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A Geometrical Formulation of the Renormalization Group Method for Global Analysis

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On the basis of the classical theory of envelopes, we formulate the renormalization group (RG) method for global analysis, recently proposed by Goldenfeld et al. It is clarified in a generic way why the RG equation improves the global nature of function obtained in the perturbation theory.

§ 1. Introduction

Recently, Goldenfeld et al.[1] have proposed a new method based on the renormalization group (RG) equation[2],[3] to get asymptotic behavior of solutions of differential equations. The method is simple and has a wide variety of applications, including the treatment of singular and reductive perturbation problems in a unified way. However, the reason why the RG equation can be relevant and useful for global analysis is obscure: The RG equation is usually related to the scale invariance of the system under consideration. The equations which can be treated using the RG method are not confined to those with scale invariance.[1] Actually, what the RG method does in Ref. 1) may be underestood as the construction of an approximate but *global* solution from those of a local nature which are obtained in the perturbation theory; the RG equation is used to improve the global behavior of the local solutions. This fact suggests that the RG method can be formulated in a purely mathematical way without recourse to the concept of the RG. A purpose of this paper is to show that this is the case, thereby revealing the mathematical structure of the method.

Our formulation is based on the classical theory of envelopes.[4] As is well known, the envelope of a family of curves or surfaces usually has an improved global nature compared with the curves or surfaces in the family. So it is natural that the theory of envelopes may have some relevance for global analysis. One will recognize that the powerfulness of the RG equation in global analysis and also in the quantum field theory[2],[3] is due to the fact that it is essentially an envelope equation. We shall also give a proof as to why the RG equation can give a globally improved solution to differential equations.

In the next section, a short review is given on the classical theory of envelopes, the notion of which is essential for the understanding of the present paper. In § 3, we formulate the RG method in the context of the theory of envelopes and give a foundation to the method. In § 4, we show a couple of other examples to apply our formulation. The last section is devoted to a brief summary and concluding remarks.

§ 2. A short review of the classical theory of envelopes

To make the discussion in the following sections clear, we here give a brief review of the theory of envelopes. Although the theory can be formulated in higher dimensions,[4] we discuss here only one-dimensional envelopes, i.e., curves, for simplicity.

Let $\{C_\tau\}_\tau$ be a family of curves parametrized by τ in the x-y plane; here C_τ is represented by the equation

$$F(x, y, \tau)=0 . \tag{2·1}$$

We suppose that $\{C_\tau\}_\tau$ has the envelope E, which is represented by the equation

$$G(x, y)=0 . \tag{2·2}$$

The problem is to obtain $G(x, y)$ from $F(x, y, \tau)$.

Now let E and a curve C_{τ_0} have the common tangent line at $(x, y)=(x_0, y_0)$, i.e., (x_0, y_0) is the point of tangency. Then x_0 and y_0 are functions of τ_0; $x_0=\phi(\tau_0)$, $y_0=\psi(\tau_0)$, and of course $G(x_0, y_0)=0$. Conversely, for each point (x_0, y_0) on E, there exists a parameter τ_0. So we can reduce the problem to obtain τ_0 as a function of (x_0, y_0); then $G(x, y)$ is obtained as $F(x, y, \tau(x, y))=G(x, y)$.*) $\tau_0(x_0, y_0)$ can be obtained as follows.

The tangent line of E at (x_0, y_0) is given by

$$\psi'(\tau_0)(x-x_0)-\phi'(\tau_0)(y-y_0)=0 , \tag{2·3}$$

while the tangent line of C_{τ_0} at the same point reads

$$F_x(x_0, y_0, \tau_0)(x-x_0)+F_y(x_0, y_0, \tau_0)(y-y_0)=0 . \tag{2·4}$$

Here $F_x=\partial F/\partial x$ and $F_y=\partial F/\partial y$. Since both equations must give the same line,

$$F_x(x_0, y_0, \tau_0)\phi'(\tau_0)+F_y(x_0, y_0, \tau_0)\psi'(\tau_0)=0 . \tag{2·5}$$

On the other hand, differentiating $F(x(\tau_0), y(\tau_0), \tau_0)=0$ with respect to τ_0, one has

$$F_x(x_0, y_0, \tau_0)\phi'(\tau_0)+F_y(x_0, y_0, \tau_0)\psi'(\tau_0)+F_{\tau_0}(x_0, y_0, \tau_0)=0 , \tag{2·6}$$

hence

$$F_{\tau_0}(x_0, y_0, \tau_0)\equiv\frac{\partial F(x_0, y_0, \tau_0)}{\partial \tau_0}=0 . \tag{2·7}$$

One can thus eliminate the parameter τ_0 to find a relation between x_0 and y_0,

$$G(x, y)=F(x, y, \tau_0(x, y))=0 \tag{2·8}$$

with the replacement $(x_0, y_0)\to(x, y)$. $G(x, y)$ is called the discriminant of $F(x, y, \tau)$.

Comments are in order here: (i) When the family of curves is given by the function $y=f(x, \tau)$, the condition Eq. (2·7) is reduced to $\partial f/\partial \tau_0=0$; the envelope is given by $y=f(x, \tau_0(x))$. (ii) The equation $G(x, y)=0$ may give not only the envelope E but also

*) Since there is relation $G(x_0, y_0)=0$ between x_0 and y_0, τ_0 is actually a function of x_0 *or* y_0.

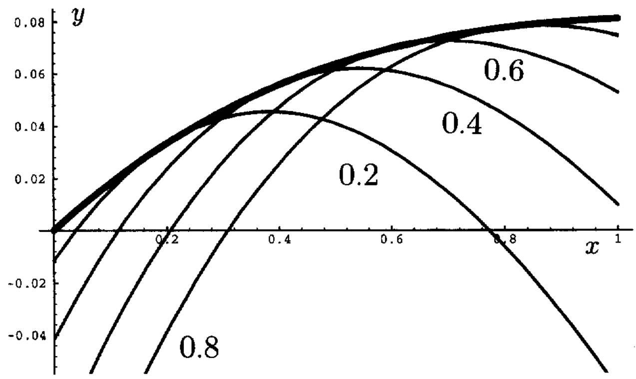

Fig. 1. A family of functions and its envelope: The thin lines represent $y=\exp(-\epsilon\tau_0)(1-\epsilon(x-\tau_0))+\exp(-x)$ with $\tau_0=0.2$, 0.4, 0.6 and 0.8, which are attached to the respective lines. The thick line represents the envelope $y=\exp(-\epsilon x)-\exp(-x)$ ($\epsilon=0.8$).

a set of singularities of the curves $\{C_\tau\}_\tau$. This is because the condition $\partial F/\partial x=\partial F/\partial y=0$ is also compatible with Eq. (2·7).

As an example, let

$$y=f(x,\tau)=e^{-\epsilon\tau}(1-\epsilon\cdot(x-\tau))+e^{-x}. \tag{2·9}$$

Note that y is unbounded for $x-\tau\to\infty$ due to the secular term. The envelope E of the curves C_τ is obtained as follows: From $\partial f/\partial\tau=0$, one has $\tau=x$. That is, the parameter in this case is the x-coordinate of the point of the tangency of E and C_τ. Thus the envelope is found to be

$$y=f(x,x)=e^{-\epsilon x}-e^{-x}. \tag{2·10}$$

One can see that the envelope is bounded even for $x\to\infty$. In short, we have obtained a function as an envelope with a better global nature than functions which are bounded only locally.

As an illustration, we show in Fig. 1 some of the curves given by $y=f(x,\tau_0)$ together with the envelope.

§ 3. Formulation of the RG method based on the theory of envelopes

In this section, we formulate and give a foundation of the RG method[1] in the context of the classical theory of envelopes sketched in the previous section. Our formulation also includes an improvement of the prescription.

Although the RG method can be applied to both (non-linear) ordinary and partial differential equations, let us take the following simplest example to show our formulation:

$$\frac{d^2x}{dt^2}+\epsilon\frac{dx}{dt}+x=0, \tag{3·1}$$

where ϵ is supposed to be small. The solution to Eq. (3·1) reads

$$x(t)=A\exp\left(-\frac{\epsilon}{2}t\right)\sin\left(\sqrt{1-\frac{\epsilon^2}{4}}t+\theta\right), \tag{3·2}$$

where A and θ are constants to be determined by an initial condition.

Now, let us blindly try to get the solution in the perturbation theory, expanding x as

$$x(t)=x_0(t)+\epsilon x_1(t)+\epsilon^2 x_2(t)+\cdots, \tag{3·3}$$

where x_n $(n=0, 1, 2, \cdots)$ satisfy

$$\ddot{x}_0+x_0=0, \quad \ddot{x}_{n+1}+x_{n+1}=-\dot{x}_n. \tag{3·4}$$

Thus $x_0=A_0\sin(t+\theta_0)$, $\ddot{x}_1+x_1=-A_0\cos(t+\theta_0)$, and so on. Then we get for x_1 and x_2 as special solutions

$$x_1(t)=-\frac{A_0}{2}\cdot(t-t_0)\sin(t+\theta_0),$$

$$x_2(t)=\frac{A_0}{8}\{(t-t_0)^2\sin(t+\theta_0)-(t-t_0)\cos(t+\theta_0)\}. \tag{3·5}$$

Here we have intentionally omitted the unperturbed solution from $x_n(t)(n=1, 2, \cdots)$. Although this prescription is not adopted in Ref. 1), subsequent calculations are simplified with this prescription; see also § 4.*) It should be noted that the secular terms have appeared in the higher order terms, which are absent in the exact solution, and invalidates the perturbation theory for t far away from t_0.

Inserting Eq. (3·5) into Eq. (3·3), we have

$$x(t, t_0)=A_0\sin(t+\theta_0)-\epsilon\frac{A_0}{2}(t-t_0)\sin(t+\theta_0)$$
$$+\epsilon^2\frac{A_0}{8}\{(t-t_0)^2\sin(t+\theta_0)-(t-t_0)\cos(t+\theta_0)\}+O(\epsilon^3). \tag{3·6}$$

Now we have a family of curves $\{C_{t_0}\}_{t_0}$ given by functions $\{x(t, t_0)\}_{t_0}$ parametrized with t_0. They are all solutions of Eq. (3·1) up to $O(\epsilon^3)$, but only valid locally, i.e., for t near t_0. Let us find a function $x_E(t)$ representing the envelope E of $\{C_{t_0}\}_{t_0}$.

According to the previous section, we only have to eliminate t_0 from

$$\frac{\partial x(t, t_0)}{\partial t_0}=0, \tag{3·7}$$

and insert the resultant $t_0(t)$ into $x(t, t_0)$. Then we have $x_E(t)=x(t, t_0(t))$. It will be shown that $x_E(t)$ satisfies the original differential equation Eq. (3·1) *uniformly* $\forall t$ up to $O(\epsilon^4)$ (see below).

Equation (3·7) is in the same form as the RG equation, hence the name of the RG method.[1] In our formulation, this is a condition for constructing the envelope.

Here comes another crucial point of the method.[1] We assume that A_0 and B_0 are

*) It is amusing to see that the unperturbed solution in the higher order terms $x_n(n=1, 2, \cdots)$ is analogous to the "dangerous" term in Bogoliubov's sense in the quantum-field theory of superfluidity and superconductivity.[5]

functionally dependent on t_0:

$$A_0 = A_0(t_0), \quad \theta_0 = \theta_0(t_0), \tag{3·8}$$

and accordingly $x(t, t_0) = x(t, A_0(t_0), \theta_0(t_0), t_0)$. Then it will be found that Eq. (3·7) gives a complicated equation involving $A_0(t_0)$, $\theta_0(t_0)$ and their derivatives as well as t_0. It turns out, however, that one can actually greatly reduce the complexity of the equation by assuming that the parameter t_0 coincides with the point of tangency, that is,

$$t_0 = t, \tag{3·9}$$

because $A_0(t_0)$ and $\theta_0(t_0)$ can be determined so that $t_0 = t$. We remark here that the meaning of setting $t_0 = t$ is not clearly explained in Ref. 1), while in our case, this has the clear meaning of choosing the point of tangency at $t = t_0$.*)

From Eqs. (3·7) and (3·9), we have

$$\frac{dA_0}{dt_0} + \epsilon A_0 = 0, \quad \frac{d\theta_0}{dt_0} + \frac{\epsilon^2}{8} = 0. \tag{3·10}$$

Solving the simple equations, we have

$$A_0(t_0) = \bar{A} e^{-\epsilon t_0/2}, \quad \theta_0(t_0) = -\frac{\epsilon^2}{8} t_0 + \bar{\theta}, \tag{3·11}$$

where $\bar{A}$ and $\bar{\theta}$ are constant numbers. Thus we get

$$x_E(t) = x(t, t) = \bar{A} \exp\left(-\frac{\epsilon}{2} t\right) \sin\left(\left(1 - \frac{\epsilon^2}{8}\right) t + \bar{\theta}\right). \tag{3·12}$$

Noting that $\sqrt{1-\epsilon^2/4} = 1 - \epsilon^2/8 + O(\epsilon^4)$, one finds that the resultant envelope function $x_E(t)$ is an approximate but *global* solution to Eq. (3·1) (see Eq. (3·2)). In short, the solution obtained in the perturbation theory with the local nature has been "improved" by the envelope equation to become a global solution.

There is another version of the RG method,[1] which involves a "renormalization" of the parameters. We shift the parameter for the local curves as follows: Let τ be close to t, and write $t - t_0 = t - \tau + \tau - t_0$. Then putting

$$A(\tau) = A_0(t_0) Z(t_0, \tau), \quad Z(t_0, \tau) = 1 - \frac{\epsilon}{2}(\tau - t_0) + \frac{\epsilon^2}{8}(\tau - t_0)^2,$$

$$\theta(\tau) = \theta_0(t_0) + \delta\theta, \quad \delta\theta = -\frac{\epsilon^2}{8}(\tau - t_0), \tag{3·13}$$

we have

$$x(t, \tau) = A(\tau)\sin(t + \theta(\tau)) - \epsilon \frac{A(\tau)}{2}(t - \tau)\sin(t + \theta(\tau))$$

*) It is interesting that the procedure to get the envelope of $x(t, A_0(t_0), \theta_0(t_0), t_0)$ assuming a functional dependence of A_0 and θ_0 on t_0 is similar to the standard method in which the general solution of a partial differential equation of first order is constructed from the complete solution.[4]

$$+\epsilon^2\frac{A(\tau)}{8}\{(t-\tau)^2\sin(t+\theta(\tau))-(t-\tau)\cos(t+\theta(\tau))\}+O(\epsilon^3), \quad (3\cdot14)$$

where

$$x(\tau,\tau)=A(\tau)\sin(\tau+\theta(\tau)). \quad (3\cdot15)$$

Then the envelope of the curves given by $\{x(t,\tau)\}_\tau$ will be found to be the same as that given in Eq. (3·12).

This may conclude the account of our formulation of the RG method based on the classical theory of envelopes. However, there is a remaining problem: Does $x_E(t) \equiv x(t,t)$ indeed satisfy the original differential equation? In our simple example, the result Eq. (3·12) shows that it does. It is also the case for all the resultant solutions worked out here and in Ref. 1). We are, however, not aware of a general proof available to show that the envelope function should satisfy the differential equation (uniformly) up to the same order as the local solutions do locally. We give here such a proof for a wide class of linear and non-linear ordinary differential equations (ODE). The proof can be easily generalized to partial differential equations (PDE).[12]

Let us assume that the differential equation under consideration can be converted to the following coupled equation of *first order*:

$$\frac{d\boldsymbol{q}(t)}{dt}=\boldsymbol{F}(\boldsymbol{q}(t),t;\epsilon), \quad (3\cdot16)$$

where ${}^t\boldsymbol{q}=(q_1,q_2,\cdots)$, and $\boldsymbol{F}$ are column vectors. It should be noted that $\boldsymbol{F}$ may be a non-linear function of $\boldsymbol{q}$ and t, although in our example,

$$q_1=x, \quad q_2=\dot{x}, \quad \boldsymbol{F}=\begin{pmatrix} q_2 \\ -q_1-\epsilon q_2 \end{pmatrix}, \quad (3\cdot17)$$

i.e., $\boldsymbol{F}$ is linear in $\boldsymbol{q}$. We also assume that we have an approximate local solution $\tilde{\boldsymbol{q}}(t,t_0)$ around $t=t_0$ up to $O(\epsilon^n)$:

$$\frac{d\tilde{\boldsymbol{q}}}{dt}=\boldsymbol{F}(\tilde{\boldsymbol{q}},t;\epsilon)+O(\epsilon^n). \quad (3\cdot18)$$

One can see for our example to satisfy this using Eq. (3·6).

The envelope equation implies

$$\frac{\partial\tilde{\boldsymbol{q}}(t,t_0)}{\partial t_0}=0 \quad (3\cdot19)$$

at $t_0=t$. With this condition, $\boldsymbol{q}_E(t)$ corresponding to $x_E(t)$ is defined by

$$\boldsymbol{q}_E(t)=\tilde{\boldsymbol{q}}(t,t). \quad (3\cdot20)$$

It is now easy to show that $\boldsymbol{q}_E(t)$ satisfies Eq. (3·16) up to the same order that $\tilde{\boldsymbol{q}}(t,t_0)$ does: In fact, for $\forall t_0$

$$\left.\frac{d\boldsymbol{q}_E(t)}{dt}\right|_{t=t_0}=\left.\frac{d\tilde{\boldsymbol{q}}(t,t_0)}{dt}\right|_{t=t_0}+\left.\frac{\partial\tilde{\boldsymbol{q}}(t,t_0)}{\partial t_0}\right|_{t=t_0}=\left.\frac{d\tilde{\boldsymbol{q}}(t,t_0)}{dt}\right|_{t=t_0}, \quad (3\cdot21)$$

where Eq. (3·19) has been used. Noting that $\boldsymbol{F}(\boldsymbol{q}_E(t_0), t_0; \epsilon)=\boldsymbol{F}(\boldsymbol{q}(t_0, t_0), t_0; \epsilon)$, we see $\forall t$

$$\frac{d\boldsymbol{q}_E}{dt}=\boldsymbol{F}(\boldsymbol{q}_E(t), t; \epsilon)+O(\epsilon^n), \tag{3·22}$$

on account of Eq. (3·18). This completes the proof. It should be stressed that Eq. (3·22) is valid uniformly $\forall t$ in contrast to Eq. (3·18) which is valid only locally around $t=t_0$.

§ 4. Examples

Let us take a couple of example to apply our formulation. These can be converted to equations in the form given in Eq. (3·16).

4.1. *A boundary-layer problem*

The first example is a typical boundary-layer problem:[6]

$$\epsilon\frac{d^2y}{dx^2}+(1+\epsilon)\frac{dy}{dx}+y=0 \tag{4·1}$$

with the boundary condition $y(0)=0$, $y(1)=1$. The exact solution to this problem is readily found to be

$$y(x)=\frac{\exp(-x)-\exp(-x/\epsilon)}{\exp(-1)-\exp(-1/\epsilon)}. \tag{4·2}$$

Now let us solve the problem in the perturbation theory. Introducing the inner variable X by $\epsilon X=x$,[6] and putting $Y(X)=y(x)$, the equation is converted to the following:

$$\frac{d^2Y}{dX^2}+\frac{dY}{dX}=-\epsilon\left(\frac{dY}{dX}+Y\right). \tag{4·3}$$

Expanding Y in a power series of ϵ as $Y=Y_0+\epsilon Y_1+\epsilon^2 Y_2+\cdots$, one has

$$\begin{aligned} &Y_0''+Y_0'=0, \\ &Y_1''+Y_1'=-Y_0'-Y_0. \\ &\quad\vdots \end{aligned} \tag{4·4}$$

Here, $Y'\equiv dY/dX$. To solve the equation, we set a boundary condition for $Y(X)$ and $Y_0(X)$ at $X=X_0$:

$$Y(X)=Y_0(X_0)=A_0, \tag{4·5}$$

where X_0 is an arbitrary constant and A_0 is supposed to be a function of X_0.

For this problem, we shall follow the prescription given in Ref. 1) for the higher order terms. Then the solutions to these equations may be written as

$$Y_0(X)=A_0-B_0e^{-(X-X_0)},$$

$$Y_1(X) = -A_0(X-X_0) - (B_0+C_0)(e^{-(X-X_0)}-1)\,. \tag{4·6}$$

Defining $A=A_0+\epsilon(B_0+C_0)$ and $B=B_0+\epsilon(B_0+C_0)$, we have

$$Y(X, X_0) = A - Be^{-(X-X_0)} - \epsilon A(X-X_0) + O(\epsilon^2)\,. \tag{4·7}$$

In terms of the original coordinate,

$$y(x, x_0) = Y(X, X_0) = A - Be^{-(x-x_0)/\epsilon} - A(x-x_0) + O(\epsilon^2) \tag{4·8}$$

with $x_0 = X_0/\epsilon$.

Now let us obtain the envelope $Y_E(X)$ of the family of functions $\{Y(X, X_0)\}_{X_0}$ each of which has the common tangent with $Y_E(X)$ at $X=X_0$. According to the standard procedure to obtain the envelope, we first solve the equation,

$$\frac{\partial Y}{\partial X_0} = 0 \quad \text{with} \quad X_0 = X\,, \tag{4·9}$$

and then make the identification $Y(X, X) = Y_E(X)$.

Equation (4·9) claims

$$A' + \epsilon A = 0\,, \quad B' + B = 0 \tag{4·10}$$

with the solutions $A(X) = \bar{A}\exp(-\epsilon X)$, $B(X) = \bar{B}\exp(-X)$, where $\bar{A}$ and $\bar{B}$ are constant. Thus one finds

$$Y_E(X) = Y(X, X) = A(X) - B(X) = \bar{A}e^{-\epsilon X} - \bar{B}e^{-X}\,. \tag{4·11}$$

In terms of the original variable x,

$$y_E(x) \equiv Y_E(X) = \bar{A}\exp(-x) - \bar{B}\exp\left(-\frac{x}{\epsilon}\right). \tag{4·12}$$

It is remarkable that the resultant $y_E(x)$ can admit both the inner and outer boundary conditions simultaneously; $y(0)=1$, $y(1)=1$. In fact, with the boundary conditions we have $\bar{A}=\bar{B}=1/(\exp(-1)-\exp(-1/\epsilon))$, hence $y_E(x)$ coincides with the exact solution

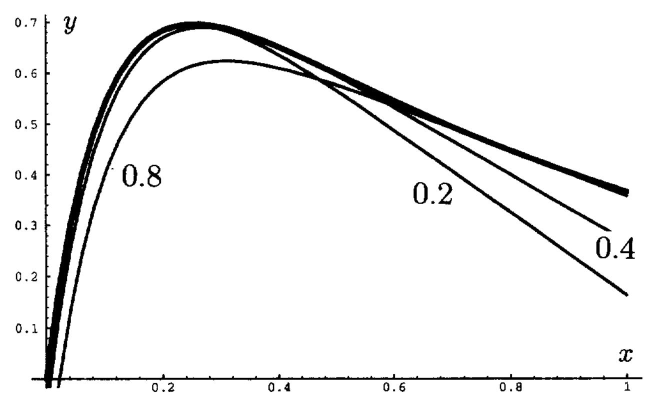

Fig. 2. The thin lines represent $y(x, x_0) = A(x_0) - B(x_0)\exp(-(x-x_0)/\epsilon) - A(x_0)(x-x_0)$ with $x_0=0.2$, 0.4 and 0.8, which are attracched to the respective lines. The thick line represents $y(x)$ given in Eq. (4·2) ($\epsilon=0.1.$).

$y(x)$ given in Eq. (4·2).

In Fig. 2, we show the exact solution $y(x)$ and the local solutions $y(x, x_0)$ for several x_0: One can clearly see that the exact solution is the envelope of the curves given by $\{y(x, x_0)\}_{x_0}$.

A comment is in order here. If we adopted the prescription given in § 3 for the higher order terms, the perturbed solution $Y_1(X)$ reads $Y_1(X)=-A_0(X-X_0)$; note the boundary condition Eq. (4·5). Then the calculations following Eq. (4·6) would be slightly simplified.

4.2. *A non-linear oscillator*

In this subsection, we consider the following Rayleigh equation,[6),1)]

$$\ddot{y}+y=\epsilon\left(\dot{y}-\frac{1}{3}\dot{y}^3\right). \tag{4·13}$$

Applying the perturbation theory with the expansion $y=y_0+\epsilon y_1+\epsilon^2 y_2+\cdots$, one has

$$y(t, t_0)=R_0\sin(t+\theta_0)+\epsilon\left\{\left(\frac{R_0}{2}-\frac{R_0^3}{8}\right)(t-t_0)\sin(t+\theta_0)\right.$$
$$\left.+\frac{R_0^3}{96}\cos 3(t+\theta_0)\right\}+O(\epsilon^2). \tag{4·14}$$

Here we have not included the terms proportional to the unperturbed solution in the higher order terms in accordance with the prescription given in § 3, so that the following calculation is somewhat simplified than in Ref. 1). Furthermore, the result with this prescription will coincide with the one given in the Krylov-Bogoliubov-Mitropolsky method,[7)] as we will see in Eq. (4·18).

Equation (4·14) gives a family of curves $\{C_{t_0}\}_{t_0}$ parametrized with t_0. The envelope E of $\{C_{t_0}\}_{t_0}$ with the point of tangency at $t=t_0$ can be obtained as

$$\frac{\partial y(t, t_0)}{\partial t_0}=0 \tag{4·15}$$

with $t_0=t$. Assuming that $\dot{R}_0$ and $\dot{\theta}_0$ are $\sim O(\epsilon)$ at most, we have

$$\dot{R}_0=\epsilon\left(\frac{R_0}{2}-\frac{R_0^3}{8}\right), \quad \dot{\theta}_0=0, \tag{4·16}$$

the solution of which reads

$$R_0(t)=\frac{\bar{R}_0}{\sqrt{\exp(-\epsilon t)+\bar{R}_0^2(1-\exp(-\epsilon t))/4}}, \tag{4·17}$$

with $\bar{R}_0=R_0(0)$ and $\theta_0=$ constant. Thus the envelope is given by

$$y_E(t)=y(t, t)=R_0(t)\sin(t+\theta_0)+\epsilon\frac{R_0(t)^3}{96}\cos 3(t+\theta_0)+O(\epsilon^2). \tag{4·18}$$

This is an approximate but global solution to Eq. (4·13) with a limit cycle.*) We note that since Eq. (4·13) can be rewritten in the form of Eq. (3·16), Eq. (4·18) satisfies Eq. (4·13) up to $O(\epsilon^2)$.

§ 5. A brief summary and concluding remarks

We have given a geometrical formulation of the RG method for global analysis recently proposed by Goldenfeld et al.[1] We have shown that the RG equation can be interpreted as an envelope equation, and given a purely mathematical foundation to the method. We have also given a proof that the envelope function satisfies the differential equation up to the same order as do the functions representing the local curves.

It is important that a geometrical meaning of the RG equation even in a generic sense has been clarified in the present work. The RG equation appears in various fields in physics. For example, let us take a model in the quantum field theory[8]

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{\lambda}{4!}\phi^4+\text{c.t.}\,, \tag{5·1}$$

where c.t. stands for counterterms. The true vacuum in the quantum field theory is determined by the minimum of the so-called effective potential $\mathcal{V}(\phi_c)$.[8),9)] In the one-loop approximation, the renormalized effective potential reads

$$\mathcal{V}(\phi_c, M)=\frac{\lambda}{4!}\phi_c{}^4+\frac{\lambda^2\phi_c{}^4}{256\pi^2}\left(\ln\frac{\phi_c{}^2}{M^2}-\frac{25}{6}\right), \tag{5·2}$$

where M^2 is the renormalization point. To see a correspondence to the envelope theory, one may parametrize as $\phi_c{}^2=\exp t$ and $M^2=\exp t_0$. Then one sees that $\ln \phi_c{}^2/M^2$ becomes a secular term $t-t_0$. In the quantum field theory, one applies the RG equation to improve the effective potential as follows:[8),10)]

$$\frac{\partial\mathcal{V}}{\partial M^2}=0 \quad \text{with} \quad M^2=\phi_c{}^2\,. \tag{5·3}$$

One sees that this is the envelope equation! The resultant "improved" effective potential is found to be

$$\mathcal{V}_{\text{impr}}(\phi_c)=\mathcal{V}(\phi_c, \phi_c)=\frac{\frac{\lambda}{4!}\phi_c{}^4}{1-\frac{3\lambda}{16\pi^2}\ln\frac{\phi_c}{\phi_{c0}}}\,. \tag{5·4}$$

Thus one can now understand that the "improved" effective potential is nothing but the envelope of the effective potential in the perturbation theory. One also sees the reason why the RG equation with $\phi_c=M$ can "improve" the effective potential. Then what is the physical significance of the envelope function $\mathcal{V}_{\text{impr}}(\phi_c)$? One can readily show that $\forall M$,

*) Equation (4·18) is slightly different from the one obtained by Cheng, Goldenfeld and Oono,[1] due to the different prescription for the treatment of the unperturbed functions in the higher order terms.

$$\left.\frac{\partial \mathcal{V}(\phi_c)_{\rm impr}}{\partial \phi_c^2}\right|_{\phi_c=M}=\left.\frac{\partial \mathcal{V}(\phi_c, M)}{\partial \phi_c^2}\right|_{\phi_c=M}, \tag{5·5}$$

owing to the envelope condition Eq. (5·3). This implies, for example, that the vacuum condensate ϕ_c that is given by $\partial \mathcal{V}_{\rm impr}/\partial \phi_c^2=0$ is correct up to the same order of the $\hbar$-expansion in which the original effective potential is calculated; this is irrespective of the size of the resultant ϕ_c. Detailed discussion of the application of envelope theory to quantum field theory will be reported elsewhere.[11),*)]

The RG equation also has had remarkable success in statistical physics, especially in critical phenomena.[3)] One may also note that there is another successful theory of critical phenomena called the coherent anomaly method (CAM).[13)] The relation between CAM and the RG equation theory is not known. Interestingly enough, CAM utilizes *envelopes* of susceptiblities and other thermodynamical quantities as a function of temperature. It might be possible to give a definite relation between CAM and RG theory, since the RG equation can be interpreted as an envelope equation, as shown in this work.

Mathematically, it is most important to give a rigorous proof for the RG method in general situations and to clarify what types of differential equations can be analyzed using this method, although we have given a simple proof for a class of ODE's. We note that the proof can be generalized to partial differential equations.[12)] One should also be able to estimate the accuracy of the envelope theory for a given equation. We hope that this paper may stimulate studies for a deeper understanding of global analysis based on the theory of envelopes.

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*) Recent renewed interest in "improving" the effective potential is motivated by the problem of how to "improve" the effective potentials with multi-scales as appearing in the standard model.[10)] The observation given here that the RG equation can be interpreted as an envelope equation may give insight into how to construct effective potentials with a global nature in multi-scale cases.

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Note added in proof: After submitting the present paper, the author was informed that S.L. Woodruff (Studies in App. Mathematics **90** (1993), 225) also developed a new method for constructing a large-scale (global) solution motivated by the fact that *the renormalization group technique of quantum field theory employs an invariance property to extend the region of validity of straightforward expansion:* Starting from the multiple-scale singular perturbation theory, he constructs *a uniformly-valid asymptotic expansion from the straightforward expansion at an arbitrary point in the large-scale domain,* which includes secular terms. Although the key equation of his method is not exactly the RG equation, nor he mentions the relevance of envelopes, we believe that his method is also best formulated in the context of envelope theory. The author thanks N. Goldenfold and Y. Oono for informing him of Woodruff's paper.