—NOTES—

A GEOMETRICAL INTERPRETATION OF THE RELAXATION METHOD*

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Let a_{ij}, B_i $(i,j=1, 2, \cdots, n)$ be given constants such that $a_{ij}=a_{ji}$ and $\sum_{i,j=1}^{n} a_{ij}x_ix_j$ is a positive definite form. Consider the equations

$$\sum_{j=1}^{n} a_{ij}x_j - B_i = 0 \qquad (i = 1, 2, \cdots, n). \tag{1}$$

The solution is easily expressed as a set of quotients of determinants. However, as n increases, the task of calculating the determinants becomes excessively burdensome. The relaxation method[1] provides a set of easy steps by which the solution of (1) is approached. The method has been compactly described by Temple.[2]

The purpose of the present note is to give a geometrical description of the relaxation method. For the trivial case $n=2$ the geometrical description may be displayed accurately in a diagram. For $n=3$ a model may be visualized. For $n>3$ we pass beyond the region of simple concrete geometrical representation, but in many ways geometry in an n-space is closely analogous to geometry in 2-space or 3-space, and the geometrical description continues to serve as a general guide to procedure.

Let us regard x_i as rectangular Cartesian coordinates in a Euclidean n-space. Let us define

$$H(x) = \tfrac{1}{2}\sum_{i,j=1}^{n} a_{ij}x_ix_j - \sum_{i=1}^{n} B_ix_i. \tag{2}$$

The equation $H(x)=$ const. represents a family of ellipsoids E; these ellipsoids have a common center, common directions for their principal axes, and common values for the ratios of their principal axes. They form, in fact, a family of similar and similarly situated ellipsoids.

The equations (1) represent a set of planes (i.e., flats of $n-1$ dimensions). The point of intersection of these planes is the common center G of E. Thus the problem of solving (1) is the problem of finding the center of an ellipsoid when its equation is given.

It is important to note that $H(x)$ takes a minimum value at G. H is constant over each ellipsoid, and increases steadily as we pass out from G.

It is not possible to define precisely what procedures are to be regarded as permissible. It is a question of ease of computation. Let us follow Southwell and consider an approach to G by steps each of which is parallel to one of the axes of coordinates x_i.

Fig. 1 shows Southwell's procedure. It is a schematic diagram in which the ellip-

* Received Dec. 29, 1943.

[1] R. V. Southwell, *Relaxation methods in engineering*, Oxford, 1940.

[2] G. Temple, Proc. Roy. Soc. London A, **169**, 476–500 (1939).

soids are represented by circles. (The ellipsoids can of course be transformed into concentric spheres by a linear transformation, which however destroys the orthogonality of perpendicular lines.) We start with an arbitrary point P_0 (the zero approximation). Let E_0 be the ellipsoid which passes through P_0. Through P_0 we draw a straight line L parallel to one of the coordinate axes. Let Q_1 be the second point in which L cuts E_0. Let P_1 be the middle point of the chord P_0Q_1. Then P_1 is the first approximation.

Since the ellipsoid is a convex surface, P_1 lies inside E_0 and so $H(P_1) < H(P_0)$. Moreover it is easy to see that P_0Q_1 is tangent at P_1 to the ellipsoid E_1 which passes through P_1. Thus, of all points on the chord P_0Q_1, the point P_1 gives the smallest value of H.

The process is repeated, starting from P_1. The second approximation P_2 is the middle point of a chord P_1Q_2 of E_1, drawn parallel to another of the coordinate axes. In this way we get a sequence of points $P_0, P_1, \cdots$. The success of the method depends on the rapidity of the convergence of this sequence to G.

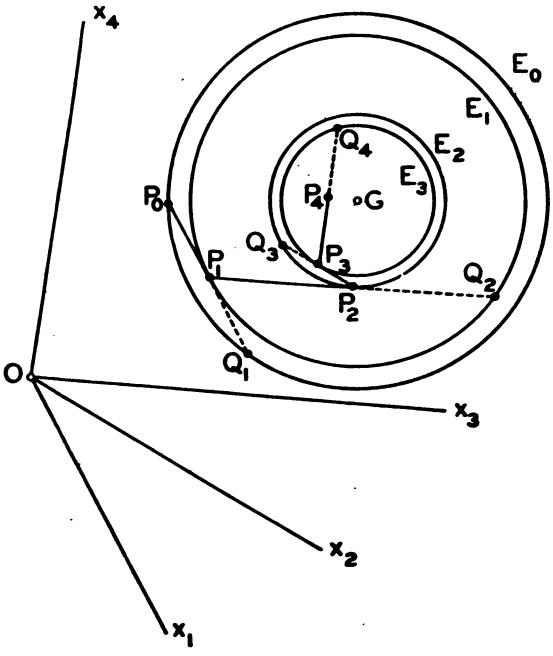

FIG. 1.

In one important respect the above procedure is incompletely defined. When we have reached P_m, in which of the directions defined by the coordinate axes are we to proceed in order to get P_{m+1}? There are n coordinate axes. Of these one cannot be used, viz., that which gave the direction of the step $P_{m-1}P_m$. But, of the remaining $n-1$ directions, which should we use?

Gaskell[3] has suggested the following plan. Write

$$C_i(x) = \sum_{j=1}^{n} a_{ij}x_j - B_i. \tag{3}$$

Having reached the point P_m, we calculate the quantities $C_i(P_m)$. Let $C_k(P_m)$ be the greatest of these in absolute value. Then we choose for the step P_mP_{m+1} the direction of the axis of x_k.

This procedure is called the liquidation of the greatest error, since we obtain $C_k(P_{m+1}) = 0$. It is interesting to see how this result fits into the geometrical discussion. The plane $C_k(x) = 0$ is the plane through G conjugate to the direction of the axis x_k. The line P_mP_{m+1} is parallel to this axis and tangent at P_{m+1} to one of the ellipsoids, E_{m+1}. But the point of contact of a line with an ellipsoid lies on the central plane conjugate to the direction of the line. Hence P_{m+1} lies on $C_k(x) = 0$, i.e., $C_k(P_{m+1}) = 0$.

[3] R. E. Gaskell, Quarterly of Applied Mathematics, 1, 237–249 (1943).

But it may well be asked whether the quantities C_i themselves possess any deep significance. It is true that G satisfies $C_i(x)=0$, but the quantity $C_i(x)$ for a general point does not represent the perpendicular distance of that point from the plane $C_i(x)=0$. This perpendicular distance is

$$f_i(x) = \frac{|C_i(x)|}{\left(\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}}. \tag{4}$$

Should we not liquidate the greatest p_i rather than the greatest C_i? Or is there a better plan than either?

The following plan is suggested. Having reached the point P_m, we have an option on $n-1$ next points. Each of these points lies on an ellipsoid of the family E. *Choose that point which lies on the innermost ellipsoid.* This is equivalent to saying: Choose that point which gives the smallest value to H.

Now[4] for a step in the direction of the axis x_i the decrease in H is $\frac{1}{2}C_i^2/a_{ii}$. This is to be made as great as possible, and so we should pick the direction of the step P_mP_{m+1} according to the following rule: *Proceed in the direction of the axis of x_k where C_k^2/a_{kk} is the greatest of the quantities C_i^2/a_{ii} $(i=1, 2, \cdots, n)$.*

Thus C_i^2/a_{ii} is made the criterion rather than Gaskell's C_i. The calculation of the former quantities involves slightly more computation, but this may be taken care of by making the initial transformation

$$x_i' = (a_{ii})^{1/2}x_i. \tag{5}$$

Then

$$H = \tfrac{1}{2}\sum_{i,j=1}^{n} a_{ij}'x_i'x_j' - \sum_{i=1}^{n} B_i'x_i', \tag{6}$$

where

$$a_{ii}' = 1, \qquad a_{ij}' = a_{ij}/(a_{ii}a_{jj})^{1/2}, \qquad B_i' = B_i/(a_{ii})^{1/2}. \tag{7}$$

Now, with

$$C_i'(x') = \sum_{j=1}^{n} a_{ij}'x_j' - B_i', \tag{8}$$

the criterion for the direction of the next displacement is $C_i'^2$ or $|C_i'|$, the same as Gaskell's. Moreover the transformation from P_m to P_{m+1} is now extremely simple. It is[5]

$$P_m: \quad x_1', \cdots, x_k', \cdots, x_n';$$
$$P_{m+1}: \quad x_1', \cdots, x_k' - C_k'(x'), \cdots, x_n'.$$

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[4] R. E. Gaskell, loc. cit., Eq. (23).

[5] R. E. Gaskell, loc. cit.