A Geometric Theory of Thermal Stresses*

Arkadas Ozakin† Arash Yavari‡

30 November 2009

Abstract

In this paper we formulate a geometric theory of thermal stresses. Given a temperature distribution, we associate a Riemannian material manifold to the body, with a metric that explicitly depends on the temperature distribution. A change of temperature corresponds to a change of the material metric. In this sense, a temperature change is a concrete example of the so-called referential evolutions. We also make a concrete connection between our geometric point of view and the multiplicative decomposition of deformation gradient into thermal and elastic parts. We study the stress-free temperature distributions of the finite-deformation theory using curvature tensor of the material manifold. We find the zero-stress temperature distributions in nonlinear elasticity. Given an equilibrium configuration, we show that a change of the material manifold, i.e. a change of the material metric will change the equilibrium configuration. In the case of a temperature change, this means that given an equilibrium configuration for a given temperature distribution, a change of temperature will change the equilibrium configuration. We obtain the explicit form of the governing partial differential equations for this equilibrium change. We also show that geometric linearization of the present nonlinear theory leads to governing equations that are identical to those of the classical linear theory of thermal stresses.

Contents

1 Introduction **1**

2 The Material Metric and Non-Uniform Temperature Distributions **3**
2.1 Stress-Free Temperature Distributions . . . 4
2.2 Connection with Multiplicative Decomposition of Deformation Gradient . . . 9
2.3 Anisotropic Thermal Expansion . . . 14

3 Geometric Elasticity with Temperature Changes **14**

4 Linearized Theory of Thermal Stresses **18**

5 Conclusions **23**

A Differential Geometry and Classical Geometric Elasticity **26**
A.1 Absolute Parallelizable (AP) Geometry . . . 28
A.2 Geometric Elasticity . . . 29

1 Introduction

Classical elasticity theory quantifies the amount of stretch in a body by using a specific configuration as the reference configuration. The displacements as measured from the reference configuration and the strains associated with them are then used to get the stresses via constitutive relations. This viewpoint works nicely when

*To appear in the *Journal of Mathematical Physics*.
†Georgia Tech Research Institute, Atlanta, GA 30332.
‡School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332. E-mail: arash.yavari@ce.gatech.edu.

there is a relaxed, stress-free configuration that can be used as the reference configuration. However, this is not always the case. A body may have various sources of residual stresses, e.g. defects such as dislocations and disclinations, in which case there may not exist any stress-free configuration. One can observe the existence of these residual stresses by cutting pieces from the body when there are no external forces, and see that the pieces relax upon being cut. One can deal with these residual stresses in the classical theory [5] but we will use a different approach in this paper; that of geometric elasticity, and the notion of a space that describes the intrinsic properties of a body with a residual stress distribution.

Temperature enters free energy density as a state variable. In classical linear theory of thermal stresses [5], it is assumed that there exists a reference temperature T_0 at which the body is stress free. Free energy is then expanded about T_0 and only linear and quadratic terms are kept. The governing equations of this theory consist of those of linearized elasticity and heat conduction with some coupling terms. Given an equilibrium configuration of the body at temperature T, a change in temperature will change the equilibrium configuration due to the coupling terms. Similar ideas are used in the nonlinear theory by looking at thermal stresses as a coupled nonlinear elasticity/heat conduction problem.

In this paper, we study thermal stresses geometrically by considering a material manifold that explicitly depends on temperature. Material manifold is endowed with a Riemanian metric $\mathbf{G}$. We assume that given a reference temperature distribution T_0, when the body is unloaded in a Riemannian manifold $(\mathcal{B}, \mathbf{G}_0)$ it is stress free. Change of temperature changes the metric. For similar ideas in the case of dislocations see [4; 15; 16; 17; 18; 20; 19]. We should emphasize that here we assume that temperature distribution is given. A geometric formulation of the coupled elasticity/heat conduction will be discussed in a future communication.

As a motivation for our viewpoint, consider a piece from a thin, elastic spherical shell being forced to lie on a plane, e.g., by being squeezed between two flat surfaces. This constraint will induce stresses on the shell. Let us for the moment imagine that we are observing this shell from the two-dimensional viewpoint of the plane, ignoring the third dimension. When we cut pieces from the shell (all still forced to lie on the plane), we will observe that the former relax by a certain amount, demonstrating the "residual stress" on the body. The two-dimensional, planar viewpoint dictates that there is no stress-free configuration for this piece of material. However, if this same shell is "forced" to live on the surface of a sphere of appropriate radius, there will be a stress-free configuration, which may then be used as the reference configuration for measuring the amount of stretch, etc. The surface of the sphere is intrinsically two-dimensional, in the sense that it can be described without any reference to a third dimension, by using only two coordinates, and an intrinsic measure of distance, e.g. by using the spherical coordinates:

$$ds^2 = R^2(d\theta^2 + \sin^2\theta \; d\phi^2). \tag{1.1}$$

This suggests a generalization to three dimensions: given a body with residual stresses with no apparent stress-free configuration, could it be possible to find some abstract three-dimensional space in which it would be stress-free? As stated, this is a very general question, and the answer depends on the source of the residual stresses for the case at hand and the notion of "space" one is considering.

In this paper, we answer this question affirmatively for a specific source for residual stresses: that induced by the thermal expansion due to a non-uniform temperature distribution in a solid material. Our analysis is constructive; given a temperature distribution on a previously stress-free material, we construct a Riemannian manifold (to be defined shortly) in which the material under consideration with the given thermal profile would have a stress-free state. This construction is not of purely academic interest, since given the appropriate constitutive relations, it allows us to calculate the stresses in an elastic body in a given thermal setting, and is not restricted to a linear response. The appropriate way to quantify deformations for a body with residual stresses is by considering a map ("the configuration map") from the "material manifold" to the three-dimensional spatial space that the body lives in. The constitutive relations are then written in terms of this map.

In the remaining of the paper we study the effect of changes of the material manifold on equilibrium configuration. We do this in a general setting when the material manifold is Riemannian. We also consider the case where the change in metric is small enough so that a linear approximation would be enough. In the case of small changes in material manifold we obtain the governing equations of evolution of the equilibrium configuration as a function of changes of the material manifold.

Let us pause here and explain the main ideas and conclusions. Suppose we have a flat, two-dimensional elastic material on a plane. If we heat this material in a non-uniform way, it will tend to bend out from the plane, and take the shape of a curved surface. If we then force this curved surface to live in the flat plane by perhaps squeezing it between two flat, rigid surfaces, we will induce some stresses on it. This suggests that for a given non-uniform temperature distribution, there may be a curved, stress-free shape that an "originally flat" elastic material wants to take. When we force this material to live on a flat plane, we induce "thermal stresses". Of course in real life, we have three-dimensional elastic bodies, forever bound to live in flat, three-dimensional Euclidean space.

Here is the main idea: given a material metric $\mathbf{G}$ describing the stress-free state of an elastic material at constant temperature, and a spatial metric $\mathbf{g}$ (for simplicity, both of these metrics can be taken as flat, three-dimensional metrics), we claim that a non-uniform temperature distribution on the material should be represented as a change in the material metric $\mathbf{G}$. In particular, for a material with isotropic thermal properties, we claim that the required change in $\mathbf{G}$ is just a pointwise rescaling (i.e. multiplying the metric with a scalar function on the material manifold), and the way the scaling factor depends on the temperature is determined by the physical properties of the material. Non-isotropic thermal expansion will also be considered.

We believe this is an example where a change in the material manifold can be clearly understood conceptually. This may be very helpful in understanding the role of the change in material connection in defect mechanics, where one has to consider not only a change in the metric, but also the "torsion part" of the connection. In particular, we have the hope that an analogy with thermal expansion will help clarify the often encountered (and, in our opinion, confusing) discussions of decomposing the deformation gradient $\mathbf{F}$ into "elastic" and "plastic" parts. The analogue of a plastic deformation in our case is a change in the temperature, resulting in a change in the material connection. Such a change in the material connection may induce stresses on a previously stress-free configuration, and as a result, change the equilibrium configuration. The resultant stresses in the new equilibrium configuration should be explored with the "usual" geometric elasticity.

This paper is structured as follows. In §2 we motivate the connection between thermal stresses and changes in material metric. We do this by looking at the example of a two-dimensional disk and study the possibility of existence of a relaxed configuration embedded in the three-dimensional Euclidean space for an arbitrary radial temperature distribution. We also study the stress-free temperature distributions in nonlinear elasticity. In §3 we present the main ideas of a geometric formulation of thermal stresses. We study the effect of a change of material manifold on the equilibrium configuration. In §4 the geometric linearization of the nonlinear theory is presented. Conclusions are given in §5. To make the paper self contained, we briefly review the basic concepts of differentoial geometry, parralelizable manifolds, and geometric theory of elasticity in the appendix.

2 The Material Metric and Non-Uniform Temperature Distributions

Motivation. Suppose we start with a stress-free isotropic material with a uniform temperature distribution T_1 and free boundary conditions, and increase its temperature to T_2. The material will expand, and the original distance δL_1 between two neighboring points A and B in the solid body will increase to δL_2. The quantity $(\delta L_2 - \delta L_1)/\delta L_1$ turns out to be independent of the two points, i.e., the expansion is uniform. Note that

$$\frac{\delta L_2 - \delta L_1}{\delta L_1} = \alpha(T_2 - T_1), \tag{2.1}$$

where α is the coefficient of thermal expansion. Let us now assume that we use a Lagrangian coordinate system, X^1, X^2, X^3,[1]i.e., assume that the same material points have the same coordinates before and after the expansion. Then, the distance between the two points A and B is given approximately in terms of the metric tensor G_{ij} as follows:

$$\delta l \approx \sqrt{G_{ij}(X_B^i - X_A^i)(X_B^j - X_A^j)}, \tag{2.2}$$

where the components G_{ij} are evaluated at a point between X_A and X_B. This shows that G_{ij} should somehow depend on temperature. In other words, this suggests that, in this Lagrangian setting, we should be using different metric tensors for T_1 and T_2. Note that this relation between the Lagrangian coordinates and the material manifold works only because the material is in a relaxed, stress-free state in both temperatures. Otherwise, the distance that the material metric $\mathbf{G}$ measures would not be the spatial distance.

[1]The superscripts denote coordinate labels.

The thermal expansion coefficient. Let us now connect the above description of the material metric in terms of a temperature distribution to the thermal expansion coefficient used in the classical theory. We imagine a material with a nonuniform temperature distribution, for various values of the constant temperature. The thermal expansion coefficient is defined by looking at the equilibrium volume of the material at different temperatures. Let us therefore look at the volume element of a material at a given constant temperature. We assume that the material metric is given by

$$G_{IJ}(\mathbf{X},T) = H_{IJ}(\mathbf{X})e^{2\omega(T)}, \tag{2.3}$$

where H_{IJ} is independent of temperature and $T = T(\mathbf{X})$. The volume form associated with this metric is given as

$$dV(\mathbf{X},T) = \sqrt{\det|H_{IJ}|}\ e^{N\omega(T)}d^N X, \tag{2.4}$$

where $d^N X$ is shorthand for $dX^1 \wedge dX^2 \cdots \wedge dX^N$.[2] Differentiating with respect to T, we obtain

$$\frac{d}{dT}dV(\mathbf{X},T) = \sqrt{\det|H_{IJ}|}\ e^{N\omega(T)}d^N X\ \frac{Nd\omega(T)}{dT} = dV(\mathbf{X},T)\, N\, \frac{d\omega(T)}{dT}. \tag{2.5}$$

Thus, we can read-off the thermal expansion coefficient in terms of the temperature dependence of $\omega(T)$:[3]

$$\alpha(T) = \frac{d\omega(T)}{dT}\,. \tag{2.6}$$

Remark: Suppose $H_{IJ} = \delta_{IJ}$, i.e. the initial material manifold is Euclidean. In this case G_{IJ} is conformally flat. It is known that any 2-dimensional Riemannian manifold is conformally flat and the function ω is unique [2]. See the appendix for more discussions on this.

2.1 Stress-Free Temperature Distributions

Before we go into the general geometric theory of thermal stresses, we will first demonstrate an important application of our geometric approach, that of stress-free temperature distributions.

The two-dimensional case. As in the introduction, consider a two-dimensional shell restricted to live on a flat planar surface between two rigid planes. We will assume that initially the shell is at constant temperature, and is stress-free, with no external or body forces. We would like to find the temperature distributions that will result in equilibrium configurations with zero stress. Changing the temperature uniformly will result in uniform expansion, and hence no stress. Are there other temperature distributions with this property? The answer is yes as is already known in the framework of linear thermoelasticity [5]. Due to the nature of our geometric approach, we will also be able to answer this question in a nonlinear setting. To the best of our knowledge, this has not been done in the literature.

The spatial distances between material points are measured by the ambient space metric (the "spatial metric"), which is Euclidean. A given temperature distribution will result in a change in the material metric, as described above. A configuration will be stress-free if there is no "stretch" in the material, i.e., if the material distance between two points is the same as the spatial distance. This happens only if the two types of metric tensors (spatial and material) agree on the distance measurements between nearby material points, i.e. only if they are isometric. Since the spatial metric is assumed to be Euclidean, this means that the material metric, after the change due to a given thermal distribution, has to be Euclidean.

It is worth emphasizing that one cannot simply set $G_{IJ} = \delta_{IJ}$, the precise requirement is that the pull-back of the spatial (Euclidean) metric by the deformation map φ has to be equal to the material metric. This issue is closely related to the fact that a metric may be Euclidean "in disguise", i.e., one can write the flat two-dimensional metric in different coordinate systems, and it is not always easy to recognize that the metric is flat by simply looking at its components in a given coordinate system.

[2] Note that $N = 2$ or 3.

[3] Note that for thermally isotropic materials volumetric thermal strain is N times the thermal strain.

Riemann's original work solves this problem for any dimensionality by defining the curvature tensor of the metric: a metric is flat, i.e., it can be brought into the Euclidean form δ_{IJ} *locally* by a coordinate transformation, if and only if its curvature tensor is zero [2].[4]

It turns out that in two dimensions, a weaker requirement is sufficient [2]: a metric is flat if and only if its scalar curvature (the Ricci scalar) is zero. Let us now apply this condition to a two-dimensional metric that is obtained from a non-uniform temperature distribution on an initially stress-free, planar shell, i.e., $G_{IJ} = e^{2\Omega}\delta_{IJ}$, where $\Omega(\mathbf{X}) = \omega(T(\mathbf{X}))$. The Ricci scalar for a metric of this form is given by [39]

$$\mathsf{R} = -2\, e^{-2\Omega}\nabla^2\Omega\,. \tag{2.7}$$

Thus, $\mathsf{R} = 0$ requires $\nabla^2\Omega = 0$, i.e., the exponent in the scale factor has to be a harmonic function. If we assume that $\omega(T)$ depends on temperature *linearly*, we obtain $\nabla^2 T = 0$. This is exactly the same condition encountered in linear elasticity, see, e.g. Boley and Weiner [5]. This means that for the case of constant thermal expansion coefficient in two dimensions, harmonic temperature distributions do not result in any stresses. However, our result is more general: even if the thermal expansion is non-linear, we obtain the condition $\nabla^2\Omega(X) = \nabla^2\omega(T(X)) = 0$, where $\omega(T)$ gives the general, non-linear dependence of (isotropic) thermal expansion to temperature.

It is worth emphasizing the distinction between local and global flatness, and the implications for stress-free thermal distributions. Even though the surface of a right circular cylinder in three dimensions looks curved, it is locally, *intrinsically* flat. For any given point on the cylinder, one can find a finite-sized region containing the point, and a single-valued coordinate system on this region, for which the metric has the Euclidean form. Physically, this means that for any given point, we can cut some finite-sized piece containing the point, and can lay the piece on a flat plane, without stretching it. The surface of a sphere in three dimensions, on the other hand, is intrinsically curved; it is impossible to make any finite-sized piece of the sphere, no matter how small, to lie on a flat plane without stretching it.

Curvature conditions like $\mathsf{R} = 0$, or $\nabla^2\Omega = 0$ can only detect such local issues. That it is impossible to make a full cylinder lie in a plane nicely (i.e., without tearing, folding, or stretching it) is due to the global topology of the cylinder, and local restrictions on curvature are not capable of constraining the global properties sufficiently.

In the context of thermal stresses, this subtlety is nicely demonstrated by the following example. Let us specialize to the case where Ω depends only on the radial coordinate R of an initially flat annular piece of a material, $R_0 \leq R \leq R_1$. The flatness condition gives

$$\nabla^2\Omega = \frac{1}{R}\frac{d}{dR}\left(R\,\frac{d\Omega(R)}{dR}\right) = 0\,. \tag{2.8}$$

Solving this gives

$$e^{2\Omega} = \gamma R^{2\beta}, \tag{2.9}$$

where $\gamma > 0$ and β are constants.[5] Thus, we are concerned with temperature distributions that result in metrics of the form

$$dS^2 = R^{2\beta}\left(dR^2 + R^2\, d\Theta^2\right), \tag{2.11}$$

where we have set $\gamma = 1$ by a rescaling of R. It may not be immediately obvious that these metrics are flat, but a transformation to a new radial coordinate r by $R = r^{\frac{1}{\beta+1}}$ gives

$$dS^2 = c^2 dr^2 + r^2 d\Theta^2, \tag{2.12}$$

where $c = \frac{1}{\beta+1}$. A further transformation, $\tilde{R} = |c|r$, $\tilde{\Theta} = \Theta/|c|$ gives

$$dS^2 = d\tilde{R}^2 + \tilde{R}^2 d\tilde{\Theta}^2\,, \tag{2.13}$$

[4] We have collected the definitions of various curvature-related quantities of interest in the appendix.

[5] Assuming $T(R_0) = T_0$, and that the coefficient of thermal expansion is a constant α, this solution corresponds to the temperature distribution

$$T(R) = T_0 + \frac{2\beta}{\alpha}\ln\left(\frac{R}{R_0}\right). \tag{2.10}$$

which is the flat two-dimensional metric, except for an important subtlety. In the original coordinates, a point (R, Θ) was identified with the point $(R, \Theta + 2\pi)$. In terms of the new coordinates, this means that a point $(\tilde{R}, \tilde{\Theta})$ needs to be identified with the point $(\tilde{R}, \tilde{\Theta} + 2\pi/|c|)$. The geometric meaning of this is clear: the metric (2.13), with the proper identifications, is describing an annular piece from a conical surface, with deficit angle $\alpha = 2\pi(1 - 1/|c|)$, see Fig. 2.1.

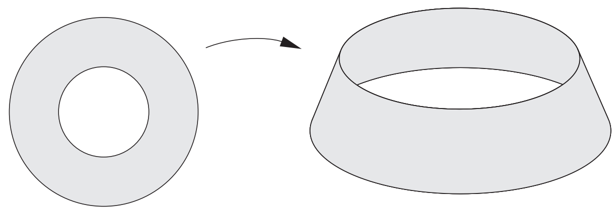

Figure 2.1: *Zero-stress deformation of an annulus to a cone.*

Now, intuitively, one can guess that it will be impossible to make such a conical surface lie on the plane without tearing, stretching, or folding it. Thus, if we start with an annular shell between two rigid planes, a temperature distribution of the form (2.10) will indeed *result in stresses*, although the related material metric is intrinsically flat. However, if the material consists only of a simply-connected piece of the annulus (say, $R_1 < R < R_2$, $0 < \Theta_1 < \Theta < \Theta_2 < 2\pi$), the temperature distribution (2.10) will just cause a stress-free expansion of the material, between the two rigid planes. See Fig. 2.2.

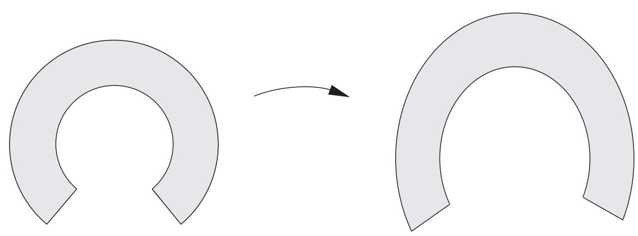

Figure 2.2: *Zero-stress deformation of a simply-connected piece of an annulus.*

This issue exists in the classical theory of elasticity, as well, see p. 256 in [5] where a set of global conditions have been given for stress-free thermal distributions in two-dimensions for multiply-connected bodies. As an application of these ideas, suppose we are given a temperature distribution $T(\mathbf{X})$, and are seeking a temperature-dependent coefficient of thermal expansion $\alpha(T)$ that would result in a stress-free equilibrium state. In general, this amounts to solving the equation

$$\nabla^2 \Omega = \frac{d\alpha}{dT} \frac{\partial T}{\partial X^A} \frac{\partial T}{\partial X^B} \delta^{AB} + \alpha \frac{\partial^2 T}{\partial X^A \partial X^B} \delta^{AB} = \frac{d\alpha}{dT} |\nabla T|^2 + \alpha \nabla^2 T = 0. \tag{2.14}$$

This equation may or may not admit a solution, depending on $T(\mathbf{X})$, but for the simple case of a radial temperature distribution $T(R)$, the solution (2.9) dictates that $\alpha(T)$ is given by

$$\alpha(T) = \frac{d\omega}{dT} = \frac{\frac{d\Omega}{dR}}{\frac{dT}{dR}} = \frac{\beta}{R T'(R)} \, . \tag{2.15}$$

As above, however, this works only for simply-connected pieces such as those described by $\Theta_1 < \Theta < \Theta_2$ and $R_1 < R < R_2$, with $\Theta_2 - \Theta_1 < 2\pi$. With this in mind, $\alpha(T)$ for some specific radial temperature distributions are as follows.

$$T(R) = \frac{T_0 R_1 - T_1 R_0}{R_1 - R_0} + \left[\frac{T_1 - T_0}{R_1 - R_0}\right] R \quad : \quad \alpha(R) = \alpha_0 \frac{R_0}{R}, \tag{2.16}$$

$$T(R) = \frac{T_1 R_1 - T_0 R_0}{R_1 - R_0} + \left[\frac{R_0 R_1 (T_0 - T_1)}{R_1 - R_0}\right] \frac{1}{R} \quad : \quad \alpha(R) = \alpha_0 \frac{R}{R_0}, \tag{2.17}$$

$$T(R) = \frac{\ln\left(R_1^{T_0}/R_0^{T_1}\right)}{\ln(R_1/R_0)} + \left[\frac{T_1 - T_0}{\ln(R_1/R_0)}\right] \ln R \quad : \quad \alpha(R) = \alpha_0. \tag{2.18}$$

Let us next consider the case where the two-dimensional material is allowed to bend into the third dimension, instead of being squeezed between two rigid flat planes. Assume, once again, that we start with a stress-free, planar piece of material at uniform temperature, and we introduce a non-uniform temperature distribution. Will the material be able to find a stress-free state by bending into the third dimension? We will investigate this problem for the special case of a radial temperature distribution, and cylindrically symmetric configurations in three dimensions.

According to our approach, the metric

$$dS^2 = e^{2\omega(T(R))}(dX^2 + dY^2) = e^{2\omega(T(R))}(dR^2 + R^2 d\Theta^2), \tag{2.19}$$

describes the natural, stress-free state of the shell, and for a given configuration in the ambient 3-dimensional space, stresses will be due to a different metric being induced on the shell by the embedding in the ambient space. We will seek embeddings for which the metric induced from the ambient space is the same as the intrinsic metric, which, as opposed to the case considered above, is not necessarily flat.

Let us begin by writing the induced metric for a given configuration with cylindrical symmetry, which is best done in cylindrical coordinates (ρ, ϕ, z). Instead of the most general configuration, we will seek a solution of the form

$$\phi(R,\Theta) = \Theta, \tag{2.20}$$

$$\rho(R,\Theta) = \rho(R), \tag{2.21}$$

$$z(R,\Theta) = z(R). \tag{2.22}$$

For such an embedding, the metric induced from the ambient space is given as

$$dS^2_{\text{induced}} = dR^2\left[\left(\frac{dz}{dR}\right)^2 + \left(\frac{d\rho}{dR}\right)^2 + \rho(R)^2\right]. \tag{2.23}$$

In order for this induced metric to be the same as the intrinsic, material metric given by (2.19), we need

$$\rho(R) = R\, e^{\Omega(R)}, \tag{2.24}$$

$$\left(\frac{dz}{dR}\right)^2 + \left(\frac{d\rho}{dR}\right)^2 = e^{2\Omega(R)}, \tag{2.25}$$

where we define $\Omega(R) = \omega(T(R))$. Substituting $\rho(R)$ in the second equation, we obtain

$$\left(\frac{dz}{dR}\right)^2 + e^{2\Omega(R)}\left[1 + R\,\Omega'(R)\right]^2 = e^{2\Omega(R)}, \tag{2.26}$$

which, in principle, lets us solve for $z(R)$. For a region where $\rho(R)$ is invertible, we can also obtain the surface in three dimensions as given by $z(\rho)$, by solving

$$\left[1 + R\,\Omega'(R(\rho))\right]^2 \left[1 + \left(\frac{dz}{d\rho}\right)^2\right] = 1. \tag{2.27}$$

Note that for a uniform temperature distribution $T(R) = T_0$, $\Omega'(R(\rho)) = 0$ and hence $z(\rho) = z_0$, which is what we expect, i.e., in this case the relaxed configuration is planar. Note also that (2.27) has a solution only if $-2/R < \Omega'(R) < 0$. In order to have some insight about the meaning of this constraint, let us specialize to conical metrics described above in (2.13), $e^{2\Omega} = \gamma R^{2\beta}$. For this case, the constraint $-2/R < \Omega'(R) < 0$ translates to $-2 < \beta < 0$, which gives, in terms of the deficit angle α, $0 < \alpha < 2\pi$. The condition $\alpha < 2\pi$ is not surprising, but there is nothing wrong with a cone with negative deficit angle in terms of intrinsic geometry. The lower bound on Ω is simply telling us that it is not possible to embed such a cone in $\mathbb{R}^3$ in a cylindrically symmetric way, which makes intuitive sense, considering the twisted shape of a saddle.

The three-dimensional case. Let us next consider the three-dimensional case. In three dimensions, a vanishing Ricci scalar is not sufficient to guarantee local flatness, however, a three-dimensional metric is flat if and only if its Ricci tensor vanishes [2]. The Ricci tensor $\mathcal{R}_{IJ}$ of the metric $G_{IJ} = e^{2\Omega} G^{(0)}_{IJ}$ is given in terms of the Ricci tensor $\mathcal{R}^{(0)}_{IJ}$ of $G^{(0)}_{IJ}$ by the following relation [39]:

$$\mathcal{R}_{IJ} = \mathcal{R}^{(0)}_{IJ} - (n-2)\nabla_I\nabla_J\Omega - G^{(0)}_{IJ}\left(G^{(0)}\right)^{KL}\nabla_K\nabla_L\Omega + (n-2)\nabla_I\Omega\nabla_J\Omega - (n-2)G^{(0)}_{IJ}\left(G^{(0)}\right)^{KL}\nabla_K\Omega\nabla_L\Omega\,, \tag{2.28}$$

where n is the dimensionality. Now, once again, assume that the initial metric $G^{(0)}_{IJ} = \delta_{IJ}$, $\mathcal{R}^{(0)}_{IJ} = 0$, and $n = 3$, and replace the covariant derivatives with partial derivatives. This gives

$$\mathcal{R}_{IJ} = -\partial_I\partial_J\Omega - \delta_{IJ}\delta^{KL}\partial_K\partial_L\Omega + \partial_I\Omega\partial_J\Omega - \delta_{IJ}\delta^{KL}\partial_K\Omega\partial_L\Omega = 0. \tag{2.29}$$

Similar to what is implicitly done in classical linear thermoelasticity, let us assume that the reference temperature T_0 is uniform, i.e. independent of position and that change of temperature is "small". This means that $\frac{\partial T}{\partial X^I}$ is small. But note that

$$\frac{\partial\Omega}{\partial X^I} = \alpha(T)\frac{\partial T}{\partial X^I}. \tag{2.30}$$

Therefore, $\partial_I\Omega$ is small too, i.e. quadratic terms in $\partial_I\Omega$ can be ignored. This gives us the condition that all the second derivatives of Ω have to vanish. This means that Ω is a linear function of the original Euclidean coordinates. If we further assume that $\omega(T)$ is a linear function of temperature, we see that temperature itself has to be a linear function of the original Euclidean coordinates. Therefore, we recover the classical result in linearized thermal elasticity that in three dimensions, the only stress-free temperature distributions for an initially stress-free material depend linearly on the coordinates [5].

In the nonlinear case $\mathcal{R}_{IJ} = 0$ is equivalent to the following system of nonlinear partial differential equations in terms of Ω:

$$\Omega_{,12} = \Omega_{,1}\Omega_{,2}, \tag{2.31}$$
$$\Omega_{,13} = \Omega_{,1}\Omega_{,3}, \tag{2.32}$$
$$\Omega_{,23} = \Omega_{,2}\Omega_{,3}, \tag{2.33}$$
$$\Omega_{,11} + \nabla^2\Omega + \Omega_{,2}^2 + \Omega_{,3}^2 = 0, \tag{2.34}$$
$$\Omega_{,22} + \nabla^2\Omega + \Omega_{,1}^2 + \Omega_{,3}^2 = 0, \tag{2.35}$$
$$\Omega_{,33} + \nabla^2\Omega + \Omega_{,1}^2 + \Omega_{,2}^2 = 0. \tag{2.36}$$

Let us first look at the following nonlinear partial differential equation for $w = w(x, y)$.

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \tag{2.37}$$

Using the change of variable $u = e^{-w}$, one obtains

$$\frac{\partial^2 u}{\partial x\partial y} = 0. \tag{2.38}$$

Thus, the most general solution is

$$w(x, y) = -\ln[f(x) + g(y)], \tag{2.39}$$

for some arbitrary functions f and g. Using (2.31)-(2.33), one can show that

$$\Omega(X^1, X^2, X^3) = -\ln[f(X^1) + g(X^2) + h(X^3)], \tag{2.40}$$

for some arbitrary functions f, g, and h. Now substituting (2.40) into (2.33)-(2.36), we obtain

$$f''(X^1) = g''(X^2) = h''(X^3). \tag{2.41}$$

Therefore

$$\begin{aligned} f(X^1) &= c_0(X^1)^2 + d_1X^1 + d_2, \\ g(X^2) &= c_0(X^2)^2 + d_3X^2 + d_4, \\ h(X^3) &= c_0(X^3)^2 + d_5X^3 + d_6, \end{aligned} \tag{2.42}$$

for some constants $c_0, d_1, ..., d_6$.[6] Plugging these back into (2.34)-(2.36), we get

$$\Omega(X^1, X^2, X^3) = -\ln\left[c_0\sum_{i=1}^{3}(X^i - b^i)^2\right]. \tag{2.43}$$

Shifting the origin $X^i \to X^i + b^i$, this becomes

$$\Omega(X^1, X^2, X^3) = -\ln\left(c_0R^2\right), \tag{2.44}$$

where $R = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}$.

If α is constant, then this corresponds to the following temperature distribution[7]

$$-\ln\left(c_0R^2\right) = \alpha(T - T_0) \quad \text{or} \quad T - T_0 = c_0 - \frac{2}{\alpha}\ln R. \tag{2.45}$$

Note that c_0 represents a uniform change in temperature. In order to understand what this solution represents physically, let us write the metric in polar coordinates.

$$dS^2 = e^{2\Omega}\left[dR^2 + R^2(d\Theta^2 + \sin^2\Theta d\Phi^2)\right] = \frac{1}{c^2R^4}\left[dR^2 + R^2(d\Theta^2 + \sin^2\Theta d\Phi^2)\right]. \tag{2.46}$$

Now let us define

$$\tilde{R} = \frac{1}{cR}. \tag{2.47}$$

In terms of $\tilde{R}$, the metric becomes

$$dS^2 = d\tilde{R}^2 + \tilde{R}^2(d\Theta^2 + \sin^2\Theta d\phi^2), \tag{2.48}$$

which is precisely the flat Euclidean metric in three dimensions. Thus, after the thermal expansion, the metric is still flat, but the radial coordinate in which it is manifestly so is related to the old radial coordinate by (2.47) (up to a simple shift of origin). This means that, particles at the two radii $R_1 < R_2$ move to the new radii $\tilde{R}_1 > \tilde{R}_2$, after the thermal expansion, i.e., the material gets "inverted". This may not be possible for a solid ball without tearing it apart, but it is perfectly possible for a piece from such a ball, as demonstrated in Fig. 2.3.

2.2 Connection with Multiplicative Decomposition of Deformation Gradient

Geometric study of thermal stresses goes back to the works of Stojanović, et al. [36]; Stojanović [37]. These researchers extended Kondo's [15; 16; 17; 18] and Bilby's [4; 3] idea of local elastic relaxation in the continuum theory of distributed defects to the case of thermal stresses. One should note that the idea of using differential geometry in anelasticity goes back to an earlier work by Eckart [7].

[6] Note that the case $c_0 = d_1 = d_3 = d_5 = 0$ corresponds to uniform temperature distributions.

[7] Note that this is similar in form to the 2D solution (2.18).

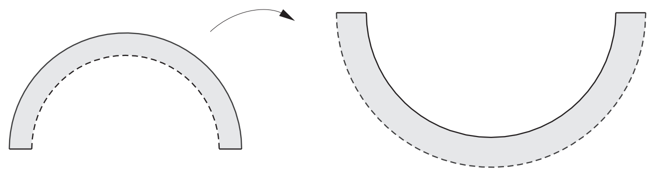

Figure 2.3: *A nonlinear stress-free thermal deformation of a ball. Note that because of symmetry only deformation of a great half circle of the projected sphere on a plane passing thorugh the center of the ball is shown in this figure.*

Stojanović's idea is similar in spirit to the approach described in this paper: a nonuniform temperature distribution in general leads to residual stresses essentially because the body is constrained to deform in Euclidean space. If one partitions the body into small pieces, each piece will individually relax, but it is impossible to realize a relaxed state for the full body by combining these pieces in Euclidean space. Any attempt to reconstruct the full body by sticking the particles together will induce deformations on them, and will result in stresses. An imaginary relaxed configuration for the full body is incompatible with the geometry of Euclidean space.

The approach taken in this paper is to ask the question: which space, as opposed to the Euclidean space, would be compatible with a relaxed state of the body? We claimed above that the answer to this question is, a Riemannian manifold whose metric is related to the nonuniform temperature distribution by (2.3). This metric describes the relaxed state of the material, and the strains in a given configuration should be measured with respect to the relaxed state, i.e., the new material metric. In this setup, the constitutive relation (e.g., a free energy function) is given in terms of the material metric, the (Euclidean) spatial metric, and the deformation gradient $\mathbf{F}$. The constitutive relation allows one to calculate the stresses induced for a given configuration, at least in principle.

Stojanović, on the other hand, takes the following viewpoint. Consider one of the imaginary relaxed pieces described above. The process of relaxation after the piece is cut corresponds to a linear deformation of this piece (linear, since the piece is small)[8]. Let us call this deformation $\mathbf{F}_T$. If this piece is deformed in some arbitrary way after the relaxation, one can calculate the induced stresses by using the tangent map of this deformation mapping in the constitutive relation.

Now, in order to calculate the stresses induced for a given deformation of the *full body*, we focus our attention to one such particular piece. The deformation gradient of the full body at this piece $\mathbf{F}$ can be decomposed as $\mathbf{F} = \mathbf{F}_e\mathbf{F}_T$, where, by definition, $\mathbf{F}_e = \mathbf{F}\mathbf{F}_T^{-1}$. Thus, as far as this piece is concerned, the deformation of the full body consists of a relaxation, followed by a linear deformation given by $\mathbf{F}_e$. The stresses induced on this piece, for an arbitrary deformation of the full body, can be calculated by substituting $\mathbf{F}_e$ in the constitutive relation.

Note that $\mathbf{F}_e$ and $\mathbf{F}_T$ are not necessarily true deformation gradients in the sense that one cannot necessarily find deformations φ_e and φ_T whose tangent maps are given by $\mathbf{F}_e$ and $\mathbf{F}_T$, respectively. This is due precisely to the incompatibility mentioned above. However, as long as we have a prescription for obtaining $\mathbf{F}_e$ and $\mathbf{F}_T$ directly for a given deformation map φ for the body and a temperature distribution, we can calculate the stresses by the following prescription.

For an isotropic material, Stojanović gives the following formula for $\mathbf{F}_T$ in terms of the temperature.

$$(F_T)^A{}_B = \vartheta(T)\delta^A_B. \tag{2.49}$$

This means that a small piece relaxes by a uniform expansion, whose magnitude is determined by a function $\vartheta(T)$ that characterizes the thermal expansion properties of the material under consideration. Given this formula for $\mathbf{F}_T$, we can calculate $\mathbf{F}_e = \mathbf{F}\mathbf{F}_T^{-1}$ for a given deformation, and utilize a constitutive relation that gives the stresses in terms of $\mathbf{F}_e$.

These two approaches seem very different philosophically, and at first sight, $\mathbf{F}_T$, the "incompatible intermediate deformation gradient," perhaps seems a little mysterious from the geometric standpoint. However, these approaches are related, as we will demonstrate next. Our discussion can easily be generalized to other sources of residual stresses; a local relaxation approach and the Riemannian approach are equivalent for a large class of settings.[9]

[8] This transformation is not necessarily uniquely determined, see below for a discussion of this issue.

[9] However, there are cases that will require a further generalization, namely, cases where the material manifold has a connection with torsion and non-metricity.

As mentioned above, Stojanović [36; 37] gives $(F_T)^A{}_B = \vartheta(T)\delta^A_B$ and relates the coefficient of thermal expansion α to $\vartheta(T)$ by

$$\alpha(T) = \vartheta(T)\frac{d\vartheta(T)}{dT}. \tag{2.50}$$

This can agree with (2.6) only if

$$e^{\omega(T)} = \vartheta(T). \tag{2.51}$$

In order to show the mechanical equivalence of the two approaches by using this identification, we need to show that for any given constitutive relation for one of the approaches, one can find a corresponding constitutive relation for the other approach that predicts the same stresses for all possible deformations when ϑ and ω are related through (2.51).

The constitutive relations of the two approaches are formulated in terms of different quantities: $\mathbf{F}_e = \mathbf{F}\mathbf{F}_T^{-1}$ on one side, and $\mathbf{G}(T)$ and $\mathbf{F}$ on the other. Let us start with our approach, namely, assume that a constitutive relation is given in terms of $\mathbf{G}(T)$ and $\mathbf{F}$. This takes the form of a scalar free energy function that depends on $\mathbf{G}(T)$, $\mathbf{F}$, as well as on the spatial metric tensor $\mathbf{g}$, and possibly $\mathbf{X}$ and $T(\mathbf{X})$ explicitly:

$$\Psi = \Psi(\mathbf{X}, T, \mathbf{G}(\mathbf{X}, T), \mathbf{F}, \mathbf{g}). \tag{2.52}$$

Now, $\mathbf{G}$, $\mathbf{F}$, and $\mathbf{g}$ are tensors, written in terms of specific bases for the material space and the ambient space. Commonly, bases associated to coordinate systems are used. A change of basis changes the components of these tensors, but Ψ, being a scalar, does not change. Let us consider a change of basis from the original coordinate basis $\mathbf{E}_A$ of the material space, satisfying

$$\langle\!\langle \mathbf{E}_A, \mathbf{E}_B \rangle\!\rangle_{\mathbf{G}} = G_{AB}\,, \tag{2.53}$$

to an orthonormal basis $\hat{\mathbf{E}}_{\hat{A}}$ that satisfies

$$\langle\!\langle \hat{\mathbf{E}}_{\hat{A}}, \hat{\mathbf{E}}_{\hat{B}} \rangle\!\rangle_{\mathbf{G}} = \delta_{\hat{A}\hat{B}}\,. \tag{2.54}$$

The transformation between the two bases is given by a matrix $\mathsf{F}_{\hat{A}}{}^B$ as

$$\hat{\mathbf{E}}_{\hat{A}} = \mathsf{F}_{\hat{A}}{}^B\ \mathbf{E}_B\,. \tag{2.55}$$

The orthonormality condition gives

$$\mathsf{F}_{\hat{A}}{}^C \mathsf{F}_{\hat{B}}{}^D G_{CD} = \delta_{\hat{A}\hat{B}}. \tag{2.56}$$

Any $\mathsf{F}_{\hat{A}}{}^C$ that satisfies this equation gives an orthonormal basis. Given such an $\mathsf{F}_{\hat{A}}{}^C$, we can also obtain an orthonormal basis for the dual space by using its inverse. Defining $\mathsf{F}^{\hat{C}}{}_D$ as the inverse of the matrix $\mathsf{F}_{\hat{A}}{}^B$, i.e., $\mathsf{F}_{\hat{A}}{}^B \mathsf{F}^{\hat{A}}{}_C = \delta^B_C$ and $\mathsf{F}_{\hat{A}}{}^B \mathsf{F}^{\hat{C}}{}_B = \delta^{\hat{C}}_{\hat{A}}$, we obtain the dual orthonormal basis $\{\hat{\mathbf{E}}^{\hat{A}}\}$ in terms of the original dual basis $\{\mathbf{E}^A\}$ by

$$\hat{\mathbf{E}}^{\hat{A}} = \mathsf{F}^{\hat{A}}{}_B \mathbf{E}^B. \tag{2.57}$$

For thermal stresses, assuming that the initial material manifold is Euclidean, $G_{CD} = e^{2\omega(T)}\delta_{CD} = \vartheta(T)^2\delta_{CD}$ gives

$$\mathsf{F}_{\hat{A}}{}^C = \delta^C_{\hat{A}}\ e^{-\omega(T)} = \delta^C_{\hat{A}}\ \vartheta^{-1}(T), \tag{2.58}$$

as a solution to (2.56). Here, $\delta^B_{\hat{A}}$ is 1 for $A = B$, and 0, otherwise, i.e., $\delta^1_{\hat{1}} = \delta^2_{\hat{2}} = \delta^3_{\hat{3}} = 1$, etc. Note that (2.56) has other solutions, too, which we will comment on below.

Now let us write the components of the total deformation gradient $\mathbf{F}$ in the orthonormal basis $\{\hat{\mathbf{E}}_{\hat{A}}\}$. The components are transformed by using F:

$$F^a{}_{\hat{A}} = \mathsf{F}_{\hat{A}}{}^B F^a{}_B\,. \tag{2.59}$$

Now, using (2.58), (2.51), and (2.49), we see that the components $F^a{}_{\hat{A}}$ are given precisely by those of $\mathbf{F}_e$, the "elastic part" of the deformation gradient in Stojanovitch's approach:

$$F^a{}_{\hat{A}} = \mathsf{F}_{\hat{A}}{}^B F^a{}_B = \delta^B_{\hat{A}} e^{-\omega(T)} F^a{}_B = (\vartheta(T))^{-1}\delta^B_{\hat{A}} F^a{}_B = (F_T^{-1})_A{}^B F^a{}_B = (F_e)^a{}_A. \tag{2.60}$$

Thus, Stojanović's $\mathbf{F}_e$ is nothing but the original deformation gradient, written in terms of an orthonormal basis in the material space. In passing, we have also shown that there is no need for a mysterious "intermediate configuration" as the target space of $\mathbf{F}_T$, the latter just gives an orthonormal frame in the material manifold, and as such, can be treated as a linear map from the tangent space of the material manifold to itself. As mentioned above, these ideas can be generalized to other problems with residual stresses.

Rewriting the constitutive relation (2.52) by using an orthonormal basis for the material manifold, we obtain

$$\Psi = \Psi(\mathbf{X}, T, G_{AB} = \delta_{AB}, F^a{}_B = (F_e)^a{}_B, g_{ab}). \tag{2.61}$$

Thus, given a constitutive relation Ψ^{Riem} in the Riemannian approach, one can obtain a constitutive relation Ψ^{LR} in the "local relaxation" approach by simply going to an orthonormal basis by (2.55) and (2.56), and ignoring the constant terms $G_{AB} = \delta_{AB}$ and $g_{ab} = \delta_{ab}$ in the functional dependence.

$$\Psi^{\mathrm{LR}}(\mathbf{X}, T, (F_e)^a{}_B) = \Psi^{\mathrm{Riem}}\left(\mathbf{X}, T, G_{AB} = \delta_{AB}, F^a{}_B = (F_e)^a{}_B, g_{ab} = \delta_{ab}\right). \tag{2.62}$$

Going in the opposite direction is also possible; starting with a free energy function for the Stojanović's approach, one can derive an equivalent free energy in the Riemannian approach. This direction may be slightly more confusing, since the metrics of the material manifold and the spatial manifold are not explicitly written out initially. One proceeds by first writing $\mathbf{F}_e$ in terms of its proper index structure $(F_e)^a{}_B$ in the Riemannian approach, and inserting δ_{ab} and δ_{AB} where necessary for tensorial consistency, and finally interpreting these as the components of metric tensors, and performing a change of basis, if desired.

Non-coordinate bases and torsion. Although a coordinate basis $\{\mathbf{E}_A = \partial/\partial X^A\}$ is not necessarily orthonormal, one can always obtain an orthonormal basis by applying a pointwise change of basis $\mathsf{F}_{\hat{A}}{}^B$. Moreover, giving an orthonormal basis in this way is equivalent to giving a metric tensor at each point; the inner product of any two vectors can be calculated by using their components in the orthonormal basis. We have seen above that in the context of thermo-elasticity, this means that a change in the material metric due to a change in temperature can be given in terms of the "thermal deformation gradient" of the local relaxation approach.

Given an orthonormal basis $\{\hat{\mathbf{E}}_{\hat{A}}\}$, it is possible to obtain another one, $\{\hat{\mathbf{E}}'_{\hat{A}}\}$, by using an orthogonal transformation $\Lambda_{\hat{A}}{}^{\hat{B}}$ as

$$\hat{\mathbf{E}}'_A = \Lambda_{\hat{A}}{}^{\hat{B}}\, \hat{\mathbf{E}}_{\hat{B}}, \tag{2.63}$$

where $\Lambda_{\hat{A}}{}^{\hat{B}}$ satisfies $\Lambda_{\hat{A}}{}^{\hat{C}}\Lambda_{\hat{B}}{}^{\hat{D}}\delta_{\hat{C}\hat{D}} = \delta_{\hat{A}\hat{B}}$. Let the relation between the original coordinate basis $\{\mathbf{E}_A\}$ and the new orthonormal basis be given by the matrix $\mathsf{F}'{}_{\hat{A}}{}^B$ as follows

$$\hat{\mathbf{E}}'_{\hat{A}} = \mathsf{F}'_{\hat{A}}{}^B \mathbf{E}_B. \tag{2.64}$$

The relation between F and F' is given as

$$\mathsf{F}'_{\hat{A}}{}^B = \Lambda_{\hat{A}}{}^{\hat{C}} \mathsf{F}_{\hat{C}}{}^B. \tag{2.65}$$

Going in the opposite direction, one can see that F and F' represent the same material metric $\mathbf{G}$, if and only if they are related through (2.65) for some orthogonal matrix $\Lambda_{\hat{A}}{}^{\hat{B}}$. This means that there is an $SO(3)$ ambiguity in the choice of F, and hence, in that of $\mathbf{F}_T$.

As opposed to a coordinate basis, the elements of an orthonormal basis do not necessarily commute with each other; whereas $[\mathbf{E}_A, \mathbf{E}_B] = \mathbf{0}$ for $E_A = \partial/\partial X^A$, for an orhonormal basis $\hat{\mathbf{E}}_{\hat{A}} = \mathsf{F}_{\hat{A}}{}^B \mathbf{E}_B$, one has [34]

$$[\hat{\mathbf{E}}_{\hat{A}}, \hat{\mathbf{E}}_{\hat{B}}] = c_{\hat{A}\hat{B}}{}^{\hat{C}}\, \hat{\mathbf{E}}_{\hat{C}}, \tag{2.66}$$

where

$$c_{\hat{A}\hat{B}}{}^{\hat{C}} = \mathsf{F}^{\hat{C}}{}_D \left(\mathsf{F}_{\hat{A}}{}^E \frac{\partial \mathsf{F}_{\hat{B}}{}^D}{\partial X^E} - \mathsf{F}_{\hat{B}}{}^E \frac{\partial \mathsf{F}_{\hat{A}}{}^D}{\partial X^E} \right). \tag{2.67}$$

The connection coefficients $\overline{\Gamma}^{\hat{C}}_{\hat{A}\hat{B}}$ for an orthonormal basis, defined through

$$\nabla_{\hat{A}} \hat{\mathbf{E}}_{\hat{B}} = \overline{\Gamma}^{\hat{C}}_{\hat{A}\hat{B}}\, \hat{\mathbf{E}}_{\hat{C}}, \tag{2.68}$$

are related to the connection coefficients $\Gamma^C_{AB}$of the coordinate basis by

$$\overline{\Gamma}^{\hat{C}}_{\hat{A}\hat{B}} = \mathsf{F}_{\hat{A}}{}^{D}\mathsf{F}^{\hat{C}}{}_{F}\left(\frac{\partial \mathsf{F}_{\hat{B}}{}^{F}}{\partial X^D} + \mathsf{F}_{\hat{B}}{}^{E}\Gamma^F_{DE}\right). \tag{2.69}$$

In a coordinate basis, the components of the torsion tensor are given by the antisymmetrization of the two lower indices of the connection coefficients, i.e. (see the appendix)

$$T_{AB}{}^{C} = \Gamma^C_{AB} - \Gamma^C_{BA}. \tag{2.70}$$

However, for a non-coordinate basis, the components are given by

$$T^{\hat{C}}{}_{\hat{A}\hat{B}} = \overline{\Gamma}^{\hat{C}}_{\hat{A}\hat{B}} - \overline{\Gamma}^{\hat{C}}_{\hat{B}\hat{A}} - c_{\hat{A}\hat{B}}{}^{\hat{C}}. \tag{2.71}$$

Our formalism is based on Riemannian geometry, and in particular, on the torsion-free Levi-Civita connection defined by the metric $\mathbf{G}(T)$. Thus, the torsion tensor for the material connection vanishes in both the coordinate basis, and the orthonormal basis. Let us show this explicitly for $\mathsf{F}_{\hat{A}}{}^{B} = \vartheta^{-1}(T)\delta^B_{\hat{A}}$. In the original coordinate basis, the metric tensor is given by $G_{AB} = e^{2\omega(T)}\delta_{AB} = \vartheta^2(T)\delta_{AB}$. Thus, the connection coefficients in this basis are

$$\Gamma^A_{BC} = \vartheta^{-1}\left(\vartheta_{,B}\delta^A_C + \vartheta_{,C}\delta^A_B - \vartheta_{,D}\delta^{AD}\delta_{BC}\right). \tag{2.72}$$

Using (2.70), we have $T^A{}_{BC} = 0$. Next, using (2.67) and (2.69) with

$$\mathsf{F}^{\hat{A}}{}_{B} = \vartheta\ \delta^{\hat{A}}_B, \quad \mathsf{F}_{\hat{A}}{}^{B} = \vartheta^{-1}\delta^B_{\hat{A}}, \tag{2.73}$$

we obtain

$$c_{\hat{A}\hat{B}}{}^{\hat{C}} = \frac{\vartheta_{,D}}{\vartheta^2}\left(\delta^{\hat{C}}_{\hat{A}}\delta^D_{\hat{B}} - \delta^{\hat{C}}_{\hat{B}}\delta^D_{\hat{A}}\right), \tag{2.74}$$

and

$$\overline{\Gamma}^{\hat{C}}_{\hat{A}\hat{B}} = -\frac{\vartheta_{,D}}{\vartheta^2}\delta^D_{\hat{A}}\delta^{\hat{C}}_{\hat{B}} + \frac{1}{\vartheta}\delta^D_{\hat{A}}\delta^E_{\hat{B}}\delta^{\hat{C}}_C\Gamma_{DE}{}^{C}. \tag{2.75}$$

Using these in (2.71), we obtain $\overline{T}_{\hat{B}\hat{C}}{}^{\hat{A}} = 0$. Note that the Riemann curvature tensor has the following form

$$\begin{aligned}\mathcal{R}^A{}_{BCD} \quad = \quad & \vartheta^{-2}\left[2\left(\vartheta_{,C}\delta^A_B - \vartheta_{,B}\delta^A_C\right)\vartheta_{,D} + 2\left(\vartheta_{,B}\delta_{CD} - \vartheta_{,C}\delta_{BD}\right)\vartheta_{,E}\delta^{AE} + \vartheta_{,E}\vartheta_{,F}\delta^{EF}\left(\delta^A_C\delta_{BD} - \delta^A_B\delta_{CD}\right)\right] \\ & +\vartheta^{-1}\left[\vartheta_{,BD}\delta^A_C - \vartheta_{,CD}\delta^A_B + \left(\vartheta_{,CE}\delta_{BD} - \vartheta_{,BD}\delta_{CD}\right)\delta^{AE}\right],\end{aligned} \tag{2.76}$$

which, in general, does not vanish.

Stojanović, on the other hand, calculates a non-vanishing torsion tensor. This discrepancy is due to the fact that he uses the following connection

$$\Gamma^A_{BC} = \left(F_T^{-1}\right)^A{}_M \frac{\partial\left(F_T\right)^M{}_C}{\partial X^B}. \tag{2.77}$$

Similar connections have been used in other contexts [15; 4]. It can be shown that this connection has vanishing curvature but has nonvanishing torsion. This is related to the so-called canonical connection in absolutely parallelizable manifolds [8; 9; 43]. See also Epstein and Elżanowski [10] for similar connections in the context of inhomogeneities and their geometric representations. In the appendix, we give some details on absolutely parallelizable manifolds and the above connection.

In summary, our geometric approach has a concrete connection with that of Stojanović: in our approach we use a Riemannian manifold with a temperature-dependent metric as the material manifold while Stojanović implicitly uses the same metric but in an absolutely parallelizable manifold that is not Riemannian. Using either approach would be fine and a matter of taste, however we believe that our approach is more straightforward as we do not introduce an unnecessary torsion in the material manifold. Representing changes of temperature by a one-parameter family of conformal Riemannian metrics enables us to find the zero-stress temperature distributions even in finite deformations.

2.3 Anisotropic Thermal Expansion

So far, we have assumed that thermal expansion is isotropic, i.e. a change of temperature results in a change of length independent of orientation. Let us now see how one should modify the theory when a temperature change results in different changes in length in different directions and in possibly a change of shape. Even in the case of anisotropic thermal expansion all one needs is a temperature-dependent material metric $\mathbf{G}(\mathbf{X},T)$ but in this case the material metric is no longer a simple rescaling of the original material metric. The physical idea is the following. Given any point in the initial stress-free material manifold with metric $\mathbf{G}_0(\mathbf{X})$, there exists a frame field $\{\mathbf{E}^1(\mathbf{X}),\mathbf{E}^2(\mathbf{X}),\mathbf{E}^3(\mathbf{X})\}$ for each material point such that in this frame the material metric is diagonal and has the following form

$$\mathbf{G}(\mathbf{X},T)=e^{2\omega_1(T)}\mathbf{E}^1\otimes\mathbf{E}^1+e^{2\omega_2(T)}\mathbf{E}^2\otimes\mathbf{E}^2+e^{2\omega_3(T)}\mathbf{E}^3\otimes\mathbf{E}^3. \tag{2.78}$$

Given a coordinate basis $\widetilde{\mathbf{E}}_I=\partial/\partial X^I$, we have

$$\mathbf{E}_I=A^J{}_I\widetilde{\mathbf{E}}_J. \tag{2.79}$$

Thus

$$\mathbf{G}(\mathbf{X},T)=\sum_I e^{2\omega_I(T)}A^J{}_I\widetilde{\mathbf{E}}_J\otimes A^K{}_I\widetilde{\mathbf{E}}_K. \tag{2.80}$$

3 Geometric Elasticity with Temperature Changes

Material manifold. As mentioned in the previous sections, the material manifold describes the intrinsic "shape" of the natural, stress-free state of the material. The geometry induced by a given configuration of the material in the spatial manifold may or may not agree with the intrinsic geometry. The discrepancy between the two geometries (the induced and the intrinsic) is in general a cause for stresses, which are described by geometric constitutive relations. In this section, we will describe this framework.[10]

The motion of an elastic body is described by a possibly non-isometric, time dependent embedding of the material manifold in the spatial manifold (see Fig. 3.1). There is, however, another possible source of time-dependency: the geometry of the material manifold itself may change in time.[11] A change in the geometry of the material manifold is sometimes known as a referential change (see [12; 31] and references therein), and the precise meaning of this has sometimes been a source of confusion in the literature. In this paper, we have a case where the change in the geometry of the material manifold is described explicitly in terms of the temperature, and we believe that the conceptual clarity brought by this simple example may provide insights to other cases of referential changes, such as those that describe the evolution of defects in a crystalline solid.

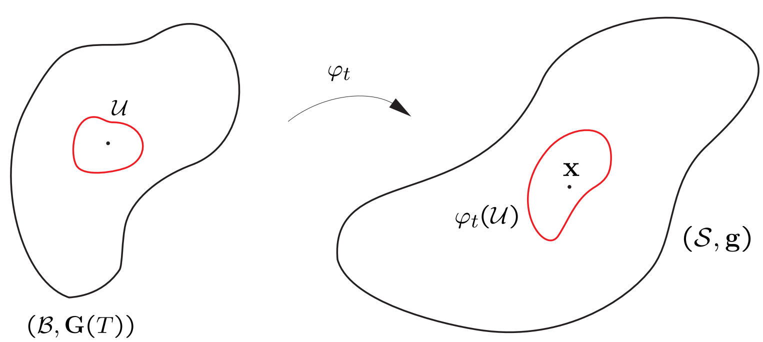

Figure 3.1: *Motion of a continuum with temperature changes.*

[10]Due to our approach to thermal stresses, in this paper, we treat the material and the spatial spaces as Riemannian manifolds, as in [29; 41], and by "geometry", we understand the Levi-Civita connection associated to the metric tensor. In general, the geometry of either of these spaces can be given by a more general connection that has torsion and/or non-metricity. Such connections have found use in the literature of defect mechanics.

[11]The geometry of the spatial manifold may also change, but we do not consider this issue in this paper.

In Section 2, we proposed to describe the thermal expansion of an isotropic material by a change in the material metric, given by equation (2.3):

$$G_{IJ}(\mathbf{X},T) = H_{IJ}(\mathbf{X})e^{2\omega(T)}\,. \tag{3.1}$$

Here, $\omega(T)$ is a function that describes the thermal expansion properties of the isotropic material under consideration. The coefficient of thermal expansion is given by equation (2.6):

$$\alpha(T) = \frac{d\omega(T)}{dT}\,. \tag{3.2}$$

We assume that the temperature distribution in the material is given. If the temperature depends on time, then the material metric describing the relaxed state of the material will also depend on time, through (3.1).

We should mention that evolution of reference configuration in the literature of continuum mechanics is more or less ambiguous. It is believed that an evolving reference configuration can model dynamics of defects. However, to our best knowledge, there are no concrete examples in the literature. We believe that the present geometric formulation of thermal stresses in terms of a temperature-dependent material manifold can make the role of reference manifold clearer and can shed light on other more complicated problems, e.g., continuum theory of solids with distributed dislocations.

Conservation of mass. Let us begin by writing the conservation of mass in this setting. If the temperature is time-independent, the usual material version of the conservation of mass holds: the material density is constant.

$$\rho_0(\mathbf{X},t) = \rho_0(\mathbf{X}). \tag{3.3}$$

If, however, temperature changes in time, the material metric will expand or contract, so the material mass density will change. The evolution of the mass density will then be given by

$$\rho_0(\mathbf{X},T)dV(\mathbf{X},T) = \mathsf{m}(\mathbf{X}), \tag{3.4}$$

where $dV(\mathbf{X},T(t))$ is the volume form of the metric $G(\mathbf{X},T)$, and $\mathsf{m}(\mathbf{X})$ is the temperature-independent (and hence time-independent) differential form representing the mass density (mass form). This equation tells us that if the material manifold expands due to a temperature change, the total mass in a material region will not change, and hence the density ρ_0 will decrease inversely with the increase in the volume of that region. Since the volume form is given by

$$dV(\mathbf{X},T) = \sqrt{\det|H_{IJ}|}\; e^{N\omega(T)}d^NX, \tag{3.5}$$

we can get the density for a given temperature T in terms of the density at an initial temperature T_0 by using $\rho_0(\mathbf{X},T)dV(\mathbf{X},T) = \rho_0(\mathbf{X},T_0)dV(\mathbf{X},T_0)$. This gives

$$\rho_0(\mathbf{X},T) = e^{N(\omega(T_0)-\omega(T))}\rho_0(\mathbf{X},T_0). \tag{3.6}$$

In terms of the coefficient of thermal expansion $\alpha = \frac{d\omega}{dT}$, this can be written as

$$\rho_0(\mathbf{X},T) = \rho_0(\mathbf{X},T_0)e^{-N\int_{T_0}^{T}\alpha(\tau)d\tau}. \tag{3.7}$$

Incompressibility. Elastic incompressibility means that elastic deformations cannot cause any changes in volume. Thus, for a given temperature distribution, the deformation map must preserve the volume element. The volume elements in the material and spatial manifolds, $dV(\mathbf{X})$ and $dv(\mathbf{x})$ are related by

$$dv(\mathbf{x}(\mathbf{X})) = J(\mathbf{X},T)dV(\mathbf{X},T), \tag{3.8}$$

where the Jacobian J is given as

$$J(\mathbf{X},T) = \det\mathbf{F}\sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}(T)}}. \tag{3.9}$$

Thus, incompressibility means that $J(\mathbf{X},T) = 1$. Given two Riemannian manifolds, distance preserving maps, i.e., isometries between them may or may not exist. A similar question may arise for volume-preserving maps: given two Riemannian manifolds (in our case, the material manifold and the spatial manifold), does there exist a volume-preserving map between them? Moser [33] answers this question in the affirmative, so the study of incompressibility in this setting is not vacuous.

Free energy. The free energy, in addition to explicitly depending on temperature, will depend on the temperature-dependent material metric tensor as well, i.e.

$$\Psi = \Psi(\mathbf{X}, T, \mathbf{G}(\mathbf{X}, T), \mathbf{F}, \mathbf{g}). \tag{3.10}$$

Therefor, the first Piola-Kirchoff stress, given by

$$\mathbf{P} = \mathbf{P}(\mathbf{X}, T) = \mathbf{g}^{-1}\frac{\partial \Psi}{\partial \mathbf{F}}\,, \tag{3.11}$$

explicitly depends on the temperature-dependent material metric.

Balance of Linear Momentum. Let us now look at the governing equations for a given temperature distribution $T = T(\mathbf{X})$. We will only study the static case[12], for which the balance of linear momentum reads

$$\operatorname{Div}\mathbf{P} = \mathbf{0} \qquad \text{or} \qquad P^{aA}{}_{|A} = \frac{\partial P^{aA}}{\partial X^A} + \Gamma^A_{AB} P^{aB} + \gamma^a_{bc} F^c{}_A P^{bA} = 0, \tag{3.12}$$

where Γ^C_{AB} are the connection coefficients for the material metric G_{AB}, and γ^c_{ab} are the connection coefficients for the metric g_{ab}. This is the standard balance of momentum in geometric elastostatics, see, e.g., Yavari, et al. [41]. For the case of thermal stresses, the material connection coefficients Γ^C_{AB} are those of the metric (2.3), $G_{IJ}(\mathbf{X}, T) = H_{IJ}(\mathbf{X})e^{2\omega(T)}$; they are given in terms of the connection coefficients $\Gamma^{(H)\ C}_{AB}$ of the metric H_{IJ} as [39]

$$\Gamma^C_{AB} = \Gamma^{(H)\ C}_{AB} + \left(\delta^C_A \partial_B \Omega + \delta^C_B \partial_A \Omega - H_{AB} H^{CD} \partial_D \Omega\right). \tag{3.13}$$

Suppose the initial material metric H_{AB} is Euclidean. Then, using Cartesian coordinates, we have

$$\Gamma^A_{AB} = 3\partial_B \Omega. \tag{3.14}$$

For a Euclidean spatial metric in Cartesian coordinates, $g_{ab} = \delta_{ab}$, we obtain

$$\frac{\partial P^{aA}}{\partial X^A} + 3\frac{\partial \Omega}{\partial X^B} P^{aB} = 0. \tag{3.15}$$

In terms of the thermal expansion coefficient $\alpha = \frac{d\omega}{dT}$, this becomes

$$\frac{\partial P^{aA}}{\partial X^A} + 3\alpha\frac{\partial T}{\partial X^B} P^{aB} = 0. \tag{3.16}$$

In the following example, we show that in the geometric framework, some nonlinear problems can be solved analytically.

Example. Let us consider a two-dimensional, incompressible neo-Hookean material in a flat two-dimensional spatial manifold. The free energy density of a neo-Hookean material in two dimensions has the form

$$\Psi = \Psi(\mathbf{X}, \mathbf{C}) = \mu(\operatorname{tr}\mathbf{C} - 2), \tag{3.17}$$

where $\mathbf{C}$ is the Cauchy-Green tensor, or equivalently, the pull-back of the spatial metric, $C_{AB} = F^a{}_A F^b{}_B g_{ab}$, and μ is a material constant. We will assume that this form holds for an isotropic material under thermal expansion, and in particular, we assume that there is no explicit temperature dependence in the free energy apart from the dependence through $\mathbf{C}$. In components

$$\Psi = \mu\left(F^a{}_A F^b{}_B g_{ab} G^{AB} - 2\right). \tag{3.18}$$

The "2" is of no particular significance: when the material metric is fixed, it just shifts the free energy by a constant. When the material metric changes as in (2.3), its contribution to the free energy is proportional to

[12] The governing equations for the dynamics case are similar. However, for the dynamic problem one has to consider an evolving temperature distribution governed by the heat equation. This will be discussed in a future communication.

the temperature-dependent material volume, which, for a given temperature distribution, is independent of the spatial configuration. We ignore this term, and use $\Psi = \Psi(\mathbf{X}, \mathbf{C}) = \mu \operatorname{tr} \mathbf{C}$ as our definition of the free energy.

Let us assume that initially the material has a flat annular shape $R_1 \leq R \leq R_2$ without any stresses, at a uniform temperature T_0. We would like to calculate the stresses that occur in the new equilibrium configuration after we change the temperature in a rotationally symmetric way, $T = T(R)$. In polar coordinates, the spatial metric and its inverse read

$$\mathbf{g} = \begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad \mathbf{g}^{-1} = \begin{pmatrix} g^{rr} & g^{r\theta} \\ g^{\theta r} & g^{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}, \tag{3.19}$$

and thus $\det \mathbf{g} = r^2$. The only nonzero connection coefficients are:

$$\gamma^r_{\theta\theta} = -r, \; \gamma^\theta_{r\theta} = \gamma^\theta_{\theta r} = 1/r. \tag{3.20}$$

For the temperature-dependent material metric we have

$$\mathbf{G} = \begin{pmatrix} G_{RR} & G_{R\Theta} \\ G_{\Theta R} & G_{\Theta\Theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix} e^{2\omega(T(R))}, \quad \mathbf{G}^{-1} = \begin{pmatrix} G^{RR} & G^{R\Theta} \\ G^{\Theta R} & G^{\Theta\Theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/R^2 \end{pmatrix} e^{-2\omega(T(R))}, \tag{3.21}$$

and thus, $\det \mathbf{G} = R^2 e^{4\omega(T(R))}$. The following nonzero connection coefficients are needed in the balance of linear momentum:

$$\Gamma^R_{RR} = \Omega'(R), \; \Gamma^R_{\Theta\Theta} = -R - R^2\Omega'(R), \; \Gamma^\Theta_{R\Theta} = \Gamma^\Theta_{\Theta R} = 1/R + \Omega'(R). \tag{3.22}$$

In terms of the thermal expansion coefficient, these are given as

$$\Gamma^R_{RR} = \alpha T'(R), \; \Gamma^R_{\Theta\Theta} = -R - R^2\alpha T'(R), \; \Gamma^\Theta_{R\Theta} = \Gamma^\Theta_{\Theta R} = 1/R + \alpha T'(R). \tag{3.23}$$

Given the temperature distribution $T = T(R)$, we are looking for solutions of the form

$$\varphi(R, \Theta) = (r, \theta) = (r(R), \Theta). \tag{3.24}$$

Thus

$$\mathbf{F} = \begin{pmatrix} r'(R) & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{F}^{-1} = \begin{pmatrix} 1/r'(R) & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.25}$$

This gives the Jacobian as

$$J = \frac{r\, r'}{Re^{2\omega(T)}}. \tag{3.26}$$

Incompressibility dictates that

$$rr' = Re^{2\omega(T)}. \tag{3.27}$$

This differential equation has the following solution

$$r^2(R) = \int_{R_1}^{R} 2\xi e^{2\omega(T(\xi))} d\xi + r_1^2(R). \tag{3.28}$$

Note that $r_1(R)$ is not known a priori and will be obtained after imposing the traction boundary conditions at r_1 and r_2.

In incompressible elasticity, P^{aA} is replaced by $P^{aA} - Jp(F^{-1})^{-A}{}_b g^{ab}$, where p is an unknown scalar field (pressure) that will be determined using the constraint $J = 1$ [29], i.e.

$$P^{aA} = 2\mu F^a{}_B G^{AB} - p(R)(F^{-1})^A{}_b g^{ab}. \tag{3.29}$$

Therefore, using (3.27), we get the nonzero stress components as

$$P^{rR} = \frac{2\mu R}{r} - p(R)\frac{r}{R} e^{-2\omega(T(R))}, \quad P^{\theta\Theta} = \frac{2\mu}{R^2} e^{-2\omega(T(R))} - \frac{p(R)}{r^2}, \tag{3.30}$$

where $p(R)$ is an unknown pressure.

Balance of linear momentum in components reads

$$P^{aA}|_A = \frac{\partial P^{aA}}{\partial X^A} + \Gamma^A_{AB} P^{aB} + P^{bA} \gamma^a_{bc} F^c{}_A = 0. \tag{3.31}$$

For the radial direction, $a = r$, we have

$$\begin{aligned} P^{rA}|_A &= \frac{\partial P^{rA}}{\partial X^A} + \Gamma^A_{AB} P^{rB} + P^{bA} \gamma^r_{bc} F^c{}_A \\ &= \frac{\partial P^{rR}}{\partial R} + \left(\Gamma^R_{RR} + \Gamma^\Theta_{\Theta R}\right) P^{rR} + P^{\theta\Theta} \gamma^r_{\theta\theta} F^\theta{}_\Theta \\ &= \frac{\partial P^{rR}}{\partial R} + \left(\frac{1}{R} + 2\alpha T'(R)\right) P^{rR} - r P^{\theta\Theta} = 0. \end{aligned} \tag{3.32}$$

This gives

$$p'(R) = \frac{2\mu R}{r^2} e^{2\omega(T(R))} \left[2\left(1 + \alpha R T'\right) - \frac{R^2}{r^2} e^{2\omega(T(R))} - \frac{r^2}{R^2} e^{-2\omega(T(R))} \right]. \tag{3.33}$$

Assuming that $p(R_i) = 0$, we obtain

$$p(R) = \int_{R_i}^{R} \frac{2\mu\xi}{r^2(\xi)} e^{2\omega(T(\xi))} \left[2\left(1 + \alpha(\xi)\xi T'(\xi)\right) - \frac{\xi^2}{r^2(\xi)} e^{2\omega(T(\xi))} - \frac{r^2(\xi)}{\xi^2} e^{-2\omega(T(\xi))} \right] d\xi. \tag{3.34}$$

For $a = \theta$, balance of momentum (3.31) gives,

$$P^{\theta A}|_A = \frac{\partial P^{\theta\Theta}}{\partial \Theta} + \Gamma^A_{A\Theta} P^{\theta\Theta} + P^{\theta R} \gamma^\theta_{rr} F^r{}_R + P^{\theta\Theta} \gamma^\theta_{\theta\theta} F^\theta{}_\Theta = \left(\Gamma^R_{R\Theta} + \Gamma^\Theta_{\Theta\Theta}\right) P^{\theta\Theta} = 0. \tag{3.35}$$

i.e. this equilibrium equation is trivially satisfied. Therefore, given the temperature distribution $T(R)$, we can calculate all the thermal stresses analytically.

4 Linearized Theory of Thermal Stresses

In this section, we linearize the governing equations of the nonlinear theory presented in the previous section about a reference motion. Geometric linearization of elasticity was first introduced by Marsden and Hughes [29] and was further developed by Yavari and Ozakin [42]. See also [32] for similar discussions. Here, we start with a temperature-dependent material manifold and its motion in an ambient space. Given a reference motion, we are interested in obtaining the linearized governing equations with respect to this motion. We will assume that the ambient space manifold is Euclidean. This is not a necessary assumption but it provides a natural setting for most practical problems of interest and will simplify the subsequent calculations. For simplicity, we will restrict attention to time-independent solutions.

Suppose a given material with a temperature distribution $T(\mathbf{X})$ and the related material metric $\mathbf{G}$ is in a static equilibrium configuration, φ. The balance of linear momentum for this material body reads

$$\operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \mathbf{0}\,. \tag{4.1}$$

Now suppose we change the temperature of this material by a small amount $\delta T(\mathbf{X})$. This will change the material metric to $\mathbf{G}' = \mathbf{G} + \delta\mathbf{G}$, and φ will no longer describe a static equilibrium configuration. A nearby equilibrium configuration may be given by $\varphi' = \varphi + \delta\varphi$, and the stress in this new equilibrium configuration will be $\mathbf{P}' = \mathbf{P} + \delta\mathbf{P}$. One would like to calculate the change in the stress (or the configuration), for a given small change in temperature (see Fig. 4.1).

While the spirit of this setup is familiar from other linearization problems, some care is needed in interpreting its meaning. $\mathbf{P}(\mathbf{X})$ is a two-point tensor (it has components in both the material and the ambient spaces: P^{aA}) based at $\mathbf{X}$ and $\varphi(\mathbf{X})$, whereas $\mathbf{P}'(\mathbf{X})$ is based at $\mathbf{X}$ and $\varphi'(\mathbf{X})$. Defining $\delta\mathbf{P}(\mathbf{X}) = \mathbf{P}' - \mathbf{P}$ is nontrivial for a general ambient space metric. This is related to the fact that subtracting tangent vectors at different points in a manifold is only defined with respect to a choice of a path connecting the two points, on which a parallel

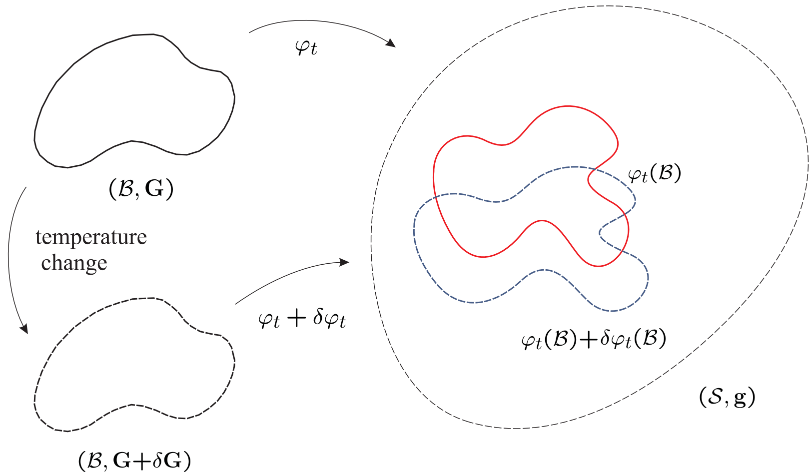

Figure 4.1: *Motion of a continuum with temperature changes.*

transport is to be performed. By restricting our attention to a Euclidean ambient space we sidestep this issue, using the natural, path-independent parallel transport in Euclidean space. Another issue is the definition of $\delta\varphi = \varphi' - \varphi$. While one can use coordinate systems to make approximate sense of this equation for two nearby maps, it is a little troublesome geometrically, since the subtraction of two maps between manifolds is not defined geometrically.

The linearization procedure can be put to firmer footing if instead of talking about two nearby configurations and the differences of various quantities for these configurations, we describe the situation in terms of a 1-parameter family of configurations around a reference configuration, and calculate the derivatives of various quantities with respect to the parameter. These derivatives will capture the behavior of the solution as a function of the parameter, for small values of the latter. Thus, let $T_\epsilon(\mathbf{X})$ be a 1-parameter family of temperature distributions on our material manifold, $\mathbf{G}_\epsilon$ be the corresponding family of material metrics, φ_ϵ be the equilibrium configurations, and $\mathbf{P}_\epsilon$ be the stresses. Let $\epsilon = 0$ describe the reference equilibrium configuration. Now, for a fixed point $\mathbf{X}$ in the material manifold, $\varphi_\epsilon(\mathbf{X})$ describes a curve in the spatial manifold, and its derivative at $\epsilon = 0$ gives a vector $\mathbf{U}(\mathbf{X})$ at $\varphi(\mathbf{X})$:

$$\mathbf{U}(\mathbf{X}) = \frac{d\varphi_\epsilon(\mathbf{X})}{d\epsilon}\Big|_{\epsilon=0}. \tag{4.2}$$

Considering $\delta\varphi \approx \epsilon \frac{d\varphi_\epsilon}{d\epsilon}$, we see that a more rigorous version of $\delta\varphi$ is the vector field $\mathbf{U}$. Similarly, one has

$$\delta\mathbf{G} \approx \epsilon \, \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathbf{G}_\epsilon. \tag{4.3}$$

When the change in $\mathbf{G}$ is due to a change in T, we have

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathbf{G}_\epsilon = 2\frac{d\omega}{dT}\frac{dT}{\partial\epsilon}\Big|_{\epsilon=0} \mathbf{G} = \beta\mathbf{G}, \tag{4.4}$$

where $\beta = 2\frac{d\omega}{dT}\frac{dT}{d\epsilon}\big|_{\epsilon=0} = 2\alpha(T_\epsilon)\frac{dT}{d\epsilon}\big|_{\epsilon=0}$.

Now consider, in the absence of body forces, the equilibrium equation $\operatorname{Div}\mathbf{P} = \mathbf{0}$ for the family of temperature distributions parametrized by ϵ:

$$\operatorname{Div}_\epsilon \mathbf{P}_\epsilon = \mathbf{0}. \tag{4.5}$$

Linearization of (4.5) is defined as [29; 42]:

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \left(\operatorname{Div}_\epsilon \mathbf{P}_\epsilon\right) = \mathbf{0}. \tag{4.6}$$

Once again, one should note that since the equilibrium configuration is different for each ϵ, $\mathbf{P}_\epsilon$ is based at different points in the ambient space for different values of ϵ, and in order to calculate the derivative with respect to ϵ, one in general needs to use the connection (parallel transport) in the ambient space. For the Euclidean case we are considering and a Cartesian coordinate system x^a, this is trivial. In components (4.6) reads

$$\frac{\partial P^{aA}(\epsilon)}{\partial X^A} + \Gamma^A_{AB}(\epsilon) P^{aB}(\epsilon) = 0. \tag{4.7}$$

Thus, the linearized balance of linear momentum reads

$$\frac{\partial}{\partial X^A} \frac{d}{d\epsilon}\Big|_{\epsilon=0} P^{aA}(\epsilon) + \left[\frac{d}{d\epsilon}\Big|_{\epsilon=0} \Gamma^A_{AB}(\epsilon) \right] P^{aB} + \Gamma^A_{AB} \frac{d}{d\epsilon}\Big|_{\epsilon=0} P^{aB}(\epsilon) = 0. \tag{4.8}$$

Note that

$$F^{aA} = g^{ac} \frac{\partial \Psi}{\partial F^c{}_A}, \tag{4.9}$$

where $\Psi = \Psi(\mathbf{X}, T, \mathbf{F}, \mathbf{G}, \mathbf{g})$ is the material free energy density. In calculating $\frac{dP^{aA}(\epsilon)}{d\epsilon}$, we need to consider the changes in $\mathbf{F}$ and $\mathbf{G}$ due to the change in the equilibrium configuration:

$$\frac{dP^{aA}(\epsilon)}{d\epsilon} = \frac{\partial P^{aA}}{\partial F^b{}_B} \frac{dF^b{}_B}{d\epsilon} + \frac{\partial P^{aA}}{\partial G_{CD}} \frac{dG_{CD}}{d\epsilon}. \tag{4.10}$$

Defining

$$\mathbb{A}^{aA}{}_b{}^B = \frac{\partial P^{aA}}{\partial F^b{}_B} = g^{ac} \frac{\partial^2 \Psi}{\partial F^b{}_B \partial F^c{}_A} \qquad \text{and} \qquad \mathbb{B}^{aACD} = \frac{P^{aA}}{G_{CD}} = \frac{\partial^2 \Psi}{\partial G_{CD} \partial F^c{}_A}, \tag{4.11}$$

where the derivatives are to be evaluated at the reference configuration $\epsilon = 0$. Noting that

$$\frac{dF^a{}_A}{d\epsilon}\Big|_{\epsilon=0} = \frac{\partial U^a}{\partial X^A}, \tag{4.12}$$

we obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} P^{aA}(\epsilon) = \mathbb{A}^{aA}{}_b{}^B U^b{}_{,B} + \mathbb{B}^{aACD} G_{CD}\ \beta. \tag{4.13}$$

Using

$$\Gamma^A_{BC} = \frac{1}{2} G^{AD} \left(\frac{\partial G_{BD}}{\partial X^C} + \frac{\partial G_{CD}}{\partial X^B} - \frac{\partial G_{BC}}{\partial X^D} \right), \tag{4.14}$$

and

$$\frac{dG^{AB}}{d\epsilon} = -G^{AC} G^{BD} \frac{\partial G_{BD}}{\partial \epsilon}, \tag{4.15}$$

and plugging in (4.4), we obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \Gamma^A_{AB}(\epsilon) = \frac{3}{2} \frac{\partial \beta}{\partial X^B}. \tag{4.16}$$

With these results, the linearized balance of linear momentum (4.6) becomes

$$\left(\mathbb{A}^{aA}{}_b{}^B U^b{}_{,B} \right)_{,A} + \left(\mathbb{B}^{aACD} G_{CD}\ \beta \right)_{,A} + \frac{3}{2} \frac{\partial \beta}{\partial X^B} P^{aB} = 0. \tag{4.17}$$

Assuming that $\mathbb{A}$ and $\mathbb{B}$ are independent of $\mathbf{X}$, the linearized equilibrium equations are simplified to read

$$\mathbb{A}^{aA}{}_b{}^B \frac{\partial^2 U^b}{\partial X^A \partial X^B} + \mathbb{B}^{aACD} G_{CD} \frac{\partial \beta}{\partial X^A} + \frac{3}{2} \frac{\partial \beta}{\partial X^B} P^{aB} = 0. \tag{4.18}$$

If the initial configuration is stress-free, we have

$$\mathbb{A}^{aA}{}_b{}^B \frac{\partial^2 U^b}{\partial X^A \partial X^B} = -\mathbb{B}^{aACD} G_{CD} \frac{\partial \beta}{\partial X^A}. \tag{4.19}$$

Let us next show that these results agree with those of classical thermoelasticity. We first consider a special class of isotropic materials.

Saint-Venant-Kirchhoff materials. Saint-Venant-Kirchhoff materials have a constitutive relation that is analogous to the linear isortropic materials, namely, the second Piola-Kirchhoff stress $\mathbf{S}$ is given in terms of the Lagrangian strain $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G})$ as [29]

$$\mathbf{S} = \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{G}^{-1} + 2\mu\mathbf{E}, \tag{4.20}$$

or in components

$$S^{CD} = \lambda E_{AB}G^{AB}G^{CD} + 2\mu E^{CD} = \frac{\lambda}{2}(C_{AB}G^{AB} - 3)G^{CD} + \mu(C_{AB}G^{AC}G^{BD} - G^{CD}), \tag{4.21}$$

where $\lambda = \lambda(\mathbf{X})$ and $\mu = \mu(\mathbf{X})$ are two scalars characterizing the material properties. This means that $\mathbf{S}$ is a linear function of $\mathbf{E}$. We next show that for this class of materials linearization of our geometric theory leads to linear governing equations that are identical to those of the classical linear theory of thermal stresses for linear, isotropic materials. We will show later in this section that this is true for any elastic material.

We can obtain the tensor $\mathbb{B}^{aCAB}$ from $\mathbf{S}$ as follows

$$\mathbb{B}^{aCAB} = \frac{\partial}{\partial G_{AB}}\left(g^{ab}\frac{\partial \psi}{\partial F^b{}_C}\right) = \frac{\partial P^{aC}}{\partial G_{AB}} = F^a{}_D\frac{\partial S^{CD}}{\partial G_{AB}}. \tag{4.22}$$

Using

$$\frac{\partial G^{AB}}{\partial G_{MN}} = -G^{AM}G^{BN}, \tag{4.23}$$

we obtain

$$\mathbb{B}^{aACD}G_{CD} = -2C_{MN}F^a{}_B\left(\lambda G^{AB}G^{MN} + 2\mu G^{AM}G^{BN}\right) + (3\lambda + 2\mu)F^a{}_BG^{AB}. \tag{4.24}$$

The initial metric is Euclidean; in Cartesian coordinates, $G_{AB} = \delta_{AB}$. Since the ambient space is also Euclidean, we can choose a Cartesian coordinate system whose axes coincide with the initial location of the material points along the material Cartesian axis. This will give, $F^a{}_A = \delta^a_A$, where a and A both range over $1, 2, 3$. This gives

$$\mathbb{B}^{aACD}G_{CD} = -\frac{3\lambda + 2\mu}{2}\,\delta^{aA}. \tag{4.25}$$

Similarly, for an initially stress-free material manifold, we obtain

$$\mathbb{A}^{aA}{}_b{}^B = F^a{}_MF^c{}_Ng_{bc}\left[\lambda G^{AM}G^{BN} + \mu(G^{AB}G^{MN} + G^{AN}G^{BM})\right]. \tag{4.26}$$

For the case of an initially Euclidean material manifold with Cartesian coordinates we have

$$\mathbb{A}^{aA}{}_b{}^B\frac{\partial^2 U^b}{\partial X^A \partial X^B} = (\lambda + \mu)U_{b,ab} + \mu U_{a,bb}. \tag{4.27}$$

Therefore, Eq. (4.19) reads

$$(\lambda + \mu)U_{b,ab} + \mu U_{a,bb} = \frac{3\lambda + 2\mu}{2}\frac{\partial \beta}{\partial x_a}. \tag{4.28}$$

Recalling $\beta = 2\alpha\frac{dT}{d\epsilon}\big|_{\epsilon=0}$, and assuming for simplicity that α is constant, we have

$$\frac{\partial \beta}{\partial x_a} = 2\alpha\frac{\partial}{\partial x_a}\frac{dT}{d\epsilon}\Big|_{\epsilon=0}. \tag{4.29}$$

Hence

$$(\lambda + \mu)U_{b,ab} + \mu U_{a,bb} = (3\lambda + 2\mu)\alpha\frac{\partial}{\partial x_a}\frac{dT}{d\epsilon}\Big|_{\epsilon=0}. \tag{4.30}$$

In the classical theory of thermal stresses, stress-strain relations in the presence of temperatures changes can be written as

$$\sigma_{ij} = \mathsf{C}_{ijkl}(\epsilon_{kl} - \alpha\delta_{kl}\Delta T), \tag{4.31}$$

where C_{ijkl} is the elasticity tensor. In this sense thermal strains are understood as "eigen strains". Equilibrium equations in the absence of body forces read

$$\mathsf{C}_{ijkl}\epsilon_{kl,j} = \mathsf{C}_{ijkk}\alpha\frac{\partial \Delta T}{\partial x_j}, \tag{4.32}$$

where once again we have assumed that the elasticity tensor and the coefficient of thermal expansion are constants. When the material is isotropic, $\mathsf{C}_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$ and hence

$$\mathsf{C}_{ijkl}\epsilon_{kl,j} = \mu u_{i,jj} + (\lambda+\mu)u_{j,ji}, \tag{4.33}$$

and

$$\mathsf{C}_{ijkk}\alpha\frac{\partial \Delta T}{\partial x_j} = (2\mu+3\lambda)\alpha\frac{\partial \Delta T}{\partial x_i}. \tag{4.34}$$

Thus, equilibrium equations read

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} = (2\mu+3\lambda)\alpha\frac{\partial \Delta T}{\partial x_i}. \tag{4.35}$$

Recalling that $\frac{dT}{d\epsilon}\big|_{\epsilon=0}$ is the linearized version of temperature change, i.e., $\delta T \approx \epsilon\frac{dT}{d\epsilon}\big|_{\epsilon=0}$ and that $\mathbf{U}$ is the linearized version of displacement, it is seen that the linearization of the geometric theory for Saint-Venant-Kirchhoff materials results in governing equations that are identical to those of the classical isotropic linear theory.

Let us now see if this holds for more general constitutive equations of the geometric theory. For a stress-free (Euclidean) initial configuration, the linearized balance of linear momentum is given by

$$\left(\mathbb{A}^{aA}{}_b{}^B\, U^b_{,B} + \mathbb{B}^{aACD}\delta_{CD}\,\beta\right)_{,A} = 0. \tag{4.36}$$

Assuming $G_{AB} = \delta_{AB}$ and $F^a{}_A = \delta^a_A$ as above, an identity proven in [29] becomes

$$\mathbb{A}^{aA}{}_b{}^B = 2\mathbb{C}^{CADB}F^c{}_D F^a{}_C g_{cb} = 2\mathbb{C}^{CADB}\delta^c_D\delta^a_C\delta_{bc} = 2\mathbb{C}^{aAbB}. \tag{4.37}$$

Noting that $\beta = 2\alpha\frac{dT}{d\epsilon}\big|_{\epsilon=0}$, (4.36) becomes

$$\left(\mathbb{C}^{aAbB}\, U^b_{,B} + \mathbb{B}^{aACC}\alpha\, \frac{dT}{d\epsilon}\Big|_{\epsilon=0}\right)_{,A} = 0. \tag{4.38}$$

Identifying superscripts and subscripts, identifying the spatial and material indices (by aligning the material and spatial Cartesian coordinates as above), and using symmetries of $\mathbb{C}$, we can write

$$\left(\mathbb{C}_{ljki}\epsilon_{ki} + \mathbb{B}_{ljkk}\alpha\, \frac{dT}{d\epsilon}\Big|_{\epsilon=0}\right)_{,j} = 0. \tag{4.39}$$

This is identical to (4.31) if

$$\mathbb{B}_{ljkk} = -\mathbb{C}_{ljki}\delta_{ki} = -\mathbb{C}_{ljkk}. \tag{4.40}$$

Let us first show that this relation always holds for isotropic materials. For isotropic materials it can be shown that [25; 26; 41]

$$\frac{\partial \Psi}{\partial \mathbf{C}}\cdot\mathbf{C} + \frac{\partial \Psi}{\partial \mathbf{G}}\cdot\mathbf{G} = \mathbf{0}. \tag{4.41}$$

Or in components

$$\frac{\partial \Psi}{\partial C_{AC}}C_{CB} + \frac{\partial \Psi}{\partial G_{AC}}G_{CB} = 0. \tag{4.42}$$

Note that

$$\mathbb{B} = \frac{\partial \mathbf{P}}{\partial \mathbf{G}} = \mathbf{F}\frac{\partial \mathbf{S}}{\partial \mathbf{G}} = 2\mathbf{F}\frac{\partial^2 \Psi}{\partial \mathbf{C}\partial \mathbf{G}}. \tag{4.43}$$

Differentiating (4.41) with respect to $\mathbf{C}$ and noting that the initial configuration is stress free we obtain

$$\frac{\partial^2\Psi}{\partial\mathbf{C}\partial\mathbf{C}}\cdot\mathbf{C}+\frac{\partial^2\Psi}{\partial\mathbf{C}\partial\mathbf{G}}\cdot\mathbf{G}=\mathbf{0}. \tag{4.44}$$

Or

$$\frac{1}{2}\mathbb{C}\cdot\mathbf{C}+\frac{1}{2}\mathbf{F}^{-1}\mathbb{B}\cdot\mathbf{G}=\mathbf{0}. \tag{4.45}$$

Noting that $F^a{}_A=\delta^a_A$ and $C_{AB}=G_{AB}$, (4.45) is identical to (4.40).

We now show that Eq.(4.40) holds even for anisotropic elastic solids. For showing this we use the fact that if the material is homogeneous (i.e., if the thermal expanstion properties do not depend on position) a uniform temperature change ΔT does not lead to any thermal stresses. Starting from a stress-free Euclidean configuration, for a uniform temperature change, one has

$$\delta F^a{}_A=U^a{}_{,A}=\alpha\Delta T\delta^a{}_A\qquad\text{and}\qquad\delta\mathbf{S}=\mathbf{0}. \tag{4.46}$$

We also know $\mathbf{S}=\mathbf{S}(\mathbf{C},\mathbf{G})$, thus

$$\delta\mathbf{S}=\frac{\partial\mathbf{S}}{\partial\mathbf{C}}\cdot\delta\mathbf{C}+\frac{\partial\mathbf{S}}{\partial\mathbf{G}}\cdot\delta\mathbf{G}=\mathbb{C}\cdot\delta\mathbf{C}+2\alpha\Delta T\ \mathbf{F}^{-1}\mathbb{B}\cdot\mathbf{G}=\mathbf{0}. \tag{4.47}$$

But note that [42]

$$\delta C_{AB}=g_{ab}F^a{}_AU^b{}_{|B}+g_{ab}F^b{}_BU^a{}_{|A}=\delta_{ab}\delta^a_A(\alpha\Delta T\delta^b_B)+\delta_{ab}\delta^b_B(\alpha\Delta T\delta^a_A)=2\alpha\Delta T\delta_{AB}. \tag{4.48}$$

Substituting (4.48) into (4.47), one obtains (4.40)! In summary, we have proved the following proposition.

Proposition. Linearization of the present geometric theory yields governing equations that are identical to those of classical linear elasticity.

5 Conclusions

In this paper, we presented a geometric theory of thermal stresses in which the material manifold is temperature dependent. Given a temperature distribution, the material metric is a Riemannian metric that is obtained by a (non-uniform) rescaling of a reference metric. In particular, starting from a Euclidean stress-free reference manifold, a non-uniform temperature distribution leads to a non-Euclidean material manifold. We studied the stress-free temperature distributions by looking at conditions that guarantee flatness of a Riemannian metric. We recovered some known facts from the linear theory of thermal stresses and obtained some new results for finite deformations. We showed that, in addition to uniform temperature distributions, there are other zero-stress temperature distributions. We obtained all such temperature distributions. We also studied the inverse problem, i.e., given a temperature distribution, what inhomogeneous coefficients of thermal expansion give zero stresses. In the present theory, there is no need to introduce an "intermediate" configuration. We made an explicit connection between our geometric theory and the previous works on multiplicative decomposition of deformation gradient in the presence of temperature changes. Given a temperature distribution, we obtained the temperature-dependent governing equations. In order to demonstrate the power of the geometric theory, we solved the example of an axisymmetric temperature distribution and obtained some exact results. We showed that linearization of the present geometric theory about a stress-free configuration results in governing equations that are identical to those of the classical linear thermoelasticity.

Geometric formulation of the coupled problem of elastic deformations with heat conduction will be studied in a future communication. The ideas presented in this paper can also be used in modeling bodies with growing mass. Growth and remodeling in biological systems is an important phenomenon and a geometric study will shed light on the coupling between growth/remodeling and elastic deformations.

References

[1] Abraham, R., J.E. Marsden and T. Ratiu [1988], *Manifolds, Tensor Analysis, and Applications*, Springer-Verlag, New York.

[2] Berger, M. [2003], *A Panoramic View of Riemannian Geometry*, Springer-Verlag, New York.

[3] Bilby, B. A., L. R. T., Gardner, and A. N., Stroh [1957], Continuous distribution of dislocations and the theory of plasticity, In: *Proceedings of the Ninth International Congress of Applied Mechanics*, Brussels, 1956. Universite de Bruxelles, pp. 35-44.

[4] Bilby, B. A., R., Bullough, and E., Smith [1955], Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. *Proceedings of the Royal Society of London* **A231**(1185): 263-273.

[5] Boley, B. A. and J. H. Weiner [1997], *Theory of Thermal Stresses*, Dover, New York.

[6] Christodoulou, D. [1998], On the geometry and dynamics of crystalline continua. *Annales De L Institut Henri Poincare-Physique Theorique* **69**:335-358.

[7] Eckart, C. [1948], The thermodynamics of irreversible processes. 4. The theory of elasticity and anelasticity. *Physical Review* **73**(4):373-382.

[8] Eisenhart, L.P. [1926], *Riemannian Geometry*, Princeton University Press, Princeton.

[9] Eisenhart, L.P. [1927], *Non-Riemannian Geometry*, Dover, New York.

[10] Epstein, M. and M. Elżanowski [2007], *Material Inhomogeneities and their Evolution*, Springer, New York.

[11] Green, A.E. and Rivilin, R. S. [1964], On Cauchy's equations of motion. *ZAMP* **15**: 290-293.

[12] Gurtin, M. E. [2000], *Configurational Forces as a Basic Concept of Continuum Physics*. Springer-Verlag.

[13] Ibrahimbegovic, A. and Chorfi, L. and Gharzeddine, F. [2001], Thermomechanical coupling at finite elastic strain: covariant formulation and numerical implementation. *Communications in Numerical Methods in Engineering* **17**:275-289.

[14] Imam, A. and Johnson, G. C. [1998], Decomposition of the deformation gradient in thermoelasticity. *Journal of Applied Mechanics* **65**:362-366.

[15] Kondo, K. [1955], Geometry of elastic deformation and incompatibility, *Memoirs of the Unifying Study of the Basic Problems in Engineering Science by Means of Geometry*, (K. Kondo, ed.), vol. 1, Division C, Gakujutsu Bunken Fukyo-Kai, 1955, pp. 5-17.

[16] Kondo, K. [1955], Non-Riemannien geometry of imperfect crystals from a macroscopic viewpoint. *Memoirs of the Unifying Study of the Basic Problems in Engineering Science by Means of Geometry*, (K. Kondo, ed.), vol. 1, Division D-I, Gakujutsu Bunken Fukyo-Kai, 1955, pp. 6-17

[17] Kondo, K. [1963], Non-Riemannian and Finslerian approaches to the theory of yielding. *International Journal of Engineering Science* **1**:71-88.

[18] Kondo, K. [1964], On the analytical and physical foundations of the theory of dislocations and yielding by the differential geometry of continua. *International Journal of Engineering Science* **2**:219-251.

[19] Kröner, E. [1996], Dislocation theory as a physical field theory. *Meccanica* **31**: 577-587.

[20] Kröner, E. [1992], The internal mechanical state of solids with defects. *International Journal of Solids and Structures* **29**: 1849-1857.

[21] Lee, J.M. [1997], *Riemannian Manifold An Introduction to Curvature*, Springer-Verlag, New York.

[22] Lee, E.H. [1967], Elastic-plastic deformation at finite strains. *Journal of Applied Mechanics* **36**:1-6.

[23] Lee, E. H. and Liu, D. T. [1967], Finite-strain elastic-plastic theory with application to plane-wave analysis. *Journal of Applied Physics* **38**:19-27.

[24] Lu, S.C.H. and Pister, K.S. [1975], Decomposition of deformation and representation of the free energy function for isotropic thermoelastic solids. *International Journal of Solids and Structures* **11**:927-934.

[25] Lu, J. and P. Papadopoulos. [2000], A covariant constitutive descroption of anisotropic non-linear elasticity. *ZAMP* **51**:204-217.

[26] Lu, J. and P. Papadopoulos. [2003], Referential Doyle-Ericksen formulae for the eshelby tensor in non-linear elasticity. *ZAMP* **54**:964–976.

[27] Lubrada, V. A. [2004], Constitutive theories based on the multiplicative decomposition of deformation gradient: Thermoelasticity, elastoplasticity, and biomechanics. *Applied Mechanics Reviews* **57**(2):95-108.

[28] Mariano, P. M. [2005], SO(3) invariance and covariance in mixtures of simple bodies. *International Journal of Non-Linear Mechanics* **40**:1023-1030.

[29] Marsden, J.E. and T.J.R. Hughes [1983], *Mathematical Foundations of Elasticity*, Dover, New York.

[30] Marsden, J.E. and T. Ratiu [2003], *Introduction to Mechanics and Symmetry*, Springer, New York.

[31] Maugin, G.A. [1993], *Material Inhomogeneities in Elasticity.* Chapman and Hall.

[32] Mazzucato, A. L. and Rachele, L. V. [2006], Partial uniqueness and obstruction to uniqueness in inverse problems for anisotropic elastic media. *Journal of Elasticity* **83**:205-245.

[33] Moser, J. [1965], On volume elements on a manifold. *Transactions of the American Mathematical Society* **120**:286-294.

[34] Nakahara, M. [2003], *Geometry, Topology and Physics*, Taylor & Francis, New York.

[35] Simo, J. C. and J. E. Marsden [1984], On the rotated stress tensor and the material version of the Doyle-Ericksen formula. *Archive for Rational Mechanics and Analysis* **86**: 213 - 231.

[36] Stojanović, R., Djurić, S., and L. Vujošević [1964], On finite thermal deformations. *Archiwum Mechaniki Stosowanej* **16**: 103 - 108.

[37] Stojanović, R. [1969], On the stress relation in non-linear thermoelasticity, *International Journal of Non-Linear Mechanics* **4**: 217-233.

[38] Vujošević, L. and Lubarda, V.A. [2002], Finite-strain thermoelasticity based on multiplicative decomposition of deformation gradient. *Theoretical and Applied Mechanics* **28-29**:379-399.

[39] Wald, R.M. [1984], *General Relativity*, The University of Chicago Press, Chicago.

[40] Wanas, M.I. [2008], Absolute parallelism geometry: developments, applications and problems. arXiv:gr-qc/0209050v1.

[41] Yavari, A., J. E. Marsden and M. Ortiz [2006], On the spatial and material covariant balance laws in elasticity. *Journal of Mathematical Physics* **47**: 042903; 85-112.

[42] Yavari, A. and A. Ozakin [2008] Covariance in linearized elasticity. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)* **59**(6):1081-1110.

[43] Youssef, N. L. and Sid-Ahmed, A. M. [2007], Linear connections and curvature tensors in the geometry of parallelizable manifolds. *Reports on Mathematical Physics* **60**:39-53.

A Differential Geometry and Classical Geometric Elasticity

In this section, in order to make the paper self-contained, we review some notation from geometric elasticity. For more details refer to [29; 1; 30]. By *classical* geometric elasticity we mean elasticity of bodies with stationary defects (if any) and a fixed material manifold. We extended this theory for thermal deformations in §3.

For a smooth n-manifold M, the tangent space to M at a point $p \in M$ is denoted T_pM and the whole tangent bundle is denoted TM. We denote by $\mathcal{B}$ a reference manifold for our body and by $\mathcal{S}$ the space in which the body moves. We assume that $\mathcal{B}$ and $\mathcal{S}$ are Riemannian manifolds with metrics $\mathbf{G}$ and $\mathbf{g}$, respectively. Local coordinates on $\mathcal{B}$ are denoted by $\{X^A\}$ and those on $\mathcal{S}$ by $\{x^a\}$.

A ***deformation*** of the body is a C^1 embedding $\varphi : \mathcal{B} \to \mathcal{S}$. The tangent map of φ is denoted $\mathbf{F} = T\varphi : T\mathcal{B} \to T\mathcal{S}$, which is often called the deformation gradient. In local charts on $\mathcal{B}$ and $\mathcal{S}$, the tangent map of φ is given by the Jacobian matrix of partial derivatives of the components of φ, as

$$\mathbf{F} = T\varphi : T\mathcal{B} \to T\mathcal{S}, \quad T\varphi(\mathbf{X}, \mathbf{Y}) = (\varphi(\mathbf{X}), \mathbf{D}\varphi(\mathbf{X}) \cdot \mathbf{Y}). \tag{A.1}$$

If $\mathbf{Y}$ is a vector field on $\mathcal{B}$, then $\varphi_*\mathbf{Y} = T\varphi \cdot \mathbf{Y} \circ \varphi^{-1}$, or using the $\mathbf{F}$ notation, $\varphi_*\mathbf{Y} = \mathbf{F} \cdot \mathbf{Y} \circ \varphi^{-1}$ is a vector field on $\varphi(\mathcal{B})$ called the ***push-forward*** of $\mathbf{Y}$ by φ. Similarly, if $\mathbf{y}$ is a vector field on $\varphi(\mathcal{B}) \subset \mathcal{S}$, then $\varphi^*\mathbf{y} = T(\varphi^{-1}) \cdot \mathbf{y} \circ \varphi$ is a vector field on $\mathcal{B}$ and is called the pull-back of $\mathbf{y}$ by φ.

The cotangent bundle of a manifold M is denoted T^*M and the fiber at a point $p \in M$ (the vector space of one-forms at p) is denoted by T_p^*M. If β is a one-form on $\mathcal{S}$, i.e., a section of the cotangent bundle $T^*\mathcal{S}$, then the one-form on $\mathcal{B}$ defined as

$$(\varphi^*\beta)_{\mathbf{X}} \cdot \mathbf{V}_{\mathbf{X}} = \beta_{\varphi(\mathbf{X})} \cdot (T\varphi \cdot \mathbf{V}_{\mathbf{X}}) = \beta_{\varphi(\mathbf{X})} \cdot (\mathbf{F} \cdot \mathbf{V}_{\mathbf{X}}) \tag{A.2}$$

for $\mathbf{X} \in \mathcal{B}$ and $\mathbf{V}_{\mathbf{X}} \in T_{\mathbf{X}}\mathcal{B}$, is called the ***pull-back*** of β by φ. Similarly, the ***push-forward*** of a one-form α on $\mathcal{B}$ is the one form on $\varphi(\mathcal{B})$ defined by $\varphi_*\alpha = (\varphi^{-1})^*\alpha$.

We can associate a vector field $\beta^\sharp$ to a one-form β on a Riemannian manifold M through the equation

$$\langle \beta_{\mathbf{x}}, \mathbf{v}_{\mathbf{x}} \rangle = \langle\!\langle \beta^\sharp_{\mathbf{x}}, \mathbf{v}_{\mathbf{x}} \rangle\!\rangle_{\mathbf{x}}, \tag{A.3}$$

where $\langle , \rangle$ denotes the natural pairing between the one form $\beta_{\mathbf{x}} \in T^*_{\mathbf{x}}M$ and the vector $\mathbf{v}_{\mathbf{x}} \in T_{\mathbf{x}}M$ and where $\langle\!\langle \beta^\sharp_{\mathbf{x}}, \mathbf{v}_{\mathbf{x}} \rangle\!\rangle_{\mathbf{x}}$ denotes the inner product between $\beta^\sharp_{\mathbf{x}} \in T_{\mathbf{x}}M$ and $\mathbf{v}_{\mathbf{x}} \in T_{\mathbf{x}}M$ induced by the metric $\mathbf{g}$. In coordinates, the components of $\beta^\sharp$ are given by $\beta^a = g^{ab}\beta_b$.

A type $\begin{pmatrix} m \\ n \end{pmatrix}$-tensor at $\mathbf{X} \in \mathcal{B}$ is a multilinear map

$$\mathbf{T} : \underbrace{T^*_{\mathbf{X}}\mathcal{B} \times ... \times T^*_{\mathbf{X}}\mathcal{B}}_{m \text{ copies}} \times \underbrace{T_{\mathbf{X}}\mathcal{B} \times ... \times T_{\mathbf{X}}\mathcal{B}}_{n \text{ copies}} \to \mathbb{R}. \tag{A.4}$$

$\mathbf{T}$ is said to be contravariant of order m and covariant of order n. In a local coordinate chart

$$\mathbf{T}(\alpha^1, ..., \alpha^m, \mathbf{V}_1, ..., \mathbf{V}_n) = T^{\ i_1 ... i_m}{}_{j_1 ... j_n} \alpha^1_{i_1} ... \alpha^m_{i_m} V_1^{j_1} ... V_n^{j_n}, \tag{A.5}$$

where $\alpha^k \in T^*_{\mathbf{X}}\mathcal{B}$ and $\mathbf{V}^k \in T_{\mathbf{X}}\mathcal{B}$.

A ***two-point tensor*** $\mathbf{T}$ of type $\begin{pmatrix} m & r \\ n & s \end{pmatrix}$ at $\mathbf{X} \in \mathcal{B}$ over a map $\varphi : \mathcal{B} \to \mathcal{S}$ is a multilinear map

$$T : \underbrace{T^*_{\mathbf{X}}\mathcal{B} \times ... \times T^*_{\mathbf{X}}\mathcal{B}}_{m \text{ copies}} \times \underbrace{T_{\mathbf{X}}\mathcal{B} \times ... \times T_{\mathbf{X}}\mathcal{B}}_{n \text{ copies}} \times \underbrace{T^*_{\mathbf{x}}\mathcal{S} \times ... \times T^*_{\mathbf{x}}\mathcal{S}}_{r \text{ copies}} \times \underbrace{T_{\mathbf{x}}\mathcal{S} \times ... \times T_{\mathbf{x}}\mathcal{S}}_{s \text{ copies}} \to \mathbb{R}, \tag{A.6}$$

where $\mathbf{x} = \varphi(\mathbf{X})$.

Let $\mathbf{y}$ be a vector field on $\mathcal{S}$ and $\varphi : \mathcal{B} \to \mathcal{S}$ a regular and orientation preserving C^1 map. The ***Piola transform*** of $\mathbf{y}$ is defined as

$$\mathbf{Y} = J\varphi^*\mathbf{y}, \tag{A.7}$$

where J is the Jacobian of φ. If $\mathbf{Y}$ is the Piola transform of $\mathbf{y}$, then the ***Piola identity*** holds:

$$\operatorname{Div} \mathbf{Y} = J(\operatorname{div} \mathbf{y}) \circ \varphi. \tag{A.8}$$

A p-form on a manifold M is a skew-symmetric $\binom{0}{p}$-tensor. The space of p-forms on M is denoted by $\Omega^p(M)$. If $\varphi : M \to N$ is a regular and orientation preserving C^1 map and $\alpha \in \Omega^p(\varphi(M))$, then

$$\int_{\varphi(M)} \alpha = \int_M \varphi^* \alpha. \tag{A.9}$$

Let $\pi : E \to \mathcal{S}$ be a vector bundle over a manifold $\mathcal{S}$ and let $\mathcal{E}(\mathcal{S})$ be the space of smooth sections of E and $\mathcal{X}(\mathcal{S})$ the space of vector fields on $\mathcal{S}$. A ***connection*** on E is a map $\nabla : \mathcal{X}(\mathcal{S}) \times \mathcal{E}(\mathcal{S}) \to \mathcal{E}(\mathcal{S})$ such that $\forall\ f, f_1, f_2 \in C^\infty(\mathcal{S}),\ \forall\ a_1, a_2 \in \mathbb{R}$

$$i)\quad \nabla_{f_1\mathbf{X}_1+f_2\mathbf{X}_2}\mathbf{Y} = f_1\nabla_{\mathbf{X}_1}\mathbf{Y} + f_2\nabla_{\mathbf{X}_2}\mathbf{Y}, \tag{A.10}$$

$$ii)\quad \nabla_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\nabla_{\mathbf{X}}(\mathbf{Y}_1) + a_2\nabla_{\mathbf{X}}(\mathbf{Y}_2), \tag{A.11}$$

$$iii)\quad \nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}. \tag{A.12}$$

A ***linear connection*** on $\mathcal{S}$ is a connection on $T\mathcal{S}$, i.e., $\nabla : \mathcal{X}(\mathcal{S}) \times \mathcal{X}(\mathcal{S}) \to \mathcal{X}(\mathcal{S})$. In a local chart

$$\nabla_{\partial_i}\partial_j = \gamma^k_{ij}\partial_k, \tag{A.13}$$

where γ^k_{ij} are Christoffel symbols of the connection and $\partial_i = \frac{\partial}{\partial x^i}$. A linear connection is said to be compatible with the metric of the manifold if

$$\nabla_{\mathbf{X}} \langle\!\langle \mathbf{Y}, \mathbf{Z} \rangle\!\rangle = \langle\!\langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle\!\rangle + \langle\!\langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle\!\rangle\,. \tag{A.14}$$

It can be shown that ∇ is compatible with $\mathbf{g}$ if and only if $\nabla\mathbf{g} = \mathbf{0}$. ***Torsion*** of a connection is defined as

$$\boldsymbol{\mathcal{T}}(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}], \tag{A.15}$$

where

$$[\mathbf{X}, \mathbf{Y}](F) = \mathbf{X}(\mathbf{Y}(F)) - \mathbf{Y}(\mathbf{X}(F)) \qquad \forall\ F \in C^\infty(\mathcal{S}), \tag{A.16}$$

is the ***commutator*** of $\mathbf{X}$ and $\mathbf{Y}$. ∇ is symmetric if it is torsion-free, i.e.

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]. \tag{A.17}$$

It can be shown that on any Riemannian manifold $(\mathcal{S}, \mathbf{g})$ there is a unique linear connection ∇ that is compatible with $\mathbf{g}$ and is torsion-free with the following Christoffel symbols

$$\gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right). \tag{A.18}$$

This is the ***Fundamental Lemma of Riemannian Geometry*** [21] and this connection is called the ***Levi-Civita connection***.

Curvature tensor $\boldsymbol{\mathcal{R}}$ of a Riemannian manifold $(\mathcal{S}, \mathbf{g})$ is a $\binom{1}{3}$-tensor $\boldsymbol{\mathcal{R}} : T^*_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{x}}\mathcal{S} \to \mathbb{R}$ defined as

$$\boldsymbol{\mathcal{R}}(\alpha, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \alpha\left(\nabla_{\mathbf{w}_1}\nabla_{\mathbf{w}_2}\mathbf{w}_3 - \nabla_{\mathbf{w}_2}\nabla_{\mathbf{w}_1}\mathbf{w}_3 - \nabla_{[\mathbf{w}_1,\mathbf{w}_2]}\mathbf{w}_3\right) \tag{A.19}$$

for $\alpha \in T^*_{\mathbf{x}}S,\ \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in T_{\mathbf{x}}S$. In a coordinate chart $\{x^a\}$

$$\mathcal{R}^a{}_{bcd} = \frac{\partial\gamma^a_{bd}}{\partial x^c} - \frac{\partial\gamma^a_{bc}}{\partial x^d} + \gamma^a_{ce}\gamma^e_{bd} - \gamma^a_{de}\gamma^e_{bc}. \tag{A.20}$$

Note that for an arbitrary vector field $\mathbf{w}$

$$w^a{}_{|bc} - w^a{}_{|cb} = \mathcal{R}^a{}_{bcd}w^d + \mathcal{T}^d{}_{cb}w^a{}_{|d}. \tag{A.21}$$

An n-dimensional Riemannian manifold is flat if it is isometric to Euclidean space. A Riemannian manifold is flat if and only if its curvature tensor vanishes. A Riemannian manifold $(\mathcal{B}, \mathbf{G})$ is conformally flat if there exists a smooth map $f : \mathcal{B} \to \mathbb{R}$ such that $\mathbf{G} = f\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the Euclidean metric. In *isothermal coordinates* the Riemannian metric has the following local form

$$\mathbf{G} = f(\mathbf{X})\left(dX_1^2 + ... + dX_n^2\right). \tag{A.22}$$

It is know that [2] any two-dimensional Riemannian manifold is conformally flat and the map f is unique. A corollary of this theorem in our theory of thermal stresses is the following. Given any smooth curved 2D solid that is stress free, there exists a unique change of temperature distribution such that in the new temperature distribution the 2D solid is flat and still stress free. Equivalently, starting from a stress free flat sheet, it is always possible to deform it to any smooth curved shape by changing temperature without imposing any residual stresses.

For a Riemannian manifold $(\mathcal{B}, \mathbf{G})$ the Weyl-Schouten tensor is defined as [34]

$$\mathsf{C}_{IJK} = \nabla_K \mathcal{R}_{IJ} - \nabla_J \mathcal{R}_{IK} - \frac{1}{4}\left(G_{IJ}\frac{\partial \mathcal{R}}{\partial X^K} - G_{IK}\frac{\partial \mathcal{R}}{\partial X^J}\right), \tag{A.23}$$

where $\boldsymbol{\mathcal{R}}$ and $\mathcal{R}$ are the Ricci tensor and the scalar curvature, respectively. A necessary and sufficient condition for a Riemannian manifold $(\mathcal{B}, \mathbf{G})$ to be conformally flat is $\mathsf{C} = \mathsf{0}$ when $\dim \mathcal{B} = 3$.

A.1 Absolute Parallelizable (AP) Geometry

In many physical problems in which deformation is coupled with other phenomena, e.g. plasticity, growth/remodeling, thermal expansion/contraction, etc. all one can hope to do is to locally decouple the elastic deformations from the inelastic deformations. This has led to many works that start from a decomposition of deformation gradient $\mathbf{F} = \mathbf{F}_e\mathbf{F}_i$, where $\mathbf{F}_e$ is the elastic deformation gradient and $\mathbf{F}_i$ is the remaining local deformation or inelastic deformation gradient.

Given an inelastic deformation gradient, here a thermal deformation gradient, a vector in the tangent space of $\mathbf{X} \in \mathcal{B}$, i.e. $\mathbf{W} \in T_{\mathbf{X}}\mathcal{B}$ is mapped to another vector $\hat{\mathbf{W}} = \mathbf{F}_T\mathbf{W}$. Traditionally these vectors are assumed to lie in the tangent bundle of an "intermediate configuration." In the literature, intermediate configuration is not clearly defined and at first glance it seems to be more or less mysterious as was explained in §2.3. These are closely related to parallelizable manifolds (or absolutely parallelizable (AP) manifolds) going back to the works by Eisenhart [8, 9]. See also Youssef and Sid-Ahmed [43] and Wanas [40]. In an n-dimensional AP-manifold M, one starts with a field of n linearly independent vectors $\left\{\mathbf{E}_{(A)}\right\}$ that span the tangent vector at each point. Let us denote the components of $\mathbf{E}_{(A)}$ by $\mathbf{E}^I_{(A)}$. The dual vectors, i.e. the corresponding basis vectors for the cotangent space are denoted by $\left\{\mathbf{E}^{(A)}\right\}$ with components $\left\{\mathbf{E}^{(A)}_I\right\}$. Note that

$$\mathbf{E}^{(A)}_I \mathbf{E}^I_{(B)} = \delta^A_B \qquad \text{and} \qquad \mathbf{E}^{(A)}_I \mathbf{E}^J_{(A)} = \delta^I_J. \tag{A.24}$$

One is then interested in equipping M with a connection Γ^I_{JK} such that the basis vectors $\left\{\mathbf{E}_{(A)}\right\}$ are covariantly constant, i.e.[13]

$$\mathbf{E}^I_{(A)|J} = 0. \tag{A.25}$$

Note that

$$\mathbf{E}^I_{(A)|JK} - \mathbf{E}^I_{(A)|KJ} = \mathcal{R}^I{}_{LJK}\mathbf{E}^L_{(A)} + \mathcal{T}^L{}_{KJ}\mathbf{E}^I_{(A)|L}. \tag{A.26}$$

Thus (A.25) implies that

$$\mathcal{R}^I{}_{LJK} = 0 \tag{A.27}$$

i.e., M is flat with respect to the connection Γ^I_{JK}. Note that

$$\mathbf{E}^I_{(A)|J} = \frac{\partial \mathbf{E}^I_{(A)}}{\partial X^J} + \Gamma^I_{JK}\mathbf{E}^K_{(A)}. \tag{A.28}$$

Thus

$$\mathbf{E}^{(A)}_L \frac{\partial \mathbf{E}^I_{(A)}}{\partial X^J} + \Gamma^I_{LK} = 0. \tag{A.29}$$

[13] Equivalently, the tangent bundle is a trivial bundle, so that the associated principal bundle of linear frames has a section on M.

Hence

$$\Gamma^I_{JK} = -\mathbf{E}^{(A)}_J \frac{\partial \mathbf{E}^I_{(A)}}{\partial X^K} = \mathbf{E}^{(A)}_I \frac{\partial \mathbf{E}^{(A)}_J}{\partial X^K}. \tag{A.30}$$

This is similar in form to the connections used by many authors, e.g. by Bilby, et al. [4] and Kondo [15] for dislocations, by Epstein and Elżanowski [10] for material inhomogeneities, and by Stojanović, et al. [36] for thermal stresses.

Looking at local charts $\{X^A\}$ and $\{U^I\}$ for the reference and intermediate configurations, we have

$$dU^I = (F_T)^I{}_A dX^A. \tag{A.31}$$

$(F_T)^I{}_A$ can be identified with $\mathbf{E}^I_{(A)}$, and hence

$$\Gamma^I_{JK} = (F_T)^I{}_A \frac{\partial \left(F_T^{-1}\right)^A{}_J}{\partial X^K}, \tag{A.32}$$

is the connection used in [36; 37]. Note that this connection is curvature free but has non-vanishing torsion.

A.2 Geometric Elasticity

Let us next review a few of the basic notions of geometric continuum mechanics. A ***body*** $\mathcal{B}$ is identified with a Riemannian manifold $\mathcal{B}$ and a ***configuration*** of $\mathcal{B}$ is a mapping $\varphi : \mathcal{B} \to \mathcal{S}$, where $\mathcal{S}$ is another Riemannian manifold. The set of all configurations of $\mathcal{B}$ is denoted $\mathcal{C}$. A ***motion*** is a curve $c : \mathbb{R} \to \mathcal{C}; t \mapsto \varphi_t$ in $\mathcal{C}$. It is assumed that the body is stress free in the material manifold.

For a fixed t, $\varphi_t(\mathbf{X}) = \varphi(\mathbf{X}, t)$ and for a fixed $\mathbf{X}$, $\varphi_{\mathbf{X}}(t) = \varphi(\mathbf{X}, t)$, where $\mathbf{X}$ is position of material points in the undeformed configuration $\mathcal{B}$. The ***material velocity*** is the map $\mathbf{V}_t : \mathcal{B} \to \mathbb{R}^3$ given by

$$\mathbf{V}_t(\mathbf{X}) = \mathbf{V}(\mathbf{X}, t) = \frac{\partial \varphi(\mathbf{X}, t)}{\partial t} = \frac{d}{dt}\varphi_{\mathbf{X}}(t). \tag{A.33}$$

Similarly, the ***material acceleration*** is defined by

$$\mathbf{A}_t(\mathbf{X}) = \mathbf{A}(\mathbf{X}, t) = \frac{\partial \mathbf{V}(\mathbf{X}, t)}{\partial t} = \frac{d}{dt}\mathbf{V}_{\mathbf{X}}(t). \tag{A.34}$$

In components

$$A^a = \frac{\partial V^a}{\partial t} + \gamma^a_{bc} V^b V^c, \tag{A.35}$$

where γ^a_{bc} is the Christoffel symbol of the local coordinate chart $\{x^a\}$. Note that $\mathbf{A}$ does not depend on the connection coefficients of the material manifold.

Here it is assumed that φ_t is invertible and regular. The ***spatial velocity*** of a regular motion φ_t is defined as

$$\mathbf{v}_t : \varphi_t(\mathcal{B}) \to \mathbb{R}^3, \quad \mathbf{v}_t = \mathbf{V}_t \circ \varphi_t^{-1}, \tag{A.36}$$

and the ***spatial acceleration*** $\mathbf{a}_t$ is defined as

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \nabla_{\mathbf{v}} \mathbf{v}. \tag{A.37}$$

In components

$$a^a = \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^b} v^b + \gamma^a_{bc} v^b v^c. \tag{A.38}$$

Let $\varphi : \mathcal{B} \to \mathcal{S}$ be a C^1 configuration of $\mathcal{B}$ in $\mathcal{S}$, where $\mathcal{B}$ and $\mathcal{S}$ are manifolds. Recall that the deformation gradient is the tangent map of φ and is denoted by $\mathbf{F} = T\varphi$. Thus, at each point $\mathbf{X} \in \mathcal{B}$, it is a linear map

$$\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \to T_{\varphi(\mathbf{X})}\mathcal{S}. \tag{A.39}$$

If $\{x^a\}$ and $\{X^A\}$ are local coordinate charts on $\mathcal{S}$ and $\mathcal{B}$, respectively, the components of $\mathbf{F}$ are

$$F^a{}_A(\mathbf{X}) = \frac{\partial \varphi^a}{\partial X^A}(\mathbf{X}). \tag{A.40}$$

The deformation gradient may be viewed as a two-point tensor

$$\mathbf{F}(\mathbf{X}) : T^*_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{X}}\mathcal{B} \to \mathbb{R}; \quad (\alpha, \mathbf{V}) \mapsto \langle \alpha, T_{\mathbf{X}}\varphi \cdot \mathbf{V} \rangle. \tag{A.41}$$

Suppose $\mathcal{B}$ and $\mathcal{S}$ are Riemannian manifolds with inner products $\langle\!\langle , \rangle\!\rangle_{\mathbf{X}}$ and $\langle\!\langle , \rangle\!\rangle_{\mathbf{x}}$ based at $\mathbf{X} \in \mathcal{B}$ and $\mathbf{x} \in \mathcal{S}$, respectively. Recall that the transpose of $\mathbf{F}$ is defined by

$$\mathbf{F}^{\mathsf{T}} : T_{\mathbf{x}}\mathcal{S} \to T_{\mathbf{X}}\mathcal{B}, \quad \langle\!\langle \mathbf{F}\mathbf{V}, \mathbf{v} \rangle\!\rangle_{\mathbf{x}} = \left\langle\!\!\left\langle \mathbf{V}, \mathbf{F}^{\mathsf{T}}\mathbf{v} \right\rangle\!\!\right\rangle_{\mathbf{X}}, \tag{A.42}$$

for all $\mathbf{V} \in T_{\mathbf{X}}\mathcal{B},\ \mathbf{v} \in T_{\mathbf{x}}\mathcal{S}$. In components

$$(F^{\mathsf{T}}(\mathbf{X}))^A{}_a = g_{ab}(\mathbf{x}) F^b{}_B(\mathbf{X}) G^{AB}(\mathbf{X}), \tag{A.43}$$

where $\mathbf{g}$ and $\mathbf{G}$ are metric tensors on $\mathcal{S}$ and $\mathcal{B}$, respectively. On the other hand, the ***dual*** of $\mathbf{F}$, a metric independent notion, is defined by

$$\mathbf{F}^*(\mathbf{x}) : T^*_{\mathbf{x}}\mathcal{S} \to T^*_{\mathbf{X}}\mathcal{B}; \quad \langle \mathbf{F}^*(\mathbf{x}) \cdot \alpha, \mathbf{W} \rangle = \langle \alpha, \mathbf{F}(\mathbf{X})\mathbf{W} \rangle, \tag{A.44}$$

for all $\alpha \in T^*_{\mathbf{x}}\mathcal{S}, \mathbf{W} \in T_{\mathbf{X}}\mathcal{B}$. Considering bases $\mathbf{e}_a$ and $\mathbf{E}_A$ for $\mathcal{S}$ and $\mathcal{B}$, respectively, one can define the corresponding dual bases $\mathbf{e}^a$ and $\mathbf{E}^A$. The matrix representation of $\mathbf{F}^*$ with respect to the dual bases is the transpose of $F^a{}_A$. $\mathbf{F}$ and $\mathbf{F}^*$ have the following local representations

$$\mathbf{F} = F^a{}_A \frac{\partial}{\partial x^a} \otimes dX^A, \qquad \mathbf{F}^* = F^a{}_A dX^A \otimes \frac{\partial}{\partial x^a}. \tag{A.45}$$

The ***right Cauchy-Green deformation tensor*** is defined by

$$\mathbf{C}(X) : T_{\mathbf{X}}\mathcal{B} \to T_{\mathbf{X}}\mathcal{B}, \qquad \mathbf{C}(\mathbf{X}) = \mathbf{F}(\mathbf{X})^{\mathsf{T}}\mathbf{F}(\mathbf{X}). \tag{A.46}$$

In components

$$C^A_B = (F^{\mathsf{T}})^A{}_a F^a{}_B. \tag{A.47}$$

It is straightforward to show that

$$\mathbf{C}^\flat = \varphi^*(\mathbf{g}) = \mathbf{F}^*\mathbf{g}\mathbf{F}, \text{ i.e. } \quad C_{AB} = (g_{ab} \circ \varphi) F^a{}_A F^b{}_B. \tag{A.48}$$

Let $\varphi_t : \mathcal{B} \to \mathcal{S}$ be a regular motion of $\mathcal{B}$ in $\mathcal{S}$ and $\mathcal{P} \subset \mathcal{B}$ a p-dimensional submanifold. The ***Transport Theorem*** says that for any p-form α on $\mathcal{S}$

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{P})} \alpha = \int_{\varphi_t(\mathcal{P})} \mathbf{L}_{\mathbf{v}}\alpha, \tag{A.49}$$

where $\mathbf{v}$ is the spatial velocity of the motion. In a special case when $\alpha = f dv$ and $\mathcal{P} = \mathcal{U}$ is an open set, one can write

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{P})} f dv = \int_{\varphi_t(\mathcal{P})} \left[\frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{v})\right] dv. \tag{A.50}$$