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A GLOBAL OPTIMIZATION APPROACH TO RATIONALLY CONSTRAINED RATIONAL PROGRAMMING†‡

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The rationally constrained rational programming (RCRP) problem is shown, for the first time, to be equivalent to the quadratically constrained quadratic programming problem with convex objective function and constraints that are all convex except for one that is concave and separable. This equivalence is then used in developing a novel implementation of the Generalized Benders Decomposition (GBDA) which, unlike **all** earlier implementations, is guaranteed to identify the global optimum of the RCRP problem. It is also shown, that the critical step in the proposed GBDA implementation is the solution of the master problem which is a quadratically constrained, separable, reverse convex programming problem that must be solved globally. Algorithmic approaches to the solution of such problems are discussed and illustrative examples are presented.

KEYWORDS Global Optimization Decomposition Reverse convex.

I. INTRODUCTION

The class of rationally constrained rational programming (RCRP) problems and its subclass of polynomially constrained polynomial programming (PCPP) problems are often encountered in chemical engineering applications. The heat exchanger network synthesis problem can be formulated as a mixed integer nonlinear programming problem (Grossmann, 1990) that can be further transformed into a PCPP through the introduction of additional nonlinear constraints and the use of polynomial approximations of the objective function. Indeed, the requirement that a variable δ be binary is equivalent to the quadratic equality $\delta(\delta - 1) = 0$, where δ is assumed to be continuous. The robust controller design problem can be formulated as a minimax optimization problem. The latter has been shown to be equivalent to a linear programming problem with several additional quadratic equality constraints (Manousiouthakis and Sourlas, 1990). Global solution of such optimization problems is being pursued by several chemical engineering researchers (Manousiouthakis *et al.*, 1990, Swaney, 1990, Visweswaran and Floudas, 1990).

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An optimization problem that belongs to one of these classes has several local minima (or maxima). There are special cases (i.e. minimization problems with convex objective and convex constraints such as: linear programming, positive semidefinite quadratic programming, etc.) where the objective value at all local minima is the same, hence any local minimum is also global. As a result, for this subclass of problems efficient large scale optimization algorithms have been developed.

However, most RCRP problems do not enjoy this property, namely not all local minima are global, and are thus more difficult to solve. In the case of the negative definite quadratic programming problem (NDQP) it has been established that the global optima are among the extreme points of the feasible region (which in this case is a convex polyhedron) (Charnes & Cooper, 1961). For the indefinite quadratic programming problem (IDQP) it has been established that the global optima lie on the feasible region's boundary (Mueller, 1970).

Another class of problems that has the same extreme point property is the class of concave minimization (or convex maximization) problems over a polyhedral feasible region. One can identify the global solution to this problem by total enumeration of the extreme points of the feasible region. These methods become computationally intensive for large scale problems. In this spirit, Cabot and Francis (1970) combined extreme point ranking techniques (Murty, 1969) with underestimating techniques to solve the quadratic concave minimization problem. Cutting plane methods have also been employed for the solution of the concave minimization problem. In that regard, Tuy (1964) introduced a cone splitting procedure (Tuy cuts) which was later demonstrated by Zwart (1973) to exhibit convergence problems. Zwart (1974) later presented a modified algorithm that is computationally finite. Several researchers have generalized the idea of Tuy cuts. Jacobsen (1981) proposed a similar algorithm and provided proof for its convergence. Glover (1973) extended the notion of the Tuy cuts and introduced so called convexity cuts.

In addition to the extreme point enumeration and the cutting plane methods, branch and bound techniques have also been used. Falk and Soland (1969) and Soland (1971) proposed algorithms applicable to separable problems. Horst (1976) presented an algorithm that can be used to solve nonseparable problems as well. Hoffman (1981) also presented a global optimization algorithm based on underestimating techniques. Pardalos and Rosen (1986) give an excellent review on the subject of concave minimization.

The extreme point property is also satisfied by the class of reverse convex programming (RCP) problems. A constraint $g(x) \geq 0$ is called reverse convex when $g(x)$ is quasiconvex (i.e. $g(\lambda x_1 + (1-\lambda)x_2) \leq \max\{g(x_1), g(x_2)\}$ for all $\lambda \in [0, 1]$). An optimization problem that involves reverse convex constraints and pseudo concave objective is a RCP problem. Ueing (1972) proposed a combinatorial procedure that yields the global optimum of RCP problems, when the objective function is strictly concave and the constraints are convex. For the solution of the general RCP problem, Hillestad and Jacobsen (1980a) proposed a cutting plane algorithm which however could exhibit convergence to infeasible points, as they demonstrated. Finally, for the special RCP problem of linear programming with one reverse convex constraint, Hillestad and Jacobsen (1980b) proposed a finite algorithm.

Linear Programming with one reverse convex constraint can also be viewed as a special case of convex minimization problems with an additional reverse convex constraint. For this general class of problems Tuy (1987) proposed a method that reduces the problem to a sequence of convex maximization problems that can be solved globally with the techniques mentioned in the previous paragraph. Tuy (1986) also established connections between this type of problems and the so called d.c. programming (DCP) problem. In fact, he demonstrated that any nonlinear programming (NLP) problem can, in principle, be approximated by a DCP problem which in turn can be further transformed, in principle, into a convex programming problem with an additional reverse convex constraint.

In this paper, it is demonstrated that the RCRP problem is equivalent to a convex quadratically constrained quadratic programming problem with an additional reverse convex, quadratic and separable constraint. It is shown, that one can *exactly* transform the former into the latter by the use of variable transformations and the introduction of new variables. Furthermore, a novel implementation of the Generalized Benders' Decomposition Algorithm (GBDA) is proposed for the solution of the latter problem. It is shown that this implementation of GBDA is *always guaranteed* to identify the global optimum of the general RCRP problem. The critical step in this procedure is the solution of the master problem, which is shown to be a quadratic, separable, RCP problem.

II. OPTIMIZATION PROBLEM EQUIVALENCE

In this work we deal with several optimization problems which we consider expedient to present next.

The rationally constrained rational programming problem (P1) can be stated as follows:

$$\min_{z \in R^n} \frac{f^0(z)}{g^0(z)} \tag{P1}$$

subject to,

$$\frac{f^j(z)}{g^j(z)} \leq 0, \qquad j = 1, \ldots, k$$

where,

$$\begin{aligned} f^j(z) \triangleq \alpha_0^j + \sum_{i_1=1}^{n} \alpha_{i_1}^j z_{i_1} + \sum_{i_1=1}^{n} \sum_{i_2 \geq i_1}^{n} \alpha_{i_1 i_2}^j z_{i_1} z_{i_2} + \cdots \\ + \sum_{i_1=1}^{n} \sum_{i_2 \geq i_1}^{n} \cdots \sum_{i_m \geq i_{m-1}}^{n} \alpha_{i_1 i_2 \cdots i_m}^j z_{i_1} z_{i_2} \cdots z_{i_m} \qquad j = 0, \ldots, k, \end{aligned}$$

$$\begin{aligned} g^j(z) \triangleq \beta_0^j + \sum_{i_1=1}^{n} \beta_{i_1}^j z_{i_1} + \sum_{i_1=1}^{n} \sum_{i_2 \geq i_1}^{n} \beta_{i_1 i_2}^j z_{i_1} z_{i_2} + \cdots \\ + \sum_{i_1=1}^{n} \sum_{i_2 \geq i_1}^{n} \cdots \sum_{i_m \geq i_{m-1}}^{n} \beta_{i_1 i_2 \cdots i_m}^j z_{i_1} z_{i_2} \cdots z_{i_m}, \qquad j = 0, \ldots, k \end{aligned}$$

and

$$z = [z_1, z_2, \ldots, z_n] \in R^n$$

Similarly, the polynomially constrained polynomial programming problem (P2) can be stated as follows:

$$\min_{z \in R^n}\left\{\alpha_0^0 + \sum_{i_1=1}^{n} \alpha_{i_1}^0 z_{i_1} + \sum_{i_1=1}^{n}\sum_{i_2 \geq i_1}^{n} \alpha_{i_1 i_2}^0 z_{i_1} z_{i_2} + \cdots + \sum_{i_1=1}^{n}\sum_{i_2 \geq i_1}^{n} \cdots \sum_{i_m \geq i_{m-1}}^{n} \alpha_{i_1 i_2 \ldots i_m}^0 z_{i_1} z_{i_2} \ldots z_{i_m}\right\} \tag{P2}$$

subject to,

$$\alpha_0^1 + \sum_{i_1=1}^{n} \alpha_{i_1}^1 z_{i_1} + \sum_{i_1=1}^{n}\sum_{i_2 \geq i_1}^{n} \alpha_{i_1 i_2}^1 z_{i_1} z_{i_2} + \cdots + \sum_{i_1=1}^{n}\sum_{i_2 \geq i_1}^{n} \cdots \sum_{i_m \geq i_{m-1}}^{n} \alpha_{i_1 i_2 \ldots i_m}^1 z_{i_1} z_{i_2} \ldots z_{i_m} \leq 0$$

$$\cdots$$

$$\alpha_0^k + \sum_{i_1=1}^{n} \alpha_{i_1}^k z_{i_1} + \sum_{i_1=1}^{n}\sum_{i_2 \geq i_1}^{n} \alpha_{i_1 i_2}^k z_{i_1} z_{i_2} + \cdots + \sum_{i_1=1}^{n}\sum_{i_2 \geq i_1}^{n} \cdots \sum_{i_m \geq i_{m-1}}^{n} \alpha_{i_1 i_2 \ldots i_m}^k z_{i_1} z_{i_2} \cdots z_{i_m} \leq 0$$

This problem formulation encompasses all polynomial programming problems since equality constraints can also be expressed as pairs of inequality constraints.

In turn, the quadratically constrained quadratic programming problem (P3) can also be stated as follows:

$$\min_{x_1, \ldots, x_n} \alpha_0^0 + \sum_{i_1=1}^{n} \alpha_{i_1}^0 x_{i_1} + \sum_{i_1=1}^{n}\sum_{i_2 \geq i_1}^{n} \alpha_{i_1 i_2}^0 x_{i_1} x_{i_2} \tag{P3}$$

subject to,

$$\alpha_0^1 + \sum_{i_1=1}^{n} \alpha_{i_1}^1 x_{i_1} + \sum_{i_1=1}^{n}\sum_{i_2 \geq i_1}^{n} \alpha_{i_1 i_2}^1 x_{i_1} x_{i_2} \leq 0$$

$$\cdots$$

$$\alpha_0^k + \sum_{i_1=1}^{n} \alpha_{i_1}^k x_{i_1} + \sum_{i_1=1}^{n}\sum_{i_2 \geq i_1}^{n} \alpha_{i_1 i_2}^k x_{i_1} x_{i_2} \leq 0$$

By defining $x \triangleq [x_1 x_2 \cdots x_n]^T$, $c_j \triangleq \alpha_0^j$, $b_j \triangleq [\alpha_1^j \alpha_2^j \cdots \alpha_n^j]^T$,

$$A_j \triangleq \begin{bmatrix} \alpha_{11}^j & 0.5\,\alpha_{12}^j & \cdots & 0.5\,\alpha_{1n}^j \\ 0.5\,\alpha_{12}^j & \alpha_{22}^j & \cdots & 0.5\,\alpha_{2n}^j \\ \vdots & \vdots & \vdots & \vdots \\ 0.5\,\alpha_{1n}^j & 0.5\,\alpha_{2n}^j & \cdots & \alpha_{nn}^j \end{bmatrix}$$

we can also represent (P3) as follows:

$$\min_x x^T A_0 x + b_0^T x + c_0 \tag{P3}$$

subject to,

$$x^T A_j x + b_j^T x + c_j \leq 0 \qquad j = 1, \ldots, k$$

Let us finally state the quadratically constrained quadratic programming problem with the following special features:

(i) the objective function is convex (A_0 is positive semi-definite)

(ii) all the constraints, except the last one, are convex (A_j, $j = 1, \ldots, k-1$ are p.s.d.)

(iii) the last constraint is concave and separable (A_k diagonal and negative s.d.)

We refer to this problem as (P4), and state it next.

$$\min_{x \in R^n} x^T A_0 x + b_0^T x + c_0 \qquad (A_0 \text{ p.s.d.}) \tag{P4}$$

subject to,

$$x^T A_j x + b_j^T x + c_j \leq 0 \qquad j = 1, \ldots, k-1 \qquad (A_j \text{ p.s.d.})$$

$$x^T A_k x + b_k^T x + c_k \leq 0 \qquad (A_k \text{ diagonal and n.s.d.})$$

The four aforementioned programming problems (P1), (P2), (P3), (P4) will be shown to be equivalent to each other, in the sense that a problem of one type can be exactly transformed to a problem of the other type through the use of variable transformations and the introduction of new variables. In that respect, the following theorem is proved.

Theorem 1:
P1 $\Leftrightarrow$ P2 $\Leftrightarrow$ P3 $\Leftrightarrow$ P4

Proof:
P1 $\Rightarrow$ P2
Define $\gamma \triangleq f^0(z)/g^0(z)$. Then the optimization problem (P1) can be rewritten in the following form:

$$\min_{z \in R^n} \gamma$$

subject to,

$$f^0(z) - \gamma g^0(z) \leq 0$$

$$-f^0(z) + \gamma g^0(z) \leq 0$$

$$f^j(z) \cdot g^j(z) \leq 0, \qquad j = 1, \ldots, k$$

This is a (P2) type problem.

P2 $\Rightarrow$ P3
Define:

$$y_{i_1,i_2} \triangleq z_{i_1} z_{i_2}, \qquad i_1 = 1, \ldots, n, \qquad i_2 = i_1, \ldots, n,$$

$$y_{i_1,i_2,i_3} \triangleq y_{i_1,i_2} z_{i_3}, \qquad i_1 = 1, \ldots, n, \qquad i_2 = i_1, \ldots, n, \qquad i_3 = i_2, \ldots, n,$$

$$y_{i_1,i_2,\ldots,i_m} \triangleq y_{i_1,i_2,\ldots,i_{m-1}} z_{i_m},$$

$$i_1 = 1, \ldots, n, \qquad i_2 = i_1, \ldots, n, \ldots, i_m = i_{m-1}, \ldots, n.$$

Under these transformations the objective function and the constraints of (P2) become linear. The only nonlinearity in the resulting optimization problems stems from the equality constraints defining the y variables. Since these constraints are quadratic in nature and since equality constraints can be replaced by inequality constraints ($f = 0 \Leftrightarrow -f \leq 0$, $f \leq 0$) the resulting optimization problem is of the form (P3).

P3 $\Rightarrow$ P4

Let $A_j = W_j \Lambda_j W_j^T$ denote the eigendecomposition of A_j (A_j is a symmetric matrix and thus Λ_j is real, W_j is orthogonal and $W_j^{-1} = W_j^T$). Let also

$$\Lambda_j = \operatorname{diag}(\{\lambda_{ji}^+\}_{i=1}^{n_j}, \{\lambda_{ji}^-\}_{i=n_j+1}^{n})$$
$$\Lambda_j^+ = \operatorname{diag}(\{\lambda_{ji}^+\}_{i=1}^{n_j}, \{0\}_{i=n_j+1}^{n})$$
$$\Lambda_j^- = \operatorname{diag}(\{0\}_{i=1}^{n_j}, \{\lambda_{ji}^-\}_{i=n_j+1}^{n}$$

where

$$\lambda_{ji}^+ \geq 0 \qquad i = 1, \ldots, n_j, \qquad j = 0, \ldots, k$$
$$\lambda_{ji}^- < 0 \qquad i = n_j + 1, \ldots, n, \qquad j = 0, \ldots, k$$

Then $A_j = W_j \Lambda_j W_j^T = W_j(\Lambda_j^+ + \Lambda_j^-)W_j^T = W_j \Lambda_j^+ W_j^T + W_j \Lambda_j^- W_j^T \triangleq A_j^+ + A_j^-$ where A_j^+ is p.s.d. and A_j^- is n.s.d.. As a result, (P3) becomes

$$\min_x x^T A_0^+ x + x^T A_0^- x + b_0^T x + c_0$$

subject to,

$$x^T A_j^+ x + x^T A_j^- x + b_j^T x + c_j \leq 0 \qquad j = 1, \ldots, k$$

Let J be the set of indices j ($j = 0, \ldots, k$) for which A_j^- has at least one nonzero eigenvalue and let n_t be the cardinality of this set. We now introduce n_t nonnegative variables t_j, $j \in J$ defined as follows:

$$t_j \triangleq -x^T A_j^- x, \qquad j \in J$$

These equalities are equivalent to the following set of inequalities

$$x^T(-A_j^-)x - t_j \leq 0, \qquad j \in J,$$
$$\sum_{j \in J} (t_j + x^T A_j^- x) \leq 0.$$

By construction, $\sum_{j \in J} A_j^-$ is symmetric. Let $\sum_{j \in J} A_j^- = W_\Sigma \Lambda_\Sigma W_\Sigma^T$ be an eigendecomposition of $\sum_{j \in J} A_j^-$, where Λ_Σ is a diagonal n.s.d. matrix and W_Σ is orthogonal ($W_\Sigma^{-1} = W_\Sigma^T$). Let n_y be the number of strictly negative eigenvalues of $\sum_{j \in J} A_j^-$ (the other $n - n_y$ eigenvalues are zero). Without loss of generality the following structure can be assumed for Λ_Σ:

$$\Lambda_\Sigma = \begin{bmatrix} \Lambda_{\Sigma 1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $\Lambda_{\Sigma 1}$ is a $n_y \times n_y$, diagonal, n.d. matrix (contains only the strictly negative diagonal elements of Λ_Σ). Based on this partition the matrix W_Σ of eigenvectors of $\sum_{j \in J} A_j^-$ can be written as:

$$W_\Sigma = [W_{\Sigma 1} \quad W_{\Sigma 2}]$$

where,

$W_{\Sigma 1}$ $n \times n_y$ real matrix containing as columns the eigenvectors associated with the strictly negative eigenvalues of $\sum_{j \in J} A_j^-$.

$W_{\Sigma 2}$ $n \times (n - n_y)$ real matrix.

Then $\sum_{j\in J} A_j^-$ can be written as:

$$\sum_{j\in J} A_j^- = W_\Sigma \Lambda_\Sigma W_\Sigma^T = W_{\Sigma 1}\Lambda_{\Sigma 1} W_{\Sigma 1}^T$$

and the inequalities defining the variables t_j, $(j \in J)$ take the form:

$$x^T(-A_j^-)x - t_j \leq 0, \qquad j \in J,$$

$$\left(\sum_{j\in J} t_j\right) + x^T W_{\Sigma 1}\Lambda_{\Sigma 1} W_{\Sigma 1}^T x \leq 0.$$

Define the vector $y \in R^{n_y}$ as:

$$y \triangleq W_{\Sigma 1}^T x$$

As a result the problem (P3) is tranformed to

$$\min_{\begin{bmatrix} x \\ t \\ y \end{bmatrix} \in R^{n+n_t+n_y}} \quad x^T A_0^+ x - e_0^T t + b_0^T x + c_0 \tag{1}$$

subject to,

$$\begin{aligned} x^T A_j^+ x + b_j^T x + c_j &\leq 0, \qquad j \notin J \\ x^T A_j^+ x - t_j + b_j^T x + c_j &\leq 0, \qquad j \in J \\ x^T(-A_j^-)x - t_j &\leq 0, \qquad j \in J \\ y - W_{\Sigma 1}^T x &\leq 0 \\ -y + W_{\Sigma 1}^T x &\leq 0 \\ y^T \Lambda_{\Sigma 1} y + \sum_{j\in J} t_j &\leq 0. \end{aligned}$$

where, $e_0 \in R^{n_t}$ is identically zero if $j = 0 \notin J$ and is such that $e_0^T t = t_0$ otherwise. Also $\Lambda_{\Sigma 1} = \mathrm{diag}\{\lambda_{\Sigma,i}\}_{i=1}^{n_y}$, $\lambda_{\Sigma,i} < 0$, $i = 1, \ldots, n_y$. As a result, $\Lambda_{\Sigma 1}$ is a negative definite diagonal matrix and the last constraint is a quadratic, separable, reverse convex constraint. Since the remaining constraints are quadratic and convex it follows that the last optimization problem belongs to the class (P4).

(P4) $\Rightarrow$ (P1) obvious. O.E.Δ.

Having established that (P1), (P2), (P3) and (P4) are equivalent, we now proceed to the discussion of solution methodologies for (P4) type problems.

III. SOLUTION METHODOLOGIES FOR (P4)

The globally optimal solution for optimization problems of the type (P4) can be obtained by several algorithms. Three such algorithms will be presented. They are iterative in nature and their ε-convergence to the globally optimal solution is guaranteed.

Algorithm 1

This solution methodology is based on the Generalized Benders Decomposition Algorithm (GBDA) (Geoffrion, 1972). Floudas *et al.* (1989) proposed a GBDA implementation as a so called "global optimum search technique" for the solution of P2 type problems, namely NLP's and MINLP's. However, Bagajewicz and Manousiouthakis (1991) demonstrated that the proposed GBDA implementation is not guaranteed to identify the global solution for such problems. Nevertheless the following theorem holds:

Theorem 2:

The GBDA, if properly implemented, is guaranteed to identify the global optimum of the general RCRP problem (P1). Furthermore, global solution of (P1) is ascertained upon global solution of a series of separable, quadratically constrained, reverse convex programming (RCP) problems.

Proof:

It has been established in Theorem 1 that (P1) is equivalent to (P4) which can take the form:

$$\min_{\substack{x\in R^n\\ t\in R^{n_t}\\ y\in R^{n_y}}} F(x, t) = x^T A_0^+ x - e_0^T t + b_0^T x + c_0 \tag{1}$$

subject to,

$$G(x, t) \le 0$$

$$L(x, y) \le 0$$

$$y^T \Lambda_{\Sigma 1} y + \sum_{j\in J} t_j \le 0$$

where:

$$G(x, t) = \begin{bmatrix} x^T A_j^+ x + b_j^T x + c_j \le 0, & j \notin J \\ x^T A_j^+ x - t_j + b_j^T x + c_j \le 0, & j \in J \\ x^T(-A_j^-)x - t_j \le 0, & j \in J \end{bmatrix},$$

$$L(x, y) = \begin{bmatrix} y - W_{\Sigma 1}^T x \le 0 \\ -y + W_{\Sigma 1}^T x \le 0 \end{bmatrix},$$

$$\Lambda_{\Sigma 1} = \operatorname{diag}\{\lambda_{\Sigma,i}\}_{i=1}^{n_y}, \qquad \lambda_{\Sigma,i} < 0, \qquad i = 1, \ldots, n_y.$$

Let the variable vector $\begin{bmatrix} x \\ t \\ y \end{bmatrix}$ be decomposed in two parts: the noncomplicating variable vector $\begin{bmatrix} x \\ t \end{bmatrix}$ and the complicating variable vector y. Based on this decomposition, (1) takes the form (Geoffrion, 1972):

$$\min_{y\in R^{n_y}\cap V} \phi(y) \tag{1a}$$

subject to,

$$\phi(y)=\left\{\begin{array}{ll} & \min\limits_{\substack{x\in R^n\\ t\in R^{n_t}}} F(x,t)\\ s.t. & G(x,t)\le 0\\ & L(x,y)\le 0\\ & y^T\Lambda_{\Sigma 1}y+\sum\limits_{j\in J} t_j\le 0\end{array}\right\} \tag{1b}$$

$$V=\left\{y: L(x,y)\le 0,\ y^T\Lambda_{\Sigma 1}y+\sum_{j\in J} t_j\le 0 \text{ for some } x, t \text{ satisfying } G(x,t)\le 0\right\}$$

For each value of y, the internal optimization problem (1b) is a convex optimization problem. Therefore, based on the strong duality theorem (Luenberger, 1969, p. 224), the value of (1b) is equal to the value of its dual. Thus (1b) is equivalent to:

$$\phi(y)=\max_{u\ge o}\min_{x,\,t}\left[F(x,t)+u_1^TG(x,t)+u_2^TL(x,y)+u_3\left(y^T\Lambda_{\Sigma 1}y+\sum_{j\in J}t_j\right)\right] \tag{1c}$$

Based on the proposed variable transformations and the presented problem decomposition the following two subproblems are created.

Primal:

$$v(\bar{y})=\min_{\substack{x\in R^n\\ t\in R^{n_t}}} F(x,t)=x^TA_0^+x-e_0^Tt+b_0^Tx+c_0 \tag{2}$$

subject to,

$$G(x,t)\le 0$$
$$L(x,\bar{y})\le 0$$
$$\bar{y}^T\Lambda_{\Sigma 1}\bar{y}+\sum_{j\in J}t_j\le 0$$

where $\bar{y}$ is fixed.

Master:

$$\min_{\substack{y_0\in R\\ y\in R^{n_y}}} y_0 \tag{3}$$

subject to,

$$L^*(y,u)=\min_{x,\,t}\left[F(x,t)+u_1^TG(x,t)+u_2^TL(x,y)+u_3\left(y^T\Lambda_{\Sigma 1}y+\sum_{j\in J}t_j\right)\right]\le y_0,$$

for all $u\ge 0$.

$$L_*(y,v)=\min_{x,\,t}\left[v_1^TG(x,t)+v_2^TL(x,y)+v_3\left(y^T\Lambda_{\Sigma 1}y+\sum_{j\in J}t_j\right)\right]\le 0,$$

for all $v\in N$

where $u=(u_1^Tu_2^Tu_3)^T$, $v=(v_1^Tv_2^Tv_3)^T$, $N=\{v\ge 0, \|v\|_1=1\}$. In this formulation, $L_*(y,v)\le 0$ is equivalent to the requirement that $y\in V$.

For each value of $\bar{y}$ the primal subproblem, (2) is an upper bound to the global minimum of (1). The global solution of the master (3) is equal to the global solution of (1). By creating a relaxed version of (3) one can develop an iterative procedure for the global solution of (1) as follows:

Step 1: Identify a feasible point $\bar{y} \in R^{n_y} \cap V$. Solve (2) and obtain a multiplier vector $\bar{u}$ and the optimal variable vector $[\bar{x}^T \bar{t}^T]^T$. Set $p = 1$, $r = 0$, $u^1 = \bar{u}$, $[x^{1T} t^{1T}]^T = [\bar{x}^T \bar{t}^T]^T$ and $UBD = \phi(\bar{y})$. The separability of the function $L^*(y, u^p)$ in y, allows its evaluation as follows:

$$\begin{aligned}
L^*(y, u^p) &= \min_{x,t}\Big[F(x, t) + u_1^{pT} G(x, t) + u_2^{pT}\begin{bmatrix} y - W_{\Sigma 1}^T x \\ -y + W_{\Sigma 1}^T x\end{bmatrix} \\
&\quad + u_3^p\Big(y^T \Lambda_{\Sigma 1} y + \sum_{j \in J} t_j\Big)\Big] \\
&= \min_{x,t}\Big[F(x, t) + u_1^{pT} G(x, t) + u_2^{pT}\begin{bmatrix} -I \\ I\end{bmatrix} W_{\Sigma 1}^T x + u_3^p \sum_{j \in J} t_j\Big] \\
&\quad + u_2^{pT}\begin{bmatrix} I \\ -I\end{bmatrix} y + u_3^p y^T \Lambda_{\Sigma 1} y \Rightarrow L^*(y, u^p) \\
&= F(x^p, t^p) + u_1^{pT} G(x^p, t^p) + u_2^{pT}\begin{bmatrix} -I \\ I\end{bmatrix} W_{\Sigma 1}^T x^p \\
&\quad + u_3^p \sum_{j \in J} t_j^p + u_2^{pT}\begin{bmatrix} I \\ -I\end{bmatrix} y + u_3^p y^t \Lambda_{\Sigma 1} y. \qquad (4)
\end{aligned}$$

The last equality is a result of the saddle point property that holds for the Langrangian of the primal (Luenberger, 1969, p. 219).

Step 2: Solve globally the relaxed master problem:

$$\min_{\substack{y_0 \in R \\ y \in R^{n_y}}} y_0 \qquad (5)$$

subject to,

$$L^*(y, u^i) \leq y_0, \qquad i = 1, \ldots, p$$

$$L_*(y, v^j) \leq 0, \qquad j = 1, \ldots, r$$

The value of $L_*(y, v^j)$ is calculated according to the procedure presented in step (3b). Let $(\hat{y}, \hat{y}_0)$ denote the global solution of the relaxed master. Then $\hat{y}_0$ is a lower bound to the global minimum of (1). If $\mathrm{UBD} \leq \hat{y}_0 + \varepsilon$, where ε is a convergence tolerance, then terminate. Otherwise continue to the next step.

Step 3: Solve (2) for $\bar{y} = \hat{y}$. Then there are two possibilities: the primal is either feasible or infeasible.

(a) *The primal is feasible*: if $\phi(\hat{y}) \leq \hat{y}_0 + \varepsilon$ then terminate. Otherwise determine a new optimal multiplier vector $\bar{u}$ and set $p = p + 1$ and $u^p = \bar{u}$. If $\phi(\hat{y}) < \mathrm{UBD}$ then set $\mathrm{UBD} = \phi(\hat{y})$. Then evaluate the function $L^*(y, u^p)$ and return to step 2.

(b) *The primal is infeasible*: Then solve the following infeasibility minimization problem:

$$\min_{\substack{x,t \\ a\in R}} a \tag{6}$$

subject to,

$$\begin{bmatrix} G(x, t) \\ L(x, \bar{y}) \\ \bar{y}^T \Lambda_{\Sigma 1} \bar{y} + \sum_{j\in J} t_j \end{bmatrix} - a\mathbf{1} \le 0.$$

where $\mathbf{1}[1\,1 \cdots 1]^T$.

Since the primal is infeasible the solution of (6) is positive. Based on the Kuhn–Tucker necessary conditions for this problem the optimal multiplier vector $\bar{v}$ can be shown to satisfy the relations: $\bar{v} \ge 0$, $1 - \|\bar{v}\|_1 = 0$. Hence $v \in N$. Once $\bar{v}$ is determined, set $r = r + 1$, $v^r = \bar{v}$ and evalute the function $L_*(y, v^r)$. Similarly to $L^*(y, u^p)$, the minimum in the definition of $L_*(y, v^r)$ can be calculated independently of y, directly from the solution of (6):

$$L_*(y, v^r) = v_1^{rT} G(x^r, t^r) + v_2^{rT} \begin{bmatrix} -I \\ I \end{bmatrix} W_{\Sigma 1}^T x^r + v_3^r \sum_{j\in J} t_j^r$$
$$+ v_2^{rT} \begin{bmatrix} I \\ -I \end{bmatrix} y + v_3^r y^T \Lambda_{\Sigma 1} y.$$

where $[x^{rT}\, t^{rT}]^T$ is the solution of (6) that corresponds to v^r. Then return to step 2.

This procedure is guaranteed to create a nondecreasing sequence of lower bounds for the global optimum of (1) iff each relaxed master is solved globally. Furthermore, since the primal is convex, and thus there is no gap between (1b) and (1c), this sequence will converge to the global optimum of (1) (Geoffrion, 1972, Bagajewicz and Manousiouthakis, 1991).

Therefore, the global solution of (1) (equivalently P1) is obtained through the global solution of a series of relaxed master problems (5). Based on (4), (7) each relaxed master problem is a separable, quadratically constrained, reverse convex programming (RCP) problem since $\Lambda_{\Sigma 1}$ is diagonal, n.d. and u_3^i, v_3^j are positive. O.E.Δ.

Remark 1: It has been shown, that the GBDA implementation we have proposed, can be used to identify the global optimum of the general RCRP problem. The unique features of this implementation are:

- The primal is convex, therefore there is no dual gap between the primal and its dual.
- The functions $L^*(y, u)$, $L_*(y, v)$ are such that the minimization problems in their definition can be solved independently of y. Furthermore, the solution to these optimization problems is readily obtained from the solution of the related

primal. The resulting relaxed master problem is a separable quadratically constrained RCP problem.

Because of its characteristics the proposed GBDA implementation coverges to the global optimum.

Remark 2: As mentioned above, each relaxed master problem may have several local minima. Thus, its global solution can be obtained only through the use of special algorithms. Several algorithms for the solution of such problems have been developed. Ueing (1972) proposed a combinatorial procedure for the solution of RCP problems and Hillestad and Jacobsen (1980) proposed a cutting plane method that utilizes Tuy type cuts but may converge to infeasible points. The separability of the relaxed master's constraints allows also application of Soland's (1971) algorithm which guarantees ε-convergence in a finite number of iterations and is described later as algorithm 2.

The relaxed master problem can be stated as follows:

$$\min_{y, y_0} y_0 \tag{M}$$

subject to,

$$f_i(y) - y_o \leq 0, \qquad i = 1, 2, \ldots, K_f$$
$$g_j(y_0) \leq 0, \qquad j = 1, 2, \ldots, K_i$$

where $y \in R^{n_y}$ and $y_0 \in R$, and $f_i(y)$, $g_j(y)$ are concave real valued functions of the complicating variables.

To apply Ueing's algorithm it is essential that the objective function be strictly concave. This requirement can be satisfied by a slightly perturbed objective that results in the following modified master:

$$\min_{y, y_0} y_0 - \alpha\left[\sum_{m=1}^{n_y} y_m^2 + y_0^2\right] \tag{M1}$$

subject to,

$$f_i(y) - y_0 \leq 0, \qquad i = 1, 2, \ldots, K_f$$
$$g_j(y) \leq 0, \qquad j = 1, 2, \ldots, K_i$$

where α is an arbitrarily small constant. Then at every local minimum (y, y_0) of this modified problem at least $n_y + 1$ constraints are active. Furthermore each local minimum can be identified as the solution of a concave maximization problem that has the same objective as the modified master and involves only $(n_y + 1)$ of the $(p + r)$ constraints of the modified master with reversed sign (Ueing, 1972):

$$\max_{y, y_0} y_0 - \alpha\left[\sum_{m=1}^{n_y} y_m^2 + y_0^2\right] \tag{M2}$$

subject, to

$$-f_i(y) + y_0 \leq 0, \qquad i = 1, 2, \ldots, p_1$$
$$-g_j(y) \leq 0, \qquad j = 1, 2, \ldots, r_1$$

where: $p_1 + r_1 = n_y + 1$. This concave maximization over a convex set (note that all the constraints are convex) has naturally a unique global maximum. If, for that maximum, all the constraints of (M2) are active and all the constraints of (M1) are satisfied then this maximum is also a local minimum of the modified master (M1). Using this procedure one can determine all the local minima of the modified master, and in the limit ($\alpha \to 0$) of the master itself. Since there is only a finite number of local minima the global minimum can be recovered in a finite number of steps.

Algorithm 2

As stated earlier (P4) is a nonconvex optimization problem with a single reverse convex constraint, $\phi(y, t)$, that is separable:

$$\phi(y, t) = \sum_{i=1}^{n_y} \phi_i(y_i) + \sum_{i=1}^{n_t} t_i ,$$

where $y \in R^{n_y}$, $t \in R^{n_t}$ and $\phi_i(.)$ is a concave function in one variable. Furthermore, the master problem in the GBDA implementation is of the same type, that is it has linear objective and concave, separable constraints. For this type of problems Soland (1971) proposed an algorithm that can identify the globally ε-optimal solution in a finite number of steps.

The algorithm assumes the existence of a "rectangular" region C where the y variables lie: $C = \{y \in R^{n_y}: l \le y \le L\}$, with l and L being vectors of upper and lower bounds. Through solution of a series of convex programming problems, the algorithm generates a sequence of lower bounds to the global optimum of (P4). Each of the intermediate convex problems, $(P4_k)$, is obtained from (P4) through substitution of $\phi_i(.)$ by its convex envelope for all $i = 1, \ldots, n_y$ over a rectangular subset of C (Soland, 1971). To obtain $(P4_{k+1})$ from $(P4_k)$ a branch and bound technique is used: first C is refined into smaller rectangles, and then the objective is minimized over the intersection of each rectangle with the feasible set and a lower bound on the objective is determined. The sequence of lower bounds produced by this procedure is guaranteed to ε-converge to the global optimum of (P4) in a finite number of iterations.

Algorithm 3

Tuy (1987) proposed an algorithm for the solution of convex problems with an additional reverse convex constraint, that can be applied to solve (P4).

Let C be the convex set defined by the $k - 1$ convex constraints of (P4). Then (P4) can be restated as:

$$\min_{x \in C} x^T A_0 x + b_0^T x + c_0 \tag{P4}$$

subject to,

$$\phi(x) = x^T A_k x + b_k^T x + c_k \le 0$$

Let v be the global minimum to this problem. It has been established that the value of the following optimization problem is zero:

$$\max_{x \in C} x^T(-A_k)x - b_k^T x - c_k$$

subject to,

$$x^T A_0 x + b_0^T x + c_0 - v \leq 0$$

The global solution to this convex maximization problem can be obtained by available algorithms (Hoffman, 1981, Horst, 1976). The complete algorithm for the solution of (P4) is comprised of the following steps:

- Solve (P4) without the reverse convex constraint and let w be the resulting global optimum. It is assumed that this optimum is finite, but this is rather a technicality than a restrictive assumption. If w satisfies the reverse convex constraint then it is the global optimum for (P4).
- If w is such that: $\phi(w) > 0$ and $w^T A_0 w + b_0^T w + c_0 < v$ identify a point x_i that belongs on the boundary of the set $G = \{x \in R^n\colon \ \phi(x) > 0\}$. Then solve the following convex maximization subproblem:

  $$\max_{x \in C} x^T(-A_k)x - b_k^T x - c_k$$

  subject to,

  $$x^T A_0 x + b_0^T x + c_0 - x_i^T A_0 x_i - b_0^T x_i - c_0 \leq 0$$

  Let z_i be the global solution to this problem. If $\phi(z_i) = 0$ then the algorithm terminates. Otherwise a one dimensional search that identifies a new point $x_i + 1$ belonging to the boundary of G is performed, and the same procedure is repeated.

The described algorithm provides a sequence of points $z_{i_{i=1,\ldots,\infty}}$. This sequence converges to the solution of (P4), thus resulting in a globally ε-optimal solution in a finite number of steps. Within the same conceptual framework, there are several improvements that can help increase the speed of convergence of this algorithm (Tuy, 1987).

IV. EXAMPLES

1. *Polynomially Constrained Polynomial Programming Problem*

Consider the following nonconvex optimization problem:

$$\min_{x_1, x_2} x_1^4 - 14x_1^2 + 24x_1 - x_2^2$$

subject to,

$$-x_1 + x_2 - 8 \leq 0$$

$$x_1 - 10 \leq 0$$

$$-x_2 \leq 0$$

$$x_2 - x_1^2 - 2x_1 + 2 \leq 0$$

This optimization problem has several local minima. The following table contains the values of the variables and the corresponding value of the objective at each local minimum.

x_1	x_2	Objective
0.84025	0.3865	10.631
0.7320	0.0000	10.354
2.7016	10.7016	−98.600
−3.1736	1.7245	−118.705

The problem was solved by both the first and the second algorithm.

Algorithm 1 (Benders Decomposition)
Employing the transformation $x_3 = x_1^2$ and $x_4 = x_2^2$ and introducing the variables $y_1 = x_1$ and $y_2 = x_2$ the original optimization problem is being transformed to the following:

$$\min_{\substack{x_1, x_2, x_3 \\ x_4, y_1, y_2}} x_3^2 - 14x_3 + 24y_1 - x_4$$

subject to,

$$-y_1 + y_2 - 8 \le 0$$
$$y_1 - 10 \le 0$$
$$-y_2 \le 0$$
$$x_2 - x_3 - 2y_1 + 2 \le 0$$
$$x_1^2 - x_3 \le 0$$
$$x_2^2 - x_4 \le 0$$
$$y_1 - x_1 = 0$$
$$y_2 - x_2 = 0$$
$$-y_1^2 + x_3 + y_2^2 + x_4 \le 0$$

The complicating variables for this problem are y_1, and y_2. Then the primal subproblem becomes:

$$\min_{\substack{x_1, x_2 \\ x_3, x_4}} x_3^2 - 14x_3 + 24\bar{y}_1 + x_4 \qquad \text{(Primal)}$$

subject to,

$$x_2 - x_3 - 2\bar{y}_1 + 2 \le 0$$
$$x_1^2 - x_3 \le 0$$
$$x_2^2 - x_4 \le 0$$
$$\bar{y}_1 - x_1 = 0$$
$$\bar{y}_2 - x_2 = 0$$
$$-\bar{y}_1^2 + x_3 - \bar{y}_2^2 + x_4 \le 0$$

The primal subproblem is an evaluation step and a check for the feasbility of the vector $[\bar{y}_1\ \bar{y}_2]$. The master subproblem has the following form:

$$\min_{y_0, y_1, y_2} y_0 \qquad \text{(Master)}$$

subject to,

$$-y_1 + y_2 - 8 \le 0$$
$$y_1 - 10 \le 0$$
$$-y_2 \le 0$$
$$(x_{3,i}^2 - 14x_{3,i} + 24y_1 - x_{4,i}) + \lambda_{1,i}(x_{2,i} - x_{3,i} - 2y_1 + 2) + \lambda_{2,i}(x_{1,i}^2 - x_{3,i}) + \lambda_{3,i}(x_{2,i}^2 - x_{4,i}) + \lambda_{4,i}(y_1 - x_{1,i}) + \lambda_{5,i}(y_2 - x_{2,i}) + \lambda_{6,i}(-y_1^2 + x_{3,i} - y_2^2 + x_{4,i}) \le y_0, \qquad i = 1, \ldots, K_f$$
$$\mu_{1,j}(x_{2,j} - x_{3,j} - 2y_1 + 2) + \mu_{2,j}(x_{1,j}^2 - x_{3,j}) + \mu_{3,j}(x_{2,j}^2 - x_{4,j}) + \mu_{4,j}(y_1 - x_{1,j}) + \mu_{5,j}(y_2 - x_{2,j}) + \mu_{6,j}(-y_1^2 + x_{3,j} - y_2^2 + x_{4,j}) \le 0, \qquad j = 1, \ldots, K_i$$

As expected, the master subproblem is a separable quadratically constrained RCP problem. The solution of an RCP problem can be obtained by several methods. In the following, a branch and bound method (Algorithm 2) is being used.

The point $(\bar{y}_1, \bar{y}_2) = (-8, 0)$ was chosen as the initial point for the Benders iterations. For $\varepsilon = 0.001$ the global optimum was identified at $(x_1, x_2) = (-3.1749, 1.7301)$ with objective value -118.706 in 43 Benders iterations. MINOS was used to solve the primal subproblem and the subproblems that were generated by the branch and bound procedure. On the average, the solution of each master required about 30 branch and bound iterations.

Algorithm 2 (Branch and Bound)
Employing the transformation $x_3 = x_1^2$ and $x_4 = x_2^2$ the original optimization problem is transformed to the following:

$$\min_{\substack{x_1, x_2 \\ x_3, x_4}} x_3^2 - 14x_3 + 24x_1 - x_4$$

subject to,

$$-x_1 + x_2 - 8 \le 0$$
$$x_1 - 10 \le 0$$
$$-x_2 \le 0$$
$$x_2 - x_3 - 2x_1 + 2 \le 0$$
$$x_1^2 - x_3 \le 0$$
$$x_2^2 - x_4 \le 0$$
$$-x_1^2 + x_3 - x_2^2 + x_4 \le 0$$

In this form, the problem has become a convex quadratically constrained quadratic programming problem with a reverse convex quadratic and separable constraint and therefore algorithm 2 can be employed. The optimization package MINOS was used for the solution of the intermediate convex subproblems. For $\varepsilon = 0.001$ the global optimum was identified as $(x_1, x_2) = (-3.173, 1.721)$ and the corresponding objective value was -118.705. The execution time for this particular problem was approximately 5.2 cpu seconds on a IBM-4381 computer. The solution required 35 branch and bound iterations.

2. *Indefinite Quadratic Programming Problem*

Consider the following indefinite quadratic optimization problem (V. Visweswaran and C.A. Floudas, 1990):

$$\min_{\mathbf{x},\mathbf{y}} \Phi_1(\mathbf{x}) + \Phi_2(\mathbf{y})$$

subject to,

$$A_1\mathbf{x} + A_2\mathbf{y} \le b$$

$$x_i \ge 0, \qquad i = 1, 2, \ldots, 10,$$

$$y_i \ge 0, \qquad i = 11, 12, \ldots, 20.$$

where

$$\Phi_1(\mathbf{x}) = \frac{-1}{2} \sum_{i=1}^{10} C_i(x_i - \bar{x}_i)^2$$

$$\Phi_2(\mathbf{y}) = \frac{1}{2} \sum_{i=11}^{20} C_i(y_i - \bar{y}_i)^2.$$

The data for this problem are:

$$C = (63, 15, 44, 91, 45, 50, 89, 58, 86, 82, 42, 98, 48, 91, 11, 63, 61, 61, 38, 26)$$

$$\bar{\mathbf{x}} = (-19, -27, -23, -53, -42, 26, -33, -23, 41, 19)$$

$$\bar{\mathbf{y}} = (-52, -3, 81, 30, -85, 68, 27, -81, 97, -73)$$

$$A_1 = \begin{bmatrix} 3 & 5 & 5 & 6 & 4 & 4 & 5 & 6 & 4 & 4 \\ 5 & 4 & 5 & 4 & 1 & 4 & 4 & 2 & 5 & 2 \\ 1 & 5 & 2 & 4 & 7 & 3 & 1 & 5 & 7 & 6 \\ 3 & 2 & 6 & 3 & 2 & 1 & 6 & 1 & 7 & 3 \\ 6 & 6 & 6 & 4 & 5 & 2 & 2 & 4 & 3 & 2 \\ 5 & 5 & 2 & 1 & 3 & 5 & 5 & 7 & 4 & 3 \\ 3 & 6 & 6 & 3 & 1 & 6 & 1 & 6 & 7 & 1 \\ 1 & 2 & 1 & 7 & 8 & 7 & 6 & 5 & 8 & 7 \\ 8 & 5 & 2 & 5 & 3 & 8 & 1 & 3 & 3 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 8 & 2 & 4 & 1 & 1 & 1 & 2 & 1 & 7 & 3 \\ 3 & 6 & 1 & 7 & 7 & 5 & 8 & 7 & 2 & 1 \\ 1 & 7 & 2 & 4 & 7 & 5 & 3 & 4 & 1 & 2 \\ 7 & 7 & 8 & 2 & 3 & 4 & 5 & 8 & 1 & 2 \\ 7 & 5 & 3 & 6 & 7 & 5 & 8 & 4 & 6 & 3 \\ 4 & 1 & 7 & 3 & 8 & 3 & 1 & 6 & 2 & 8 \\ 4 & 3 & 1 & 4 & 3 & 6 & 4 & 6 & 5 & 4 \\ 2 & 3 & 5 & 5 & 4 & 5 & 4 & 2 & 2 & 8 \\ 4 & 5 & 5 & 6 & 1 & 7 & 1 & 2 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$b = (380, 415, 385, 405, 470, 415, 400, 460, 400, 200)$$

In this optimization problem the function $\Phi_1(\mathbf{x})$ is concave while the function $\Phi_2(\mathbf{y})$ is convex. Employing the transformation $z_i = x_i^2$, $i = 1, 2, \ldots, 10$ the following reverse convex programming problem is obtained:

$$\min_{\mathbf{z},\mathbf{x},\mathbf{y}}\left\{\frac{-1}{2}\sum_{i=1}^{10} C_i(z_i - 2x_i\bar{x}_i + \bar{x}_i^2) + \frac{1}{2}\sum_{i=11}^{20} C_i(y_i - \bar{y}_i)^2\right\}$$

subject to,

$$A_1\mathbf{x} + A_2\mathbf{y} \le b$$

$$x_i^2 - z_i \le 0, \qquad i = 1, 2, \ldots, 10$$

$$\sum_{i=1}^{10}(z_i - x_i^2) \le 0$$

$$x_i \ge 0, \qquad i = 1, 2, \ldots, 10,$$

$$y_i \ge 0, \qquad i = 11, 12, \ldots, 20.$$

As in the previous example, the original nonconvex optimization problem has been transformed into an optimization problem with objective function that is quadratic and convex, and constraints that are also quadratic and convex except one that is quadratic, reverse convex and separable. For $\varepsilon = 0.001$ the ε-global optimum was identified at:

$$\mathbf{x}_{opt} = (0, 0, 0, 62.609, 0, 0, 0, 0, 0, 0)$$

$$\mathbf{y}_{opt} = (0, 0, 0, 0, 0, 4.348, 0, 0, 0, 0),$$

the objective value was 49318.078 and its determination required 4 branch and bound iterations. The execution time for this problem was 9.7 cpu seconds on an IBM-4381 computer.

3. Reactor Sequence Design with Capital Cost Constraints

Consider the reaction sequence $A \rightarrow B \rightarrow C$. Assuming first order kinetics for both reactions, design a sequence of two reactors such that the concentration of B in the exit stream of the second reactor (c_{b2}) is maximized and the investment cost does not exceed a given upper bound.

The values of the reaction constants for the first and the second reaction are given in the following table:

	Reactor 1	Reactor 2
k_a	$9.6540\ 10^{-2}\ s^{-1}$	$9.7515\ 10^{-2}\ s^{-1}$
k_b	$3.5272\ 10^{-2}\ s^{-1}$	$3.9191\ 10^{-2}\ s^{-1}$

The inlet concentration for B and C is zero. The inlet concentration for A is $c_{a0} = 1.0$ mol/l.

Problem Formulation:

Let V_1, V_2 be the residence times for the first and the second reactor respectively. Let k_{a1}, k_{a2}, k_{b1} and k_{b2} be the rate constants for the first and second reaction in the first and the second reactor respectively. Then the reactor design problem is formulated as a Nonlinear Programming Problem:

$$\max c_{b2}$$

subject to,

$$(c_{a1} - c_{a0}) + k_{a1}c_{a1}V_1 = 0$$

$$(c_{a2} - c_{a1}) + k_{a2}c_{a2}V_2 = 0$$

$$(c_{b1} - c_{a1} - c_{a0}) + k_{b1}c_{b1}V_1 = 0$$

$$(c_{b2} - c_{b1} - c_{a2} + c_{a1}) + k_{b2}c_{b2}V_2 = 0$$

Assuming that the capital cost of a reactor is proportional to the square root of its residence time, the capital cost constraint can be written as:

$$V_1^{0.5} + V_2^{0.5} \leq 4$$

Employing the transformation $z_1^2 = V_1$ and $z_2^2 = V_2$, the capital cost constraint is replaced by the following set of constraints:

$$z_1 + z_2 \leq 4$$

$$z_1^2 - V_1 = 0$$

$$z_2^2 - V_2 = 0$$

The resulting optimization problem belongs to the class (P2).

The problem has 2 local minima with objective values $c_{b2} = 0.38810$ mol/lt and $c_{b2} = 0.3746$ mol/lt respectively.

Using algorithm 2, the global optimum is identified as $c_{b2} = 0.38810$ mol/lt. The total number of branch and bound iterations for this problem was 7950, and the execution time on an Apollo DN10000 was 7950 cpu seconds.

V. CONCLUSIONS

In this paper, it has been demonstrated that any *rationally constrained rational programming* problem can be exactly transformed into a convex, quadratically constrained, quadratic programming problem with an additional separable, quadratic, reverse convex constraint. One can generate the latter through variable transformations and introduction of new variables. The single reverse convex constraint has the additional feature of being separable, something that broadens the class of optimization algorithms that can be used for the solution of this problem.

A novel implementation of the GBDA which benefits from this problem equivalence has been shown to guarantee solution to global optimality for RCRP problems. Based on this result, the global solution of an RCRP problem has been translated to the global solution of a series of quadratically constrained and separable RCP problems. This in turn suggests that new, more efficient algorithmic approaches for the global solution of RCP problems will have an immediate positive impact on the global solution of RCRP problems.

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