

A GEOMETRICALLY EXACT PLANAR COSSERAT SHELL-MODEL WITH MICROSTRUCTURE: EXISTENCE OF MINIMIZERS FOR ZERO COSSERAT COUPLE MODULUS

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The existence of minimizers to a geometrically exact Cosserat planar shell model with microstructure is proven. The membrane energy is a quadratic, uniformly Legendre–Hadamard elliptic energy in contrast to traditional membrane energies. The bending contribution is augmented by a curvature term representing the interaction of the rotational microstructure in the Cosserat theory. The model includes non-classical size effects, transverse shear resistance, drilling degrees of freedom and accounts implicitly for thickness extension and asymmetric shift of the midsurface. Upon linearization with zero Cosserat couple modulus $\mu_c = 0$, one recovers the infinitesimal-displacement Reissner–Mindlin model. It is shown that the Cosserat shell formulation admits minimizers even for $\mu_c = 0$, in which case the drill-energy is absent. The midsurface deformation m is found in $H^1(\omega, \mathbb{R}^3)$. Since the existence of energy minimizers rather than equilibrium solutions is established, the proposed analysis includes the large deformation/large rotation buckling behaviour of thin shells.

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1. Introduction

1.1. Aspects of shell theory

The dimensional reduction of a given continuum-mechanical model is an old subject and has seen many “solutions”. The investigated model herein falls within the so-called *derivation approach*, i.e. reducing a given three-dimensional model via physically reasonable constitutive assumptions on the kinematics to a two-dimensional model. This is opposed to either the *intrinsic approach* which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the *asymptotic methods* which try to establish two-dimensional

equations by formal expansion of the three-dimensional solution in power series in terms of a small thickness parameter. The intrinsic approach is closely related to the *direct approach* which takes the shell to be a two-dimensional medium with additional *extrinsic directors* in the sense of a *restricted Cosserat surface*.^{a,12} There, two-dimensional equilibrium in appropriate resultant stress and strain variables is postulated ab-initio more or less independent of three-dimensional considerations.^{2,14,15,22,28,33,58,61}

A detailed presentation of the different approaches in classical shell theories can be found in the monograph, Ref. 45. A thorough mathematical analysis of linear, infinitesimal-displacement shell theory, based on asymptotic methods is found in Ref. 18, see also Refs. 2, 17, 20, 21, 24 and 26. Excellent reviews of the modelling and finite element implementation may be found in Refs. 6, 9, 11, 35, 36, 62, 63, 65, 74, and in the series of papers, Refs. 66–71. Properly invariant, geometrically exact, elastic plate (planar shell) theories are derived by formal asymptotic methods in Ref. 30. This formal derivation is extended to curvilinear shells in Refs. 39 and 42. Apart from the pure bending case,²⁹ which is rigorously justified as the variational Γ -limit⁴¹ of the three-dimensional model and which can be shown to be intrinsically well-posed, the obtained finite-strain models have not yet been shown to be well-posed. Indeed, the membrane energy contribution is notoriously not Legendre–Hadamard elliptic in the compression range while membrane action is a dominant deformation mode in many situations of interest. The membrane model justified in Ref. 25 by Γ -convergence is geometrically exact and quasiconvex/elliptic but does not coincide upon linearization with the infinitesimal-displacement membrane model. Moreover, this model does not describe the detailed geometry of deformation in compression but reduces to a tension-field theory.⁷²

The “rational” of descend from three to two dimensions should always be complemented by an investigation of the intrinsic mathematical properties of the lower dimensional models. Today, the need to simulate the mechanical response of highly flexible thin structures allowing easily for finite rotations excludes the use of classical infinitesimal-displacement models, either of Reissner–Mindlin or Kirchhoff–Love type. Also, certain “intermediary” models allowing for buckling like the “nonlinear” von Kármán plates, see p. 403 of Ref. 17 and penalized “nonlinear” Reissner–Mindlin models^{b,23} or “semilinear” Kirchhoff–Love plate models⁴⁴ are not geometrically exact (not frame-indifferent). Nevertheless, the nonlinear von Kármán plate has been successfully applied to the delamination problem of thin films.^{32,34,57}

^aRestricted, since no material length scale enters the direct approach, only the *nondimensional relative characteristic thickness (aspect ratio) h* appears in the model. In terminology we distinguish between a “true” Cosserat shell model operating on $SO(3)$ and theories with any number of directors. Moreover, even a “true” Cosserat shell model does not necessarily feature an interacting microstructure with length scale effects—the case treated here.

^bA von Kármán plate with one independent director \mathbf{d} and addition of a penalization term $\mu_c \left(\langle \mathbf{d}, \partial_x m \rangle^2 + \langle \mathbf{d}, \partial_y m \rangle^2 \right)$, $\mu_c \rightarrow \infty$ with m the sought midsurface deformation.

Mielke⁴³ established in the infinitesimal-displacement context that by using more than five ansatz-functions in a director model it is possible to obtain exponential decay estimates for the boundary layer and to establish a St. Venant principle for linearized plates. While it is not clear how this method can be transferred to the finite-strain case, they provide, independent of mechanical or physical considerations, a strong motivation to use a director ansatz also in the finite-strain case to capture the boundary layer phenomena.

Indeed, the so-called shear-deformable theories with independent directors are often preferred in the engineering community.^{1,11} In view of an efficient finite element implementation one considers a hyperelastic, variationally based formulation with second-order Euler–Lagrange equations and uses low order mixed interpolation⁵ or discontinuous Galerkin methods.^{8,40} The prototype examples are models based on the Reissner–Mindlin kinematical assumption. There are numerous proposals in the engineering literature for such a finite-strain, geometrically exact shell formulation, see e.g. Refs. 6, 9, 31, 63–65, 74. In many cases the need has been felt to devote attention to rotations $R \in \text{SO}(3)$, since rotations are the dominant deformation mode of a thin flexible structure. This has led to the *drill-rotation formulation* which means that proper rotations either appear in the formulation as independent fields (leading to a restricted Cosserat surface without size-effect) or they are an intermediary ingredient³⁷ in the numerical treatment (constraint Cosserat surface, also without size-effect). While the computational merit of this approach is well documented, a mathematical analysis for such a family of finite-strain shell models is yet missing, both for the Cosserat surface and the constraint model. It may be speculated that the restricted geometrically exact Cosserat shells (obtained from classical non-polar bulk models or from direct modelling), might not be well posed for certain membrane strain measures either, notably if Green-strains: $F^T F - \mathbf{1}$ or Hencky-strains: $\ln F^T F$ are used. Another drawback from a modelling point of view is that the inclusion of drill-rotations is mostly often done in an ad hoc fashion.

While the classical infinitesimal-displacement plate models based on the Reissner–Mindlin kinematical assumption can lead to effective numerical schemes even for very small aspect ratio³⁸ $h > 0$ if mixed interpolation⁵ is used, it remains open whether the same is true for the finite-strain shell-models proposed in the literature. However, there is an abundance of applications where very thin structures are used, e.g. very thin metal layers on a substrate (in computer hardware, for the aspect ratio $h \leq 5 \times 10^{-4}$). In these cases, classical bending energy alone, which comes with a factor of h^2 compared with the membrane energy contribution, might not play a stabilizing role for non-vanishing membrane energy. See Ref. 7 for such a problem occurring in thin films. But, as noted, the *membrane terms* in a finite-strain, invariant Kirchhoff–Love shell^{30,50} or finite-strain Reissner–Mindlin model^{31,50} are *non-elliptic* and the remaining minimization problem is not well-posed even if classical bending is present. Addressing partly this problem, in Ref. 53 a geometrically exact, viscoelastic membrane formulation has been proposed, where

the viscoelastic effect, operative there through an independent local field of rotations, is driven by transverse shear. This viscoelastic formulation has been shown to be locally well-posed.⁵²

However, comparing physical shells with different thicknesses $d[\text{m}]$ and in-plane lengths $L[\text{m}]$ but same nondimensional aspect ratio $h = \frac{d}{L}$ shows experimentally that the response of the smaller shell is stiffer, cf. the discussion of scaling effects in Sec. A.1. These *non-classical size effects*, due to curvature effects of the microstructure cannot be neglected for very thin structures.¹³ Size-effects are not accounted for either in classical theories or in the aforementioned viscoelastic case. In addition, classical infinitesimal-displacement or finite-strain shell models predict unrealistically high levels of smoothness, typically $m \in W^{1,4}(\omega, \mathbb{R}^3)$ for the midsurface m in both finite-strain Kirchhoff–Love and Reissner–Mindlin models and $m \in H^2(\omega, \mathbb{R}^3)$ in the finite-strain pure bending problem²⁹ and the von Kármán model. This implies at least $C^{0,\alpha}(\omega)$ for the midsurface deformation m which rule out the description of boundary layer effects and possible failure along asymptotic lines of the surface.

The author has proposed a planar shell model, see (4.1), intended to be useful for very thin materials with interacting rotational microstructure, which might resolve some of the aforementioned shortcomings with a view towards a stringent mathematical analysis and stable finite element implementation. It is the goal to provide a model which is both theoretically and physically sound, such that its numerical implementation can concentrate on real convergence issues. Let us summarize what could be required of a general, all purpose, consistent first approximation, large deformation/large rotation thin shell model. Necessary requirements could be

- (1) A formulation which is geometrically exact and allows for finite rotations.
- (2) The description of transverse shear, drill rotations, thickness stretch and asymmetric shift of the midsurface. No normality assumptions for a director.
- (3) A qualitative resolution of the boundary layer and edge effect compared with the bulk model.
- (4) Well-posedness: existence, but not unqualified uniqueness to describe buckling due to membrane forces, e.g. under lateral compression or shear and avoiding smoothness for the midsurface, requiring only $m \in H^{1,2}(\omega, \mathbb{R}^3)$.
- (5) A hyperelastic, variational formulation with second-order Euler–Lagrange equations in view of an efficient finite element implementation with low order elements and mixed interpolation.
- (6) A reduced energy density which is defined in terms of two-dimensional quantities with a clear physical meaning.
- (7) The incorporation of non-classical size effects without leading to trivial compactness arguments for the midsurface m .^c

^cAdding a second derivative $L_c^p \|D^2 m\|^p$ to the energy density would “resolve” all mathematical difficulties but lead to $m \in W^{2,p}(\omega, \mathbb{R}^3)$.

- (8) The consistency with classical plate models (infinitesimal displacement Reissner–Mindlin, infinitesimal-displacement Kirchhoff–Love) upon linearization and consistency with rigorously justified finite-strain Kirchhoff–Love bending model^{29,30} in pure bending for large samples.

1.2. Outline of this contribution

The basic idea here to meet these requirements for a shell model is to descend from a three-dimensional Cosserat bulk model with rotationally interacting microstructure. First, we recall in Sec. 2 the underlying “parent” three-dimensional finite-strain frame-indifferent Cosserat model with *size effects* and already appearing *independent microrotations* \overline{R} , i.e. a *triad of rigid directors* $(\overline{R}_1|\overline{R}_2|\overline{R}_3) = \overline{R} \in \text{SO}(3)$. We then provide the restriction of the bulk model to a thin domain in Sec. 3 on which the dimensional reduction is based. This reduction is given in Refs. 48, 50 and we recall in Sec. 4 the two-field minimization problem for the new Cosserat shell model. It should be observed that the resulting Cosserat shell model cannot be obtained from a naive *energy projection* and the already obtained three-dimensional existence results do not apply.

The corresponding equilibrium problem defined over the two-dimensional planar referential domain $\omega \subset \mathbb{R}^2$ has six **degrees of freedom** (three for the midsurface deformation $m : \omega \mapsto \mathbb{R}^3$ and three for the independent rotations $\overline{R} : \omega \mapsto \text{SO}(3)$, **6 dof**) and constitutes a nonlinear, partial differential elliptic system of six equations for basically six unknown functions. The model includes naturally one-drilling degree of freedom for in-plane rotations and accounts for thickness stretch and transverse shear. The drilling degree is strictly related to the size-effect and microstructure of the bulk model and not specifically introduced in an *ad hoc* manner by the dimensional reduction. The model features also a *non-standard boundary condition*, called *consistent coupling*, which precludes polar effects at the Dirichlet boundary by (roughly) requiring the symmetry of the second Piola–Kirchhoff stresses there. This development is based on the results obtained in Ref. 50. The novelty in this contribution is the mathematical treatment of the zero Cosserat couple modulus case, $\mu_c = 0$. This case is especially interesting since it provides the direct link to the classical infinitesimal Reissner–Mindlin model via linearization and this case is shown to be physically motivated in Ref. 55. Moreover, $\mu_c = 0$ sets the *drill-energy to zero*.

As a preparation of the existence proofs we derive a new extended Korn’s first inequality for plates and elasto-plastic shells in Sec. 5 which is needed for the mathematical treatment in a variational context. Depending on material constants and boundary conditions, mathematical existence theorems are proposed in Sec. 6. We obtain for the minimizing midsurface deformation $m \in H^{1,2}(\omega, \mathbb{R}^3)$. For these results the direct methods of the calculus of variations are used. Since we establish the existence of energy minimizers rather than the existence of equilibrium solutions, the proposed analysis includes the large deformation/large rotation

buckling behaviour of thin shells where the buckled state is identified as energy minimizer.

The *quasiconvexity* of the reduced energy functional $I(m, \bar{R})$ in the pair (m, \bar{R}) is easy to see, however, *unqualified coercivity*^{59,d} w.r.t. the midsurface deformation m depends on the *uniform positivity* of the *Cosserat couple modulus* $\mu_c > 0$, which goes along with a nonzero drill-energy contribution. The much simpler existence of minimizers in this case has been established previously in Refs. 48 and 50. For *zero Cosserat couple modulus* $\mu_c = 0$, the lack of unqualified coercivity can only be overcome by a control of the microstructural curvature in conjunction with the new Korn's inequality for shells.

In order to be able to treat external loads for zero Cosserat couple modulus $\mu_c = 0$, the resultant dead load loading functional Π has to be adapted. This modification, which is already needed in the Cosserat bulk model, has been termed there "*principle of bounded external work*"⁵⁴ and expresses the observation that by arbitrary translation of a solid in a force field only a finite amount of energy can be gained which is certainly true for any classical physical field. If we want to make sense of the non-standard boundary condition of consistent coupling, we need to relax this requirement into a symmetry condition in a boundary layer.

In Sec. 7 the mathematical analysis is also extended to a *polyconvex Cosserat shell* model appropriate for large stretch which has appealing physical features. The present analysis is easily extended from planar shells to curved shells provided that the initial parametrization of the curved shell space is smooth enough.

1.2.1. Notation for bulk material

Let $\Omega \subset \mathbb{R}^3$ be a bounded, open domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing two-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b \rangle_{\mathbb{R}^3}$ denote the scalar product on \mathbb{R}^3 with vector norm $\|a\|_{\mathbb{R}^3}^2 = \langle a, a \rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3 \times 3}$ the set of real 3×3 second-order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{M}^{3 \times 3}} = \text{tr}[XY^T]$, and the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{M}^{3 \times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3 \times 3}$. The identity tensor on $\mathbb{M}^{3 \times 3}$ will be denoted by $\mathbb{1}$, so that $\text{tr}[X] = \langle X, \mathbb{1} \rangle$ and $\text{tr}[X]^2 = \langle X, \mathbb{1} \rangle^2$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e. $\text{GL}(3) := \{X \in \mathbb{M}^{3 \times 3} \mid \det X \neq 0\}$ the general linear group, $\text{SL}(3) := \{X \in \text{GL}(3) \mid \det X = 1\}$, $\text{O}(3) := \{X \in \text{GL}(3) \mid X^T X = \mathbb{1}\}$, $\text{SO}(3) := \{X \in \text{GL}(3) \mid X^T X = \mathbb{1}, \det X = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3 \times 3} \mid X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3 \times 3} \mid \text{tr}[X] = 0\}$ of traceless tensors. With $\text{Adj}X$ we denote the tensor of transposed cofactors $\text{Cof}(X)$ such that $\text{Adj}X = \det X X^{-1} = \text{Cof}(X)^T$ if $X \in \text{GL}(3)$. We set $\text{sym}(X) = \frac{1}{2}(X^T + X)$ and

^dIn finite-strain elasticity: $W(F) \geq c_1^+ \|F\|^p - c_2^+$, $p \geq 2$.

$\text{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \text{sym}(X) + \text{skew}(X)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$. We write the polar decomposition in the form $F = RU = \text{polar}(F) U$ with $R = \text{polar}(F)$ the orthogonal part of F . For a second-order tensor X we define the third-order tensor $\mathfrak{h} = D_x X(x) = (\nabla(X(x) \cdot e_1), \nabla(X(x) \cdot e_2), \nabla(X(x) \cdot e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in \mathbb{M}^{3 \times 3} \times \mathbb{M}^{3 \times 3} \times \mathbb{M}^{3 \times 3} \cong \mathfrak{T}(3)$. Quantities with a bar, e.g. the micropolar rotation \overline{R} , represent the micropolar replacement of the corresponding classical continuum rotation R . We work in the context of finite-strain elasticity. For the deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3 \times 3})$. Furthermore, $S_1(F) = D_F W(F)$ and $S_2(F) = F^{-1} D_F W(F)$ denote the 1. and 2. Piola-Kirchhoff stress tensors, respectively. The first and second differential of a scalar valued function $W(F)$ are written as $D_F W(F) \cdot H$ and $D_F^2 W(F) \cdot (H, H)$, respectively. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H_0^{1,2}(\Omega), W^{1,q}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. The set $W^{1,q}(\Omega, \text{SO}(3))$ denotes orthogonal tensors whose components are in $W^{1,q}(\Omega)$. Moreover, we set $\|X\|_\infty = \sup_{x \in \Omega} \|X(x)\|$. We define $H_0^{1,2}(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi|_\Gamma = 0\}$, where $\phi|_\Gamma = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω . We use capital letters to denote possibly large positive constants, e.g. C^+, K and lower case letters to denote possibly small positive constants, e.g. c^+, d^+ .

1.2.2. Notation for planar shells

Let $\omega \subset \mathbb{R}^2$ be a bounded, open domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with non-vanishing one-dimensional Hausdorff measure. The nondimensional relative characteristic thickness of the plate (aspect ratio) is taken to be $h > 0$ (contrary to Ciarlet's definition of the characteristic thickness to be 2ϵ , which difference leads only to various modified constants in the resulting formulas). We denote by $\mathbb{M}^{m \times n}$ the set of matrices mapping $\mathbb{R}^n \mapsto \mathbb{R}^m$. For $H \in \mathbb{M}^{3 \times 2}$ and $\xi \in \mathbb{R}^3$ we employ also the notation $(H|\xi) \in \mathbb{M}^{3 \times 3}$ to denote the matrix composed of H and the column ξ . Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η . This allows us to write for $\varphi \in C^1(\mathbb{R}^3, \mathbb{R}^3) : \nabla\varphi = (\varphi_x|\varphi_y|\varphi_z) = (\partial_x\varphi|\partial_y\varphi|\partial_z\varphi)$. The identity tensor on $\mathbb{M}^{2 \times 2}$ will be denoted by $\mathbb{1}_2$. The mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ is the deformation of the midsurface, $\nabla m \in \mathbb{M}^{3 \times 2}$ is the corresponding deformation gradient and \mathbf{n}_m is the outer unit normal on the surface m .

2. The Cosserat Bulk Model with Microstructure

In Ref. 49 a finite-strain, fully frame-indifferent, three-dimensional Cosserat micropolar model is introduced. The *two-field* problem has been posed in a variational setting. The task is to find a pair $(\varphi, \overline{R}) : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \text{SO}(3)$ of deformation φ

and independent microrotation $\bar{R} \in \text{SO}(3)$ minimizing the energy functional I ,

$$\begin{aligned}
 I(\varphi, \bar{R}) = & \int_{\Omega} W_{\text{mp}}(\bar{R}^T \nabla \varphi) + W_{\text{curv}}(\bar{R}^T D_x \bar{R}) - \Pi_f(\varphi) - \Pi_M(\bar{R}) dV \\
 & - \int_{\Gamma_S} \Pi_N(\varphi) dS - \int_{\Gamma_C} \Pi_{M_c}(\bar{R}) dS \mapsto \min \text{ w.r.t. } (\varphi, \bar{R}), \quad (2.1)
 \end{aligned}$$

together with the Dirichlet boundary condition of place for the deformation φ on Γ : $\varphi|_{\Gamma} = g_d$ and three possible alternative boundary conditions for the microrotations \bar{R} on Γ ,

$$\bar{R}|_{\Gamma} = \begin{cases} \bar{R}_d, & \text{the case of rigid prescription,} \\ \text{polar}(\nabla \varphi), & \text{the case of strong consistent coupling,} \\ \text{no condition for } \bar{R} \text{ on } \Gamma, & \text{induced Neumann-type relations for } \bar{R} \text{ on } \Gamma. \end{cases}$$

The constitutive assumptions on the densities are

$$\begin{aligned}
 W_{\text{mp}}(\bar{U}) &= \mu \|\text{sym}(\bar{U} - \mathbb{1})\|^2 + \mu_c \|\text{skew}(\bar{U} - \mathbb{1})\|^2 + \frac{\lambda}{2} \text{tr} [\text{sym}(\bar{U} - \mathbb{1})]^2, \\
 \bar{U} &= \bar{R}^T F, \quad F = \nabla \varphi, \quad (2.2)
 \end{aligned}$$

$W_{\text{curv}}(\mathfrak{K}) = \mu L_c^{1+p} \|\mathfrak{K}\|^{1+p}$, curvature energy,

$$\mathfrak{K} = \bar{R}^T D_x \bar{R} := \left(\bar{R}^T \nabla(\bar{R} \cdot e_1), \bar{R}^T \nabla(\bar{R} \cdot e_2), \bar{R}^T \nabla(\bar{R} \cdot e_3) \right), \text{ curvature tensor,}$$

under the minimal requirement $p \geq 1$. The total elastically stored energy $W = W_{\text{mp}} + W_{\text{curv}}$ is quadratic in the stretch \bar{U} and possibly super-quadratic in the curvature \mathfrak{K} . The strain energy W_{mp} depends on the deformation gradient $F = \nabla \varphi$ and the microrotations $\bar{R} \in \text{SO}(3)$, which do not necessarily coincide with the continuum rotations $R = \text{polar}(F)$. The curvature energy W_{curv} depends moreover on the space derivatives $D_x \bar{R}$ which describe the self-interaction of the microstructure.^e The micropolar stretch tensor \bar{U} is not symmetric and does not coincide with the symmetric continuum stretch tensor $U = R^T F = \sqrt{F^T F}$. By abuse of notation we set $\|\mathfrak{K}\|^2 := \sum_{i=1}^3 \|\mathfrak{K}^i\|^2$ for third order tensors \mathfrak{K} , cf. (1.2.1).

Here $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions g_d, \bar{R}_d for deformations and microrotations or coupling conditions for microrotations, are prescribed. $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where the potential of external surface couples Π_{M_c} are applied with $\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$ the corresponding natural boundary conditions for (φ, \bar{R}) apply. The

^eObserve that $\bar{R}^T \nabla(\bar{R} \cdot e_i) \neq \bar{R}^T \partial_{x_i} \bar{R} \in \mathfrak{so}(3)$.

potential of the external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples. For simplicity we assume

$$\Pi_f(\varphi) = \langle f, \varphi \rangle, \Pi_M(\overline{R}) = \langle M, \overline{R} \rangle, \Pi_N(\varphi) = \langle N, \varphi \rangle, \Pi_{M_c}(\overline{R}) = \langle M_c, \overline{R} \rangle, \quad (2.3)$$

for the potentials of applied loads with given functions $f \in L^2(\Omega, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathbb{M}^{3 \times 3})$, $N \in L^2(\Gamma_S, \mathbb{R}^3)$, $M_c \in L^2(\Gamma_C, \mathbb{M}^{3 \times 3})$.

The parameters $\mu, \lambda > 0$ are the Lamé constants of classical isotropic elasticity, the additional parameter $\mu_c \geq 0$ is the *Cosserat couple modulus*. For $\mu_c > 0$ the elastic strain energy density $W_{mp}(\overline{U})$ is *uniformly convex* in \overline{U} and satisfies the *standard growth condition* for all $F \in GL^+(3)$:

$$W_{mp}(\overline{R}^T F) \geq \min(\mu, \mu_c) \|\overline{R}^T F - \mathbb{1}\|^2 \geq \min(\mu, \mu_c) \text{dist}^2(F, SO(3)). \quad (2.4)$$

In contrast, for the interesting case here $\mu_c = 0$ the strain energy density is *only convex* w.r.t. F and does not satisfy (2.4).

The parameter $L_c > 0$ (with dimension length) introduces an *internal length* which is *characteristic* for the material, e.g. related to the grain size in a polycrystal and governs the interaction of the rotational microstructure. The internal length $L_c > 0$ is responsible for *size effects* in the sense that smaller samples are relatively stiffer than larger samples. Since we are interested in the case $\mu_c = 0$, the model features only 4-parameters: the two Lamé constants μ, λ , the internal length $L_c > 0$ and the curvature exponent p . As a rule of thumb, L_c will be small compared to dimensions of the bulk and the value of p determines the level of smoothness of the microrotations.

The non-standard boundary condition of *strong consistent coupling* ensures that no unwanted non-classical, polar effects may occur at the Dirichlet boundary Γ . It implies for the micropolar stretch that $\overline{U}|_\Gamma \in \text{Sym}$ and for the second Piola–Kirchhoff stress tensor $S_2 := F^{-1} D_F W_{mp}(\overline{U}) \in \text{Sym}$ on Γ as in the classical, non-polar case. We refer to the weaker boundary condition $\overline{U}|_\Gamma \in \text{Sym}$ as *weak consistent coupling*. In general, the consistent coupling condition needs a higher level of regularity in the deformation φ to lead to a well-posed problem.

It is important to realize that a linearization of this Cosserat bulk model with $\mu_c = 0$ for small displacement and small microrotations completely decouples the two fields of deformation φ and microrotations \overline{R} and leads to the classical linear elasticity problem for the deformation.^f For more details on the modelling of the three-dimensional Cosserat model as well as available existence results for $(\mu_c = 0, p > 2)$ or $(\mu_c > 0, p \geq 2)$ we refer the reader to Refs. 49, 50, 54 and 56.

^fThinking of the infinitesimal-displacement Cosserat theory one might believe that $\mu_c > 0$ is strictly necessary for a finite-strain Cosserat theory.

3. Dimensional Reduction of the Cosserat Bulk Model

3.1. The Cosserat bulk problem on a nondimensional thin domain

The basic task of any shell theory is a consistent reduction of a given 3D-theory to 2D. The problem (2.1) will be adapted to a shell-like theory. Let us assume that the problem is already presented in nondimensional form, see Sec. A.1.3. This means we are given a three-dimensional (nondimensional) *thin domain*

$$\Omega_h := \omega \times \left[-\frac{h}{2}, \frac{h}{2}\right], \quad \omega \subset \mathbb{R}^2, \tag{3.1}$$

with *transverse boundary* $\partial\Omega_h^{\text{trans}} = \omega \times \{-\frac{h}{2}, \frac{h}{2}\}$ and *lateral boundary* $\partial\Omega_h^{\text{lat}} = \partial\omega \times [-\frac{h}{2}, \frac{h}{2}]$, where ω is a bounded open domain \mathfrak{S} in \mathbb{R}^2 with smooth boundary $\partial\omega$ and $h > 0$ is the *nondimensional relative characteristic thickness*, $h \ll 1$.^h Moreover, assume that we are given a deformation φ and microrotation \overline{R}^{3d} ,

$$\varphi : \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3, \quad \overline{R}^{3d} : \Omega_h \subset \mathbb{R}^3 \mapsto \text{SO}(3), \tag{3.2}$$

solving the following two-field minimization problem on the thin domain Ω_h :

$$\begin{aligned} I(\varphi, \nabla\varphi, \overline{R}^{3d}, D_x\overline{R}^{3d}) &= \int_{\Omega_h} W_{\text{mp}}(\overline{U}) + W_{\text{curv}}(\mathfrak{K}) - \langle f, \varphi \rangle dV \\ &\quad - \int_{\partial\Omega_h^{\text{trans}} \cup \{\gamma_s \times [-\frac{h}{2}, \frac{h}{2}]\}} \langle N, \varphi \rangle dS \mapsto \min \text{ w.r.t. } (\varphi, \overline{R}), \\ \overline{U} &= \overline{R}^{3d, T} F, \end{aligned} \tag{3.3}$$

$$\varphi|_{\Gamma_0^h} = g_d(x, y, z), \quad \Gamma_0^h = \gamma_0 \times \left[-\frac{h}{2}, \frac{h}{2}\right],$$

$$\gamma_0 \subset \partial\omega, \quad \gamma_s \cap \gamma_0 = \emptyset,$$

$$\overline{U}|_{\Gamma_0^h} = \overline{R}^{3d, T} \nabla\varphi|_{\Gamma_0^h} \in \text{Sym}(3), \text{ weak consistent coupling or}$$

$$\overline{R}^{3d} : \text{ free on } \Gamma_0^h, \text{ alternative Neumann-type}$$

boundary condition,

$$W_{\text{curv}}(\mathfrak{K}) = \mu \widehat{L}_c^{1+p} \|\mathfrak{K}\|^{1+p},$$

where $\widehat{L}_c = \frac{L_c}{L}$ is a *nondimensional ratio* in terms of L_c in (2.2) and L a typical in-plane length of the underlying physical shell, see Sec. A.1. We want to find an approximation $(\varphi_s, \overline{R}_s)$ of $(\varphi, \overline{R}^{3d})$ involving only two-dimensional quantities.

^gFor definiteness, one can think of $\omega = [0, 1] \times [0, 1]$, without dimensions of length.

^hThe aspect ratio $h = \frac{d}{L}$ with d the thickness of the real shell and L a typical in-plane length.

4. The Cosserat Thin Planar Shell Model with Size Effects

4.1. Statement of the formal Cosserat shell model

In Ref. 50 a method of dimensional reduction is proposed which leads us to *postulate* the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the planar shell $\bar{R} : \omega \subset \mathbb{R}^2 \mapsto \text{SO}(3)$ on ω :

$$I(m, \bar{R}) = \int_{\omega} h W_{\text{mp}}(\bar{U}) + h W_{\text{curv}}(\mathfrak{K}_s) + \frac{h^3}{12} W_{\text{bend}}(\mathfrak{K}_b) d\omega - \Pi(m, \bar{R}_3) \mapsto \min \text{ w.r.t. } (m, \bar{R}), \tag{4.1}$$

under the constraints

$$\bar{U} = \bar{R}^T \hat{F}, \quad \hat{F} = (\nabla m | \bar{R}_3) \in \text{GL}^+(3), \tag{4.2}$$

$$\mathfrak{K}_s = \left(\bar{R}^T (\nabla(\bar{R} \cdot e_1) | 0), \bar{R}^T (\nabla(\bar{R} \cdot e_2) | 0), \bar{R}^T (\nabla(\bar{R} \cdot e_3) | 0) \right) \in \mathfrak{T}(3), \quad \mathfrak{K}_b = \mathfrak{K}_s^3,$$

and the boundary conditions of place (simple support) for the midsurface deformation m on the Dirichlet part of the lateral boundary $\gamma_0 : m|_{\gamma_0} = g_d(x, y, 0)$. Possible *alternative boundary conditions* for the microrotations \bar{R} on γ_0 are

$\bar{R}|_{\gamma_0} = \text{polar}((\nabla m | \nabla g_d(x, y, 0) \cdot e_3))|_{\gamma_0}$, strong reduced consistent coupling, or

$$\forall A \in C_0^\infty(\gamma_0, \mathfrak{so}(3)) : \int_{\gamma_0} \langle \bar{R}^T (\nabla m(x, y) | \nabla g_d(x, y, 0) \cdot e_3), A(x, y) \rangle ds = 0, \tag{4.3}$$

very weak consistent coupling, or

$$\bar{R}_3|_{\gamma_0} = \frac{\nabla g_d(x, y, 0) \cdot e_3}{\|\nabla g_d(x, y, 0) \cdot e_3\|}, \quad \text{rigid director prescription.}$$

The constitutive assumptions on the reduced densities areⁱ

$$\begin{aligned} W_{\text{mp}}(\bar{U}) &= \mu \|\text{sym}(\bar{U} - \mathbb{1})\|^2 + \mu_c \|\text{skew}(\bar{U} - \mathbb{1})\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \text{tr} [\text{sym}(\bar{U} - \mathbb{1})]^2 \\ &= \mu \underbrace{\|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)\|^2}_{\text{shear-stretch energy}} + \mu_c \underbrace{\|\text{skew}((\bar{R}_1 | \bar{R}_2)^T \nabla m)\|^2}_{\text{first order drill energy}} \\ &\quad + \underbrace{\frac{\kappa(\mu + \mu_c)}{2} \left(\langle \bar{R}_3, m_x \rangle^2 + \langle \bar{R}_3, m_y \rangle^2 \right)}_{\text{classical transverse shear energy}} \\ &\quad + \frac{\mu\lambda}{2\mu + \lambda} \underbrace{\text{tr} [\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)]^2}_{\text{elongational stretch energy}}, \end{aligned} \tag{4.4}$$

ⁱ $\|\text{skew}((\bar{R}_1 | \bar{R}_2)^T \nabla m)\|^2 = (\langle \bar{R}_1, m_y \rangle - \langle \bar{R}_2, m_x \rangle)^2$.

and

$$\begin{aligned}
 W_{\text{curv}}(\mathfrak{K}_s) &= \mu \widehat{L}_c^{1+p} \|\mathfrak{K}_s\|^{1+p}, \text{ reduced curvature energy,} \\
 \mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R} \cdot e_1)|0), \overline{R}^T(\nabla(\overline{R} \cdot e_2)|0), \overline{R}^T(\nabla(\overline{R} \cdot e_3)|0) \right), \\
 \mathfrak{K}_s &= (\mathfrak{K}_s^1, \mathfrak{K}_s^2, \mathfrak{K}_s^3) \in \mathfrak{T}(3), \text{ the reduced third order curvature tensor,} \\
 W_{\text{bend}}(\mathfrak{K}_b) &= \mu \|\text{sym}(\mathfrak{K}_b)\|^2 + \mu_c \|\text{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \text{tr} [\text{sym}(\mathfrak{K}_b)]^2, \\
 \mathfrak{K}_b &= \overline{R}^T(\nabla\overline{R}_3|0) = \mathfrak{K}_s^3, \text{ second order non-symmetric bending tensor.}
 \end{aligned}
 \tag{4.5}$$

The elastically stored energy density due to *membrane-strain*, *microstructural curvature* and *bending*

$$W = \underbrace{h W_{\text{mp}}}_{\text{membrane}} + \underbrace{h W_{\text{curv}}}_{\text{curvature}} + \underbrace{\frac{h^3}{12} W_{\text{bend}}}_{\text{bending}},
 \tag{4.6}$$

depends on the midsurface deformation gradient ∇m and microrotations \overline{R} together with their space derivatives only through the frame-indifferent measures \overline{U} and \mathfrak{K}_s . The *micropolar stretch tensor* \overline{U} of the planar shell is *non-symmetric*, neither is the *micropolar reduced third order curvature tensor* \mathfrak{K}_s . The three-dimensional shell deformation is reconstructed as

$$\varphi_s(x, y, z) = m(x, y) + \left(z \varrho_m(x, y) + \frac{z^2}{2} \varrho_b(x, y) \right) \overline{R}(x, y) \cdot e_3,
 \tag{4.7}$$

where

$$\begin{aligned}
 \varrho_m &= 1 - \underbrace{\frac{\lambda}{2\mu + \lambda} [\langle (\nabla m|0), \overline{R} \rangle - 2]}_{\text{first order thickness change due to membrane stretch}} = 1 - \frac{\lambda}{2\mu + \lambda} \text{tr} [\overline{U} - \mathbb{1}], \\
 \varrho_b &= \underbrace{-\frac{\lambda}{2\mu + \lambda} \langle (\nabla\overline{R}_3|0), \overline{R} \rangle}_{\text{non-symmetric shift of the midsurface due to bending}} = -\frac{\lambda}{2\mu + \lambda} \text{tr} [\mathfrak{K}_b].
 \end{aligned}
 \tag{4.8}$$

To first order, the reconstructed deformation gradient is given by $F_s = (\nabla m|_{\varrho_m} \overline{R}_3)$.

The reduced external loading functional $\Pi(m, \overline{R}_3)$ is a linear form in (m, \overline{R}_3) defined in (5.7) in terms of the underlying three-dimensional loads. We have included the *shear correction factor* κ ($0 < \kappa \leq 1$) to keep in line with classical infinitesimal-displacement Reissner–Mindlin plate models. In the derivation, however, one obtains $\kappa \equiv 1$. The dimensionally reduced model (4.1) is fully *frame-indifferent*, meaning that

$$\forall Q \in \text{SO}(3) : W_{\text{mp}}(Q\widehat{F}, Q\overline{R}) = W_{\text{mp}}(\widehat{F}, \overline{R}), \quad \mathfrak{K}_s(Q\overline{R}) = \mathfrak{K}_s(\overline{R}).
 \tag{4.9}$$

Strain and curvature parts are *additively decoupled*, as in the “parent” Cosserat bulk model (2.1). Note the appearance of the *harmonic mean* \mathcal{H} and *arithmetic mean* \mathcal{A}

$$\frac{1}{2}\mathcal{H}\left(\mu, \frac{\lambda}{2}\right) = \frac{\mu\lambda}{2\mu + \lambda}, \quad \kappa \mathcal{A}(\mu, \mu_c) = \kappa \frac{\mu + \mu_c}{2}. \tag{4.10}$$

Following Ref. 50 we refer to $0 < p < 1$ as the *sub-critical case*, to $p = 1$ as the *critical case* and to $p > 1$ as the *super-critical case*. In this contribution we will treat mathematically exclusively the super-critical case for zero Cosserat couple modulus $\mu_c = 0$. The simpler critical case for positive Cosserat couple modulus $\mu_c > 0$ with rigid director prescription at the boundary is already dealt with in Ref. 50.

It is easy to see (Ref. 52) that the membrane energy part W_{mp} in (4.1) is *uniformly Legendre-Hadamard elliptic* with ellipticity constant $\mu > 0$ independent of the value of the Cosserat couple modulus μ_c at given rotations \bar{R} . In Ref. 50 it is shown that a linearization of (4.1) with $\mu_c = 0$ and $p > 1$ (super-quadratic curvature energy W_{curv}) for small displacement and small microrotation does not decouple the fields, as would be the case in the three-dimensional situation, but leads formally to the infinitesimal-displacement, classical linear Reissner–Mindlin model for one infinitesimal director without drill-energy contribution.

5. The Coercivity Inequality in Two Dimensions

In this section we show how to use the three-dimensional extended Korn’s first inequality^{47,60} in our reduced two-dimensional context of *planar and curved shells* in order to improve Legendre–Hadamard ellipticity to uniform positivity. In order to show that the elastic membrane energy is uniformly convex for zero Cosserat couple modulus $\mu_c = 0$ we look at the second differential of $W_{\text{mp}}(\bar{R}^T \hat{F})$ with respect to the midsurface m

$$D_{\nabla m}^2 W_{\text{mp}}(\bar{R}^T \hat{F}) \cdot (\nabla\phi, \nabla\phi) \geq \frac{\mu}{2} \|(\nabla\phi|0)^T \bar{R} + \bar{R}^T (\nabla\phi|0)\|^2. \tag{5.1}$$

Set for simplicity $\mu = 2$ and consider the slightly more general quadratic form (appropriate for curved elastic shells: $F_p = \nabla\Theta \in \text{GL}(3)$ with Θ a regular parametrization of the stress-free initial curvilinear shell surface and curved elastoplastic shells: $F_p, \bar{R}_e \in \text{GL}^+(3)$ arbitrary, but independent of the transverse variable z)

$$\begin{aligned} & \|F_p^{-T}(\nabla\phi|0)^T \bar{R}_e + \bar{R}_e^T(\nabla\phi|0)F_p^{-1}\|^2 \\ &= \|\bar{R}_e \left(F_p^{-T}(\nabla\phi|0)^T \bar{R}_e + \bar{R}_e^T(\nabla\phi|0)F_p^{-1} \right) \bar{R}_e^T\|^2 \\ &= \|(\bar{R}_e F_p)^{-T}(\nabla\phi|0)^T + (\nabla\phi|0)(\bar{R}_e F_p)^{-1}\|^2, \end{aligned} \tag{5.2}$$

where $\phi : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and $\phi|_{\gamma_0} = 0$ for $\gamma_0 \subset \partial\omega$. We can prove

Theorem 5.1. (Extended Korn’s inequality for curved shells) *Let $\omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $\gamma_0 \subset \partial\omega$ be a part of the boundary*

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with non-vanishing one-dimensional Hausdorff measure. Define $H_o^{1,2}(\omega, \mathbb{R}^3; \gamma_0) := \{\phi \in H^{1,2}(\omega, \mathbb{R}^3) | \phi|_{\gamma_0} = 0\}$ and let $F_p, F_p^{-1} \in W^{1,2+\delta}(\bar{\omega}, \text{GL}(3))$. Then

$$\exists c^+ > 0 \quad \forall \phi \in H_o^{1,2}(\omega, \mathbb{R}^3; \gamma_0) : \\ \|(\nabla\phi|_0)F_p^{-1}(x) + F_p^{-T}(x)(\nabla\phi|_0)^T\|_{L^2(\omega)}^2 \geq c^+ \|\phi\|_{H^{1,2}(\omega)}^2,$$

and the constant is bounded away from zero for F_p, F_p^{-1} bounded in $W^{1,2+\delta}(\omega, \text{GL}(3))$.

Proof. The proof is based on a suitable lifting of the function ϕ from two dimensions to three dimensions and on the strengthening of the extended Korn's first inequality⁴⁷ proposed in Ref. 60. More precisely let $\phi : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and $\phi|_{\gamma_0} = 0$ for $\gamma_0 \subset \partial\omega$. Extend now ϕ to $\bar{\phi} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ through

$$\bar{\phi}(x, y, z) := \phi(x, y) \Rightarrow \bar{\phi}(x, y, z)|_{\gamma_0 \times [-\frac{h}{2}, \frac{h}{2}]} = 0 \quad \text{and} \\ \nabla_{(x,y,z)} \bar{\phi}(x, y, z) = (\nabla_{(x,y)} \phi|_0). \tag{5.3}$$

For $\bar{\phi}$ it is possible to use the 3D-extended Korn's first inequality.⁶⁰ To this end consider $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$ and the lateral Dirichlet boundary $\Gamma_0^h = \gamma_0 \times [-\frac{h}{2}, \frac{h}{2}] \subset \partial\Omega_h$. Then Γ_0^h has non-vanishing two-dimensional Hausdorff measure. Set by abuse of notation $F_p = (\bar{R}_e F_p)$ for the moment. With smooth enough, invertible F_p it holds on applying the extended Korn's first inequality that

$$\int_{\Omega_h} \|\nabla \bar{\phi}^T F_p^{-1} + F_p^{-T} \nabla \bar{\phi}\|^2 dV \geq c_{3D}^+ \cdot \int_{\omega \times [-\frac{h}{2}, \frac{h}{2}]} \|\bar{\phi}\|^2 + \|\nabla \bar{\phi}\|^2 dV \\ \Rightarrow \int_{\omega} \int_{-\frac{h}{2}}^{\frac{h}{2}} \|\nabla \bar{\phi}^T F_p^{-1} + F_p^{-T} \nabla \bar{\phi}\|^2 d\omega dz \geq c_{3D}^+ \cdot \int_{\omega} \int_{-\frac{h}{2}}^{\frac{h}{2}} \|\bar{\phi}\|^2 + \|\nabla \bar{\phi}\|^2 d\omega dz. \tag{5.4}$$

Since $\bar{\phi}$ and F_p are in fact independent of z we may carry out the integration with respect to the transverse variable and get,

$$\int_{\omega} \|\nabla \bar{\phi}^T F_p^{-1} + F_p^{-T} \nabla \bar{\phi}\|^2 d\omega \geq c_{3D}^+ \cdot \int_{\omega} \|\bar{\phi}\|^2 + \|\nabla \bar{\phi}\|^2 d\omega, \tag{5.5}$$

or back in terms of ϕ

$$\int_{\omega} \|(\nabla\phi|_0)^T F_p^{-1} + F_p^{-T}(\nabla\phi|_0)\|^2 d\omega \geq c_{3D}^+ \cdot \int_{\omega} \|\phi\|^2 + \|(\nabla\phi|_0)\|^2 d\omega. \tag{5.6}$$

Observe that the constant c_{3D}^+ is in fact independent of the characteristic thickness h (we could set $h = 1$ in the lifting from two to three dimensions) which might be surprising at first glance. This observation allows one to bound $m \in H_o^{1,2}(\omega, \mathbb{R}^3; \gamma_0)$, independent of the relative thickness h only in terms of the membrane energy $\int_{\omega} W(\nabla m, \bar{R}) d\omega$ if $\bar{R} \in \text{SO}(3)$ is smooth enough. This shows the first claim.

For the second claim the Sobolev embedding shows that $F_p \in W^{1,2+\delta}(\omega, \text{GL}(3))$ may be identified with a continuous function. In order to show that the constant

is uniformly bounded away from zero for bounded $F_p, F_p^{-1} \in W^{1,2+\delta}(\omega, \text{GL}(3))$ a contradiction argument as in Ref. 51 is employed which uses the fact that $W^{1,2+\delta}(\omega, \text{GL}(3))$ is compactly embedded in $C^0(\bar{\omega}, \text{GL}(3))$. \square

5.1. The external resultant loading functional Π^\sharp

The mathematical analysis of the case with zero Cosserat couple modulus $\mu_c = 0$ necessitates a modification of the classical resultant dead load loading functional Π given by the linear form

$$\Pi(m, \bar{R}_3) = \int_{\omega} \langle \bar{f}, m \rangle + \langle \bar{M}, \bar{R}_3 \rangle d\omega + \int_{\gamma_s} \langle \bar{N}, m \rangle + \langle \bar{M}_c, \bar{R}_3 \rangle ds, \tag{5.7}$$

where terms with a bar denote resultant quantities, e.g. \bar{f} is the resultant body force, see Ref. 50. We replace (5.7) by a live load resultant loading functional Π^\sharp :

$$\begin{aligned} \Pi^\sharp(m, \bar{R}_3) &= \int_{\omega} \left\langle \bar{f}, \frac{m}{1 + [||m|| - K]_+} \right\rangle + \langle \bar{M}, \bar{R}_3 \rangle d\omega \\ &+ \int_{\gamma_s} \left\langle \bar{N}, \frac{m}{1 + [||m|| - K]_+} \right\rangle + \langle \bar{M}_c, \bar{R}_3 \rangle ds. \end{aligned} \tag{5.8}$$

Here $K > 0$ is a possibly large constant and $[\cdot]_+$ denotes the positive part of its scalar argument. Note that (5.8) is always bounded, provided $\bar{f}, \bar{M} \in L^1(\omega, \mathbb{R}^3)$, $\bar{M}_c, \bar{N} \in L^1(\gamma_s, \mathbb{R}^3)$ and the linearization of Π^\sharp coincides with the linearization of (5.7). In the three-dimensional theory this replacement is called the “principle of bounded external work”^{54,56}

6. Analysis for Zero Cosserat Couple Modulus

The following results provide existence theorems for geometrically exact deduced elastic Cosserat shell models for the physically more realistic super-critical case $\mu_c = 0, p > 1$.^j

Theorem 6.1. (Existence for 2D-elastic Cosserat model) *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_d \in H^1(\omega, \mathbb{R}^3)$ and $\text{polar}(\nabla g_d) \in W^{1,1+p}(\omega, \text{SO}(3))$. Moreover, let $\bar{f} \in L^1(\omega, \mathbb{R}^3)$ and suppose $\bar{N} \in L^1(\gamma_s, \mathbb{R}^3)$ together with $\bar{M} \in L^1(\omega, \mathbb{R}^3)$ and $\bar{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (5.7). Then (4.1) with $\mu_c = 0$ and $p > 1$, boundary conditions for \bar{R} of rigid director prescription on γ_0 and modified external potential Π^\sharp (5.8) admits at least one minimizing solution pair $(m, \bar{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p}(\omega, \text{SO}(3))$.*

^jThe proposed results determine the macroscopic midsurface deformation $m \in H^1(\omega, \mathbb{R}^3)$ and not more. This means that discontinuous macroscopic deformations by cavities or the formation of holes are not excluded (possible mode I failure), in this sense point defects are included. If $\mu_c > 0$ fracture is effectively ruled out, which is, however, somewhat unrealistic. All results remain true for arbitrary shear correction factor $\kappa > 0$. For $\kappa = 0$, however, uniform Legendre–Hadamard ellipticity is lost.

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Proof. We apply the direct methods of the calculus of variations. First, the requirement on the data shows that

$$\forall m \in H^1(\omega, \mathbb{R}^3), \bar{R} \in W^{1,1+p}(\omega, \text{SO}(3)) : \quad \Pi^\sharp(m, \bar{R}_3) \leq C, \tag{6.1}$$

i.e. a uniform bound on the external loading functional. Let us define the admissible set

$$\mathcal{A} := \left\{ m \in H^1(\omega, \mathbb{R}^3), \bar{R} \in W^{1,1+p}(\omega, \text{SO}(3)) \mid m|_{\gamma_0} = g_d(x, y, 0), \right. \\ \left. \bar{R}_{3|_{\gamma_0}} = \frac{\nabla g_d(x, y, 0) \cdot e_3}{\|\nabla g_d(x, y, 0) \cdot e_3\|} \right\}. \tag{6.2}$$

For the boundary data g_d it is clear that $I(g_d(x, y, 0), \text{polar}(\nabla g_d(x, y, 0))) < \infty$, hence I is bounded above on \mathcal{A} . Consider a sequence of pairs of deformation m_k and rotations \bar{R}^k in the admissible set \mathcal{A} with bounded energy I . For such a sequence we have

$$\infty > I(m_k, \bar{R}^k) = \int_{\omega} h W_{\text{mp}}(\bar{U}_k) + h W_{\text{curv}}(\mathfrak{K}_{s,k}) + \frac{h^3}{12} W_{\text{bend}}(\mathfrak{K}_{b,k}) d\omega - \Pi^\sharp(m_k, \bar{R}_3^k) \\ \geq \int_{\omega} h W_{\text{mp}}(\bar{U}_k) + h W_{\text{curv}}(\mathfrak{K}_{s,k}) + \frac{h^3}{12} W_{\text{bend}}(\mathfrak{K}_{b,k}) d\omega - C \geq C_3, \tag{6.3}$$

which implies that I is bounded from below on \mathcal{A} and the positive curvature energy $\int_{\omega} h W_{\text{curv}}(\mathfrak{K}_{s,k}) d\omega$ can be bounded independent of $k \in \mathbb{N}$. Observe now that the curvature energy bounds the sequence of curvature tensors $\mathfrak{K}_{s,k}$ in $L^{1+p}(\omega, \mathfrak{T}(3))$. Since $\|\mathfrak{K}_s\| = \|\bar{R}^T D_x \bar{R}\| = \|D_x \bar{R}\|$ pointwise, this implies that $\|D_x \bar{R}^k\|_{L^{1+p}(\omega)}$ is bounded as well. Since $\|\bar{R}^k\| = \sqrt{3}$ pointwise, this shows the boundedness of $\bar{R}^k \subset W^{1,1+p}(\omega, \text{SO}(3))$, even without specific Dirichlet boundary conditions on the remaining “free” columns $\bar{R} \cdot e_1, \bar{R} \cdot e_2$.^k This is a distinctive feature for exact rotations. A subsequence can be chosen such that $\mathfrak{K}_{s,k} \rightharpoonup \widehat{\mathfrak{K}}_s$ in $L^{1+p}(\omega, \mathfrak{T}(3))$, weakly. Since the boundedness of the rotations \bar{R}^k holds true in the space $W^{1,1+p}(\omega, \text{SO}(3))$ with $1 + p > N = 3$, it is possible to extract a subsequence, not relabeled, such that \bar{R}^k converges strongly to $\widehat{\bar{R}} \in C^0(\bar{\omega}, \text{SO}(3))$ in the topology of $C^0(\bar{\omega}, \text{SO}(3))$ on account of the Sobolev-embedding theorem.

Since I is bounded below on \mathcal{A} we may consider from now on infimizing sequences of midsurface deformations m_k and rotations \bar{R}^k with

$$\lim_{k \rightarrow \infty} I(m_k, \bar{R}^k) = \inf_{(m, \bar{R}) \in \mathcal{A}} I(m, \bar{R}). \tag{6.4}$$

Along the strongly convergent sequence of rotations, the corresponding sequence of mid-surface deformations m_k is also bounded in $H^1(\omega, \mathbb{R}^3)$. However, this is

^kWithout independent curvature control, nothing can be shown for $\mu_c = 0$. This is the reason for the modification of the external loads.

not due to a simple pointwise estimate as in case I ($\mu_c > 0$),⁵⁰ but only true after integration over the domain ω : at face value we only control certain mixed symmetric expressions in the reconstructed deformation gradient. Let us therefore define $v_k \in H^{1,2}(\omega, \mathbb{R}^3)$ by $m_k = g_d + (m_k - g_d) = g_d + v_k$. Then we have (constants may change from line to line)

$$\begin{aligned}
 \infty > I(m_k, \bar{R}^k) &= \int_{\omega} h W_{\text{mp}}(\bar{U}_k) \\
 &\quad + h W_{\text{curv}}(\mathfrak{R}_{s,k}) + \frac{h^3}{12} W_{\text{bend}}(\mathfrak{R}_{b,k}) d\omega - \Pi^{\sharp}(m_k, \bar{R}_3^k) \\
 &\geq \int_{\omega} h W_{\text{mp}}(\bar{U}_k) - \Pi^{\sharp}(m_k, \bar{R}_3^k) d\omega \geq \int_{\omega} h W_{\text{mp}}(\bar{U}_k) d\omega - C \\
 &\geq \int_{\omega} h \frac{\mu}{4} \|\bar{R}^{k,\text{T}}(\nabla m_k | \bar{R}_3^k) + (\nabla m_k | \bar{R}_3^k)^{\text{T}} \bar{R}^k - 2\mathbb{1}\|^2 d\omega - C \\
 &= \int_{\omega} h \frac{\mu}{4} \|\bar{R}^{\text{T}}(\nabla m_k | \bar{R}_3) + (\nabla m_k | \bar{R}_3)^{\text{T}} \bar{R}\|^2 \\
 &\quad - 4h \frac{\mu}{4} \text{tr} \left[\bar{R}^{\text{T}}(\nabla m_k | \bar{R}_3) + (\nabla m_k | \bar{R}_3)^{\text{T}} \bar{R} \right] + 4h \frac{\mu}{4} \|\mathbb{1}\|^2 d\omega - C \\
 &\geq \int_{\omega} h \frac{\mu}{4} \|\bar{R}^{k,\text{T}}(\nabla m_k | 0) + (\nabla m_k | 0)^{\text{T}} \bar{R}^k\|^2 d\omega - C_1 \|m_k\|_{H^{1,2}(\omega)} + C_2 \\
 &= \int_{\omega} h \frac{\mu}{4} \|\bar{R}^{k,\text{T}}(\nabla v_k | 0) + (\nabla v_k | 0)^{\text{T}} \bar{R}^k + \bar{R}^{k,\text{T}}(\nabla g_d | 0) + (\nabla g_d | 0)^{\text{T}} \bar{R}^k\|^2 d\omega \\
 &\quad - C_1 \|v_k + g_d\|_{H^{1,2}(\omega)} + C_2 \\
 &= \int_{\omega} h \frac{\mu}{4} \|\bar{R}^{k,\text{T}}(\nabla v_k | 0) + (\nabla v_k | 0)^{\text{T}} \bar{R}^k\|^2 \\
 &\quad + h \frac{2\mu}{4} \langle \bar{R}^{k,\text{T}}(\nabla v_k | 0) + (\nabla v_k | 0)^{\text{T}} \bar{R}^k, \bar{R}^{k,\text{T}}(\nabla g_d | 0) + (\nabla g_d | 0)^{\text{T}} \bar{R}^k \rangle \\
 &\quad + h \frac{\mu}{4} \|\bar{R}^{k,\text{T}}(\nabla g_d | 0) + (\nabla g_d | 0)^{\text{T}} \bar{R}^k\|^2 d\omega - C_1 \|v_k\|_{H^{1,2}(\omega)} + C_2 . \\
 &\geq \int_{\omega} h \frac{\mu}{4} \|\bar{R}^{k,\text{T}}(\nabla v_k | 0) + (\nabla v_k | 0)^{\text{T}} \bar{R}^k\|^2 \tag{6.5} \\
 &\quad - h \frac{\mu}{4} \left(\varepsilon \|\bar{R}^{k,\text{T}}(\nabla v_k | 0) + (\nabla v_k | 0)^{\text{T}} \bar{R}^k\|^2 + \frac{1}{\varepsilon} \|\bar{R}^{k,\text{T}}(\nabla g_d | 0) + (\nabla g_d | 0)^{\text{T}} \bar{R}^k\|^2 \right) \\
 &\quad + h \frac{\mu}{4} \|\bar{R}^{k,\text{T}}(\nabla g_d | 0) + (\nabla g_d | 0)^{\text{T}} \bar{R}^k\|^2 d\omega - C_1 \|v_k\|_{H^{1,2}(\omega)} + C_2 . \\
 &\geq \int_{\omega} h \frac{\mu}{8} \|\bar{R}^{k,\text{T}}(\nabla v_k | 0) + (\nabla v_k | 0)^{\text{T}} \bar{R}^k\|^2 \\
 &\quad + h \frac{\mu}{8} \|\bar{R}^{k,\text{T}}(\nabla g_d | 0) + (\nabla g_d | 0)^{\text{T}} \bar{R}^k\|^2 d\omega - C_1 \|v_k\|_{H^{1,2}(\omega)} + C_2 .
 \end{aligned}$$

$$\begin{aligned} &\geq \int_{\omega} h \frac{\mu}{8} \|\overline{R}^{k,T}(\nabla v_k|_0) + (\nabla v_k|_0)^T \overline{R}^k\|^2 d\omega - C_1 \|v_k\|_{H^{1,2}(\omega)} + C_2 \\ &= \int_{\omega} h \frac{\mu}{8} \|(\overline{R}^k - \widehat{R} + \widehat{R})^T(\nabla v_k|_0) + (\nabla v_k|_0)^T(\overline{R}^k - \widehat{R} + \widehat{R})\|^2 d\omega \\ &\quad - C_1 \|v_k\|_{H^{1,2}(\omega)} + C_2. \end{aligned}$$

Continuing the estimate we have

$$\begin{aligned} I(m_k, \overline{R}^k) &\geq \int_{\omega} h \frac{\mu}{8} \underbrace{\|\widehat{R}^T(\nabla v_k|_0) + (\nabla v_k|_0)^T \widehat{R}\|^2}_{\text{combinations of derivatives}} d\omega - C_3 \|\widehat{R} - \overline{R}^k\|_{\infty} \|v_k\|_{H^{1,2}(\omega)}^2 \\ &\quad - (C_1 + 2 \|\widehat{R} - \overline{R}^k\|_{\infty}) \|v_k\|_{H^{1,2}(\omega)} + C_2 \\ &\geq \left(h \frac{\mu}{8} c_K^+ - C_3 \|\widehat{R} - \overline{R}^k\|_{\infty} \right) \|v_k\|_{H^{1,2}(\omega)}^2 \\ &\quad - (C_1 + 2 \|\widehat{R} - \overline{R}^k\|_{\infty}) \|v_k\|_{H^{1,2}(\omega)} + C_2, \end{aligned}$$

where we made use of the zero boundary conditions for v_k on γ_0 , used Young's inequality with $\varepsilon = \frac{1}{2}$, the essential boundedness of exact rotations and applied finally the extended Korn's inequality Theorem 5.1 (note that $\overline{R}^{-T} = \overline{R}$ for exact rotations) yielding the positive constant c_K^+ for the continuous microrotation \widehat{R} . Since $\|\widehat{R} - \overline{R}^k\|_{\infty} \rightarrow 0$ we conclude the boundedness of v_k in $H^1(\omega, \mathbb{R}^3)$. Hence, m_k is bounded as well in $H^1(\omega, \mathbb{R}^3)$.

From the boundedness of m_k in $H^1(\omega, \mathbb{R}^3)$ we may extract a subsequence, not relabelled, such that $m_k \rightharpoonup \widehat{m} \in H^1(\omega, \mathbb{R}^3)$. Furthermore, we may always obtain a subsequence of (m_k, \overline{R}^k) such that $\overline{U}_k = \overline{R}^{k,T} \widehat{F}^k = \overline{R}^{k,T}(\nabla m_k|_{\widehat{R}_3})$ converges weakly in $L^2(\omega)$ to $\widehat{U} = \widehat{R}^T(\nabla \widehat{m}|_{\widehat{R}_3})$. Weak convergence of $D_x \overline{R}^k$ in $L^{1+p}(\omega, \mathfrak{T}(3))$ and strong convergence of \overline{R}^k in $L^2(\omega)$ together show that the sequence of the third order curvature tensors $\widehat{\mathfrak{R}}_{s,k} = \overline{R}^{k,T} D_x \overline{R}^k$ converges indeed weakly to the correct limit $\widehat{R}^T D_x \widehat{R} = \widehat{\mathfrak{R}}_s$ in $L^1(\omega, \mathfrak{T}(3))$. But from above we know already that weak convergence for $\widehat{\mathfrak{R}}_{s,k}$ takes place in $L^2(\omega, \mathfrak{T}(3))$. Gathering the obtained statements we have

$$\begin{aligned} \overline{U}_k = \overline{R}^{k,T} \widehat{F}^k &\rightharpoonup \widehat{U} = \widehat{R}^T(\nabla \widehat{m}|_{\widehat{R}_3}) && \text{in } L^2(\omega)\text{-weak,} \\ \widehat{\mathfrak{R}}_{s,k} = \overline{R}^{k,T} D_x \overline{R}^k &\rightharpoonup \widehat{\mathfrak{R}}_s = \widehat{R}^T D_x \widehat{R} && \text{in } L^2(\omega, \mathfrak{T}(3))\text{-weak,} \\ \widehat{\mathfrak{R}}_{b,k} &\rightharpoonup \widehat{\mathfrak{R}}_b && \text{in } L^2(\omega, \mathbb{M}^{3 \times 3})\text{-weak,} \\ m_k &\rightarrow \widehat{m} && \text{in } L^2(\omega, \mathbb{R}^3)\text{-strong,} \\ \overline{R}^k &\rightarrow \widehat{R} && \text{in } C(\omega, \text{SO}(3))\text{-strong.} \end{aligned} \tag{6.6}$$

Since the total energy is convex in the combined terms $(\bar{U}, \mathfrak{K}_s, \mathfrak{K}_b)$ we get

$$\begin{aligned}
 I(\widehat{m}, \widehat{R}) &= \int_{\omega} h W_{\text{mp}}(\widehat{U}) + h W_{\text{curv}}(\widehat{\mathfrak{K}}_s) + \frac{h^3}{12} W_{\text{bend}}(\widehat{\mathfrak{K}}_b) d\omega - \Pi^{\sharp}(\widehat{m}, \widehat{R}_3) \\
 &\leq \liminf_{k \rightarrow \infty} \int_{\omega} h W_{\text{mp}}(\bar{U}_k) + h W_{\text{curv}}(\mathfrak{K}_{s,k}) + \frac{h^3}{12} W_{\text{bend}}(\mathfrak{K}_{b,k}) d\omega - \Pi^{\sharp}(m_k, \bar{R}_3^k) \\
 &= \lim_{k \rightarrow \infty} I(m_k, \bar{R}^k) = \inf_{(m, \bar{R}) \in \mathcal{A}} I(m, \bar{R}), \tag{6.7}
 \end{aligned}$$

which implies that the limit pair $(\widehat{m}, \widehat{R})$ is a minimizer and the Dirichlet boundary conditions for either midsurface deformation \widehat{m} and “director” \widehat{R}_3 are satisfied strongly by compact embedding in the sense of traces on γ_0 . This finishes the argument. \square

Let us turn now to a slightly modified formulation which gives the very weak consistent coupling boundary condition in (4.3) a certain sense. The problem with the very weak consistent coupling boundary condition consists in that it is making a statement for quantities, which may not be defined properly on the boundary γ_0 if higher regularity is missing (∇m might not have a trace on γ_0 for $m \in H^1(\omega)$). Therefore we relax the condition on the boundary γ_0 into a symmetry condition to be satisfied in a *boundary layer* $\omega_{\text{bdl},h}$ adjacent to γ_0 of thickness $h > 0$. First we define this boundary layer. We set

$$\omega_{\text{bdl},h} := \{ \bar{x} \in \bar{\omega} \mid \bar{x} = x - \varrho \mathbf{n}_{\gamma_0}, x \in \gamma_0, 0 \leq \varrho \leq h \}, \tag{6.8}$$

with \mathbf{n}_{γ_0} the outer unit normal on γ_0 . We require then the symmetry condition

$$\forall A \in C^\infty(\omega_{\text{bdl},h}, \mathfrak{so}(3)) : \int_{\omega_{\text{bdl},h}} \langle \bar{R}^T (\nabla m(x, y) | \nabla g_d(x, y, 0) \cdot e_3), A(x, y) \rangle d\omega = 0. \tag{6.9}$$

The interest in this formulation stems from the fact that finite element computations with very weak consistent coupling for the Cosserat bulk were implemented by a penalty formulation in such a boundary layer. With this modification it is possible to show

Corollary 6.2. (Existence for very weak consistent coupling) *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_d \in H^1(\omega, \mathbb{R}^3)$, $\text{polar}(\nabla g_d) \in W^{1,1+p}(\omega, \text{SO}(3))$, $\text{polar}(\nabla g_d)|_{\gamma_0} \in W^{1,1+p}(\gamma_0, \text{SO}(3))$ and $\partial_z g_d|_{\gamma_0} \in L^2(\gamma_0, \mathbb{R}^3)$. Moreover, let $\bar{f} \in L^1(\omega, \mathbb{R}^3)$ and suppose $\bar{N} \in L^1(\gamma_s, \mathbb{R}^3)$ together with $\bar{M} \in L^1(\omega, \mathbb{R}^3)$ and $\bar{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (5.7). Then (4.1) with $\mu_c = 0$ and $p > 1$, relaxed weak consistent coupling (6.9) and modified external potential Π^{\sharp} (5.8) admits at least one minimizing solution pair $(m, \bar{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p}(\omega, \text{SO}(3))$.*

Proof. We repeat the argument of Theorem 6.1. First, we define the admissible set

$$\mathcal{A} := \left\{ m \in H^1(\omega, \mathbb{R}^3), \bar{R} \in W^{1,1+p}(\omega, \text{SO}(3)) \mid m|_{\gamma_0} = g_d(x, y, 0), \right. \\ \left. \int_{\omega_{\text{bdl},h}} \langle \bar{R}^T (\nabla m(x, y) | \nabla g_d(x, y, 0) \cdot e_3), A(x, y) \rangle d\omega = 0 \right. \\ \left. \forall A \in C^\infty(\omega_{\text{bdl},h}, \mathfrak{so}(3)) \right\}, \tag{6.10}$$

which incorporates the weak consistent coupling condition in its relaxed form. In order to see that \mathcal{A} is not empty take $\bar{R} = \text{polar}(\nabla g_d)$ and $m = g_d$. As in Theorem 6.1 one shows that I is bounded above and below on \mathcal{A} . We then choose minimizing sequences of midsurface deformations m_k and rotations \bar{R}^k in \mathcal{A} . Thus, along the minimizing sequence (m_k, \bar{R}^k)

$$\forall k \in \mathbb{N} : \int_{\omega_{\text{bdl},h}} \langle \bar{R}^{k,T} x (\nabla m_k(x, y) | \nabla g_d(x, y, 0) \cdot e_3), A(x, y) \rangle d\omega = 0, \tag{6.11}$$

for all test functions $A \in C^\infty(\omega_{\text{bdl},h}, \mathfrak{so}(3))$. We need to investigate in which sense the weak/strong limits found in Theorem 6.1 satisfy this additional relation on $\omega_{\text{bdl},h}$. However, we have already observed weak convergence of ∇m_k and strong convergence of \bar{R}^k . Hence the minimizing solution $(\hat{m}, \hat{\bar{R}}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p}(\omega, \text{SO}(3))$ satisfies the relaxed weak consistent coupling condition and the proof is finished. \square

7. A Cosserat Shell for Large Stretch and Local Invertibility

While the preceding models have been motivated from a “parent” model which itself is appropriate for small elastic strain and finite rotations, let us present a modified model,¹ which allows for arbitrary large elastic stretch and which preserves local invertibility of the shell surface if the reconstructed deformation is smooth. It is clear that such an extension is not unique. We consider the large stretch/large rotation model

$$I(m, \bar{R}) = \int_{\omega} h W_{\text{mp}}(\bar{U}) + h W_{\text{curv}}(\mathfrak{R}_s) \\ + \frac{h^3}{12} W_{\text{bend}}(\mathfrak{R}_b) d\omega - \Pi(m, \bar{R}_3) \mapsto \min \quad \text{w.r.t. } (m, \bar{R}), \\ \bar{U} = \bar{R}^T \hat{F}, \quad \hat{F} = (\nabla m | \bar{R}_3), \quad F_s = (\nabla m | \varrho_m \bar{R}_3), \tag{7.1}$$

¹It is clear that a modification to large stretch does not concern the bending and curvature terms since these modes are only relevant for small membrane stretch.

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where now

$$\begin{aligned}
 W_{\text{mp}}(\bar{U}) &= \mu \|\text{sym}(\bar{U} - \mathbf{1})\|^2 + \mu_c \|\text{skew}(\bar{U} - \mathbf{1})\|^2 \\
 &\quad + \frac{\mu\lambda}{2\mu + \lambda} \frac{1}{2} \left((\det \bar{U} - 1)^2 + \left(\frac{1}{\det \bar{U}} - 1 \right)^2 \right) \tag{7.2} \\
 \varrho_m &= \frac{1}{1 + \frac{\lambda}{2\mu + \lambda} (\det \bar{U} - 1)}, \quad \text{modified thickness stretch.}
 \end{aligned}$$

Let us summarize the features of this new model: First, $W_{\text{mp}}(\bar{U}) \rightarrow \infty$ if $\det \bar{U} \rightarrow 0$. Thus, if minimizers exist, then $\det \bar{U} > 0$ a.e. and the minimizing surface is locally regular. The modified membrane energy contribution W_{mp} is *polyconvex*⁴ w.r.t. ∇m at given \bar{R} and indeed *uniformly Legendre–Hadamard elliptic*, independent of $\mu_c \geq 0$. If $\bar{R}_3 = \mathbf{n}_m$, then

$$\det [\bar{U}^2] = \|\text{Cof}(\nabla m|_0)\|^2 = \|m_x \times m_y\|^2 = \|m_x\|^2 \|m_y\|^2 - \langle m_x, m_y \rangle^2 = \det [I_m],$$

with \mathbf{n}_m the outer unit normal of the surface m and I_m the *first fundamental form* of the surface. This formula represents a pure, intrinsic measure of the surface stretch. If $W_{\text{mp}}(\bar{U}) = 0$ then $\bar{U} = \mathbf{1}$ even for $\mu_c = 0$ without gradient constraint.^m Moreover, it can be shown that for zero Cosserat couple modulus $\mu_c = 0$ and zero internal length $L_c = 0$, the pure bending problem coincides with the rigorously justified (by means of Γ -convergence) classical finite-strain bending problem given in Ref. 29. The thickness stretch ϱ_m , used for the *a posteriori* reconstruction of the bulk deformation, has such an analytical form, that at finite energy one has $0 < \varrho_m < \infty$, in line with the underlying physical description without restriction on the kinematics. In addition transverse fibers will always be monotonically elongated upon action of opposite transverse surface tractions. Moreover, $\varrho_m \equiv 1$ for $\lambda = 0$ (extreme compressibility, Poisson-ratio $\nu = 0$) and $\varrho_m = \frac{1}{\det \bar{U}}$ for $\lambda = \infty$ (exact incompressibility, $\nu = \frac{1}{2}$) such that $\det F_s = \det(\nabla m|_{\varrho_m \bar{R}_3}) \equiv 1$, i.e. exact incompressibility for the reconstructed deformation $\varphi_s(x, y, z)$.

The formulation (7.1) still has the same linearized behaviour as the initial model (4.1) and reduces to the classical infinitesimal-displacement Reissner–Mindlin model for the choice of parameters $\mu_c = 0, p > 1$.ⁿ We can prove

Theorem 7.1. (Existence for thin Cosserat shell with large stretch) *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_d \in H^1(\omega, \mathbb{R}^3)$*

^mIt is easy to see, that $\text{sym}(\bar{U} - \mathbf{1}) = 0$ implies $\bar{R}_3 = \mathbf{n}_m$. The remaining consideration leads to $X \in \mathbb{M}^{2 \times 2} : \text{sym} X = \mathbf{1}_2, \det X = 1 \Rightarrow X = \mathbf{1}_2$.

ⁿBecause $\left((\det \bar{U} - 1)^2 + \left(\frac{1}{\det \bar{U}} - 1 \right)^2 \right) = 2 \text{tr} [\bar{U} - \mathbf{1}]^2 + O(\|\bar{U} - \mathbf{1}\|^3)$. Note that the left-hand side is a convex function in the determinant.

and $\overline{R}_d \in W^{1,1+p}(\omega, \text{SO}(3))$. Moreover, let $\overline{f} \in L^1(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^1(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (5.7). Then (7.1) with $\mu_c = 0$ and $p > 1$, rigid director prescription for \overline{R} on γ_0 and modified external potential Π^\sharp (5.8) admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p}(\omega, \text{SO}(3))$ with $\det(\nabla m|_{\overline{R}_3}) > 0$ a.e. $(x, y) \in \omega$. □

Proof. The proof mimics the arguments of Theorem 6.1. We observe in addition, that the modified membrane energy W_{mp} is in fact polyconvex⁴ at given \overline{R} w.r.t. ∇m since $((\det \overline{U} - 1)^2 + (\frac{1}{\det \overline{U}} - 1)^2)$ is convex in $\det \overline{U}$. The modified membrane strain energy term provides us with the information that $\det(\nabla m_k|_{\overline{R}_3^k})$ is uniformly bounded in $L^2(\omega)$ for minimizing sequences. Hence we may always choose a minimizing sequence, such that $\det(\nabla m_k|_{\overline{R}_3^k}) \rightharpoonup \zeta \in L^2(\omega)$, converges weakly to some ζ . A further subsequence may be chosen, not relabelled, such that $\overline{R}^k \rightharpoonup \overline{R} \in C^0(\omega, \text{SO}(3))$, due to the compact embedding $W^{1,1+p}(\omega) \subset C^0(\omega)$ for $p > 1$. Moreover, $\nabla m_k \rightharpoonup \nabla \hat{m} \in L^2(\omega, \mathbb{M}^{2 \times 3})$, weakly, as in Theorem 6.1. For two space dimensions, this implies the strong convergence of $\text{Cof}(\nabla m_k|_0)$ in the sense of distributions, cf. Theorem 3.4 in Ref. 3:

$$\forall \psi \in C_0^\infty(\omega) : \int_\omega \text{Cof}(\nabla m_k|_0) \psi d\omega \rightarrow \int_\omega \text{Cof}(\nabla \hat{m}|_0) \psi d\omega, \quad k \rightarrow \infty. \tag{7.3}$$

Let us analyze in more detail the term $\det(\nabla m_k|_{\overline{R}_3^k})$. One has upon expansion of the determinant

$$\begin{aligned} \det(\nabla m_k|_{\overline{R}_3^k}) &= \sum_{i=1}^3 \overline{R}_{3,i}^k \text{Cof}(\nabla m_k|_0)_{3,i} = \sum_{i=1}^3 (\overline{R}_{3,i}^k - \widehat{\overline{R}}_{3,i} + \widehat{\overline{R}}_{3,i}) \text{Cof}(\nabla m_k|_0)_{3,i} \\ &= \sum_{i=1}^3 (\overline{R}_{3,i}^k - \widehat{\overline{R}}_{3,i}) \text{Cof}(\nabla m_k|_0)_{3,i} + \sum_{i=1}^3 \widehat{\overline{R}}_{3,i} \text{Cof}(\nabla m_k|_0)_{3,i} \\ &= \sum_{i=1}^3 (\overline{R}_{3,i}^k - \widehat{\overline{R}}_{3,i}) \text{Cof}(\nabla m_k|_0)_{3,i} \\ &\quad + \sum_{i=1}^3 (\widehat{\overline{R}}_{3,i} - \widehat{\overline{R}} + \widehat{\overline{R}}) \text{Cof}(\nabla m_k|_0)_{3,i} \\ &= \sum_{i=1}^3 (\overline{R}_{3,i}^k - \widehat{\overline{R}}_{3,i}) \text{Cof}(\nabla m_k|_0)_{3,i} + \sum_{i=1}^3 (\widehat{\overline{R}}_{3,i} - \widehat{\overline{R}}_{3,i}^\varepsilon) \text{Cof}(\nabla m_k|_0)_{3,i} \\ &\quad + \widehat{\overline{R}}_{3,i}^\varepsilon \text{Cof}(\nabla m_k|_0)_{3,i}, \end{aligned} \tag{7.4}$$

where $\widehat{R}^\varepsilon \in C^\infty$ is introduced as a mollification of \widehat{R} . Now we integrate $\det(\nabla m_k | \overline{R}_3^k)$ over ω against an arbitrary test function $\psi \in C_0^\infty(\omega)$:

$$\begin{aligned} \int_\omega \det \left[(\nabla m_k | \overline{R}_3^k) \right] \psi d\omega &= \int_\omega \sum_{i=1}^3 \left(\overline{R}_{3,i}^k - \widehat{R}_{3,i} \right) \text{Cof}(\nabla m_k | 0)_{3,i} \psi \\ &\quad + \sum_{i=1}^3 \left(\widehat{R}_{3,i} - \widehat{R}_{3,i}^\varepsilon \right) \text{Cof}(\nabla m_k | 0)_{3,i} \psi \\ &\quad + \widehat{R}_{3,i}^\varepsilon \text{Cof}(\nabla m_k | 0)_{3,i} \psi d\omega. \end{aligned} \tag{7.5}$$

Since $\text{Cof}(\nabla m_k | 0)$ is bounded in $L^1(\omega)$, the first sum converges to zero because of strong convergence of \overline{R}^k . The second term can be made arbitrarily small for $\varepsilon \rightarrow 0$ and the third term converges because $\widehat{R}_{3,i}^\varepsilon \psi \in C_0^\infty(\omega)$ is an admitted test-function in (7.3). Altogether, the strong convergence of \overline{R}_3^k in $C^0(\omega)$ and the strong convergence of $\text{Cof}(\nabla m_k | 0)$ in the sense of distributions³ for two space-dimensions show that

$$\forall \psi \in C_0^\infty(\omega) : \int_\omega \det \left[(\nabla m_k | \overline{R}_3^k) \right] \psi d\omega \rightarrow \int_\omega \det \left[(\nabla \widehat{m} | \widehat{R}_3) \right] \psi d\omega, \quad k \rightarrow \infty. \tag{7.6}$$

Thus, $\det(\nabla m_k | \overline{R}_3^k) \rightarrow \det(\nabla \widehat{m} | \widehat{R}_3)$, strongly in the sense of distributions as well. This implies for the weak limit ζ found above that $\zeta = \det(\nabla \widehat{m} | \widehat{R}_3)$. The remainder proceeds as in Theorem 6.1. □

This shows that (7.1) represents an improvement of the initially proposed shell model (4.1), although (7.1) itself is not strictly obtained from a “parent” model in the framework of dimensional descend. The extension of Theorem 7.1 to relaxed weak consistent coupling is possible along the lines of Corollary 6.2.

8. Conclusion

We have investigated a frame-indifferent Cosserat shell model derived in Refs. 48 and 50. Only for vanishing Cosserat couple modulus $\mu_c = 0$, the formulation is downwards compatible with traditional infinitesimal-displacement linear Reissner–Mindlin shell theories.⁵⁰ A mathematical analysis for $\mu_c = 0$ of the shell model is proposed. Existence of minimizers in appropriate Sobolev-spaces is shown despite the nonlinearity of the problem and the lack of unqualified coercivity. The main tool is a novel two-dimensional version of an extended Korn’s first inequality.

Compared to traditional, non-elliptic finite-strain Reissner–Mindlin and Kirchhoff–Love shell models,⁵⁰ it is the influence of the explicitly appearing rotations in conjunction with the nonclassical internal length $L_c > 0$, governing the

microstructural interaction, which stabilizes the Cosserat thin shell model by introducing a curvature contribution which augments the classical bending energy. Comparing with other shell models from the literature with constraint or independent proper rotations, the additional burden for the new Cosserat shell models with rotational microstructure seems to be small compared to the possible gain of having a well-posed model. Limit cases ($\mu_c = 0, p = 1$) related to critical Sobolev-exponents remain mathematically open for the moment. They leave open challenging mathematical problems.

At the lateral Dirichlet-boundary γ_0 we prefer a generalization of the three-dimensional consistent coupling condition which provides maximal consistency with the classical “symmetric” situation. One might expect that this coupling condition reduces the strength of boundary layers. However, only a relaxed version of this requirement, including the introduction of an artificial boundary layer, could be shown to admit minimizers. Further research should clarify, under what circumstances the Cosserat shell model (4.1) can be obtained as a Γ -limit of the Cosserat bulk problem for vanishing characteristic thickness h . A first answer in this direction has been achieved in Ref. 46.

Appendix

A.1. Transformation of the domain and scaling

Since the investigated Cosserat model has nonclassical size-effects incorporated we recall the scaling and transformation relations. We distinguish between \mathbb{E}^3 , the ambient three-dimensional physical space with measurement units and \mathbb{R}^3 , the corresponding nondimensional mathematical vector space.

A.1.1. Classical finite-strain elasticity

Set $\Omega_L^{\text{rel.thin}} = [0, L] \times [0, L] \times [-\frac{h}{2} \cdot L, \frac{h}{2} \cdot L] \subset \mathbb{E}^3$ with h a small nondimensional parameter indicating the *relative characteristic thickness* of the domain, e.g. $h \in (0, \frac{1}{20}]$. The three-dimensional problem with respect to the *relatively thin domain* $\Omega_L^{\text{rel.thin}} \subset \mathbb{E}^3$ reads

$$\int_{\xi \in \Omega_L^{\text{rel.thin}}} W_{3D}(\nabla_{\xi} \varphi_L(\xi)) - \langle f_L(\xi), \varphi_L(\xi) \rangle d\xi - \int_{\partial \Omega_L^{\text{rel.thin}}} \langle N_L, \varphi_L \rangle dS_L \mapsto \min \quad \text{w.r.t. } \varphi_L,$$

where we are looking for a *dimensional* function $\varphi_L : \Omega_L^{\text{rel.thin}} \subset \mathbb{E}^3 \mapsto \mathbb{E}^3$. Introducing the *scaling transformation* $\zeta : \mathbb{R}^3 \mapsto \mathbb{E}^3$

$$\zeta : \Omega_h = [0, 1] \times [0, 1] \times \left[-\frac{h}{2}, \frac{h}{2}\right] \subset \mathbb{R}^3 \mapsto \Omega_L^{\text{rel.thin}} \subset \mathbb{E}^3, \quad \zeta(x) = L \cdot x,$$

(note again that $x \in \mathbb{R}^3$ is free of dimensions) this turns into

$$\int_{x \in \Omega_h} [W_{3D}(\nabla\zeta(x)\nabla\varphi(x)\nabla\zeta^{-1}(x)) - \langle f_L(\zeta(x)), L \cdot \varphi(x) \rangle] \det \nabla\zeta(x) dV - \int_{\partial\Omega_h} \langle N_L(\zeta(x)), L \cdot \varphi(x) \rangle \| \text{Cof} \nabla\zeta \cdot e_3 \| dS_h \mapsto \min \quad \text{w.r.t. } \varphi, \quad (\text{A.1})$$

for a *nondimensional* function $\varphi : \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined implicitly through $\varphi_L(\xi) = \zeta(\varphi(\zeta^{-1}(\xi)))$. With $f(x) = L \cdot f_L(\zeta(x))$, $N(x) = N_L(\zeta(x))$ we have

$$\int_{x \in \Omega_h} [W_{3D}(\nabla\varphi) - \langle f, \varphi \rangle] L^3 dV - \int_{\partial\Omega_h} L \langle N, \varphi \rangle L^2 dS \mapsto \min \quad \text{w.r.t. } \varphi, \quad (\text{A.2})$$

which shows how the scaling from a physical domain with dimensions which is relatively thin to a nondimensional domain which is absolutely thin is to be performed in order to apply the subsequent dimensional reduction procedure.

A.1.2. Scaling relations for Cosserat bulk models with internal length

Now we relate the response of large and small samples of the same material and asses the influence of the characteristic length L_c [m]. The *characteristic length* L_c is a given material parameter, corresponding e.g. to the *smallest discernable distance* to be accounted for in the model. A simple consequence is that all geometrical dimensions L of the bulk material must be larger than L_c , indeed for a continuum theory to apply L should be significantly larger than L_c .

Now let $\Omega_L = [0, L] \times [0, L] \times [0, L] \subset \mathbb{E}^3$ be the cube with edge length L [m], representing the bulk material in physical space. Consider a deformation $\varphi_L : \xi \in \Omega_L \mapsto \mathbb{E}^3$ and microrotation $\bar{R}_L(\xi) : \Omega_L \mapsto \text{SO}(3)$ as solution of the simplified two-field minimization problem

$$\int_{\xi \in \Omega_L} \mu \| \bar{R}_L^T(\xi) F(\xi) - \mathbb{1} \|^2 + \mu L_c^q \| D_\xi \bar{R}_L(\xi) \|^q d\xi \mapsto \min \quad \text{w.r.t. } (\varphi_L, \bar{R}_L). \quad (\text{A.3})$$

The scaling transformation $\zeta : \mathbb{R}^3 \mapsto \mathbb{E}^3$, $\zeta(x) = L \cdot x$ maps the nondimensional unit cube $\Omega_1 = [0, 1] \times [0, 1] \times [0, 1]$ into Ω_L . Defining the related scaled deformation $\varphi : x \in \Omega_1 \mapsto \mathbb{R}^3$ and microrotation $\bar{R}(x) : \Omega_1 \mapsto \text{SO}(3)$ as

$$\varphi(x) := \zeta^{-1}(\varphi_L(\zeta(x))), \quad \bar{R}(x) := \bar{R}_L(\zeta(x)), \quad (\text{A.4})$$

implies

$$\begin{aligned} \nabla_x \varphi(x) &= \frac{1}{L} \nabla_\xi \varphi_L(\zeta(x)) \nabla_x \zeta(x) = \nabla_\xi \varphi_L(\xi), \\ D_x \bar{R}(x) &= D_\xi \bar{R}_L(\zeta(x)) \cdot \nabla_x \zeta(x) = D_\xi \bar{R}_L(\xi) \cdot L. \end{aligned} \quad (\text{A.5})$$

Hence, the minimization problem in the physical space can be transformed

$$\begin{aligned}
 & \int_{\xi \in \Omega_L} \mu \|\bar{R}_L^T(\xi) \nabla_\xi \varphi_L(\xi) - \mathbb{1}\|^2 + \mu L_c^q \|D_\xi \bar{R}_L(\xi)\|^q d\xi \\
 &= \int_{x \in \Omega_1} \mu \|\bar{R}^T(x) \nabla_x \varphi(x) - \mathbb{1}\|^2 \det \nabla_x \zeta(x) \\
 &\quad + \mu L_c^q \left\| \frac{1}{L} D_x \bar{R}(x) \right\|^q \det \nabla_x \zeta(x) dx \\
 &= \int_{x \in \Omega_1} \mu \|\bar{R}^T(x) \nabla_x \varphi(x) - \mathbb{1}\|^2 L^3 + \mu L_c^q L^{3-q} \|D_x \bar{R}(x)\|^q dx, \quad (A.6)
 \end{aligned}$$

into nondimensional form and we may equivalently consider the minimization problem defined on the nondimensional unit cube Ω_1 :

$$\int_{x \in \Omega_1} \mu \|\bar{R}^T(x) \nabla_x \varphi(x) - \mathbb{1}\|^2 + \mu L_c^q L^{3-q-3} \|D_x \bar{R}(x)\|^q dx \mapsto \min \text{ w.r.t. } (\varphi, \bar{R}).$$

Comparison of different sample sizes is afforded by transformation to the unit cube respectively, e.g. we compare two samples of the same material with bulk sizes $L_1 > L_2$. In nondimensional form it can be seen that the response of the sample with size L_2 is stiffer than the response of the sample with size L_1 . It is plain to see that for L large compared to L_c , the influence of the rotations will be small and in the limit $\frac{L_c}{L} \rightarrow 0$, classical behaviour results. Otherwise, the larger $\frac{L_c}{L} < 1$, the more pronounced the Cosserat effects become and a small sample is relatively stiffer than a large one.

A.1.3. Scaling relations for Cosserat shells

For relatively thin shells (in physical space \mathbb{E}^3) we consider the finite-strain problem on the *relatively thin domain* $\Omega_L^{\text{rel.thin}} \subset \mathbb{E}^3$ in simplified form:

$$\int_{\xi \in \Omega_L^{\text{rel.thin}}} \mu \|\bar{R}_L^T(\xi) \nabla_\xi \varphi_L(\xi) - \mathbb{1}\|^2 + \mu L_c^q \|D_\xi \bar{R}_L(\xi)\|^q d\xi \mapsto \min \text{ w.r.t. } (\varphi_L, \bar{R}_L).$$

This implies on the *nondimensional thin domain* $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$ for the correspondingly transformed variables

$$\int_{x \in \Omega_h} \mu \|\bar{R}^T(x) \nabla_x \varphi(x) - \mathbb{1}\|^2 + \mu \frac{L_c^q}{L^q} \|D_x \bar{R}(x)\|^q dx \mapsto \min \text{ w.r.t. } (\varphi, \bar{R}). \quad (A.7)$$

Inserting the reduced kinematics and integrating over the thickness we should consider on $\omega \subset \mathbb{R}^2$ with $\hat{L}_c = \frac{L_c}{L}$

$$\begin{aligned}
 & \int_\omega \mu h \|\bar{R}^T(\nabla m|_{\bar{R}_3}) - \mathbb{1}\|^2 + \mu \frac{h^3}{12} \|\bar{R}^T(\nabla \bar{R}_3|_0)\|^2 \\
 & \quad + \mu h \hat{L}_c^q \|D_x \bar{R}(x)\|^q d\omega \mapsto \min \text{ w.r.t. } (m, \bar{R}). \quad (A.8)
 \end{aligned}$$

Comparing domains with the *same relative characteristic thickness (aspect ratio)* $h > 0$, but different global characteristic dimension L , we see that the *smaller*

sample in L is relatively stiffer. For very large samples with the same relative characteristic thickness, the classical bending terms are retrieved. In this sense, *classical shell formulations represent the limit behaviour of ever larger, thin structures with the same small relative characteristic thickness.*

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