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LETTER TO THE EDITOR

A geometrical inverse problem

A G Ramm

Mathematics Department, Kansas State University, Cardwell Hall, Manhattan, KS 66506, USA

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Abstract. A simple solution is given to the problem of finding the unknown boundary from the extra boundary condition.

Let

$$\Delta u=0 \quad \text{in} \quad \mathrm{D}, \qquad u=u_0 \quad \text{on} \quad \Gamma_0, \qquad u_N=u_1 \quad \text{on} \quad \Gamma_0 \tag{1}$$

$$u=0 \quad \text{on} \quad \Gamma_1 \tag{2a}$$

or

$$u_N=0 \quad \text{on} \quad \Gamma_1. \tag{2b}$$

Here $u=u(x, y)$, D is a domain on the plane and is homeomorphic to an annulus, Γ_0 and Γ_1 are the boundaries of D, Γ_0 is the inner curve and Γ_1 is the outer curve. We assume that Γ_0 is known and that $|u_0|+|u_1|\not\equiv 0$, i.e. at least one of the functions u_0 or u_1 does not vanish identically. The problem is to find Γ_1 given Γ_0, u_0, u_1 and the fact that the harmonic function u vanishes on the unknown boundary Γ_1. We assume that Γ_0 and Γ_1 are closed smooth star-like curves, i.e. their equations in polar coordinates can be written as $r=f(\varphi)$, where f is a differentiable 2π-periodic function.

This type of problem can be quite useful practically. For example, problem (1)–(2b) can be interpreted as follows: if u is the velocity potential of an incompressible fluid, then the knowledge of u and u_N on Γ_0 implies that the velocity and the pressure are known on Γ_0, and we want to find the surface Γ_1 on which the normal component of the velocity vanishes, i.e. the bottom of the reservoir. A numerical method for solving problem (1)–(2a) is given in [1], where one can find other references. In [1] the existence and uniqueness of the solution to the problem (1)–(2a) and the convergence of the numerical procedure proposed there are not discussed. The numerical procedure is complicated: it is an iterative process, at each step of which one has to solve a linear boundary value problem and a Fredholm integral equation of the first kind. Here we give a simple solution to the problems (1)–(2) and discuss the questions of uniqueness and existence.

The uniqueness of the solution to (1)–(2a) is easy to establish. Indeed, if Γ_1 and Γ_1' are the solutions, then there exists a domain D′, bounded by some parts of Γ_1 and Γ_1', in which there exists a harmonic function, u, which vanishes on ∂D′. Therefore, $u\equiv 0$ in D′. This and the unique continuation property for harmonic functions imply that $u\equiv 0$ in its domain of definition. Therefore $u_0=u_1=0$, which contradicts our assumption. For problem (1)–(2b) uniqueness of the solution does not hold in general; see example 3 later in this article. But if the boundary condition is $u_N=0$ on Γ_1, then the uniqueness of Γ_1 is guaranteed if $u_0 \neq$ constant or $u_1 \not\equiv 0$. Indeed, suppose there are two different surfaces, Γ_1 and Γ_1', on which $u_N=0$. Then there is a domain D′ bounded by parts of Γ_1 and Γ_1' in which $\Delta u=0$ and $u_N=0$ on $\partial D'$. Thus, $u=$constant in D′, and, by the unique continuation property for solutions to elliptic equations, $u=$constant everywhere in its domain of definition. Therefore $u_0=$constant, $u_1=0$. Thus, there is at most one Γ_1 on which $u_N=0$ provided that $u_0 \neq 1$ or $u_1 \not\equiv 0$. Our method is applicable to other boundary conditions and other differential equations.

The existence of a solution to (1)–(2) cannot be guaranteed for arbitrary u_0 and u_1. These functions, being the Cauchy data for an elliptic equation, should satisfy some compatibility conditions. For example, if $u_0=0$ and $u=0$ on Γ_1 then $u \equiv 0$ in D, and $u_1=0$. Therefore problem (1)–(2*a*) has no solution if $u_0=0$ and $u_1=1$, for example.

A method for solving problem (1)–(2) is as follows.

(i) Without loss of generality, assume that Γ_0 is the unit circle. Otherwise one first maps conformally the exterior of Γ_0 onto the exterior of the unit circle.

(ii) Any harmonic function in the exterior of the unit circle can be written in polar coordinates as

$$u(r,\varphi)=a_0+b_0 \ln r+\sum_{n=1}^{\infty}\left[r^n(a_n \cos n\varphi+b_n \sin n\varphi)+r^{-n}(c_n \cos n\varphi+d_n \sin n\varphi)\right]. \tag{3}$$

The constants a_n, b_n, c_n, d_n are uniquely determined by the two functions u_0 and u_1 from the equations

$$a_0=A_0 \qquad a_n+c_n=A_n \qquad b_n+d_n=B_n \qquad n\geqslant 1 \tag{4}$$

$$b_0=A_0' \qquad n(a_n-c_n)=A_n' \qquad n(b_n-d_n)=B_n' \qquad n\geqslant 1 \tag{5}$$

where

$$u_0=A_0+\sum_{n=1}^{\infty}(A_n \cos n\varphi+B_n \sin n\varphi)$$

and

$$u_1=A_0'+\sum_{n=1}^{\infty}(A_n' \cos n\varphi+B_n' \sin n\varphi).$$

Let $r=f(\varphi)$ be the equation of Γ_1. Then the unknown function $f(\varphi)$ can be found numerically from the equation

$$u(f(\varphi),\varphi)=0 \tag{6}$$

if condition (2*a*) holds, and from the equation

$$\left.\frac{\partial u}{\partial N}\right|_{r=f(\varphi)}=\left.\frac{\partial u}{\partial r}-\frac{1}{r^2}\frac{\partial u}{\partial \varphi}\frac{\partial f}{\partial \varphi}\right|_{r=f(\varphi)}=0 \tag{7}$$

if condition (2*b*) holds, where we used the formulae

$$\nabla u=\frac{\partial u}{\partial r}e_r+\frac{1}{r}\frac{\partial u}{\partial \varphi}e_\varphi \qquad N=e_r-\frac{1}{r}f'(\varphi)e_\varphi$$

and e_r and e_φ are coordinate unit vectors in polar coordinates. Here $u(r,\varphi)$ is given by (3) with the coefficients determined by (4)–(5). Equation (6) is just a transcendental equation for $f(\varphi)$. In some cases it can be solved analytically.

Example 1. Let $u_0=1$, $u_1=-1$. Then $A_n=B_n=A'_n=B'_n=0$, $n\geqslant 1$, $A_0=1$, $A'_0=-1$. From (4), (5) one finds $a_n=c_n=b_n=d_n=0$, $n\geqslant 1$, $a_0=1$, $b_0=-1$. Thus $u=1-\ln r$. Equation (6) takes the form $1=\ln f(\varphi)$. Therefore $f(\varphi)=e$, and Γ_1 is a circle of radius e. Equation (7) takes the form $-1/f(\varphi)=0$ and has no solutions.

Example 2. Let $u_0=\cos\varphi$, $u_1=\cos\varphi$. Then one can easily check that $u=r\cos\varphi$. Equation (7) becomes

$$\cos\varphi+\frac{f\sin\varphi}{f^2}f'=0 \qquad f=c|\sin\varphi|^{-1} \qquad c=\text{constant}.$$

Here there is no bounded $f(\varphi)$ that solves equation (7). Therefore, the problem has no solutions.

Example 3. Let $u_0=1$, $u_1=0$. Then $u(r,\varphi)=1$, and any closed curve Γ_1 solves problem (1)–(2*b*).

The basic idea in this paper is as follows. Suppose that a certain function satisfies an elliptic equation for which the uniqueness of the solution to the Cauchy problem is established. Then by measuring the Cauchy data on a certain surface one can uniquely determine the function everywhere in its domain of definition. In particular, one can find the sets on which this function (or a certain combination of its derivatives) vanishes. This set is often of practical interest. For example, one can measure the pressure and velocity on the surface of the water and determine the surface of a submarine. Of course, the Cauchy problem for elliptic equations is ill-posed and its numerical solution is very difficult.

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Reference

[1] Jia-qi Liu and Ying-wei Lu 1985 A new algorithm for solving the geometrical inverse problems for a partial differential equation and its numerical simulation *Proc. 11th IMACS Congress, Oslo* vol. 1 pp 221–4