A Geometrical Interpretation of Frequency

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***Abstract*—The letter provides a geometrical interpretation of frequency in electric circuits. According to this interpretation, the frequency is defined as a multivector with symmetric and antisymmetric components. The conventional definition of frequency is shown to be a special case of the proposed theoretical framework. Several examples serve to show the features, generality as well as practical aspects of the proposed approach.**

***Index Terms*—Frequency, differential geometry, curvature, inner product, wedge product, geometrical product.**

I. Introduction

In power system applications, the frequency of an ac signal is conventionally defined as the time derivative of the argument of the cosine function of the signal itself [1]. This definition appears to have some issues. First, it depends on the representation of the signal itself. This has led to a tremendous number of publications, each of which using as starting point a different representation [2]. Second, the value of the frequency often depends on the transformation utilized to represent the ac signal. A good criterion to decide if a transformation is robust is to check whether a signal can be fully reconstructed to its original state if the inverse transformation is applied to the transformed signal [3]. This is a sensible criterion but does not guarantee the correctness and consistency of the estimation of the frequency itself. The value of the estimated frequency should be always the same (invariant) independently from the transformation. A third issue with the common definition of frequency and a large number of existing techniques to estimate the frequency is that they do not account for variations of the magnitude of the signal. This assumption poses serious issues for the estimation of the frequency from measurements. The conventional definition, in fact, implicitly assumes that one is able to measure the phase angle independently from the magnitude and that the measured signal is a sine wave. But this is not always the case, in particular, in transient conditions. The theoretical framework and definition of generalized frequency proposed in this letter address the issues above.

II. Outlines of Vector Operations and Space Curves

We first provide some outlines of operations with vectors and space curves. In the remainder of the letter, vectors are indicated in bold face (e.g., $\boldsymbol{v}$), whereas scalar quantities are in normal face (e.g., v). Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ be two n-dimensional vectors in $\mathbb{R}^n$.

The *inner product* is defined as:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i \,. \tag{1}$$

For example, in $\mathbb{R}^3$, $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$. The inner product is symmetric, associative, and commutative. In particular, the inner product of a vector by itself gives:

$$x = |\boldsymbol{x}| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} \,, \tag{2}$$

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where x is the magnitude of $\boldsymbol{x}$.

The *outer product* is defined as:

$$\boldsymbol{x} \otimes \boldsymbol{y} = \begin{bmatrix} x_1 y_1 & \dots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_n y_1 & \dots & x_n y_n \end{bmatrix} . \tag{3}$$

The *wedge product* is defined as:

$$\boldsymbol{x} \wedge \boldsymbol{y} = \boldsymbol{x} \otimes \boldsymbol{y} - \boldsymbol{y} \otimes \boldsymbol{x} \,. \tag{4}$$

For example, in $\mathbb{R}^3$, the wedge product gives:

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{bmatrix} 0 & b_{12} & -b_{31} \\ -b_{12} & 0 & b_{23} \\ b_{31} & -b_{23} & 0 \end{bmatrix} , \tag{5}$$

where $b_{ij} = x_i y_j - y_i x_j$. The result of the wedge product is a *bivector*. In the remainder of this letter it will be indicated with an uppercase bold symbol, e.g., $\mathbf{B} = \boldsymbol{x} \wedge \boldsymbol{y}$, where $\mathbf{B}$ is a skew-symmetric matrix. The wedge product is antisymmetric, associative, and anti-commutative. The latter means that $\boldsymbol{x} \wedge \boldsymbol{y} = -\boldsymbol{y} \wedge \boldsymbol{x}$ and, consequently $\boldsymbol{x} \wedge \boldsymbol{x} = \mathbf{0}$. In $\mathbb{R}^3$, the wedge product is similar to the *cross product* $\boldsymbol{x} \times \boldsymbol{y}$, although the result of the cross product is a vector not a tensor. For the developments of this letter, it is relevant to note that in Euclidean metric, the magnitude of a bivector is given by:

$$|\boldsymbol{x} \wedge \boldsymbol{y}| = |\mathbf{B}| = \sqrt{\sum_{i=1}^{n} \sum_{j>i}^{n} b_{ij}^2} \,. \tag{6}$$

For example, in $\mathbb{R}^3$:

$$|\boldsymbol{x} \wedge \boldsymbol{y}| = \sqrt{b_{12}^2 + b_{23}^2 + b_{31}^2} \,. \tag{7}$$

The *geometric product* is defined as:

$$\boldsymbol{x}\boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \wedge \boldsymbol{y} \,. \tag{8}$$

The result of the geometric product, which is called *multivector*, consists of two components. The first component, $\boldsymbol{x} \cdot \boldsymbol{y}$, is a scalar that represents the projection of $\boldsymbol{y}$ onto the vector $\boldsymbol{x}$. The second component, $\boldsymbol{x} \wedge \boldsymbol{y}$, represents a bivector orthogonal to the space defined by the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. It may seem strange at first to sum a scalar with a bivector but this is exactly the same kind of operation that is intended when one writes a complex number as $a + \jmath b$. Section IV shows that, in fact, the algebra of complex numbers is a special case of the algebra of multivectors.

In this work, we are interested in time-dependent n-dimensional *curves* (or trajectories), i.e., $\boldsymbol{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))$, where t is time. The time derivative of $\boldsymbol{x}$ is defined as:

$$\boldsymbol{x}' = \frac{d\boldsymbol{x}}{dt} = (x_1', x_2', \dots, x_n') \,. \tag{9}$$

From the geometrical point of view, $\boldsymbol{x}'$ is the *tangent vector* of the curve $\boldsymbol{x}$. Let us define s as the *arc length* of the curve $\boldsymbol{x}$, then the following property holds:

$$s' = \frac{ds}{dt} = \sqrt{\boldsymbol{x}' \cdot \boldsymbol{x}'} = x' \,. \tag{10}$$

It is important to note that the arc length s and thus also its derivatives are invariant with respect to the system of coordinates.

It is relevant to define the derivative of the curve $\boldsymbol{x}$ with respect to s, as follows:

$$\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{ds} = \frac{d\boldsymbol{x}}{dt}\frac{dt}{ds} = \frac{\boldsymbol{x}'}{x'}\,, \tag{11}$$

where we have used (10) and the identity $dt/ds = 1/s'$. From (11), it follows that $\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} = 1$. The vector $\dot{\boldsymbol{x}}$ is tangent to $\boldsymbol{x}$ and that the tangent vector to a curve is the unit vector if the arc length is chosen as a parameter.

Finally, it is relevant to recall the definition of another invariant quantity, namely the *curvature*, that plays an important role in differential geometry. The curvature is defined as:

$$\kappa = |\dot{\boldsymbol{x}} \wedge \ddot{\boldsymbol{x}}| = \frac{|\boldsymbol{x}' \wedge \boldsymbol{x}''|}{(x')^3}\,, \tag{12}$$

where

$$\ddot{\boldsymbol{x}} = \frac{d\dot{\boldsymbol{x}}}{ds} = \frac{\boldsymbol{x}''}{(x')^2} - \frac{x''\,\boldsymbol{x}'}{(x')^3} \tag{13}$$

is the tangent vector to $\dot{\boldsymbol{x}}$ and satisfies the condition $\dot{\boldsymbol{x}} \cdot \ddot{\boldsymbol{x}} = 0$.

We are now ready to present the main contribution of this work.

III. Frequency as a Multivector

Let us start with the vector of the magnetic flux, $\boldsymbol{\varphi}$. According to the Faraday's law of induction, one has:

$$-\,\boldsymbol{\varphi}' = \boldsymbol{v}\,, \tag{14}$$

where $\boldsymbol{v}$ is the vector of the voltage and the minus accounts for the Lenz's law but is not crucial for the discussion below. On the other hand, it is important to note that $\boldsymbol{\varphi}$ does not need to be known or to be measurable. In the context of this work, $\boldsymbol{\varphi}$ serves only to define the macroscopic effect of the magnetic field. In this context, the most important property of $\boldsymbol{\varphi}$ is that its time derivative is the vector of the voltage. If one interprets the components of the vector of the flux as the coordinates of a *curve*, say $\boldsymbol{x} = -\boldsymbol{\varphi}$, then the voltage $\boldsymbol{v} = \boldsymbol{x}'$ is the tangent vector to this curve.

According to the definitions given in Section II, one has:

$$s' = |\boldsymbol{\varphi}' \cdot \boldsymbol{\varphi}'| = \varphi' = v\,, \tag{15}$$

and

$$\dot{\boldsymbol{x}} = -\dot{\boldsymbol{\varphi}} = \frac{\boldsymbol{\varphi}'}{s'} = \frac{\boldsymbol{v}}{v}\,, \tag{16}$$

and

$$\ddot{\boldsymbol{x}} = -\ddot{\boldsymbol{\varphi}} = \frac{\boldsymbol{v}'}{v^2} - \frac{v'\,\boldsymbol{v}}{v^3}\,. \tag{17}$$

Since $\dot{\boldsymbol{x}} \cdot \ddot{\boldsymbol{x}} = 0$, one obtains:

$$0 = \dot{\boldsymbol{\varphi}} \cdot \ddot{\boldsymbol{\varphi}} = \frac{\boldsymbol{v} \cdot \boldsymbol{v}'}{v^3} - \frac{v'\,\boldsymbol{v} \cdot \boldsymbol{v}}{v^4} \tag{18}$$

which leads to:

$$\rho_v = \frac{v'}{v} = \frac{\boldsymbol{v}}{v} \cdot \frac{\boldsymbol{v}'}{v}\,. \tag{19}$$

Similarly, from the definition of curvature in (12), one obtains:

$$\kappa_v = |\dot{\boldsymbol{\varphi}} \wedge \ddot{\boldsymbol{\varphi}}| = \left|\frac{\boldsymbol{v}}{v} \wedge \left(\frac{\boldsymbol{v}'}{v^2} - \frac{v'\,\boldsymbol{v}}{v^3}\right)\right|, \tag{20}$$

and remembering that $\boldsymbol{v} \wedge \boldsymbol{v} = 0$, one obtains:

$$\kappa_v = \left|\frac{\boldsymbol{v}}{v} \wedge \frac{\boldsymbol{v}'}{v^2}\right|, \tag{21}$$

We define the magnitude of the frequency of $\boldsymbol{v}$, say ω_v, as:

$$\omega_v = v\,\kappa_v\,. \tag{22}$$

This definition, while admittedly a little obscure at this point, will be apparent in the examples presented in Section IV. From (21) and (22), one obtains:

$$\omega_v = \left|\frac{\boldsymbol{v}}{v} \wedge \frac{\boldsymbol{v}'}{v}\right|, \tag{23}$$

and, hence, we can define the bivector $\boldsymbol{\Omega}_v$ as:

$$\boldsymbol{\Omega}_v = \frac{\boldsymbol{v}}{v} \wedge \frac{\boldsymbol{v}'}{v}\,. \tag{24}$$

Based on (19) and (22) and on the definition of geometric product given in (8), we can finally provide the following novel and most important expression of this work:

$$\boxed{\rho_v + \boldsymbol{\Omega}_v = \frac{\boldsymbol{v}\boldsymbol{v}'}{v^2}}\,, \tag{25}$$

where we define the term $\rho_v + \boldsymbol{\Omega}_v$ as the *generalized frequency* of the voltage $\boldsymbol{v}$. The left-hand side of (25) depends only on geometric invariants, namely $v = s'$, $v' = s''$ and the components of the bivector that define the magnitude of the curvature κ_v. The generalized frequency, thus, does not depend on the system of coordinates with which $\boldsymbol{v}$ is represented or measured, nor the number of "dimensions" where $\boldsymbol{v}$ is defined. It is also interesting to observe that frequency is defined in (25) as the sum of a symmetric (ρ_v) and an antisymmetric term ($\boldsymbol{\Omega}_v$). Finally, we note that (25) has been obtained without any assumption on the dynamic behavior of the components of $\boldsymbol{v}$. Unbalanced and/or non-sinusoidal conditions, multi-phase systems and even dc systems are consistent with this definition.

IV. Examples

The examples presented below are aimed at illustrating the features of the generalized frequency. The first two examples show that, in stationary conditions, (25) leads to the well-known and widely accepted definition of frequency in ac systems. Examples 3 and 4 illustrate the special cases of transient balanced three-phase systems and dc systems, respectively. Example 5 extends the definition of generalized frequency to the current. Example 6 illustrates the link between the generalized frequency and the generalized instantaneous reactive power proposed in [4]–[6] and shows a simple way to estimate the generalized frequency in practice. Finally, Example 7 compares the estimation of the frequency as obtained with a synchronous reference frame phase-locked loop (SRF-PLL) and the one obtained with (25) based on a simulation of a detailed EMT model of the well-known New England 39-bus system.

Example 1: Let us consider a stationary single-phase voltage with constant angular frequency ω_o and magnitude V. The voltage vector can be defined as:

$$\boldsymbol{v} = V\cos(\theta)\mathbf{e}_1 + V\sin(\theta)\mathbf{e}_2 = v_1\mathbf{e}_1 + v_2\mathbf{e}_2\,,$$

where $\theta = \omega_o t + \phi$ and $(\mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis of the system, with $\mathbf{e}_1$ and $\mathbf{e}_2$ orthonormal vectors. Then, $v = |\boldsymbol{v}| = V$ and:

$$\boldsymbol{v}' = v_1'\mathbf{e}_1 + v_2'\mathbf{e}_2 = -\omega_o v_2\mathbf{e}_1 + \omega_o v_1\mathbf{e}_2\,,$$

from which one can deduce that, as expected:

$$\rho_v = \frac{1}{V^2}(v_1 v_1' + v_2 v_2') = \frac{\omega_o}{V^2}(-v_1 v_2 + v_2 v_1) = 0\,,$$
$$\omega_v = \frac{1}{V^2}|v_1 v_2' - v_2 v_1'| = \frac{\omega_o}{V^2}|v_1^2 + v_2^2| = \omega_o\,.$$

It is relevant to note that, in $\mathbb{R}^2$, multivectors are isomorphic to complex numbers. In fact, the bivector $\boldsymbol{\Omega}_v$ is:

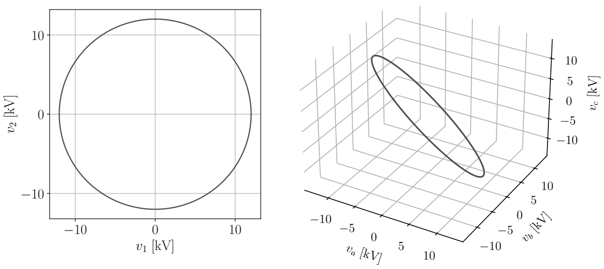
$$\boldsymbol{\Omega}_v = \omega_o(\mathbf{e}_1 \wedge \mathbf{e}_2) = \jmath\omega_o\,.$$

where the imaginary unit $\jmath$ is defined as $\jmath \equiv \mathbf{e}_1 \wedge \mathbf{e}_2$.

It is also worth observing that, in two dimensions, the curvature is defined as [7]:

$$\kappa_v = \frac{d\theta}{ds} = \frac{d\theta}{dt}\frac{dt}{ds} = \frac{\theta'}{s'} = \frac{\theta'}{v}\,, \tag{26}$$

which is valid in any transient and non-sinusoidal conditions. Using the chain rule and recalling (22), one has that $\theta' = \omega$, which is the commonly accepted definition of frequency [1]. Figure 1a illustrates the voltage "curve" for a single-phase stationary voltage with $V = 12$ kV and $\omega_o = 120\pi$ rad/s. As expected, the curve is a circle, which, as it is well-known, has constant curvature.



(a) Example 1

(b) Example 2

Fig. 1: Voltage "curves."

Example 2: We consider a stationary balanced three-phase system. The voltage vector is:

$$\begin{aligned}\boldsymbol{v} &= V\sin(\theta_a)\mathbf{e}_a + V\sin(\theta_b)\mathbf{e}_b + V\sin(\theta_c)\mathbf{e}_c \\ &= v_a\mathbf{e}_a + v_c\mathbf{e}_b + v_c\mathbf{e}_c\,,\end{aligned}$$

where V is constant; $\theta_a = \omega_o t$, $\theta_b = \theta_a - \alpha$ and $\theta_c = \theta_a + \alpha$ with ω_o constant and $\alpha = \frac{2\pi}{3}$; and $(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$ is the canonical basis of the system, with $\mathbf{e}_a$, $\mathbf{e}_b$ and $\mathbf{e}_c$ orthonormal vectors. Then:

$$v^2 = |\boldsymbol{v}|^2 = V^2(\sin^2\theta_a + \sin^2\theta_b + \sin^2\theta_c) = \frac{3}{2}V^2\,,$$

and

$$\begin{aligned}\boldsymbol{v}' &= \omega_o V\cos(\theta_a)\mathbf{e}_a + \omega_o V\cos(\theta_b)\mathbf{e}_b + \omega_o V\cos(\theta_c)\mathbf{e}_c \\ &= v_a'\mathbf{e}_a + v_c'\mathbf{e}_b + v_c'\mathbf{e}_c\,.\end{aligned}$$

Then, one has:

$$\begin{aligned}\rho_v &= \frac{v_a v_a' + v_b v_b' + v_c v_c'}{v^2} \\ &= \frac{\omega_o V^2 \frac{1}{2}(\sin 2\theta_a + \sin 2\theta_b + \sin 2\theta_c)}{v^2} = 0\,, \\ \boldsymbol{\Omega}_v &= \frac{1}{v^2}\begin{bmatrix} 0 & v_a v_b' - v_b v_a' & v_a v_c' - v_c v_a' \\ v_b v_a' - v_a v_b' & 0 & v_b v_c' - v_c v_b' \\ v_c v_a' - v_a v_c' & v_c v_b' - v_b v_c' & 0 \end{bmatrix}, \\ \omega_v &= \frac{1}{v^2}\sqrt{(v_a v_b' - v_b v_a')^2 + (v_b v_c' - v_c v_b')^2 + (v_c v_a' - v_a v_c')^2} \\ &= \frac{\sqrt{\omega_o^2 V^4 (2\sin^2(\alpha) + \sin^2(2\alpha))}}{\frac{3}{2}V^2} = \omega_o\,.\end{aligned}$$

Figure 1b illustrates the voltage "curve" for three-phase balanced and stationary voltages with $V = 12$ kV and $\omega_o = 120\pi$ rad/s. The curve is a circle in the 3D space and, as for Example 1, has constant curvature.

Example 3: We consider a balanced three-phase system in transient conditions. For illustration, we use the dqo reference frame. The voltage vector is $\boldsymbol{v} = v_d\mathbf{e}_d + v_q\mathbf{e}_q + v_o\mathbf{e}_o$, where $(\mathbf{e}_d, \mathbf{e}_q, \mathbf{e}_o)$ is the canonical basis of the system, with $\mathbf{e}_d$, $\mathbf{e}_q$ and $\mathbf{e}_o$ orthonormal vectors, and the vectors $\mathbf{e}_d$ and $\mathbf{e}_q$ are rotating at angular speed ω_o. Since the system is balanced, $v_o = 0$. Then, $v^2 = v_d^2 + v_q^2$, and:

$$\boldsymbol{v}' = (v_d' - \omega_o v_q)\mathbf{e}_d + (v_q' + \omega_o v_d)\mathbf{e}_q = \tilde{v}_d'\mathbf{e}_d + \tilde{v}_q'\mathbf{e}_q\,,$$

where, assuming that the q-axis leads the d-axis, one has:

$$\mathbf{e}_d' = \omega_o\mathbf{e}_q\,, \quad \mathbf{e}_q' = -\omega_o\mathbf{e}_d\,.$$

The components of the generalized frequency are:

$$\begin{aligned}\rho_v &= \frac{v_d v_d' + \omega_o v_d v_q + v_q v_q' - \omega_o v_d v_q}{v^2} = \frac{v_d v_d' + v_q v_q'}{v^2}\,, \\ \boldsymbol{\Omega}_v &= \begin{bmatrix} 0 & v_d\tilde{v}_q' - v_q\tilde{v}_d' \\ v_q\tilde{v}_d' - v_d\tilde{v}_q' & 0 \end{bmatrix}, \\ \omega_v &= \frac{v_q' v_d + \omega_o v_d^2 - v_d' v_q + \omega_o v_q^2}{v^2} = \omega_o + \frac{v_q' v_d - v_d' v_q}{v^2}\,,\end{aligned}$$

The equations above show that the definition of the Park vector as $\boldsymbol{v} = v_d + \jmath v_q$ and the time derivative in the Park reference frame, namely $\frac{d}{dt} + \jmath\omega_o$, are an equivalent formulation for balanced three-phase systems in transient conditions [8]. Figure 2 illustrates the expressions above for ρ_v and ω_v assuming $v_d = 10 + \exp(-t)\cos(2\pi t))$ kV, $v_q = \exp(-t)\sin(2\pi t))$ kV, $v_o = 0$, and $\omega_o = 120\pi$ rad/s. The figure also shows the components of the voltage and their time derivatives, which confirm that the curvature and thus the frequency are not constant in this case.

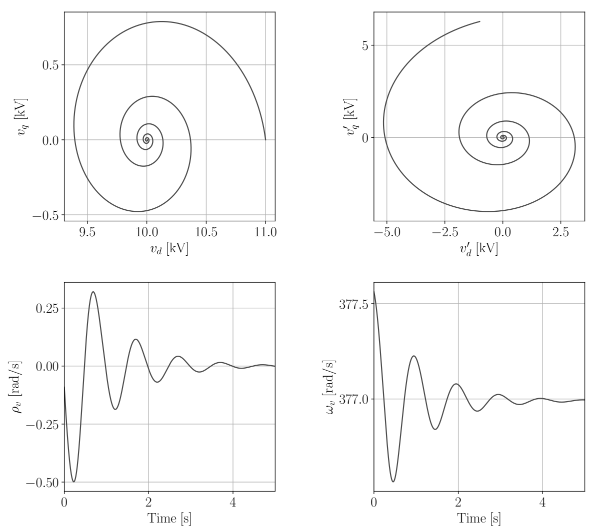


Fig. 2: Illustration of Example 3.

Example 4: We show that (25) is valid also for dc voltages. In dc circuits, the voltage has only one component along the unique basis of the system, say $\mathbf{e}_{\rm dc}$, hence, $\boldsymbol{v} = v_{\rm dc}\,\mathbf{e}_{\rm dc}$ and $\boldsymbol{v}' = v_{\rm dc}'\,\mathbf{e}_{\rm dc}$. From the definitions of inner and wedge product one has:

$$\boldsymbol{v}\cdot\boldsymbol{v}' = v_{\rm dc}\,v_{\rm dc}'\,, \quad \boldsymbol{v}\wedge\boldsymbol{v}' = \mathbf{0}\,.$$

In dc, then, the generalized frequency is equal to $\rho_v = v_{\rm dc}'/v_{\rm dc}$ and, as expected, $\omega_v = |\boldsymbol{\Omega}_v| = 0$.

Example 5: Similarly to the voltage, one can define the generalized frequency of the current. Consider the vector of the electric charge $\boldsymbol{q}$ as an abstract curve in $\mathbb{R}^n$. This vector does not have to be intended as a charge moving in space, but rather as the macroscopic effect of the electric field in a given part of a circuit. Then:

$$\boldsymbol{q}' = \boldsymbol{\imath}\,, \tag{27}$$

and, analogously to the discussion on the voltage, the generalized frequency associated with the current is given by:

$$\rho_\imath + \boldsymbol{\Omega}_\imath = \frac{\boldsymbol{\imath}\boldsymbol{\imath}'}{\imath^2}\,. \tag{28}$$

In general, for any given element of a circuit, the generalized frequency of the voltage is not equal to that of the current. Relevant exceptions are resistances. For a balanced resistive branch,

$\boldsymbol{v}\boldsymbol{v}' = R^2 \boldsymbol{\imath}\boldsymbol{\imath}'$, which indicates that, from a geometrical point of view, resistances are scaling factors.

Example 6: We further elaborate on the link between voltage and current vectors. For balanced capacitive elements, one has:

$$\boldsymbol{\imath} = C\boldsymbol{v}' \,, \tag{29}$$

Merging (25) and (29), one obtains:

$$\rho_v + \boldsymbol{\Omega}_v = \frac{\boldsymbol{v}\boldsymbol{\imath}}{Cv^2} = \frac{p - \mathbf{Q}}{Cv^2} \,, \tag{30}$$

where $p = \boldsymbol{v} \cdot \boldsymbol{\imath}$ is the instantaneous active power, which is not null only in transient conditions, and $\mathbf{Q} = \boldsymbol{\imath} \wedge \boldsymbol{v}$ is the generalized instantaneous reactive power as defined in [6]. It is interesting to note that (30) provides an expression to calculate the frequency of an electric circuit in any transient condition through instantaneous voltage and current measurements. Interestingly, no discrete Fourier transforms with mobile windows or other standard numerical techniques are required.

Example 7: This last example presents a comparison of the estimation of the frequency as obtained with a standard SRF-PLL and with (25). Figure 3 shows the results obtained with PowerFactory and the New England 39-bus system following a phase-to-phase fault at bus 3 applied at $t = 0.2$ s and cleared at $t = 0.3$ s. The simulation utilizes the fully-fledged EMT model provided by PowerFactory.

PLLs can only estimate $\omega_v = |\boldsymbol{\Omega}_v|$, i.e., the magnitude of the bivector defined in (25). Despite this limitation of the PLL, the comparison shows that the frequency obtained with the proposed approach is consistent with the conventional SRF-PLL. To obtain the estimation of ω_v shown in Fig. 3 a simple discrete first-order filter is used to smooth the numerical noise of the time derivative of the voltages and calculate ω_v. This is enough in this case to obtain good results, which are affected by less delay than those obtained with the PLL. Finally, the right panel of Fig. 3 shows that the trajectory described by the voltage changes plane during the fault and is not perfectly circular, thus leading to a time-varying curvature/frequency.

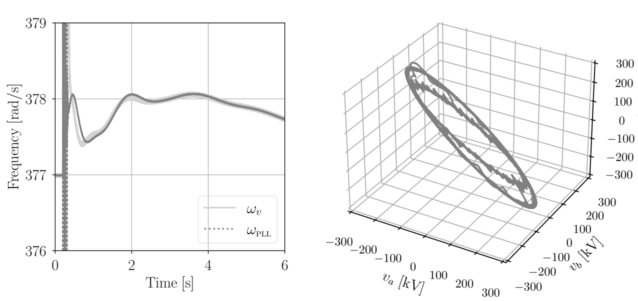


Fig. 3: Examples 7: Voltage and frequency at bus 26.

V. Remarks on the Estimation of the Frequency

There exist two main broad approaches for the estimation of the frequency: transformation-based methods (e.g., Fourier-transform-based approaches and more recently, Hilbert-transform-based approaches) and time-domain methods (e.g., PLLs). The approach proposed in this letter falls in the second category. In general, all conventional approaches define *a priori* a model of the measured signal. For example, a sine wave with constant magnitude in [1], a set of sine waves with constant magnitude and frequency in a given window in the case of the Fourier transform, or a sine wave with pulsating amplitude in the case of the Hilbert transform [3]. If the actual signal does not fit the given model, the estimation obtained with these methods might not be accurate.

The definition of frequency that is proposed in this letter has two relevant advantages, as follows.

- It is intrinsically model agnostic, i.e., no assumption is made on the time dependency of the elements of the voltage vector. This allows providing a definition independent from the transient/stationary conditions of the circuit where the frequency is to be estimated.
- Equation (25) suggests a way to calculate the frequency based directly on the measured signal. Other approaches require to process the measurements before being able to do the estimation.

Note that the issue indicated in the second point above involves also conventional time-domain approaches, as the signals that feed the PLLs are the voltage measurements processed through the Park transform, which is known to be particularly sensitive to noise [3].

Of course, also (25) presents some challenges. It requires in fact to calculate the time derivative of the voltage, which can lead to numerical issues. But this is done directly on the measured quantities not on transformed ones. Moreover, Example 7 shows that the issues deriving from numerical differentiation can be resolved with proper filtering.

In summary, the proposed approach appears useful in two ways: (i) to build a "theory" based on the geometrical interpretation of frequency and electric quantities in general; and (ii) in estimation and control, to define the frequency in a unambiguous way. This also suggests that (25), if standardized, can be utilized to compare the results obtained with other techniques, e.g., to evaluate the accuracy of PLLs and other devices that estimate the frequency, e.g., phasor measurements units.

VI. Conclusions

The proposed formal framework generalizes and solves known issues of the conventional definition of frequency. A strength of the proposed approach is that it is based on invariant quantities, hence it is compatible with any reference frame, e.g., *abc*, *dqo* and even dc circuits. It is interesting to note that the proposed approach defines the frequency as a geometrical object with symmetric and antisymmetric parts. In an example, the letter also shows the link between the generalized frequency and the power of a circuit. This link between geometry and energy appears worth further research.

VII. Acknowledgments

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