.$$

To bound $e(T, \bar{T})$, the number of edges crossing the cut, we divide the edges into two types. We call an edge $\{u, v\}$ in G_n *long* if $|u - v| \ge \eta$; otherwise we call it *short*. We will show that whp the number of long edges is small, and therefore we just need to consider short edges in a cut. Let Z denote the number of long edges. Then,

$$\begin{aligned} E[Z] &\le mt_0 + m\sum_{t\ge t_0}\sum_{v\in V_t}\frac{\deg_t(v)J_\eta}{\alpha m I t} \\ &\le mt_0 + m\sum_{t\ge t_0}\frac{J_\eta}{\alpha I} \\ &= mt_0 + O(mn\varphi). \end{aligned}$$

Now whp there are at most $E[Z]/\varphi^{1/2}$ long edges. Apart from these, edges only appear between vertices within distance η, so only edges incident with vertices appearing in the strip within distance η of the great circle can appear in the cut. Since $\eta = o(1)$, this strip has area less than $3\eta\sqrt{\pi}$, and, letting U denote the vertices appearing in this strip, we have

$$\Pr\left[|U| \geq 4\sqrt{\pi}\eta n\right] \leq e^{-\sqrt{\pi}\eta n/9} = o(1).$$

Even if every one of the vertices chooses its m neighbors on the opposite side of the cut, this will yield at most $4\sqrt{\pi}\eta nm$ edges whp. So the graph has a cut with

$$e(T, \bar{T}) = \tilde{O}\big((\eta + \varphi^{1/2})mn\big)$$

with probability at least $1 - o(1)$.

5. Connectivity and Diameter

Here we prove Theorem 1.3. Let μ be such that F is smooth for μ, and let $\rho = \rho(\mu)$. Fix $u \in S$. Let

$$B_\rho = \{v \in S : \ |v - u| \leq \rho\},$$

and let $A_\rho = \int_{v \in B_\rho} dv \in [c_1\rho^2, c_2\rho^2]$ denote the area of B_r. Here, c_1, c_2 are some absolute constants, independent of ρ.

We denote the diameter of G by $\operatorname{diam}(G)$, and follow the convention of defining $\operatorname{diam}(G) = \infty$, when G is disconnected. In particular, when we say that a graph has finite diameter, this implies it is connected.

Let

$$T = \frac{K_1 \ln n}{A_\rho} \leq \frac{K_1 n}{c_1 L},$$

where L is as in Property (S2), K_1 is sufficiently large, and $L^{2/3} \ll K_1 \ll K, L$.

Lemma 5.1.

$$\Pr\left[\operatorname{diam}(G_n(B_\rho)) \geq 2(K_1 + 1)\ln n\right] = O(n^{-1}),$$

where $G_n(B_\rho)$ is the induced subgraph of G_n in B_ρ.

Proof. Let $N = |G_n(B_\rho)|$ and let $V(G_n(B_\rho)) = \{x_{t_1}, \dots, x_{t_N}\}$, where $t_s < t_{s+1}$ for all $s < N$ and $t_N \leq n$. For $s = 1, \dots, N$, let $H_s = G_{t_s}(B_\rho)$. We concentrate our attention to the evolution of H_s.

Notice that s is the number of steps for which $x_t \in B_\rho$ with $t \leq t_s$, and so $s \sim \mathrm{Bi}(t_s, A_\rho)$. By the Chernoff bound we have that if $t_s \geq T$, then

$$\Pr\left[\frac{1}{2} < \frac{t_s A_\rho}{s} < \frac{3}{2}\right] \geq 1 - n^{-K_1/13}.$$

Therefore, if N_0 is the number of vertices in B_ρ at time T, we may assume for all $s \geq N_0$, $s/2 < t_s A_\rho < 3s/2$. In particular, $N \geq 2nA_\rho/3 \geq c_1 L \ln n/2$ and $N_0 \leq 2TA_\rho \leq 2K_1 \ln n$.

Let X_s be the number of connected components of H_s. Then,

$$X_{s+1} = X_s - Y_s + 1, \qquad X_0 = 0,$$

where $Y_s \geq 0$ is the number of components conected to x_{t_s}.

The ball B_ρ is contained in $B_{2\rho}(x_{t_s})$, the ball of radius 2ρ centered at x_{t_s}. Therefore, if $v \in B_\rho \cap V_{t_s}$ and $t_s > T$,

$$\begin{aligned}\Pr[x_{t_s} \text{ chooses } v] &\geq \frac{\deg_{t_s}(v) F(|x_{t_s} - v|)}{\alpha m I t_s} \geq \frac{F(2\rho)}{\alpha I t_s} \\ &\geq \frac{2A_\rho F(2\rho)}{3\alpha I s} \geq \frac{2c_1 \rho^2 F(2\rho)}{3\alpha I s} \geq \frac{2c_1 c_3}{3\alpha s}.\end{aligned}$$

The last inequality uses Property (S3).

Now, we can bound the probability of generating a new component,

$$\begin{aligned}\Pr\left[Y_s = 0 | H_{s-1}\right] &= \left(1 - \sum_{v \in H_{s-1}} \Pr\left[x_{t_s} \text{ chooses } v\right]\right)^m \\ &\leq \left(1 - \frac{2c_1 c_3}{3\alpha}\right)^m \leq \exp\left(-\frac{2c_1 c_3 m}{3\alpha}\right) \leq n^{-10}.\end{aligned}$$

If $s < 2K_1 \ln n$, as $m \geq K \ln n$, we can bound the probability of not collapsing components,

$$\begin{aligned}\Pr\left[Y_s = 1 | X_s \geq 2\right] &\leq \Pr\left[Y_s = 1 | X_s \geq 2,\, Y_s > 0\right] + \Pr\left[Y_s = 0 | X_s \geq 2\right] \\ &\leq 2\left(1 - \frac{2c_1 c_3}{3\alpha s}\right)^m + n^{-10} \\ &\leq 2\exp\left(-\frac{2mc_1 c_3}{3\alpha s}\right) + n^{-10} \leq 1/10\end{aligned}$$

Therefore, X_s is stochastically dominated by the random variable $\max\{1, N_0 - Z_s\}$ where $Z_s \sim \mathrm{Bi}(s, 9/10)$. We then have

$$\Pr[X_{4K_1 \ln n} > 1] \leq \Pr[Z_{4K_1 \ln n} < N_0] \leq \Pr[Z_{4K_1 \ln n} < 2K_1 \ln n \leq n^{-3}].$$

And therefore

$$\Pr[H_{4K_1 \ln n} \text{ is not connected}] \leq n^{-3}.$$

Now, to obtain an upper bound on the diameter, we run the process of construction of H_N by rounds. The first round consists of $4K_1 \ln n$ steps and in each new round we double the size of the graph, i.e., it consists of as many steps as the total number of steps of all the previous rounds. Notice that we have less than $\log_2 n$ rounds in total. Let $\mathcal{A}$ be the event for all $i > 0$ that every vertex created in the $(i+1)$th round is adjacent to a vertex in $H_{2^{i+1}K_1 \ln n}$, the graph at the end of the ith round.

On the event $\mathcal{A}$, every vertex in H_N is at distance at most $\log_2 n$ of $H_{2K_1 \ln n}$, whose diameter is not greater than $2K_1 \ln n$. Thus, the diameter of H_N is smaller than $2(K_1 + 2) \ln n$.

Now, if v is created in the $(i+1)$st round,

$$\Pr[v \text{ is not adjacent to } H_{2^{i-1}K_1 \ln n}] \leq \left(1 - \frac{2c_1c_3}{3\alpha}\right)^m.$$

Therefore,

$$\Pr[\neg\mathcal{A}] \leq \left(1 - \frac{2c_1c_3}{3\alpha}\right)^m n(\ln n) \leq n^{1+o(1)-2Kc_1c_3/(3\alpha)}.$$

□

To finish the proof of connectivity and the diameter, let u, v be two vertices of G_n. Let $C_1, C_2, \ldots, C_M$, $M = O(1/\rho)$, be a sequence of spherical caps of radius ρ such that u is the center of C_1, v is the center of C_M, and the centers of C_i, C_{i+1} are distance $\leq \rho/2$ apart. The intersections of C_i, C_{i+1} have area at least $A_\rho/10$ and so whp each intersection contains a vertex. Using Lemma 5.1 we deduce that whp there is a path from u to v in G_n of size at most $O(\ln n/\rho)$.

6. Proof of Theorem 1.4

For a set $K \subseteq V_n$, we define $\deg_n(K) = \sum_{v \in K} \deg_n(v)$.

Lemma 6.1. *There is an absolute constant* $0 < \xi < 1/4$ *such that*

$$\Pr(\exists K \subseteq V_n, |K| \geq (1-\xi)n : \deg_n(K) \leq (1+\xi)mn) = o(n^{-3}).$$

Proof. Let ζ be a small positive constant, and divide V_n into approximately $1/\zeta$ sets $S_1, S_2, \ldots$ of size $s = \lceil \zeta n \rceil$ plus a set of $n - \lfloor 1/\zeta \rfloor s$, where $S_i = \{x_{(i-1)s+1}, \ldots, x_{is}\}$, $i = 1, 2, \ldots$. We put a high-probability upper bound on $\deg_n(S_1)$. Now consider the random variables $\beta_k, k = 2, \ldots,$ where $\beta_k = \deg_{\tau_k}(S_2 \cup \cdots S_k)/ms$, and $\tau_k = ks$. Now $\beta_2 \geq ms$ and conditional on the value of $\beta_k \geq (k-1)ms$,

$$\beta_{k+1} ms \text{ dominates } ms + \beta_k ms + \text{Bi}\left(ms, \frac{\beta_k \lambda}{2(k+1)}\right),$$

where $\lambda = C_1/C_2$ and C_1, C_2 are in the definition of tameness.

So, there exist constants γ_1, γ_2 (independent of ζ) such that

$$\Pr\left(\frac{\beta_{k+1}}{ms} \leq 1 + (1+\gamma_2)\frac{\beta_k}{ms}\right) \leq e^{-m\gamma_1 n}.$$

So, after some calculations, we find that with probability $1 - O(e^{-m\gamma_1 n})$,

$$\deg_n(V_n \setminus S_1) \geq ms(1+\gamma_2)\gamma_2^{-1}((1+\gamma_2)^{\lfloor 1/\zeta \rfloor - 3} - 1) \geq mn(1+\zeta/2)$$

for small enough ζ.

Now $\deg_n(S_1)$ dominates $\deg_n(L)$ for any set L of size $\lceil \zeta n \rceil$. So, if $m > 1/\gamma_1$, then the probability there is a set of size $\lceil \zeta n \rceil$ that has total degree exceeding $mn(1-\gamma_2)$ is exponentially small ($\leq \binom{n}{\lceil \zeta n \rceil} e^{-n}$)). In this case, every set K of size at least $n - \lceil \zeta n \rceil$ has total degree $\deg_n(K) \geq mn(1+\gamma_2/2)$, and the lemma follows by taking $\xi = \min\{\zeta, \gamma_2/2, 1/4\}$. □

We have to estimate $\Phi(K)$ for all K with $\deg_n(K) \leq mn$. The above lemma shows that we can restrict our attention to sets K with $|K| \leq (1-\xi)n$.

We now observe that for $K \subseteq V_n$,

$$\deg_n(K) = m|K| + |E(K : \bar{K})|$$

and so to bound $\Phi(K)$, it suffices to prove lower bounds $|E(K : \bar{K})| \geq \eta m |K|$ for some positive constant η.

Lemma 6.2. *If $m \geq C \ln n$, where C is sufficiently large, then there exists an absolute constant $\kappa > 0$ such that*

$$\Pr(\Phi(G_n) < \kappa) = O(n^{-3}).$$

Proof.

Case 1: $1 \leq |K| \leq A_0 n$.

Here A_0 is a sufficiently small constant. Let $K_1 = K \cap V_{n/2}$ and $K_2 = K \setminus K_1$. Let $W_1 = V_{n/2} \setminus K_1$ and $W_2 = V_n \setminus (V_{n/2} \cup K_2)$. The number of edges between K_1 and W_2 dominates $\mathrm{Bi}(m(n/2 - |K_2|), \lambda |K_1|/(\alpha n))$. This is because each edge chosen by $V_j, j \in W_2$ has probability at least $m\lambda|K_1|/(\alpha mn)$ of being in K_1. Similarly, the number of edges between K_2 and W_1 dominates $\mathrm{Bi}(m|K_2|, \lambda(n/2 - |K_1|)/(\alpha n))$. Thus $E[|E(K : \bar{K})|] \geq m\lambda|K|/(3\alpha)$, and so by Hoeffding's inequality we see that $|E(K : \bar{K})| \geq m\lambda|K|/(4\alpha)$ with probability $1 - e^{-cm\lambda|K|}$ for some constant $c = c(\alpha)$. Thus,

$$\Pr(\exists K,\, 1 \leq |K| \leq A_0 n,\, |E(K : \bar{K})| < m\lambda|K|/(4\alpha)) \leq \sum_{k=1}^{A_0 n} \binom{n}{k} e^{-cC\lambda k \ln n} = o(1)$$

if $C \geq 2/(c\lambda)$.

Case 2: $A_0 n \leq |K| \leq (1-\xi)n$.

Here, ξ is as in Lemma 6.1. Let K_1, K_2, W_1, W_2 be as in Case 1. Let $q = |K_1|$ and $r = |K_2|$. We calculate the expected number of edges $\mu(K_1, K_2)$ of $L = (K_2 \times W_1 \cup W_2 \times K_1)$ generated at steps τ, $n/2 \leq \tau \leq n$, which are directed into K. At step τ the number of such edges falling in L is an independent random variable with distribution dominating

$$1_{\tau \in W_2} \mathrm{Bi}\left(m, \frac{\lambda q}{\alpha \tau}\right) + 1_{\tau \in K_2} \mathrm{Bi}\left(m, \frac{\lambda(n/2 - q)}{\alpha \tau}\right).$$

Thus,

$$\begin{aligned} \mu(K_1, K_2) &\geq \frac{m\lambda q}{\alpha} \sum_{\tau \in W_2} \frac{1}{\tau} + \frac{m\lambda(n/2 - q)}{\alpha} \sum_{\tau \in K_2} \frac{1}{\tau} \\ &= \frac{m\lambda}{\alpha} \left((k - r) \sum_{\tau \in W_2} \frac{1}{\tau} + (n/2 - (k - r)) \sum_{\tau \in K_2} \frac{1}{\tau} \right). \end{aligned}$$

Let $\mu(k) = \min_{K_1, K_2} \mu(K_1, K_2)$. Then "somewhat crudely,"

$$\begin{aligned} \sum_{\tau \in W_2} \frac{1}{\tau} &\geq \ln \frac{n}{n/2 + r}, \\ \sum_{\tau \in K_2} \frac{1}{\tau} &\geq \ln \frac{n}{n - r}. \end{aligned}$$

Thus,

$$\mu(k) \geq \frac{m\lambda}{\alpha} \left((k - r) \ln \frac{2n}{n + 2r} + \left(\frac{n}{2} - (k - r) \right) \ln \frac{n}{n - r} \right).$$

Putting $k = \kappa n$ and $r = \rho n$, we see that

$$\mu(k) \geq \frac{\lambda mn}{\alpha} g(\kappa, \rho),$$

where

$$g(\kappa, \rho) = (\kappa - \rho) \ln \frac{2}{1+2\rho} + \left(\frac{1}{2} - \kappa + \rho\right) \ln \frac{1}{1-\rho}.$$

We put a lower bound on g:

$$\rho \leq \frac{\xi}{2} \text{ implies } \kappa - \rho \geq \frac{\xi}{2} \text{ and so } g(\kappa, \rho) \geq \frac{\xi}{2} \ln \frac{2}{1+\xi}.$$

So we can assume that $\rho \geq \xi/2$. Then,

$$\begin{aligned}
\kappa - \rho \leq \frac{1-\xi}{2} \quad &\text{implies} \quad g(\kappa, \rho) \geq \frac{\xi}{2} \ln \frac{2}{2-\xi}, \\
\kappa - \rho > \frac{1-\xi}{2} \text{ and } \rho \leq \frac{1-\xi}{2} \quad &\text{implies} \quad g(\kappa, \rho) \geq \frac{1-\xi}{2} \ln \frac{2}{2-\xi}, \\
\kappa - \rho > \frac{1-\xi}{2} \text{ and } \rho > \frac{1-\xi}{2} \quad &\text{implies} \quad \kappa > 1 - \xi.
\end{aligned}$$

We deduce that within our range of interest,

$$\mu(k) \geq \eta mn$$

for some absolute constant η.

Let Z be the number of edges generated within L, so that Z counts a subset of the edges between K and $\overline{K}$. Then,

$$\Pr\left(\exists K_1, K_2 \subseteq N : \; Z \leq \frac{1}{2}\eta mn\right) \leq 2^n e^{-\eta mn/8} \leq e^{-\eta mt/10} = o(1).$$

This completes the proof of Theorem 1.4(a). Part (b) is an immediate consequence of part (a).

To prove part (c), we need to prove some vertex expansion properties of G_n. So fix $K \subseteq V_n$ with $1 \leq |K| \leq A_0 n$, and go back to Case 1. We see that the number of neighbors of K_1 in W_2 dominates $B_1 = \text{Bi}(n/2 - |K_2|, 1 - (1 - \lambda|K_1|/(\alpha n))^m)$ and the number of neighbours of K_2 in W_1 dominates $B_2 = \text{Bi}(n/2 - |K_1|, 1 - (1 - \lambda/(\alpha n))^{m|K_2|})$. So, for $i = 1, 2$,

$$E[B_i] \geq \begin{cases} \frac{\lambda m|K_i|}{3\alpha} & \text{if } \frac{\lambda m|K_i|}{\alpha n} \leq \frac{1}{10} \\ \frac{n}{60} & \text{otherwise.} \end{cases}$$

Therefore, using the Chernoff bounds, we have

$$\Pr\left(\exists K,i:\ 1\le |K_i|\le \frac{\alpha n}{10\lambda m}\ \text{and}\ B_i\le \frac{\lambda m|K_i|}{6\alpha}\right) \le \sum_{k=1}^{\alpha n/(10\lambda m)} \binom{n}{k} e^{-\lambda mk/(24\alpha)} = o(1). \tag{6.1}$$

$$\Pr\left(\exists K,i:\ \frac{\alpha n}{10\lambda m}\le |K_i|\le A_0 n\ \text{and}\ B_i\le \frac{n}{120}\right) \le \sum_{k=1}^{A_0 n} \binom{n}{k} e^{-n/1000} = o(1). \tag{6.2}$$

Now fix $x, y \in V_n$. For $a = x, y$, let $S_{i,a} = \{z \in V_n : \text{dist}(a, z) = i\}$. Here, $\text{dist}(a, z)$ is the graph distance between a and z in G_n. It follows from (6.1) and (6.2) that there exists $j_a = O(\log_m n)$ such that $|S_{j,a}| \ge n/120$. It follows from the proof of Lemma 6.2 that if $|S_{j_a}| \le (1-\xi)n$, then $|E(S_{j_a} : \bar{S}_{j_a})| \ge \eta mn/120$. It follows that there exists $l_a \le 240/\eta$ such that $|S_{j_a+l_a}| \ge (1-\xi)n \ge 3n/4$. It follows that $S_{j_x+l_x} \cap S_{j_y+l_y} \ne \emptyset$ and $\text{dist}(x, y) \le j_x + j_y + l_x + l_y = O(\log_m n)$. This completes the proof of Theorem 1.4. □

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Abraham D. Flaxman, Institute for Health Metrics and Evaluation, University of Washington, 2301 5th Avenue, Suite 600, Seattle, WA 98121 (abie@u.washington.edu)

Alan M. Frieze, Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA, 15213 (alan@random.math.cmu.edu)

Juan Vera, Department of Management Sciences, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada (jvera@uwaterloo.ca)

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