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A GEOMETRY FOR E_7

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Abstract. A geometry is defined by the 56-dimensional representation $\mathfrak{M}$ of a Lie algebra of type E_7. Every collineation is shown to be induced by a semisimilarity of $\mathfrak{M}$, and the image of the automorphism group of $\mathfrak{M}$ in the collineation group is shown to be simple.

Using the 56-dimensional ternary algebra $\mathfrak{M}$ with an alternating bilinear form introduced in [2], we define here a geometry and investigate its collineation group. The objects of the geometry are, in the real case, essentially the planes of the symplectic geometry for E_7 introduced by H. Freudenthal [4]. In §1, the notion of semisimilarities of $\mathfrak{M}$ is introduced, some semisimilarities are exhibited, and some identities in the group of semisimilarities are demonstrated. In §2, we define the geometry, show that semisimilarities induce collineations, derive some transitivity results, and prove that every collineation is induced by a semisimilarity. Finally, in §3, we show that the image of the automorphism group of $\mathfrak{M}$ in the collineation group is a simple group.

1. **Semisimilarities.** If $\mathfrak{J}=\mathfrak{J}(N, 1)$ is a quadratic Jordan algebra over a field Φ constructed as in [6] from an admissible nondegenerate cubic form N with basepoint 1, then $yU_x=T(x, y)x-x^{\#}\times y$ where $T(\ ,\ )$ and $x\to x^{\#}$ are respectively the associated nondegenerate bilinear form and quadratic mapping, and $x\times y=(x+y)^{\#}-x^{\#}-y^{\#}$. As in [2, pp. 399–401], we may construct $\mathfrak{M}=\mathfrak{M}(\mathfrak{J})=\Phi u_1\oplus\Phi u_2\oplus\mathfrak{J}_{12}\oplus\mathfrak{J}_{21}$ with elements

$$x = \alpha u_1+\beta u_2+a_{12}+b_{21}; \qquad \alpha, \beta\in\Phi; a, b\in\mathfrak{J}; \tag{1.1}$$

with a nondegenerate alternate bilinear form $\langle\ ,\ \rangle$, and with a ternary product $\langle\ ,\ ,\ \rangle$ defined by

$$\langle x_1, x_2\rangle = \alpha_1\beta_2-\alpha_2\beta_1-T(a_1, b_2)+T(a_2, b_1), \tag{1.2}$$

$$\langle x_1, x_2, x_3\rangle = \gamma u_1+\delta u_2+c_{12}+d_{21}, \tag{1.3}$$

where

$$\begin{aligned}\gamma &= \alpha_1\beta_2\alpha_3+2\alpha_1\alpha_2\beta_3-\alpha_3T(a_1, b_2)-\alpha_2T(a_1, b_3)-\alpha_1T(a_2, b_3)+T(a_1, a_2\times a_3),\\ c &= (\alpha_2\beta_3+T(b_2, a_3))a_1+(\alpha_1\beta_3+T(b_1, a_3))a_2+(\alpha_1\beta_2+T(b_1, a_2))a_3\\ &\quad -\alpha_1b_2\times b_3-\alpha_2b_1\times b_3-\alpha_3b_1\times b_2-\{a_1b_2a_3\}-\{a_1b_3a_2\}-\{a_2b_1a_3\},\\ \delta &= -\gamma^{\sigma}, \qquad d = -c^{\sigma},\end{aligned}$$

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where σ is the permutation $\sigma=(\alpha\beta)(ab)$ with $x_i=\alpha_i u_1+\beta_i u_2+(a_i)_{12}+(b_i)_{21}\in\mathfrak{M}$. In [2, pp. 399–401], it was shown that

(T1) $\langle x, y, z\rangle=\langle y, x, z\rangle+\langle x, y\rangle z$,

(T2) $\langle x, y, z\rangle=\langle x, z, y\rangle+\langle y, z\rangle x$,

(T3) $\langle\langle x, y, z\rangle, w\rangle=\langle\langle x, y, w\rangle, z\rangle+\langle x, y\rangle\langle z, w\rangle$,

(T4) $\langle\langle x, y, z\rangle, v, w\rangle=\langle\langle x, v, w\rangle, y, z\rangle+\langle x, \langle y, v, w\rangle, z\rangle+\langle x, y, \langle z, w, v\rangle\rangle$,

for $x, y, z, w\in\mathfrak{M}$. We also wish to recall that we have a nondegenerate four-linear form $q(x_1, x_2, x_3, x_4)=\langle\langle x_1, x_2, x_3\rangle, x_4\rangle$ for $x_i\in\mathfrak{M}$.

If $\mathfrak{J}'=\mathfrak{J}(N', 1')$ where N' is an admissible nondegenerate cubic form with basepoint $1'$ on $\mathfrak{J}'$ over a field Φ', if $\mathfrak{M}'=\mathfrak{M}(\mathfrak{J}')$, and if s is an isomorphism of Φ onto Φ', then an s-semilinear mapping W of $\mathfrak{M}$ onto $\mathfrak{M}'$ satisfying

$$(1.4)\qquad q'(x_1W, x_2W, x_3W, x_4W) = \rho q(x_1, x_2, x_3, x_4)^s,\qquad x_i\in\mathfrak{M},$$

for a fixed $0\neq\rho\in\Phi'$ is called an *s-semisimilarity* of $\mathfrak{M}$ to $\mathfrak{M}'$ with *multiplier* ρ. If $s=1$, W is a *similarity*. If $s=1$ and $\rho=1$, then W is a *form preserving map*. If $\mathfrak{M}=\mathfrak{M}'$, we denote the group of semisimilarities (respectively, similarities, form preserving maps) by $\Gamma=\Gamma(\mathfrak{M})$ (respectively, $G=G(\mathfrak{M})$, $S=S(\mathfrak{M})$).

Lemma 1. *An s-semilinear map W of $\mathfrak{M}$ onto $\mathfrak{M}'$ is an s-semisimilarity with multiplier ρ if and only if $\rho=\lambda^2$, $\lambda\in\Phi'$, and $\langle x_1W, x_2W, x_3W\rangle'=\lambda(\langle x_1, x_2, x_3\rangle W)$, for $x_i\in\mathfrak{M}$. In this case, $\langle x_1W, x_2W\rangle'=\lambda\langle x_1, x_2\rangle^s$.*

Proof. If W is an s-semilinear map of $\mathfrak{M}$ onto $\mathfrak{M}'$, we may define an s^{-1}-semilinear map W^* of $\mathfrak{M}'$ onto $\mathfrak{M}$ by

$$(1.5)\qquad \langle xW, y'\rangle' = \langle x, y'W^*\rangle^s,\qquad x\in\mathfrak{M},\ y'\in\mathfrak{M}'.$$

If W satisfies (1.4), then $\langle x_1W, x_2W, x_3W\rangle'W^*=\rho^{s^{-1}}\langle x_1, x_2, x_3\rangle$ for $x_i\in\mathfrak{M}$. By (T1) we have $\langle x_1W, x_2W\rangle'^{s^{-1}}(x_3WW^*)=\rho^{s^{-1}}\langle x_1, x_2\rangle x_3$. Hence, $xWW^*=\lambda^{s^{-1}}x$ where $\lambda\in\Phi'$ is given by $\lambda\langle x_1W, x_2W\rangle'=\rho\langle x_1, x_2\rangle^s$, $x_i\in\mathfrak{M}$. Now $\lambda\langle x_1W, x_2W\rangle'$ $=\lambda\langle x_1, x_2WW^*\rangle^s=\lambda^2\langle x_1, x_2\rangle^s$, $x_i\in\mathfrak{M}$, so $\rho=\lambda^2$. We see

$$\begin{aligned}\langle x_1W, x_2W, x_3W\rangle' &= \rho(\langle x_1, x_2, x_3\rangle W^{*-1}) = \rho\lambda^{-1}(\langle x_1, x_2, x_3\rangle W)\\ &= \lambda(\langle x_1, x_2, x_3\rangle W),\qquad x_i\in\mathfrak{M}.\end{aligned}$$

Conversely, if $\langle x_1W, x_2W, x_3W\rangle'=\lambda(\langle x_1, x_2, x_3\rangle W)$, then (T1) yields

$$\langle x_1W, x_2W\rangle'(x_3W) = \lambda\langle x_1, x_2\rangle^s(x_3W)$$

and

$$\langle x_1W, x_2W\rangle' = \lambda\langle x_1, x_2\rangle^s.$$

Clearly,

$$\begin{aligned}q'(x_1W, x_2W, x_3W, x_4W) &= \langle\langle x_1W, x_2W, x_3W\rangle', x_4W\rangle'\\ &= \lambda^2 q(x_1, x_2, x_3, x_4)^s\end{aligned}$$

for $x_i\in\mathfrak{M}$ as desired.

It is now clear that if W is an *automorphism* of $\mathfrak{M}$ (i.e. $\langle x_1W, x_2W, x_3W\rangle = \langle x_1, x_2, x_3\rangle$, $x_i \in \mathfrak{M}$), then $W \in S(\mathfrak{M})$. We denote the group of automorphisms by Aut ($\mathfrak{M}$).

We shall now exhibit some semisimilarities of $\mathfrak{M}$ to $\mathfrak{M}'$. If W is an s-semisimilarity of $\mathfrak{J}$ to $\mathfrak{J}'$ (with respect to N and N'; see [3, p.10]) with multiplier λ, then we define $\tilde{W}$ by

$$x\tilde{W} = \alpha^s u_1 + \lambda\beta^s u_2 + (aW)_{12} + (\lambda b\hat{W})_{21}, \qquad x \text{ as in (1.1)}, \tag{1.6}$$

where $\hat{W} = W^{*-1}$ and $T(x'W^*, y)^s = T'(x', yW)$, $x' \in \mathfrak{J}'$, $y \in \mathfrak{J}$. An easy calculation using (1.22) and (1.23) of [3] shows

$$\langle x_1\tilde{W}, x_2\tilde{W}, x_3\tilde{W}\rangle' = \lambda(\langle x_1, x_2, x_3\rangle\tilde{W}), \qquad x_i \in \mathfrak{M}, \tag{1.7}$$

so $\tilde{W}$ is an s-semisimilarity with multiplier λ^2 by Lemma 1.

If $\mathfrak{J} = \mathfrak{J}'$, we may define ε by

$$x\varepsilon = \beta u_1 - \alpha u_2 - b_{12} + a_{21}, \qquad x \text{ as in (1.1)}. \tag{1.8}$$

Clearly $\varepsilon^2 = -1$ and one checks that ε is an automorphism of $\mathfrak{M}$.

If $c \in \mathfrak{J} = \mathfrak{J}'$, we may define t_c by

$$xt_c = \alpha u_1 + (\beta + T(b, c) + T(a, c^\#) + \alpha N(c))u_2 + (a + \alpha c)_{12} + (b + a \times c + \alpha c^\#)_{21}, \tag{1.9}$$

for x as in (1.1). We shall show that $t_c \in$ Aut ($\mathfrak{M}$), but first we introduce S_c defined by

$$xS_c = T(b, c)u_2 + (\alpha c)_{12} + (a \times c)_{21}, \tag{1.10}$$

for x as in (1.1). Clearly, $S_c^4 = 0$, and if Φ is not of characteristic two or three, then $t_c = \exp(S_c) = 1 + S_c + \frac{1}{2}S_c^2 + \frac{1}{6}S_c^3$. One checks that $\langle x, u_1, -c_{21}\rangle = xS_c$ and $\langle u_1, -c_{21}\rangle = 0$ for $x \in \mathfrak{M}$. Thus by [2, p. 404], we see that S_c is an inner derivation of $\mathfrak{M}$. It is now clear that t_c is an automorphism of $\mathfrak{M}$, if Φ is of characteristic zero.

A lengthy calculation would verify that t_c is an automorphism for arbitrary fields. However, we shall be content to show this for $\mathfrak{J}$ a 27 dimensional exceptional simple Jordan algebra by the following trick. By extending Φ, we may assume that $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3)$ where $\mathfrak{D}$ is the split octonion (Cayley-Dickson) algebra. $\mathfrak{D}$ has a basis $x = e_i, e_ij, e_il, (e_ij)l$, $i = 1, 2$, with the involution given by $\bar{e}_1 = e_2$, $\bar{j} = -j$, $\bar{l} = -l$, and multiplication given by $e_i^2 = e_i$, $j^2 = l^2 = 1 = e_1 + e_2$, $al = l\bar{a}$, $a(bl) = (ba)l$, $(al)b = (a\bar{b})l$, $(al)(bl) = \bar{b}a$, for a, b either e_i or e_ij, $i = 1, 2$. $\mathfrak{J}$ has a basis $a = 1[ii]$, $x[ij]$, x as above, and $i < j = 1, 2, 3$. $\mathfrak{M}$ has a basis u_i, a_{ij}, $i, j = 1, 2$, $i \neq j$, a as above. Using (1.3), one sees that the multiplication table for $\mathfrak{M}$ relative to this basis is integral. The action of t_c on $\mathfrak{M}$ given by (1.9) is also integral for c belonging to the basis for $\mathfrak{J}$. For such a c, the automorphism condition for t_c follows from the one in which $\mathfrak{D}$ is the split octonion algebra over the integers, which follows in turn from the one in which $\mathfrak{D}$

is the split octonion algebra over the reals. Thus, t_c is an automorphism, if c belongs to the basis for $\mathfrak{J}$. Easy calculations show

$$t_c t_d = t_{c+d} \quad \text{for } c, d \in \mathfrak{J}, \tag{1.11}$$

$$t_c \tilde{W} = \tilde{W} t_{cW} \quad \text{for } c \in \mathfrak{J},\ W \in \Gamma(\mathfrak{J}), \tag{1.12}$$

where $\Gamma(\mathfrak{J})$ is the group of semisimilarities of $\mathfrak{J}$. Since W may be taken to be $\alpha 1$, $0 \neq \alpha \in \Phi$, we see that $t_c, t_d \in \operatorname{Aut}(\mathfrak{M})$ imply $t_{\alpha c + \beta d} \in \operatorname{Aut}(\mathfrak{M})$ for $\alpha, \beta \in \Phi$. Thus, $t_c \in \operatorname{Aut}(\mathfrak{M})$ for all $c \in \mathfrak{J}$.

We now list two more identities which may be checked directly. The second is an analogue of Hua's identity (see [5, p. 144]).

(1.13) $\varepsilon \tilde{W} = (\hat{W})^{\sim} \lambda \varepsilon$, for $W \in \Gamma(\mathfrak{M})$ with multiplier λ.

(1.14) $\varepsilon t_c \varepsilon t_{c^{-1}} \varepsilon t_c = -N(c)^{-1}(U_c)^{\sim}$, for $c \in \mathfrak{J}$ with $N(c) \neq 0$.

2. **The geometry and collineations.** We denote by $\Pi(\mathfrak{M})$ the set of $0 \neq x \in \mathfrak{M}$, x as in (1.1) with

$$\begin{gathered} a^{\#} = \alpha b, \quad b^{\#} = \beta a, \quad N(a) = \alpha^2 \beta, \quad N(b) = \alpha \beta^2, \\ T(a, b) = 3\alpha\beta, \quad V_{a,b} = 2\alpha\beta 1. \end{gathered} \tag{2.1}$$

We say $x \in \Pi(\mathfrak{M})$ is an element of *rank one*.

LEMMA 2. *If $x \in \Pi(\mathfrak{M})$ and $\varphi = t_c$, $\tilde{W}$, ε, or $\lambda 1$, where $c \in \mathfrak{J}$, $W \in \Gamma(\mathfrak{J})$, $0 \neq \lambda \in \Phi$, then $x^{\varphi} \in \Pi(\mathfrak{M})$.*

Proof. We may assume that the field Φ is infinite. The set $S = \{x \in \mathfrak{M} \mid x$ as in (1.1) with $\alpha \neq 0 \neq \beta \in \Phi$, $a \neq 0 \neq b \in \mathfrak{J}\}$ is open in the Zariski topology on $\mathfrak{M}$ and $\Pi \cup \{0\}$ is closed. Hence $\Pi^{\varphi} \cup \{0\}$ is also closed and $\Pi^{\varphi} \cap S$ is dense in $\Pi^{\varphi} \cup \{0\}$. Thus, we need only show $\Pi^{\varphi} \cap S \subseteq \Pi \cup \{0\}$. If $x \in S$ with $a^{\#} = \alpha b$ and $b^{\#} = \beta a$, then $x \in \Pi$, since $u^{\#\#} = N(u)u$, $T(u, u^{\#}) = 3N(u)$, $V_{u,u^{\#}} = 2N(u)1$ for $u \in \mathfrak{J}$. Thus, if $x' = \alpha' u_1 + \beta' u_2 + a'_{12} + b'_{21} = x^{\varphi}$ for $x \in \Pi$, we need only show $(a')^{\#} = \alpha' b'$, $(b')^{\#} = \beta' a'$. These follow by direct calculation from the definitions and (1.1), (1.1a), (1.1b), (1.21), and (1.22) of [3].

If $\Gamma'(\mathfrak{M})$ (respectively, $G'(\mathfrak{M})$; $\operatorname{Aut}'(\mathfrak{M})$) denotes the group generated by t_c, $\tilde{W}$, ε, $\lambda 1$ where $c \in \mathfrak{J}$, $W \in \Gamma(\mathfrak{J})$, $0 \neq \lambda \in \Phi$ (respectively, $W \in G(\mathfrak{J})$; $W \in S(\mathfrak{J})$, $\lambda = 1$), then $\Gamma'(\mathfrak{M}) \subseteq \Gamma(\mathfrak{M})$, $G'(\mathfrak{M}) \subseteq G(\mathfrak{M})$, and $\operatorname{Aut}'(\mathfrak{M}) \subseteq \operatorname{Aut}(\mathfrak{M})$ by (1.7). *From now on, we shall assume $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, \gamma)$, a reduced exceptional simple Jordan algebra* (see [6]).

LEMMA 3. *If $x \in \mathfrak{J}$, then the following are equivalent*:

(a) *x is of rank one.*

(b) *$\langle \mathfrak{M}, x, x \rangle = 0$ and* $\dim \mathfrak{M}_x = \dim \mathfrak{M}_{u_1}$ *where $\mathfrak{M}_x = \{y \mid \langle \mathfrak{M}, y, x \rangle = 0,\ y \in \mathfrak{M}\}$.*

(c) *$x^{\varphi} = \alpha u_1$ for some $\varphi \in \operatorname{Aut}'(\mathfrak{M})$, $0 \neq \alpha \in \Phi$.*

Proof. Since u_1 is of rank one, we have (c) implies (a) by Lemma 2. Since $\langle \mathfrak{M}, u_1, u_1 \rangle = 0$ and $\operatorname{Aut}'(\mathfrak{M}) \subseteq \operatorname{Aut}(\mathfrak{M})$, we see that (c) implies (b).

We shall next show that if $0 \neq x \in \mathfrak{M}$ with x as in (1.1), then after replacing x by

x^φ for some $\varphi \in \mathrm{Aut}'(\mathfrak{M})$, we may assume $\alpha \neq 0$ and $a=0$. First, we shall get $\alpha \neq 0$. If $\alpha=0$ and $\beta \neq 0$, apply ε. If $\alpha=\beta=0$, then one of a or b is nonzero, and by applying ε, we may assume $b \neq 0$. Since the elements $0 \neq c \in \mathfrak{J}$ with $c^{\#}=0$ span $\mathfrak{J}$, we may find such a c with $T(b, c) \neq 0$ and replace x by $xt_c\varepsilon$ to get $\alpha \neq 0$. If $\alpha \neq 0$, replacing x by $xt_{-\alpha^{-1}a}$ allows us to assume $\alpha \neq 0$ and $a=0$ as desired.

If (a) holds, we may normalize x as above so $\alpha \neq 0$ and $a=0$. Since $T(a, b) = 3\alpha\beta$ and $V_{a,b}=2\alpha\beta 1$, we see $\beta=0$. Also, $a^{\#}=\alpha b$ implies $b=0$, so $x=\alpha u_1$ and (c) holds.

If (b) holds, we again normalize x so $\alpha \neq 0$ and $a=0$. The condition $\langle y, x, x\rangle=0$ for all $y=c_{12}$, $c \in \mathfrak{J}$, yields $(\alpha\beta+T(a, b))1=2V_{b,a}$. Hence, $\beta=0$.

If $y=\rho u_1+\eta u_2+r_{12}+s_{21}$, $\rho, \eta \in \Phi$; $r, s \in \mathfrak{J}$, then using (1.3) one checks that $y \in \mathfrak{M}_x$ if and only if $\eta\alpha-T(r, b)=0$, $s+\rho b=0$, $s \times b=0$, $V_{b,r}=0$ (since $V_{b,r}=0$ implies $V_{r,b}=0$ and $2T(b, r)=T(1V_{b,r})=0$; and $\{scb\}=T(s, c)b+T(c, b)s-(s \times b) \times c$, $c \in \mathfrak{J}$). Thus, $\mathfrak{M}_{u_1}=\{\rho u_1+r_{12} \mid \rho \in \Phi, r \in \mathfrak{J}\}$ and $\dim \mathfrak{M}_{u_1}=28$. If $b \neq 0$, then $y \in \mathfrak{M}_x$ implies $\eta=\alpha^{-1}T(r, b)$, $s=-\alpha^{-1}\rho b$ where $V_{b,r}=0$ and $2\rho b^{\#}=0$. Since y depends linearly on the choice of ρ and r, and since $\dim \mathfrak{M}_x=28$, all $r \in \mathfrak{J}$ are possible for y; but $V_{b,r}=0$ for all $r \in \mathfrak{J}$ implies $b=0$ (use (1.35) of [3]), a contradiction. Thus, $x=\alpha u_1$.

We are now in a position to define a "geometry" from $\mathfrak{M}$. If $x \in \Pi(\mathfrak{M})$, let $\hat{x}=\{\alpha x \mid 0 \neq \alpha \in \Phi\}$ and let $\mathscr{P}(\mathfrak{M})=\{\hat{x} \mid x \in \Pi(\mathfrak{M})\}$. Define $\hat{x}$ *incident to* $\hat{y}$ (denoted $\hat{x}|\hat{y}$) if $R(x, y)=0$, where $zR(u, v)=\langle z, u, v\rangle$, $u, v, z \in \mathfrak{M}$; and define $\hat{x}$ *connected to* $\hat{y}$ (denoted $\hat{x} \simeq \hat{y}$) if $\langle x, y\rangle=0$. Since $\langle x, y\rangle 1=R(x, y)-R(y, x)$ and since $\langle uR(x, y), v\rangle+\langle u, vR(y, x)\rangle=0$ (see (2.7) of [2]), we see that $\hat{x}|\hat{y}$ implies $\hat{y}|\hat{x}$ and $\hat{x} \simeq \hat{y}$.

If $\mathfrak{J}'=\mathfrak{H}(\mathfrak{D}_3', \gamma')$, if $x \in \Pi(\mathfrak{M})$, and if W is a semisimilarity of $\mathfrak{M}$ onto $\mathfrak{M}'=\mathfrak{M}(\mathfrak{J}')$, then xW satisfies condition (b) of Lemma 3, so $xW \in \Pi(\mathfrak{M}')$. Thus, we may define a map $\ulcorner W \urcorner$ of $\mathscr{P}(\mathfrak{M})$ onto $\mathscr{P}(\mathfrak{M}')$ by $\hat{x}\ulcorner W \urcorner=(xW)^\wedge$. It is clear that W is a *collineation* in the sense $\hat{x}|\hat{y}$ if and only if $\hat{x}\ulcorner W \urcorner|\hat{y}\ulcorner W \urcorner$ and $\hat{x} \simeq \hat{y}$ if and only if $\hat{x}\ulcorner W \urcorner \simeq \hat{y}\ulcorner W \urcorner$. If H is a subgroup of $\Gamma(\mathfrak{M})$, then we denote the image of H in the collineation group of $\mathscr{P}(\mathfrak{M})$ under $W \to \ulcorner W \urcorner$ by PH. The kernel of $W \to \ulcorner W \urcorner$ in $\Gamma(\mathfrak{M})$ is easily seen to be $\{\alpha 1 \mid 0 \neq \alpha \in \Phi\}$.

One checks immediately from (1.3) that $\hat{x}|\hat{u}_1$ if and only if $x=\alpha u_1+a_{12} \in \Pi(\mathfrak{M})$, $\alpha \in \Phi$, $a \in \mathfrak{J}$. Hence, $\hat{x}|\hat{u}_1$ and $\hat{x} \simeq \hat{u}_2$ if and only if $x=a_{12}$ and $a \in \Pi(\mathfrak{J})$, where $\Pi(\mathfrak{J})=\{0 \neq a \in \mathfrak{J} \mid a^{\#}=0\}$. Similarly, $\hat{x}|\hat{u}_2$ and $\hat{x} \simeq \hat{u}_1$ if and only if $x=b_{21}$ and $b \in \Pi(\mathfrak{J})$. If $a \in \Pi(\mathfrak{J})$, we shall set $a_*=(a_{12})^\wedge$ and $a^*=(a_{21})^\wedge$. Using (1.1) and (1.2), we see $a_* \simeq b_*$ and $a^* \simeq b^*$ always hold, $a_* \simeq b^*$ holds if and only if $T(a, b)=0$; each of $a_*|b_*$ and $a^*|b^*$ hold if and only if $a \times b=0$; while $a_*|b^*$ if and only if $V_{b,a}=0$ (since $V_{b,a}=0$ implies $V_{a,b}=0$ and $T(a, b)=0$), for $a, b \in \Pi(\mathfrak{J})$. Hence, $\{a_*, b^* | a, b \in \Pi(\mathfrak{J})\}$ may be identified with $\mathscr{P}(\mathfrak{J})$ as defined in [3, p. 32] (with a slight change of notation). Moreover, if $W \in \Gamma(\mathfrak{J})$ so $\tilde{W} \in \Gamma(\mathfrak{M})$, we see that if we abuse notation and set $\ulcorner W \urcorner=\ulcorner \tilde{W} \urcorner$, we get $a_*\ulcorner W \urcorner=(aW)_*$ and $a^*\ulcorner W \urcorner=(a\hat{W})^*$ for $a \in \Pi(\mathfrak{J})$, which agrees with the action of $\Gamma(\mathfrak{J})$ on $\mathscr{P}(\mathfrak{J})$ given in [3, p. 32].

LEMMA 4. *P Aut$'$ ($\mathfrak{M}$) is transitive on*
(a) $\hat{x} \in \mathscr{P}(\mathfrak{M})$.
(b) $\hat{x}, \hat{y} \in \mathscr{P}(\mathfrak{M})$ *with* $\hat{x} \not\simeq \hat{y}$.
(c) $\hat{x}, \hat{y} \in \mathscr{P}(\mathfrak{M})$ *with* $\hat{x} \simeq \hat{y}$, $\hat{x} \nmid \hat{y}$.
(d) $\hat{x}, \hat{y} \in \mathscr{P}(\mathfrak{M})$ *with* $\hat{x} | \hat{y}$, $\hat{x} \neq \hat{y}$.

Proof. Lemma 3 yields (a). In the remaining cases we may assume $y = u_2$ and x is as in (1.1). In case (b), $\langle x, y \rangle \neq 0$ implies $\alpha \neq 0$, and we may assume $\alpha = 1$. The condition $x \in \Pi(\mathfrak{M})$ yields $a^{\#} = b$ and $N(a) = \beta$. Hence, $x = u_1 t_a$. Since $u_2 t_a = u_2$, we are done in this case. In case (c), $\alpha = 0$ and $a \neq 0$. If $\beta \neq 0$, then $b^{\#} = \beta a$ implies $b \neq 0$. One may choose $c \in \Pi(\mathfrak{J})$ with $T(b, c) = -\beta$. Replacing x by xt_c, we may assume $\alpha = \beta = 0$ and $a \neq 0$. If $b \neq 0$, then $a, b \in \Pi(\mathfrak{J})$ and $a_* | b^*$ since $V_{a,b} = 0$. By [3, Lemmas 3.6 and 3.3] we know that there exists $c_* | b^*$ with $c_* \nmid a_*$, and that we may choose c such that $a \times c = -b$. Replacing x by xt_c, we may assume $\alpha = \beta = 0$, $b = 0$, and $a \neq 0$ (since $V_{b,c} = 0$ implies $T(b, c) = 0$). Since $PS(\mathfrak{J})$ is transitive on points of $\mathscr{P}(\mathfrak{J})$, we may choose $W \in S(\mathfrak{J})$ such that $\hat{x}^{\ulcorner} W^{\urcorner} = a_*{}^{\ulcorner} W^{\urcorner} = e_*$, where $e \in \Pi(\mathfrak{J})$ is fixed. Since $u_2 \tilde{W} = u_2$, we are done in this case. In case (d), we have $\alpha = 0$ and $a = 0$. Since $\hat{y} \neq \hat{x}$, we see $b \neq 0$. We may choose $c \in \Pi(\mathfrak{J})$ with $T(b, c) = -\beta$ and replace y by yt_c to assume $\alpha = \beta = 0$, $a = 0$, $b \neq 0$. Since $PS(\mathfrak{J})$ is transitive on lines of $\mathscr{P}(\mathfrak{J})$, we may choose $W \in S(\mathfrak{J})$ with $\hat{y}^{\ulcorner} W^{\urcorner} = b^{*\ulcorner} W^{\urcorner} = e^*$, where $e \in \Pi(\mathfrak{J})$ is fixed. This completes the proof of the lemma.

We shall need the following result about $\mathfrak{H}(\mathfrak{O}_3, \gamma)$.

LEMMA 5. *If W is an s-semilinear map of $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, \gamma)$ to itself such that for $a \in \Pi(\mathfrak{J})$ there is $0 \neq \lambda_a \in \Phi$ with $aW = \lambda_a a$, then $W = \lambda 1$ for some $0 \neq \lambda \in \Phi$.*

Proof. We may assume $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$ and $\lambda \hat{e}_1 = 1$. If $u \in \mathfrak{O}$ and $0 \neq \mu \in \Phi$, then $x = \mu e_1 + \mu^{-1} n(u) e_2 + u[12] \in \Pi(\mathfrak{J})$. Since $e_1 W = e_1$ and $e_2 W = \lambda_{e_2} e_2$, we see that $u[12] W = \xi_1 e_1 + \xi_2 e_2 + \lambda_x u[12]$ where $0 \neq \lambda = \lambda_x$ is independent of u and

$$\lambda\mu = \mu^s + \xi_1 \quad \text{for } 0 \neq \mu \in \Phi, \tag{2.2}$$

$$\lambda\mu^{-1} n(u) = (\mu^{-1} n(u))^s \lambda \hat{e}_2 + \xi_2 \quad \text{for } 0 \neq u \in \Phi. \tag{2.3}$$

If Φ has two elements, then $\lambda_a \neq 0$ implies $\lambda_a = 1$ for all $a \in \Pi(\mathfrak{J})$. Since $\Pi(\mathfrak{J})$ generates $\mathfrak{J}$ under addition, $W = 1$. If $0 \neq \mu_1, \mu_2 \in \Phi$, with $\mu_1 \neq \mu_2$, then (2.2), with $\mu = \mu_1, \mu_2, \mu_1 - \mu_2$, gives $\xi_1 = 0$. Thus, $\mu = 1$ yields $\lambda = 1$ and hence $s = 1$. Similarly, (2.3) yields $\xi_2 = 0$ and $\lambda \hat{e}_2 = 1$. Thus, W is linear and $e_2 W = e_2$, $u[12] W = u[12]$ for $u \in \mathfrak{O}$. Similarly, $e_i W = e_i$, $u[ij] W = u[ij]$, $u \in \mathfrak{O}$, $i \neq j = 1, 2, 3$ and $W = 1$.

THEOREM 1. *If $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, \gamma)$ and $\mathfrak{M} = \mathfrak{M}(\mathfrak{J})$, then $\Gamma(\mathfrak{M})$ (respectively, $G(\mathfrak{M})$; Aut ($\mathfrak{M}$)) is generated by t_c, $\tilde{W}$, ε, $\lambda 1$ where $c \in \mathfrak{J}$, $W \in \Gamma(\mathfrak{J})$, $0 \neq \lambda \in \Phi$ (respectively, $W \in G(\mathfrak{J})$; $W \in S(\mathfrak{J})$, $\lambda = 1$).*

Proof. Using the notation preceding Lemma 3, we need only show that $\Gamma(\mathfrak{M}) \subseteq \Gamma'(\mathfrak{M})$, $G(\mathfrak{M}) \subseteq G'(\mathfrak{M})$, and Aut ($\mathfrak{M}$) $\subseteq$ Aut$'$ ($\mathfrak{M}$). If $W \in \Gamma(\mathfrak{M})$, then ${}^{\ulcorner} W^{\urcorner}$ is a collineation of $\mathscr{P}(\mathfrak{M})$, and by Lemma 4(b) there is $W_2 \in$ Aut$'$ ($\mathfrak{M}$) with $\hat{u}_i{}^{\ulcorner} W^{\urcorner}$

$=\hat{u}_i \ulcorner W_2 \urcorner$, $i=1, 2$. Since $\ulcorner WW_2^{-1} \urcorner$ induces a collineation of $\mathscr{P}(\mathfrak{J})$, we may apply the fundamental theorem of octonion planes (see [3, p. 40]) to find $W_3 \in \Gamma(\mathfrak{J})$ such that $\ulcorner W_3 \urcorner$ agrees with $\ulcorner WW_2^{-1} \urcorner$ on $\mathscr{P}(\mathfrak{J})$. Since $u_1 WW_2^{-1}\tilde{W}_3^{-1}=\lambda u_1$ and, by Lemma 5, $a_{12}WW_2^{-1}\tilde{W}_3^{-1}=(\eta a)_{12}$, $a \in \mathfrak{J}$, for some $0\neq\lambda$, $\eta \in \Phi$, we may set $W_1=\lambda^{-1}\eta W_3$ to get $W'=WW_2^{-1}\tilde{W}_1^{-1}\lambda^{-1}$ satisfying $xW'=\alpha u_1+\rho\beta u_2+a_{12}+(\mu b)_{21}$ for some fixed $0\neq\rho$, $\mu \in \Phi$, x as in (1.1). (Note: W' is linear on $\mathfrak{J}_{12}$ and hence on all of $\mathfrak{M}$.) If $\alpha=\beta=1$ and $a=b=1$, then $xW' \in \Pi(\mathfrak{M})$ which implies $\rho=\mu=1$ by (2.1). Hence, $W'=1$ and $W=\lambda\tilde{W}_1W_2$, $0\neq\lambda\in\Phi$, $W_1 \in \Gamma(\mathfrak{J})$, $W_2 \in \mathrm{Aut}'(\mathfrak{M})$. Clearly, $\Gamma(\mathfrak{M})\subseteq\Gamma'(\mathfrak{M})$.

If $W \in G(\mathfrak{M})$, then W_1 must be linear so $G(\mathfrak{M})\subseteq G'(\mathfrak{M})$. If $W \in \mathrm{Aut}(\mathfrak{M})$, then $\lambda\tilde{W}_1 \in \mathrm{Aut}(\mathfrak{M})$. If $W_1 \in G(\mathfrak{J})$ has multiplier ρ, then $1=\langle\lambda u_1\tilde{W}_1, \lambda u_2\tilde{W}_1\rangle=\lambda^2\rho$. Set $c=(\lambda-1)e+1 \in \mathfrak{J}$ where $e \in \mathfrak{J}$ is a primitive idempotent. Since $c^{\#\#}=\lambda c$, we see that $N(c)=\lambda$ and $U_c \in G(\mathfrak{J})$ with multiplier $\lambda^2=\rho^{-1}$. By (1.14), we see that $\lambda^{-1}(U_c)^\sim \in \mathrm{Aut}'(\mathfrak{M})$. But $\lambda\tilde{W}_1\lambda^{-1}(U_c)^\sim=(W_1U_c)^\sim \in \mathrm{Aut}'(\mathfrak{M})$ since $W_1U_c \in S(\mathfrak{J})$. Thus, $\lambda\tilde{W}_1 \in \mathrm{Aut}'(\mathfrak{M})$ and $\mathrm{Aut}(\mathfrak{M})\subseteq\mathrm{Aut}'(\mathfrak{M})$.

We shall need the following result on the plane $\mathscr{P}(\mathfrak{J})$.

LEMMA 6. *If $x_*, y_* \in \mathscr{P}(\mathfrak{J})$ and $x_*\simeq z^*$ if and only if $y_*\simeq z^*$, then $x_*=y_*$.*

Proof. Since $P\Gamma(\mathfrak{J})$ is transitive on points of $\mathscr{P}(\mathfrak{J})$, we may assume $x=e_1$, where e_1, e_2, e_3 are pairwise orthogonal primitive idempotents for $\mathfrak{J}$. If $\mathfrak{J}$ is split, then there is a basis for $\mathfrak{J}$ of elements of rank one of the form $z=e_i$, $a_i[jk]$, $i, j, k\neq$; $a_i \in \mathfrak{O}$, $n(a_i)=0$. The condition $T(y, z)=0$ if and only if $T(e_1, z)=0$ yields $y \in \Phi e_1$, as desired. If $\mathfrak{J}$ is not split, then $\mathscr{P}(\mathfrak{J})$ is a projective plane and $u_*\simeq v^*$ if and only if $u_*|v^*$ (see [3, p. 50]). Thus, $y_*|e_2^*$ and $y_*|e_3^*$ implies $y_*=e_{1*}$.

We shall eventually show that every collineation of $\mathscr{P}(\mathfrak{M})$ is in $P\Gamma(\mathfrak{M})$, but first we must demonstrate the following two characterizations of the identity collineation.

LEMMA 7. *If σ is a collineation of $\mathscr{P}(\mathfrak{M})$ such that σ fixes $\hat{u}_2$ and all points incident to $\hat{u}_1$, then σ is the identity.*

Proof. We have $(\alpha u_1+a_{12})^\wedge$ and $\hat{u}_2$ fixed by σ, for $\alpha \in \Phi$, $a \in \Pi(\mathfrak{J})$. Since by (1.3) $\hat{u}_1$ is the unique point of $\mathscr{P}(\mathfrak{M})$ incident to all a_*, $a \in \Pi(\mathfrak{J})$, $\hat{u}_1^\sigma=\hat{u}_1$. Since σ stabilizes $\mathscr{P}(\mathfrak{J})$ and fixes the points a_*, $a \in \Pi(\mathfrak{J})$, σ also fixes a^*. Let $y=\gamma u_1+\delta u_2+c_{12}+d_{21} \in \Pi(\mathfrak{M})$ and let $\hat{y}^\sigma=\hat{y}'$ where $y'=\gamma' u_1+\delta' u_2+c'_{12}+d'_{21}$. If $\delta=0$, then $\hat{y}'\simeq\hat{u}_1$ implies $\delta'=0$. In this case, $d, d' \in \Pi(\mathfrak{J})$. The condition $\hat{y}\simeq a_*$ if and only if $\hat{y}'\simeq a_*$, $a \in \Pi(\mathfrak{J})$, implies $d'=\lambda d$ for some $0\neq\lambda\in\Phi$ by Lemma 6. If $\delta\neq 0$, then $\delta'\neq 0$, and we may assume $\delta=\delta'=1$. Then $\hat{y}\simeq(\alpha u_1+a_{12})^\wedge$ if and only if $\hat{y}'\simeq(\alpha u_1+a_{12})^\wedge$, $\alpha \in \Phi$, $a \in \Pi(\mathfrak{J})$, implies $T(d, a)=T(d', a)$, $a \in \Pi(\mathfrak{J})$ or $d=d'$. In either case, $\hat{y}^\sigma=(\xi u_1+\delta u_2+h_{12}+d_{21})^\wedge$ for some $\xi \in \Phi$, $h \in \mathfrak{J}$. In particular, σ fixes all points $(\delta u_2+d_{21})^\wedge$. Using the above argument, for some $0\neq\rho\in\Phi$, one sees $\hat{y}^\sigma=(\gamma u_1+\rho\delta u_2+c_{12}+\rho d_{21})^\wedge$. Since $c^\#=\gamma d=\rho\gamma d$ and since $c=0$ implies $\gamma=0$ or $y=\gamma u_1$ (and σ fixes y in either case), we may assume $c \in \Pi(\mathfrak{J})$. Similarly, we may also assume $d \in \Pi(\mathfrak{J})$ so $\gamma=\delta=0$ and $c_*|d^*$. If $b \in \mathfrak{J}$ is such that $T(d, b)=T(c, b^\#)$

$\neq 0$, then $\hat{y} \simeq \hat{w}$ implies $\hat{y}^\sigma \simeq \hat{w}^\sigma = \hat{w}$ where $w = u_1 + N(b)u_2 + b_{12} + (b^\#)_{21}$. Thus, $\rho T(d, b) = T(c, b^\#)$ and $\rho = 1$. To show such a b exists, we choose $c_{2*} | d^*$ and c_{3*} such that $c = c_1, c_2, c_3$ form a three-point (see [3, p. 33]). If $b = c_1 + c_2 + c_3$, then $T(c_i, d) = 0$, $i = 1, 2$, and $T(c_3, d) \neq 0$ since $d^* = (c_1 \times c_2)^*$. Thus, $T(b, d) \neq 0$. Also, $T(c_1 \times c_i, c_1) = 0$, $i = 2, 3$, and $T(c_2 \times c_3, c_1) \neq 0$, so $T(b^\#, c) \neq 0$. Replacing b by $T(b, d) T(b^\#, c)^{-1} b$, we get b as desired.

LEMMA 8. *If σ is a collineation of $\mathscr{P}(\mathfrak{M})$ fixing a_*, $a \in \Pi(\mathfrak{J})$, and $(u_1 + e_{12})^\wedge$ for some $e \in \Pi(\mathfrak{J})$, then σ is the identity.*

Proof. As in the proof of Lemma 7, we see that $\hat{u}_1$ is fixed by σ. The condition, $\hat{u}_2^\sigma \not\simeq \hat{u}_1$ implies $\hat{u}_2^\sigma = (\alpha u_1 + u_2 + a_{12} + b_{21})^\wedge$ for some $\alpha \in \Phi$; $a, b \in \mathfrak{J}$. Since $\hat{u}_2^\sigma \simeq c_*$, $c \in \Pi(\mathfrak{J})$, we see $b = 0$ and $a = 0$, $\alpha = 0$, by (2.1). Thus, $\hat{u}_2^\sigma = \hat{u}_2$ and $a^{*\sigma} = a^*$ for $a \in \Pi(\mathfrak{J})$, since σ stabilizes $\mathscr{P}(\mathfrak{J})$ and fixes its points. Since $\hat{x}^\sigma | \hat{u}_1$ and $\hat{x}^\sigma \simeq a^*$ if and only if $\hat{x} \simeq a^*$, $a \in \Pi(\mathfrak{J})$, for $x = u_1 + c_{12}$, $c \in \Pi(\mathfrak{J})$, we see that $\hat{x}^\sigma = (u_1 + \rho(c) c_{12})^\wedge$ for some $0 \neq \rho(c) \in \Phi$. Similarly, $(u_2 + d_{21})^{\wedge\sigma} = (u_2 + \lambda(d) d_{21})^\wedge$ for $d \in \Pi(\mathfrak{J})$, $0 \neq \lambda(d) \in \Phi$. Since there is a norm similarity W of $\mathfrak{J}'$ to $\mathfrak{J}$ with $e_1 W = e$, where e_1, e_2, e_3 are the diagonal idempotents of $\mathfrak{J}' = \mathfrak{H}(\mathfrak{O}_3)$, we may assume $\mathfrak{J} = \mathfrak{J}'$ and $e = e_1$, after replacing σ by $\ulcorner W \urcorner \sigma \ulcorner W^{-1} \urcorner$. If $T(c, d) = 1$, then $(u_1 + \rho(c) c_{12})^\wedge \simeq (u_2 + \lambda(d) d_{21})^\wedge$ implies $\rho(c) \lambda(d) = 1$. Since $\rho(e_1) = 1$, we see $\lambda(e_1 + u[1j] + n(u) e_j) = 1$ for $j = 2, 3$, $u \in \mathfrak{O}$. Letting $u = 1$, we get $\rho(e_j) = 1$, $j = 2, 3$. Since $\lambda(e_i) = 1$, $i = 1, 2, 3$, we have $\rho(e_i + u[ij] + n(u) e_j) = 1$ for all $u \in \mathfrak{O}$, $i = 1, 2, 3$. If $x = u_2 - n(u)(e_k)_{12} + (u[ij])_{21}$ then $x \in \Pi(\mathfrak{M})$ since $N(u[ij]) = 0$, and $\hat{x}^\sigma = (u_2 + (v^\#)_{12} + v_{21})$ for some $v \in \mathfrak{J}$. Since $\hat{x}^\sigma \simeq a_*$ if and only if $\hat{x} \simeq a_*$ for $a \in \Pi(\mathfrak{J})$, we see $T(u[ij], a) = 0$ if and only if $T(v, a) = 0$ for $a \in \Pi(\mathfrak{J})$. Taking $a = e_i$, $e_i + s[ij] + n(s) e_j$, $s \in \mathfrak{O}$; $i, j = 1, 2, 3$, $i \neq j$, we get $v = \xi u[ij]$ for some $0 \neq \xi \in \Phi$. Choose $s \in \mathfrak{O}$ with $n(s, u) = -1$. Then $\hat{x} \simeq \hat{y}$ implies $\hat{x}^\sigma \simeq \hat{y}^\sigma = \hat{y}$ for $y = u_1 + (e_i + s[ij] + n(s) e_j)_{12}$. Thus, $\xi n(s, u) = -1$ and $\xi = 1$. Hence, $\hat{x}^\sigma = \hat{x}$. If $a = \sum \alpha_i e_i + \sum a_i [jk]$ with $a_k[ij] \neq 0$ and $a \in \Pi(\mathfrak{J})$, then choose $u \in \mathfrak{O}$ with $n(u, a_k) = -1$. Then $(u_1 + a_{12})^{\wedge\sigma} \simeq \hat{x}$ and $\rho(a) = 1$. If $a = \alpha e_i$, $0 \neq \alpha \in \Phi$, then what was just proved shows $\rho(\alpha e_i + \alpha[ij] + \alpha e_j) = 1$ which implies $\lambda(\alpha^{-1} e_i) = 1$ and $\rho(a) = 1$. Thus, $\rho(a) = 1$ for all $a \in \Pi(\mathfrak{J})$, and σ fixes $\hat{u}_2$ and all points incident to $\hat{u}_1$. By Lemma 7, σ is the identity.

THEOREM 2. *If $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, \gamma)$ and $\mathfrak{J}' = \mathfrak{H}(\mathfrak{O}_3', \gamma')$, then σ is a collineation of $\mathscr{P}(\mathfrak{M}(\mathfrak{J}))$ onto $\mathscr{P}(\mathfrak{M}(\mathfrak{J}'))$ if and only if $\sigma = \ulcorner W \urcorner$ for some semisimilarity W of $\mathfrak{M}(\mathfrak{J})$ onto $\mathfrak{M}(\mathfrak{J}')$.*

Proof. By Lemma 4(b), we may assume that $\hat{u}_i^\sigma = \hat{u}_i'$, $i = 1, 2$. Thus, σ induces a collineation of $\mathscr{P}(\mathfrak{J})$ onto $\mathscr{P}(\mathfrak{J}')$. By the fundamental theorem of octonion planes (see [3, p. 40]), σ agrees with $\ulcorner W \urcorner$ on $\mathscr{P}(\mathfrak{J})$ for some semisimilarity W of $\mathfrak{J}$ onto $\mathfrak{J}'$. Replacing σ by $\sigma \ulcorner W \urcorner^{-1}$, we may assume $\mathfrak{J} = \mathfrak{J}'$, $\hat{u}_i^\sigma = \hat{u}_i$, $i = 1, 2$, $a_*^\sigma = a_*$, $a^{*\sigma} = a^*$, for $a \in \Pi(\mathfrak{J})$. If $e \in \Pi(\mathfrak{J})$, then $(u_1 + e_{12})^{\wedge\sigma} | \hat{u}_1$, and $(u_1 + e_{12})^{\wedge\sigma} \simeq a^*$ if and only if $(u_1 + e_{12})^\wedge \simeq a^*$ imply by Lemma 6 that $(u_1 + e_{12})^{\wedge\sigma} = (u_1 + \rho e_{12})^\wedge$ for some $0 \neq \rho \in \Phi$. Replacing σ by $\sigma \ulcorner (\rho^{-1} 1)^\wedge \urcorner$, we may assume that σ fixes a_*, $a \in \Pi(\mathfrak{J})$ and $(u_1 + e_{12})^\wedge$. By Lemma 8, σ is the identity.

3. **Simplicity of** P Aut ($\mathfrak{M}$). The purpose of this section is to prove the following

THEOREM 3. *P Aut ($\mathfrak{M}$) is a simple group.*

We shall first establish some facts about P Aut ($\mathfrak{M}$).

LEMMA 9. *P Aut ($\mathfrak{M}$) is a primitive permutation group of $\mathscr{P}(\mathfrak{M})$.*

Proof. By Lemma 4(a), P Aut ($\mathfrak{M}$) is transitive. Suppose $M_1, M_2, \ldots$ is a system of imprimitivity for P Aut ($\mathfrak{M}$) with $\hat{u}_1 \in M_1$. If $\hat{u}_1 \neq \hat{x} \in M_1$, then either $\hat{x} \not\simeq \hat{u}_1$; $\hat{x} \simeq \hat{u}_1$ and $\hat{x} \nmid \hat{u}_1$; or $\hat{x} | \hat{u}_1$. By Lemma 4, we may assume $\hat{x} = \hat{u}_2$, $\hat{x} = e_1^*$, or $\hat{x} = e_{1*}$ where e_1, e_2, e_3 are orthogonal primitive idempotents for $\mathfrak{J}$. Let $z = u_1 + (e_1)_{12}$ and $w = u_1 + (e_1 + e_2)_{12} + (e_3)_{21}$, so $z, w \in \Pi(\mathfrak{M})$. If $\hat{x} = \hat{u}_2$, then $\hat{z} \not\simeq \hat{u}_2 \not\simeq \hat{w}$ so $\hat{z}, \hat{w} \in M_1$. But $\hat{z} | \hat{u}_1$ and $\hat{w} \simeq \hat{u}_1$ with $\hat{w} \nmid \hat{u}_1$, so elements of all three types are in M_1, and $M_1 = \mathscr{P}(\mathfrak{M})$. If $\hat{x} = e_1^*$, then $\hat{w} \simeq \hat{u}_1$ with $\hat{w} \nmid \hat{u}_1$ implies $\hat{w} \in M_1$. But $\hat{w} \not\simeq e_1^*$, and we may apply the previous case. If $\hat{x} = e_{1*}$, then $e_2^* | e_{1*}$ implies $e_2^* \in M_1$. But $e_2^* \simeq \hat{u}_1$ and $e_2^* \nmid \hat{u}_1$ and we may apply the second case.

We denote by $T(\mathfrak{J})$ the group $\{t_a \mid a \in \mathfrak{J}\}$.

LEMMA 10. *$PT(\mathfrak{J})$ is a normal abelian subgroup of the subgroup H of P Aut ($\mathfrak{M}$) fixing $\hat{u}_2$.*

Proof. $PT(\mathfrak{J})$ is abelian by (1.11) and fixes $\hat{u}_2$ by (1.9). If $\sigma \in H$, write $\hat{u}_1^\sigma = (u_1 + a_{12} + (a^\#)_{21} + N(a)u_2)^\wedge$. Replacing σ by $\sigma \ulcorner t_{-a} \urcorner$, we may assume $\hat{u}_i^\sigma = \hat{u}_i$, $i = 1, 2$. By the proof of Theorem 2, we have $\sigma = \ulcorner W \urcorner$ for some $W \in \Gamma(\mathfrak{J})$. We see $\sigma^{-1} \ulcorner t_c \urcorner \sigma = \ulcorner \tilde{W}^{-1} t_c \tilde{W} \urcorner = \ulcorner t_{cW} \urcorner$, $c \in \mathfrak{J}$, by (1.12), so $PT(\mathfrak{J})$ is normal in H.

Set $v_c = \varepsilon^{-1} t_c \varepsilon$ for $c \in \mathfrak{J}$ so

$$
\begin{aligned}
xv_c &= (\alpha - T(a, c) + T(b, c^\#) - \beta N(c))u_1 \\
&\quad + \beta u_2 + (a - b \times c + \beta c^\#)_{12} + (b - \beta c)_{21} \quad \text{for } x \text{ as in (1.1).}
\end{aligned} \tag{3.1}
$$

A direct calculation verifies, for $u \in \mathfrak{O}$,

$$
t_{e_1} v_{u[12]} t_{-e_1} v_{-u[12]} v_{n(u)e_2} = (T_{u[12], e_1})^\sim \tag{3.2}
$$

where in general $T_{a,b} = 1 + V_{a,b} + U_a U_b$ (see [3, p. 17]).

LEMMA 11. *Aut ($\mathfrak{M}$) is generated by conjugates of $T(\mathfrak{J})$ in Aut ($\mathfrak{M}$).*

Proof. If G is the group generated by conjugates of $T(\mathfrak{J})$ in Aut ($\mathfrak{M}$), then by (1.14) with $c = 1$, we see $\varepsilon = -(t_1 \varepsilon t_1 \varepsilon t_1)^{-1} = (t_1 \varepsilon^{-1} t_1 \varepsilon t_1)^{-1} \in G$. Since $v_c \in G$ for $c \in \mathfrak{J}$, (3.2) shows that $(T_{u[12], e_1})^\wedge \in G$. Theorem 4.7 of [3] implies that conjugates of $T_{u[12], e_1}$, $u \in \mathfrak{O}$, in $S(\mathfrak{J})$ generate $S(\mathfrak{J})$. By Theorem 1, we see $G = $ Aut ($\mathfrak{M}$).

LEMMA 12. *P Aut ($\mathfrak{M}$) $= \mathscr{D}(P$ Aut ($\mathfrak{M}$)), the derived group.*

Proof. By Lemma 11, we need only show $\ulcorner t_a \urcorner \in D = \mathscr{D}(P$ Aut ($\mathfrak{M}$)), $a \in \mathfrak{J}$. By (1.11), (1.12), the fact that $\Pi(\mathfrak{J})$ spans $\mathfrak{J}$ and the transitivity of $PS(\mathfrak{J})$ on points of $\mathscr{P}(\mathfrak{J})$, we need only show $t_{\alpha e_2} \in D$ for all $\alpha \in \Phi$. Since $T_{u[12], e_1} \in \mathscr{D}(PS(\mathfrak{J}))$ by Lemma 4.6 of [3], we see by (3.2) that $v_{n(u)e_2} \in D$ for all $u \in \mathfrak{O}$. Hence, $t_{n(u)e_2} \in D$.

If $\alpha \in \Phi$, there exist $u, v \in \mathfrak{D}$ with $n(u, v) = n(u+v) - n(u) - n(v) = \alpha$. Thus, by (1.11), $t_{\alpha e_2} \in D$, as desired.

Proof of Theorem 3. This follows immediately from Lemmas 9, 10, 11, 12 and Lemma 4, p. 39 of [1].

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