A GLOBAL THEORY OF STEADY VORTEX RINGS IN AN IDEAL FLUID

BY

L. E. FRAENKEL and M. S. BERGER

Dept. of Appl. Mathematics,
University of Cambridge,
Cambridge, England

Institute for Advanced Study,
Princeton, New Jersey, USA (1)

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1. Introduction

(i) In this paper we give an affirmative answer 'in the large' to the question: Do the equations governing the motion of an ideal fluid (that is, of an inviscid fluid with uniform density) admit solutions representing steady vortex rings? By a steady vortex ring we mean a figure of revolution $\mathcal{J}$ that is expected to be homeomorphic to a solid torus in most cases, and is associated with a continuous, axi-symmetric, solenoidal vector field $\mathbf{q}$ (the fluid velocity) defined, in the case of an unbounded fluid, on the real three-dimensional Euclidean space $\mathbf{R}^3$ and having the following properties when we take axes fixed in the ring $\mathcal{J}$. (a) Both $\mathcal{J}$ and $\mathbf{q}$ do not vary with time; (b) the vorticity $\boldsymbol{\omega} \equiv \operatorname{curl} \mathbf{q}$ has positive magnitude in $\mathcal{J}$, vanishes in $\mathbf{R}^3 - \mathcal{J}$, and satisfies a non-linear equation of motion which, among other things, determines the boundary of $\mathcal{J}$; (c) $\mathbf{q}$ tends to a constant value at infinity in $\mathbf{R}^3$.

One can also adopt the following viewpoint, which is perhaps closer to the vortex rings that occur in the physical world (for example: smoke rings, 'thermals' in meteorology, and, regrettably, the heads of the 'mushrooms' created by big explosions). If the constant velocity mentioned in (c) is vertically downwards (say), an observer fixed with respect to the fluid at infinity sees a vortex ring that is moving upwards at constant velocity, without change of size or shape, through fluid that is at rest at infinity.

(ii) The mathematical problem can be formulated in terms of the Stokes stream function Ψ (which is closely related to the vector potential of the velocity field) as a problem for a semi-linear elliptic partial differential equation ((2.1 a) below) which takes different forms inside and outside the unknown boundary $\partial \mathcal{J}$ of the vortex ring. Thus one has a non-linear free-boundary problem, and previous treatments (outlined in section 2.2) have been of a *local* character in that only solutions near two particular limiting cases have been established. The present *global* theory results from the following observation, which may also find application in other free-boundary problems.

If one considers (initially) only a restricted form of the non-linear term $f(\Psi)$ in the differential equation, and takes a finite cylinder V, in place of $\mathbf{R}^3$, as the domain of the velocity field $\mathbf{q}$ and the stream function Ψ, then *the problem can be reduced to a semi-linear Dirichlet problem ((2.6) below) that is untroubled by the free boundary and can be solved by modern techniques of the calculus of variations* [28], [2], [3]. The variational principle is necessarily of the 'constrained' or 'isoperimetric' type; for, although the known solutions are also unconstrained critical points (stationary points) of a certain functional on the Hilbert space appropriate to the problem, they do not maximize or minimize that functional over the whole Hilbert space.

The isoperimetric variational characterization, Steiner symmetrization [23] and the generalized maximum principle of Littman [17] together imply properties of the solution,

and lead to estimates, that allow one to *remove the additional restrictions that first made the problem tractable*. Thus limiting procedures yield a solution of the more general problem in which (a) the non-linear term $f(\Psi)$ can be discontinuous, corresponding to a jump in vorticity at the boundary $\partial \mathcal{J}$ of the vortex ring, and (b) the flow field (the domain of $\mathfrak{q}$) can be the whole of $\mathbf{R}^3$.

In connection with (b), it is shown that the *vortex ring lies within a bounded set that can be estimated from the data of the problem*. It also turns out that the *variational principle survives the transition to the more general situation*, despite the loss, in the case of the unbounded domain, of the usual compactness theorems characterizing isoperimetric variational methods in Hilbert space.

(iii) The background of the problem is as follows. In his celebrated paper of 1858 on the general equations of vortex motion, Helmholtz [11] considered rings of very small cross-section as one of two examples, and inferred on the basis of plausible approximations that such rings move (through fluid at rest at infinity) with 'approximately constant and very large velocity'. The subject was taken up in a series of papers by Kelvin [27]. Taking for granted the existence of certain steady rings of small cross-section, Kelvin calculated an explicit approximation to their propagation speed; stated a variational principle(1); conjectured the existence of steady non-axi-symmetric and knotted vortex configurations; and proposed a theory of atomic structure based on vortex rings. Hicks [12] and Dyson [5], also assuming the existence of steady vortex rings of small cross-section and constant ω/r, (where ω is the magnitude of the vorticity and r is distance from the axis of symmetry), computed the propagation speed more systematically than Kelvin had; in particular, Dyson calculated expansions for the shape and speed up to the fourth order in a small parameter. At the other extreme of steady rings of large cross-section, Hill [13] discovered in 1894 an explicit exact solution with constant ω/r for which the 'ring' is actually a ball in $\mathbf{R}^3$.

Kelvin's discussion of knotted configurations led Tait to write pioneering papers ([26], pp. 273–347) on topological knot theory; and the theory of vortex atoms has perhaps a less short-lived successor in modern discussions of superfluidity (for example, by Feynman [6]); indeed, recent results [8] for steady rings in an ideal fluid have already been applied in this field [4].

The question of a mathematical existence proof was first taken up by Lichtenstein [16], again for the case of small cross-section and constant ω/r. This work contains a constructive mathematical technique, but serious errors in hydrodynamics and arithmetic;

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(1) Kelvin's variational principle for steady vortex flows ([27], pp. 116 and 172) is not restricted to axi-symmetric flows, but it is merely stated in physical terms and its analytical basis is obscure.

it has been corrected and extended to general vorticity functions f in [18] and [7]. Recently, Norbury [21] has given a constructive existence proof for steady rings with constant ω/r that are close to Hill's vortex but are homeomorphic to a solid torus; and the existence of further steady vortex rings is suggested by numerical 'solutions' [22] for the one-parameter family of steady rings with constant ω/r, extending from those of small cross-section to Hill's vortex.

(iv) Some virtues of our method have been noted in (ii) above; it must be added that the following questions are left open.

(a) The nature of the functional which our solution maximizes over an energy surface in Hilbert space suggests that the corresponding vortex ring $\mathcal{J}$ should be homeomorphic either to a solid torus (when a certain parameter $k > 0$) or to a ball (when $k = 0$). However, this conjecture is proved (in Theorem 3G) only for convex and exceptionally smooth vorticity functions f.

(b) The paper contains no uniqueness result of any kind.

(c) While Hill's vortex and the steady rings of small cross-section described in section 2.2 are expected to be particular cases of the solution established here (that is, to have the maximizing property just mentioned), such a result has not been proved. However, we have in hand a number of partial results in this direction, and hope to return to the matter in a later paper.

2. Preliminaries
2.1. The governing equations

Let $X = [X_1, X_2, X_3] = [r \cos \theta, r \sin \theta, z]$ denote a point in the three-dimensional real Euclidean space $\mathbf{R}^3$, so that r, θ, z are cylindrical coordinates; we shall use *square* brackets for the components of a vector field (such as the fluid velocity) in the directions X_j increasing, and *round* brackets for its components in the directions r, θ, z increasing. Consider the axi-symmetric flow of an inviscid fluid, of uniform density ϱ, with velocity $\mathbf{q}$ and vorticity $\boldsymbol{\omega} \equiv \operatorname{curl} \mathbf{q}$. The equation of mass conservation, $\operatorname{div} \mathbf{q} = 0$, allows us to introduce a vector potential $\boldsymbol{\Phi} = (0, \Psi/r, 0)$, where $\Psi = \Psi(r, z)$ is the Stokes stream function, such that

$$\mathbf{q} = \operatorname{curl} \boldsymbol{\Phi} = (-\Psi_z/r, 0, \Psi_r/r), \quad \operatorname{div} \boldsymbol{\Phi} = 0,$$

$$\boldsymbol{\omega} = -\Delta \boldsymbol{\Phi} = (0, -(L\Psi)/r, 0),$$

where
$$\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial X_j^2} \quad \text{and} \quad L = r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2}.$$

The function Ψ is constant on each stream surface because $\mathbf{q} \cdot \operatorname{grad} \Psi = 0$; also, $2\pi(\Psi_2 - \Psi_1)$ is the volumetric flow rate between two stream surfaces $\Psi(r, z) = \text{const.} = \Psi_j$, $(j = 1, 2)$.

To write the partial differential equation of the problem we let $\boldsymbol{\omega} = (0, \omega, 0)$ and define the half-plane $\Pi = \{(r, z) \,|\, r > 0\}$. The dynamical equations of steady, axi-symmetric inviscid flow demand that ω/r be constant on each stream surface; accordingly, we set

$$\omega = \lambda r f(\Psi),$$

where f is a prescribed (non-dimensional) function that we call the *vorticity function*, and shall specify more precisely later; the positive constant λ is a *vortex-strength parameter*, which we also regard as prescribed for the moment.

To formulate the problem of finding a steady vortex ring in $\mathbf{R}^3$, with cross-section $A \subset \Pi$ in a meridional plane ($\theta = \text{const.}$), we take axes fixed with respect to the ring and, equating the two forms of ω above, demand that

$$L\Psi \equiv r \left(\frac{1}{r} \Psi_r \right)_r + \Psi_{zz} = \begin{cases} - \lambda r^2 f(\Psi) & \text{in } A, \\ 0 & \text{in } \Pi - \bar{A}, \end{cases} \tag{2.1a}$$

where $\operatorname{grad} \Psi$ is to be continuous on the (unknown) cross-sectional boundary ∂A of the ring. In addition, ∂A and the axis of symmetry $r = 0$ must be streamlines; we set

$$\Psi|_{\partial A} = 0 \quad \text{and} \quad \Psi|_{r=0} = -k \leqslant 0, \tag{2.1b, c}$$

where k is a prescribed *flux constant* such that $2\pi k$ is the flow rate between the axis of symmetry and the boundary of the vortex ring. Finally, we demand that the ring move relative to the fluid at infinity with velocity $(0, 0, W)$, where $W > 0$ is another prescribed constant; with axes fixed in the ring, this implies the condition

$$\Psi_z/r \to 0 \quad \text{and} \quad \Psi_r/r \to -W \quad \text{as } r^2 + z^2 \to \infty. \tag{2.1d}$$

An equivalent formulation is as follows: since the differential operator L has the fundamental solution

$$P(r_0, z_0, r, z) = \frac{r_0 r}{4\pi} \int_{-\pi}^{\pi} \frac{\cos \theta \, d\theta}{\{r^2 + r_0^2 - 2 r r_0 \cos \theta + (z - z_0)^2\}^{\frac{1}{2}}},$$

we seek ∂A and Ψ such that

$$\left. \begin{aligned} \Psi(r_0, z_0) &= -\tfrac{1}{2} W r_0^2 - k + \lambda \iint_A P(r_0, z_0, r, z) \, f(\Psi) \, r \, dr \, dz, \\ \Psi|_{\partial A} &= 0. \end{aligned} \right\} \tag{2.2}$$

(The function $-\tfrac{1}{2} W r^2 - k$ is the stream function of a uniform flow with velocity $(0, 0, -W)$, and $L(\tfrac{1}{2} W r^2 + k) = 0$.)

2.2 Known solutions: Hill's vortex and rings of small cross-section

Hill's solution [13] of the problem (2.1) for the case $f(\Psi) \equiv 1$, $k = 0$ proceeds as follows. Let R, Θ be spherical coordinates such that $r = R \sin \Theta$ and $z = R \cos \Theta$. If one guesses that a sphere $R = a$ can serve as the boundary of a steady vortex 'ring', and (taking $f = 1$ and $k = 0$) solves (2.1a) to (2.1d) separately for $R \leqslant a$ and for $R \geqslant a$, one obtains

$$\Psi = \begin{cases} \frac{1}{10} \lambda R^2 \sin^2 \Theta \, (a^2 - R^2), & R \leqslant a, \\[2mm] -\frac{1}{2} W R^2 \sin^2 \Theta \left(1 - \dfrac{a^3}{R^3}\right), & R \geqslant a, \end{cases}$$

where the exterior solution represents the classical irrotational flow past a sphere. The choice $\lambda a^2 / W = 15/2$ then makes $\partial \Psi / \partial R$ continuous on $R = a$, and all the conditions of the problem are satisfied. Moreover, neighbouring solutions that are genuine rings are now known to exist [21] for $f(\Psi) \equiv 1$ and sufficiently small values of k.

A ring of small cross-section may be defined, in terms of a small positive parameter ε, as one whose cross-section A contains a point $r = l$, $z = 0$ at which grad $\Psi = 0$, and has area $\pi \varepsilon^2 l^2 + o(\varepsilon^2)$. One prescribes l and ε in place of W and k, (the latter are to be determined as part of the solution), and makes the vortex-strength parameter λ proportional to $1/\varepsilon^2$. If coordinates s, t are defined by

$$r - l = \varepsilon l s \cos t, \quad z = \varepsilon l s \sin t,$$

the fundamental solution of L becomes, for bounded values of s and s_0,

$$P(r_0, z_0, r, z) = \frac{l}{2\pi} \left\{ \log \frac{8}{\varepsilon |\mathbf{s} - \mathbf{s}_0|} - 2 + O(\varepsilon \log \varepsilon) \right\},$$

where $$|\mathbf{s} - \mathbf{s}_0|^2 = s^2 + s_0^2 - 2 s s_0 \cos (t - t_0).$$

It is essentially this formula which Helmholtz [11] combined with [2.2] to infer a large and nearly constant propagation speed W; Kelvin [27], Hicks [12] and Dyson [5] in effect added the condition $f = $const., and the assumption that ∂A is approximately circular, to estimate this speed.

In the rigorous theory [18], [7] of steady rings of small cross-section, one sets $\Psi = 0$ at $s = 0$ and leaves the constant $\Psi|_{\partial A}$ free; f is taken to be positive and continuously differentiable on $(-\infty, 0]$, with a uniformly bounded derivative. (The positivity condition can be weakened slightly.) It is sufficient to solve (2.2) on $\bar{A}$, for (2.2) itself then defines Ψ

elsewhere, and a formal first approximation $\Psi_0(s)$ is obtained by solving the ordinary differential equation

$$\Psi_{ss} + \frac{1}{s}\Psi_s = \begin{cases} -\lambda_0 f(\Psi), & s < 1, \\ 0, & s > 1 \end{cases}$$

which then represents the dominant terms of (2.1a) and in which $\lambda_0 = \varepsilon^2 \lambda l^4$ is independent of ε. This function $\Psi_0(s)$ is made the first of a sequence of successive approximations that converge on $\bar{A}$ for sufficiently small values of ε.

It turns out that the *flux constant k is large in this case.* More precisely, if $\pi\varrho\eta$ is the kinetic energy of the vortex motion, as in (2.6c) below, then the non-dimensional ratio $kW^{1/3}\eta^{-2/3} \sim \frac{3}{8}\{(1/\pi)\log(8/\varepsilon)\}^{2/3}$ to the lowest order.

2.3. Reformulation of the problem

The difficulties of the global problem (2.1) are (a) the unbounded domain Π, associated with a lack of compactness theorems, and (b) the free boundary ∂A, at which the nonlinear term in (2.1a) is to have a simple discontinuity in cases of interest.

To overcome (a), we consider in the first instance flow in a cylinder V of radius a and length $2b$, represented in a meridional plane by the domain D; here

$$V = \{X \,|\, X_1^2 + X_2^2 < a^2, \quad |X_3| < b\}$$

and

$$D = \{(r, z) \,|\, 0 < r < a, \quad |z| < b\}.$$

Then Π is replaced by D in (2.1a), and the conditions (2.1c,d) are replaced by

$$\Psi|_{\partial D} = -\tfrac{1}{2}Wr^2|_{\partial D} - k. \tag{2.3}$$

(Thus the axial velocity component is $-W$ at the 'entry and exit planes' $r < a$, $z = \pm b$, and the radial velocity component is zero at the cylinder wall $r = a$, $|z| \leqslant b$.) The resulting problem, illustrated in Figure 1, p. 29, is physically reasonable and of interest in its own right. Ultimately, we shall pass from D to Π by a limiting procedure.

To overcome the difficulty (b), we first note an implication of the maximum principle [24] and conditions (2.1b), (2.3): if $f(\Psi) > 0$ in A, then $\Psi > 0$ in A, and $\Psi < 0$ in $D - \bar{A}$. Accordingly, we consider vorticity functions f defined on the whole real line, such that $f(t) = 0$ for $t \leqslant 0$ and $f(t) > 0$ for $t > 0$, and define

$$A = \{(r, z) \,|\, \Psi(r, z) > 0\}.$$

In the first instance we also take f to be Hölder continuous on $\mathbf{R}$; then the equation

$$L\Psi = -\lambda r^2 f(\Psi) \quad \text{in } D \tag{2.4}$$

is equivalent to (2.1a), and the free boundary ∂A is no longer a source of difficulty for the Dirichlet problem (2.4), (2.3). Ultimately, we shall pass by a limiting procedure to the case when $f(t)$ has a simple discontinuity at $t=0$.

It is convenient to decompose the total stream function Ψ by writing

$$\Psi(r, z) = \psi(r, z) - \tfrac{1}{2}Wr^2 - k, \tag{2.5}$$

where ψ is to be the stream function of the velocity field induced by the vortex ring, and $-\tfrac{1}{2}Wr^2 - k$ represents a uniform stream (as was noted after (2.2)). We add the further restriction that f be non-decreasing, which is believed to include the cases of primary interest. We also prescribe the kinetic energy of the vortex motion, instead of prescribing λ; this is mathematically convenient and physically at least as natural. In terms of ψ, the first form of our problem is then as follows.

Given the stream velocity $W>0$, the flux constant $k \geqslant 0$, the kinetic energy $\pi\varrho\eta$ of the vortex motion (where ϱ is the density of the fluid) and a vorticity function f as in (2.7) below, we seek a vortex stream function ψ such that

$$L\psi \equiv r\left(\frac{1}{r}\psi_r\right)_r + \psi_{zz} = -\lambda r^2 f(\Psi) \quad \text{in } D, \tag{2.6a}$$

$$\psi|_{\partial D} = 0, \tag{2.6b}$$

where Ψ and ψ are related by (2.5), and the vortex-strength parameter λ is to be such that

$$\iint_D \frac{1}{r^2}(\psi_r^2 + \psi_z^2)\, r\, dr\, dz = \eta > 0, \tag{2.6c}$$

(η being prescribed). Note that this last condition excludes the trivial solution $\psi = 0$ of (2.6 a, b).

The vorticity function $f: \mathbf{R} \to [0, \infty)$ is assumed to be non-decreasing and (in the first instance) locally Hölder continuous; more precisely,

$$f(t) = 0 \quad \text{for } t \leqslant 0, \quad f(t) > 0 \quad \text{for } t > 0, \tag{2.7a, b}$$

and $\qquad 0 \leqslant f(t) - f(s) \leqslant \tfrac{1}{2}M^\mu\{1 + (Mt)^{m-1}\}(t-s)^\mu \quad \text{for } t \geqslant s \geqslant 0, \tag{2.7 c}$

where $M>0$, $m \geqslant 1$ and $\mu \in (0, 1)$ are give constants, and M has the dimensions of $1/\Psi$, that is, of $W^{1/3}\eta^{-2/3}$. Note also that doubling f merely halves λ in (2.6a).

Setting $s=0$ in (2.7c) and considering the cases $Mt \leqslant 1$ and $Mt > 1$ separately, one easily obtains the crude estimate

$$0 < f(t) \leqslant 1 + (Mt)^m \quad \text{for } t > 0. \tag{2.8}$$

The three-dimensional viewpoint. In connection with regularity properties of a solution, and for applications of the generalized maximum principle, it will be convenient to consider, in place of ψ, the vector potential $\boldsymbol{\alpha}$ of the velocity induced by the vortex ring; this is defined by

$$\boldsymbol{\alpha}(X) = \left(0, \frac{\psi}{r}, 0\right) = \left[-\frac{\psi}{r}\sin\theta, \frac{\psi}{r}\cos\theta, 0\right] = [\alpha_1, \alpha_2, 0], \tag{2.9a}$$

so that
$$\psi(r, z) = X_1\,\alpha_2(X) - X_2\,\alpha_1(X). \tag{2.9b}$$

In place of (2.6a, b), in which L is singular for $r=0$, one has the system

$$\Delta[\alpha_1, \alpha_2, 0] = -\lambda[-X_2, X_1, 0]f(\Psi) \quad \text{in } V, \tag{2.10a}$$

$$\alpha_1\big|_{\partial V} = \alpha_2\big|_{\partial V} = 0. \tag{2.10b}$$

We shall use the notation $\boldsymbol{\alpha} \in S$ to indicate that α_1 and α_2 belong to some set or space S.

2.4. The Hilbert space $H(D)$

Let $C_0^\infty(D)$ denote the set of real valued functions with derivatives of every order and compact support in D. We define $H(D)$ to be the completion of $C_0^\infty(D)$ in the norm implied by the inner product

$$\langle u, v\rangle = \iint_D \frac{1}{r^2}\,(u_r v_r + u_z v_z),$$

the element $d\tau = rdrdz$ of measure being implied wherever no other element is written.

The elements of $H(D)$, which we shall often call functions, are, of course, equivalence classes of functions equal almost everywhere in D. The space $H(D)$ is a natural setting for the problem (2.6), because functions in $H(D)$ have zero *trace* on ∂D (i.e. vanish in a certain generalized sense on ∂D), so that

$$\langle u, v\rangle = -\iint_D \frac{1}{r^2}\,uLv \quad \text{if} \quad u \in H(D) \quad \text{and} \quad v \in H(D) \cap C^2(\bar{D});$$

also, the energy condition (2.6c) becomes $\|\psi\|^2 = \eta$. The transformation (2.9) implies that

$$\int_V \{|\text{grad } \alpha_1|^2 + |\text{grad } \alpha_2|^2\}\, dX = 2\pi\|\psi\|^2 \tag{2.11}$$

if $\psi \in H(D)$. Let $\overset{\circ}{W}_{1,2}(V)$ denote the completion of the set $C_0^\infty(V)$ in the norm whose square appears on the left of (2.11); it follows that if $\psi \in H(D)$, then $\boldsymbol{\alpha} \in \overset{\circ}{W}_{1,2}(V)$.

It will prove convenient to introduce the notation, for any open subset E of D,

$$\|u\|_E = \left\{\iint_E \frac{1}{r^2}\left(u_r^2 + u_z^2\right)\right\}^{\frac{1}{2}}, \tag{2.12a}$$

$$\|u\|_{p,E} = \left\{\iint_E |u|^p\right\}^{1/p}. \tag{2.12b}$$

When $E = D$, the label D will be omitted from the norm symbols on the left. The space $L_p(D, \tau)$, where τ is the measure defined by $d\tau = rdrdz$, is then the completion of the set $C_0^\infty(D)$ in the norm $\|.\|_p$.

LEMMA 2A. *Consider a function $u \in H(D)$ on the rectangles $R = (r_0 - \alpha, r_0) \times (0, 2\beta)$ and $R' = (r_0 - \alpha, r_0) \times (0, \beta)$ in D. If u has zero trace on ∂R, then*

$$\|u\|_{p,R} \leqslant c_p r_0^{1/p + \frac{1}{2}} (2\alpha\beta)^{1/p} \|u\|_R \quad \text{for every } p \geqslant 1, \tag{2.13}$$

where $c_p = 2^{-\frac{2}{3}}$ for $1 \leqslant p \leqslant 2$ and $c_p = 2^{-\frac{5}{3}}p$ for $p \geqslant 2$.

If u has zero trace on a vertical side of R (that is, on $\partial R \cap \{r = r_0 - \alpha\}$ or on $\partial R \cap \{r = r_0\}$) then

$$\|u\|_{p,R'} \leqslant 2c_p r_0^{1/p + \frac{1}{2}} (2\alpha\beta)^{1/p} \left(1 + \frac{\alpha}{2^{\frac{1}{2}}\beta}\right) \|u\|_R \quad \text{for every } p \geqslant 1. \tag{2.14}$$

Remark. The following extensions are immediate. The rectangles need not be in D, provided that $r_0 \geqslant \alpha$; we extend u by defining $u(r, z) = 0$ for $(r, z) \notin D$, $r \geqslant 0$. Also, the interval $(0, \beta)$ may be replaced by $(z_0, z_0 + \beta)$, with $(0, 2\beta)$ then replaced by either $(z_0, z_0 + 2\beta)$ or $(z_0 - \beta, z_0 + \beta)$.

Proof of Lemma 2A. Let $|\nabla v|^2 = v_r^2 + v_z^2$ be integrable on R, and let v have zero trace on *two non-parallel sides* of R. Adapting a well known calculation of Nirenberg ([20], p. 128), one can easily show that

$$\left\{\iint_R |v|^p\, dr\, dz\right\}^{1/p} \leqslant 2c_p(2\alpha\beta)^{1/p} \left\{\iint_R |\nabla v|^2\, dr\, dz\right\}^{\frac{1}{2}}. \tag{2.15}$$

If v has zero trace on ∂R, the factor $2c_p$ becomes c_p. Noting the simple inequalities

$$\|v\|_{p,R} \leqslant \left\{r_0 \iint_R |v|^p\, dr\, dz\right\}^{1/p}, \tag{2.16a}$$

$$\left\{\iint_R |\nabla v|^2\, dr\, dz\right\}^{\frac{1}{2}} \leqslant \left\{\iint_R \frac{r_0}{r} |\nabla v|^2\, dr\, dz\right\}^{\frac{1}{2}} = r_0^{\frac{1}{2}} \|v\|_R, \tag{2.16b}$$

we obtain (2.13).

Given a function $u \in H(D)$ with zero trace on *one vertical side* of R, we form a function v to which (2.15) applies by defining

$$v(r, z) = u(r, z)\,\varphi(z), \quad \text{where} \quad \varphi(z) = \begin{cases} 1, & z \leqslant \beta, \\ 2 - \dfrac{z}{\beta}, & \beta \leqslant z \leqslant 2\beta, \\ 0, & z \geqslant 2\beta. \end{cases}$$

Then $u = v$ on R', and on the bigger rectangle R

$$|\nabla v|^2 = (u_r \varphi)^2 + (u_z \varphi + u \varphi_z)^2 \leqslant u_r^2 + u_z^2 + C u_z^2 + \frac{u^2}{C\beta^2} + \frac{u^2}{\beta^2},$$

for any constant $C > 0$. Also, if u has zero trace on $\partial R \cap \{r = r_0\}$, then

$$u^2(r_1, z) = \left\{ \int_{r_1}^{r_0} u_r \, dr \right\}^2 \leqslant (r_0 - r_1) \int_{r_1}^{r_0} u_r^2 \, dr \quad \text{for } (r_1, z) \in R,$$

so that

$$\iint_R u^2(r_1, z)\, dr_1\, dz \leqslant \frac{\alpha^2}{2} \iint_R u_r^2 \, dr\, dz,$$

and this remains true if u has zero trace on $\partial R \cap \{r = r_0 - \alpha\}$. Accordingly,

$$\iint_R |\nabla v|^2 \, dr\, dz \leqslant \left\{ 1 + C + \frac{\alpha^2}{2\beta^2}\left(\frac{1}{C} + 1\right) \right\} \iint_R |\nabla u|^2 \, dr\, dz;$$

we choose $C = \alpha/2^{\frac{1}{2}}\beta$, and use (2.15) and (2.16) to obtain the result (2.14).

LEMMA 2B. *Sets bounded in* $H(D)$ *are relatively compact in* $L_p(D, \tau)$ *for every* $p \geqslant 1$.

Proof. Let $L_p(D)$ and $\overset{\circ}{W}_{1.2}(D)$ denote the completions of the set $C_0^\infty(D)$ in the norms

$$\left\{ \iint_D |u|^p \, dr\, dz \right\}^{1/p} \quad \text{and} \quad \left\{ \iint_D (u_r^2 + u_z^2)\, dr\, dz \right\}^{\frac{1}{2}},$$

respectively. It is a standard result ([25], p. 84; [14], p. 43) that $\overset{\circ}{W}_{1.2}(D)$ is embedded compactly in $L_p(D)$ for every $p \geqslant 1$; also using (2.16), we have

$$H(D) \subset \overset{\circ}{W}_{1.2}(D) \subset L_p(D) \subset L_p(D, \tau),$$

where the second embedding is compact.

LEMMA 2C. *Let* $u \in H(D)$, *with* $u(r, z) \geqslant 0$ *almost everywhere in* D, *and let* u^* *be the Steiner symmetrization (defined in Appendix I) of* u *about the line* $z = 0$. *Then* $u^* \in H(D)$ *and*

$$\|u^*\|_p = \|u\|_p, \quad \|u^*\| \leqslant \|u\|.$$

Also, if $g: [0, \infty) \times [0, a] \to \mathbf{R}$ is continuous and (for all $s \geqslant 0$, $t \geqslant 0$ and $r \in [0, a]$)

$$|g(s, r) - g(t, r)| \leqslant \text{ const. } (1 + s^c + t^c) |s - t|, \quad (c \geqslant 0),$$

then
$$\iint_D g(u^*(r, z), r) \, dr \, dz = \iint_D g(u(r, z), r) \, dr \, dz.$$

Proof. See Appendix I.

2.5. Further notation

Let Ω be an open set in $\mathbf{R}^n$; let $\beta = \{\beta_1, ..., \beta_n\}$ be a multi-index of order $|\beta| = \beta_1 + ... + \beta_n$, where $\beta_1, ..., \beta_n$ are non-negative integers; and write $\partial^\beta = (\partial/\partial X_1)^{\beta_1} ... (\partial/\partial X_n)^{\beta_n}$. By $C^l(\Omega)$ we mean the set of all real-valued functions u such that $\partial^\beta u$, $0 \leqslant |\beta| \leqslant l$, are continuous in Ω. The space $C^l(\overline{\Omega})$ is the set of real-valued functions u such that $\partial^\beta u$, $0 \leqslant |\beta| \leqslant l$, can be extended to be continuous on $\overline{\Omega}$, with the norm

$$\|u\|_{C^l(\overline{\Omega})} = \sup_{X \in \Omega, \, 0 \leqslant |\beta| \leqslant l} |\partial^\beta u(X)|.$$

The space $C^{l+\nu}(\Omega)$, where $0 < \nu < 1$, consists of functions having derivatives of order l that are uniformly Hölder continuous in Ω with exponent ν; with the notation

$$[v]_\nu = \sup_{X, X^0 \in \Omega} \frac{|v(X) - v(X^0)|}{|X - X^0|^\nu}, \quad (0 < |X - X^0| \leqslant 1),$$

$C^{l+\nu}(\overline{\Omega})$ is that subset of $C^l(\overline{\Omega})$ for which

$$\|u\|_{C^{l+\nu}(\overline{\Omega})} = \|u\|_{C^l(\overline{\Omega})} + \max_{|\beta|=l} [\partial^\beta u]_\nu < \infty.$$

We say that u is $C^{l+\nu}$ in Ω if $u \in C^{l+\nu}(K)$ for every compact subset K of Ω.

The symbol $(.)^+$ denotes the non-negative part of a function:

$$u^+(X) = \max \{u(X), 0\}.$$

3. The solution for continuous vorticity and a bounded domain

3.1. Existence of a generalized solution in D

We adopt the notation

$$x = (r, z) \quad \text{and} \quad S(\eta) = \{u \in H(D) \mid \|u\|^2 = \eta\}.$$

A generalized solution of the problem (2.6) is an element $\psi \in S(\eta)$ and a constant λ such that

$$\langle \varphi, \psi \rangle = \lambda \iint_D \varphi f(\Psi) \quad \text{for all test functions } \varphi \in H(D). \tag{3.1}$$

When ψ is also in $C^2(\bar{D})$, integration by parts relates (3.1) and (2.6a, b); also, (3.1) characterizes a critical point ψ of the restriction to the sphere $S(\eta)$ of the functional

$$J(u) = \iint_D F(u(x) - \tfrac{1}{2}Wr^2 - k), \quad \text{where } F(t) = \int_0^t f(s)\,ds. \tag{3.2}$$

Here the integral over D need be taken only over the set

$$A_u = \{x \,|\, u(x) > \tfrac{1}{2}Wr^2 + k\}, \tag{3.3}$$

which becomes the cross-section A_ψ of the vortex ring when $u = \psi$; a similar remark applies to (3.1).

THEOREM 3A. *The variational problem* $\max_{u \in S(\eta)} J(u)$ *has a solution* ψ *such that* $J(\psi) > 0$ *and* $\psi(x) \geqslant 0$ *almost everywhere in* D. *With* λ *as in* (3.7) *below, the pair* (ψ, λ) *is a generalized solution of* (2.6), *and there exist numbers* c, C *depending only on the data of the problem such that* $0 < c < \lambda < C$.

Proof. (i) First we show that, for any set of data, there exist functions $u \in S(\eta)$ such that $J(u) > 0$. (This would not be true for analogous problems involving a one-dimensional domain D.) It is sufficient to observe that functions with small norm in $H(D)$ can have large values on a small set. For example, given a point $x_0 = (r_0, z_0)$ in D, define

$$s = |x - x_0| = \{(r - r_0)^2 + (z - z_0)^2\}^{1/2},$$

$$v(x) = \begin{cases} \beta \log\left(1 + \log\dfrac{\delta}{s}\right), & s \leqslant \delta \leqslant \tfrac{1}{2}d(x_0, \partial D), \\ 0, & s \geqslant \delta, \end{cases} \tag{3.4}$$

where β, δ are positive constants and $d(.,.)$ denotes distance. Since $v(x) \to \infty$ as $s \to 0$, we have $J(v) > 0$; and since

$$\|v\|^2 = 2\pi\beta^2 g(r_0, \delta), \quad \text{where} \quad (r_0 + \delta)^{-1} < g(r_0, \delta) < (r_0 - \delta)^{-1} \quad \text{and} \quad \delta \leqslant \tfrac{1}{2}r_0,$$

we can choose β to make $\|v\|^2 = \eta$ for any $\eta > 0$.

(ii) Next, we prove that the functional J is continuous with respect to weak convergence in $H(D)$, and bounded above on $S(\eta)$. For any $u \in H(D)$, define $U(x) = u(x) - $

$\frac{1}{2}Wr^2 - k$, abbreviate $U(x)$ to U, and note that $J(u)$ depends only on the non-negative part U^+ of U because $F(U) = F(U^+)$. Then by (2.8)

$$|J(u) - J(v)| = \left| \iint_D \{F(U^+) - F(V^+)\} \right| \leqslant \iint_D \{1 + (MU^+)^m + (MV^+)^m\} |U^+ - V^+|. \quad (3.5)$$

It is clear that $U^+ \leqslant |u|$ and $|U^+ - V^+| \leqslant |u - v|$; hence

$$|J(u) - J(v)| \leqslant \|u - v\|_1 + M^m(\|u\|_{2m}^m + \|v\|_{2m}^m) \|u - v\|_2 \quad (3.6)$$

for all u and v in $H(D)$. Now if $u_n \to u$ weakly in $H(D)$, then $u_n \to u$ (strongly) in $L_p(D, \tau)$ for each $p \geqslant 1$, since the embedding operator is compact by Lemma 2B; hence (3.6) shows that $J(u_n) \to J(u)$. To obtain an upper bound for $J(u)$ on $S(\eta)$, we set $v = 0$ in (3.6) and note that $J(0) = 0$.

(iii) Write $\sup_{u \in S(\eta)} J(u) = \sigma > 0$ and let $\{u_n\}$ be a sequence in $S(\eta)$ such that $J(u_n) \to \sigma$ as $n \to \infty$. Since $\|u_n\|^2 = \eta$, this sequence is weakly compact in $H(D)$: there exists a subsequence, say $\{\psi_n\}$, and an element $\psi \in H(D)$ such that $\psi_n \to \psi$ weakly in $H(D)$. By (ii), $J(\psi) = \lim J(\psi_n) = \sigma$. To prove that $\psi \in S(\eta)$, we first note that $\|\psi\| \leqslant \lim \inf \|\psi_n\|$ by a general result for weak convergence in a Hilbert space. Hence $\|\psi\|^2 \leqslant \eta$; also $\|\psi\| > 0$ because $J(\psi) = \sigma > 0$. If $\|\psi\|^2 < \eta$, define $\tilde{\psi} = \eta^{1/2} \psi / \|\psi\|$, so that $\tilde{\psi} \in S(\eta)$ and $\tilde{\psi}(x) > \psi(x)$ wherever $\psi(x) > 0$, in particular, on the set A_ψ. But $F(t)$ is strictly increasing for $t > 0$; hence $J(\tilde{\psi}) > J(\psi) = \sigma$ and this contradicts the definition of σ.

(iv) The non-negative part ψ^+ of ψ belongs to $H(D)$ (cf. [14], p. 50); clearly $J(\psi^+) = J(\psi)$ and $\|\psi^+\| \leqslant \|\psi\|$. If $\psi(x) < 0$ on a set of positive measure, then $\|\psi^+\| < \|\psi\|$ and we obtain a contradiction as in (iii) by considering $\eta^{1/2} \psi^+ / \|\psi^+\|$; hence $\psi = \psi^+$ almost everywhere in D.

(v) The functional J has a Fréchet derivative $J'(u)$ defined by

$$\langle J'(u), \varphi \rangle = \iint_D \varphi f(u - \tfrac{1}{2}Wr^2 - k),$$

since (2.7 c), the Hölder inequality and (2.13) imply that

$$\left| J(u + \varphi) - J(u) - \iint_D \varphi f(u - \tfrac{1}{2}Wr^2 - k) \right| \leqslant \{\text{const.} + \text{const.} \, (\|u\|^{m-1} + \|\varphi\|^{m-1})\} \|\varphi\|^{1+\mu}$$

for all u and φ in $H(D)$. It is then a standard result ([28], p. 96; [2], [3]) that the maximizer ψ is a critical point of the restriction of J to $S(\eta)$ in the sense that the vectors ψ and $J'(\psi)$ are parallel. In other words, there exist constants μ_1 and μ_2, not both equal to zero, such that

$$\mu_1 \langle \varphi, \psi \rangle = \mu_2 \int\!\!\int_D \varphi f(\Psi') \quad \text{for all } \varphi \in H(D).$$

Thus (3.1) is satisfied, with $\lambda = \mu_2/\mu_1$, provided that $\mu_1 \neq 0$. Choose $\varphi = \psi$; the possibilities $\mu_1 = 0$ or $\mu_2 = 0$ are ruled out by the estimates that follow, and

$$\lambda = \eta \Big/ \int\!\!\int_{A_\psi} \psi f(\Psi'). \tag{3.7}$$

First we bound the integral from above; since $0 < \Psi'(x) < \psi(x)$ in A_ψ, we have

$$\int\!\!\int_{A_\psi} \psi f(\Psi') < \int\!\!\int_D \psi \{1 + (M\psi)^m\} \leqslant \text{const. } \eta^{\frac{1}{2}} + \text{const. } \eta^{(m+1)/2},$$

where the constants are implied by (2.13). On the other hand, $F(t) \leqslant t f(t)$ because f is non-decreasing, so that

$$\int\!\!\int_{A_\psi} \psi f(\Psi') > \int\!\!\int_{A_\psi} \Psi' f(\Psi') \geqslant \int\!\!\int_{A_\psi} F(\Psi') \geqslant J(u) \quad \text{for any } u \in S(\eta), \tag{3.8}$$

and the functions mentioned in (i) provide a lower bound for the integral and an upper bound for λ.

Remark. The simple function v in (3.4) gives a very crude upper bound for λ in (3.8); one can do better, in general, by using more elaborate functions suggested by the known particular solutions for Hill's vortex and for rings of small cross-section.

COROLLARY 3B. *Define* $D_+ = \{x \in D \,|\, z > 0\}$. *With* ψ *as in Theorem* 3A, $\psi(r, -z) = \psi(r, z)$ *almost everywhere in* D, *and* $\psi_z \leqslant 0$ *almost everywhere in* D_+.

Proof. Let ψ^* be the Steiner symmetrization of ψ with respect to the line $z = 0$ in D; then Lemma 2C, with $g(\psi, r) = r F(\Psi')$, states that $\|\psi^*\|^2 \leqslant \|\psi\|^2 = \eta$ and $J(\psi^*) = J(\psi)$. If $\|\psi^*\|^2 < \eta$, we consider $\eta^{1/2} \psi^* / \|\psi^*\|$ and obtain a contradiction, as in the proof of Theorem 3A, step (iii). If $\|\psi^*\|^2 = \eta$ but $\|\psi^* - \psi\| \neq 0$, then ψ^* is a second maximizer of J on $S(\eta)$, and we choose to adopt it, denoting it by ψ henceforth.

3.2. Regularity of the generalized solution

The next step is to prove that the equivalence class ψ has a smooth representative that satisfies (2.6 a, b) pointwise; we follow convention in calling this representative ψ also. It is shown in Appendix II that the definition (3.1) of a generalized solution implies the representation

$$\alpha(X^0) = \lambda \int_V \Gamma(X^0, X) [-X_2, X_1, 0] f(\Psi(x)) \, dX \tag{3.9}$$

of the vector potential defined by (2.9a); here Γ is the Green function of the Dirichlet problem for the Laplace operator Δ in the cylinder V. This representation is used (in Appendix II) to show that $\boldsymbol{\alpha}$ is a smooth solution of (2.10a, b), and one then returns to ψ by means of (2.9b), thus obtaining the behaviour of ψ for $r \to 0$ in a painless way, and proving the next theorem. The edges $r = a$, $z = \pm b$ of ∂V cause no difficulty because Γ is in fact analytic if X^0 is at or sufficiently near an edge while X is bounded away from the edge, and because the boundary condition of our problem makes $\Psi|_{r=a} < 0$ and ensures that the support of $f(\Psi(x))$ is indeed bounded away from $r = a$.

One can also obtain smoothness properties of $\boldsymbol{\alpha}$ by transforming (3.1) to the definition of a generalized solution of (2.10) and then appealing to general regularity theory [1], [14] for elliptic equations; however, such an approach leaves in doubt the differentiability of $\boldsymbol{\alpha}$ and ψ at the edges.

THEOREM 3C. *The generalized solution has a representative $\psi \in C^{2+\mu}(\bar{D})$ that satisfies the differential equation* (2.6a) *and the boundary condition* (2.6b) *pointwise; here μ is the Hölder exponent of f. Also, $\psi = O(r^2)$ and $\psi_r = O(r)$ for $r \to 0$. (The corresponding vector potential $\boldsymbol{\alpha} \in C^{2+\mu}(\bar{V})$ and satisfies* (2.10a, b) *pointwise.)*

3.3. Properties of the solution and of the associated vortex ring

Henceforth the symbol ψ refers to the representative in Theorem 3C; we proceed to some descriptive properties of this function and of the cross-section A_ψ of the vortex ring.

THEOREM 3D. $\psi(r, -z) = \psi(r, z)$ on $\bar{D}$; $\psi_z < 0$ in D_+; and $\psi > 0$ in D.

Proof. Now that $\psi \in C^{2+\mu}(\bar{D})$, Corollary 3B implies that ψ is even in z on $\bar{D}$, and that $\psi_z \leqslant 0$ on $\bar{D}_+$. We sharpen this latter result to $\psi_z < 0$ in D_+ by means of the generalized maximum principle [17]. Consider the function $\alpha_2 = \psi \cos \theta / r$ in the quarter-cylinder

$$V' = \{X \in V \,|\, X_1 > 0, \quad X_3 > 0\};$$

clearly $\psi_z < 0$ in D_+ if and only if $\alpha_{2,z} < 0$ in V'. We claim that $\alpha_{2,z}$ is weakly subharmonic in V' in the following sense:

$$\int_{V'} \alpha_{2,z} \Delta \Phi \, dX \geqslant 0 \quad \text{for all non-negative } \Phi \in C_0^\infty(V'). \tag{3.10}$$
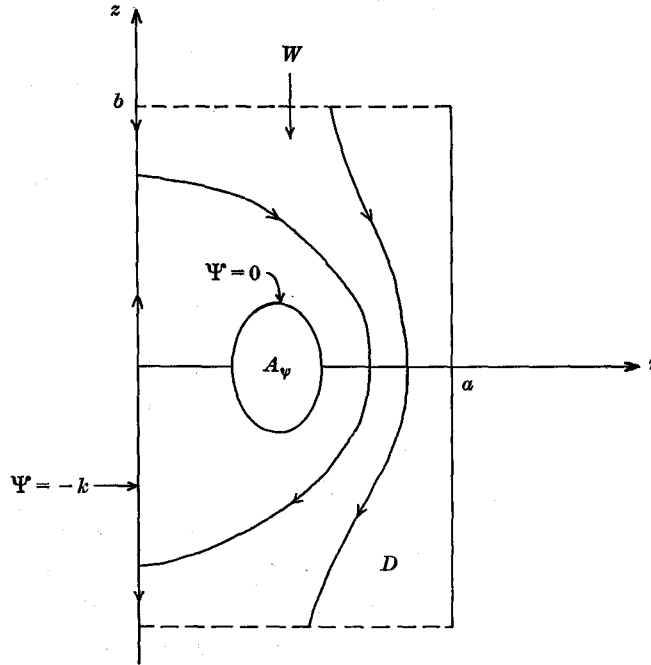
Fig. 1. Notation and expected streamline pattern for a steady vortex ring.

For, multiplying (2.10 a) by Φ_z and integrating by parts, we obtain

$$\int_{V'} \alpha_{2,z} \Delta \Phi \, dX = \lambda \int_{V'} X_1 f(\Psi) \, \Phi_z \, dX \equiv \lambda I, \text{ say,}$$

and, upon 'mollifying' or 'averaging' f, we obtain a non-decreasing function $f_\varepsilon \in C^\infty (\mathbb{R})$ such that

$$I_\varepsilon \equiv \int_{V'} X_1 f_\varepsilon(\Psi) \, \Phi_z \, dX \to I \quad \text{as } \varepsilon \to 0,$$

and

$$I_\varepsilon = - \int_{V'} X_1 f_\varepsilon'(\Psi) \, \psi_z \Phi \, dX \geqslant 0,$$

since $f_\varepsilon'(\Psi) \geqslant 0$ and $\psi_z \leqslant 0$ on $\bar{D}_+$. Thus $I \geqslant 0$; therefore $\alpha_{2,z}$ is weakly subharmonic in V', and its maximum over $\bar{V}'$ is zero. Applied to the continuous function $\alpha_{2,z}$, the generalized maximum principle in [17] now states (like its classical counterpart) that if $\alpha_{2,z}$ vanishes somewhere in V' then it vanishes everywhere on $\bar{V}'$, and this is clearly a contradiction because $\alpha_2(X) = 0$ for $z = b$.

Finally, the positivity of ψ in D follows from the boundary condition and the previous two results of the theorem (or else from the classical maximum principle [24]).

LEMMA 3E. *If the flux constant $k > 0$, then the set A_ψ is a finite or denumerable union of disjoint simply connected domains*

$$E_n = \{(r, z) \,|\, p_n < r < q_n, \; -h(r) < z < h(r)\}, \quad n = 1, 2, \ldots,$$

where $p_n > 0$, $q_n < a$ and $h \in C^{2+\mu}(p_n, q_n)$ for each n. Define $h(r) = 0$ if $r \notin \bigcup_n (p_n, q_n)$; then $h \in C[0, a]$.

If $k = 0$, the only change is that the value $p_n = 0$ is possible for one n, and the existence of a limiting value $h(0+)$ has not been proved for that case.

Proof. Assume that $k > 0$. Then $\Psi(0, 0) = -k < 0$ and $\Psi(a, 0) = -\frac{1}{2}Wa^2 - k < 0$. Hence the set $A' = \{r \,|\, \Psi(r, 0) > 0\}$, which is non-empty by Theorems 3A and 3D, is a finite or denumerable union of disjoint open intervals, say (p_n, q_n), within $(0, a)$. If $r_0 \in A'$, there exists a unique point $(r_0, h(r_0))$ in D_+ at which $\Psi = 0$, because $\Psi(r_0, b) < 0$ and $\Psi_z = \psi_z < 0$ in D_+, and then $h \in C^{2+\mu}$ in an open interval containing r_0, by the implicit function theorem.

We now show that h is also continuous at any point $p \in (0, a)$ such that $\Psi(p, 0) \leqslant 0$ and $h(p) = 0$, and it follows that $h \in C[0, a]$ for $k > 0$. For any $\varepsilon > 0$, we have $\Psi(p, \varepsilon) < 0$ because $\Psi_z < 0$ in D_+. Then there exists a number $\delta > 0$ such that $|r - p| < \delta$ implies $\Psi(r, \varepsilon) < 0$ and hence $h(r) < \varepsilon$; also, $h(r) \geqslant 0$ for all r.

If $k = 0$, then $\Psi(0, z) = 0$ for $|z| \leqslant b$ and we have the uncertainty at $r = 0$ that is mentioned in the lemma; there are no changes for $r > 0$, and $h \in C(0, a]$.

THEOREM 3F. *If, in addition to satisfying (2.7), the function $f(t)$ is Lipschitz continuous at $t = 0$, then A_ψ has only a finite number of components E_n.*

Proof. The new condition implies that (for some constant $M > 0$)

$$0 \leqslant f(t) \leqslant Mt\{1 + (Mt)^{m-1}\} \quad \text{for } t \geqslant 0.$$

Let $E = E_n$ by any component of A_ψ, and let $(p, p+\delta) = (p_n, q_n)$ be its intersection with the line $z = 0$; we shall obtain a lower bound for δ. Define a function φ by: $\varphi(x) = \Psi(x)$ in E, and $\varphi(x) = 0$ elsewhere; then $\varphi \in H(D)$, and

$$\langle \varphi, \psi - \varphi \rangle = W \iint_E \varphi_r \, dr \, dz = 0, \tag{3.11}$$

where the integration could also be over D. Hence (3.1) yields the estimate

$$\|\varphi\|^2 = \lambda \iint_D \varphi f(\Psi) < CM' \iint_E \varphi^2, \tag{3.12}$$

where
$$M' = M\{1 + (M\|\psi\|_{C(\bar{D})})^{m-1}\},$$

and where $\lambda < C$ by Theorem 3 A. On the other hand, integration of the estimate

$$\varphi^2(r_0, z) = \left\{\int_p^{r_0} \varphi_r\, dr\right\}^2 \leqslant \tfrac{1}{2}(r_0^2 - p^2)\int_p^{r_0} r^{-1}\varphi_r^2\, dr$$

yields
$$\iint_E \varphi^2 < \frac{1}{8}\delta^2(2p+\delta)^2\|\varphi\|^2. \tag{3.13}$$

Since $p+\delta < a$, (3.12) and (3.13) show that $\delta^2 > 2/CM'a^2$ (and a sharper estimate is possible for specified values of p).

THEOREM 3G. *If, in addition to satisfying* (2.7), *the function f is convex and continuously differentiable on* **R**, *then A_ψ is simply connected: $A_\psi = E_1$.*

Proof. The new condition implies that $f'(0) = 0$, $f'(t) > 0$ for $t > 0$, and

$$f(t) = \int_0^t f'(s)\, ds < tf'(t) \quad \text{for } t > 0. \tag{3.14}$$

Assume that A_ψ has at least two components E_1 and E_2; we obtain a contradiction by considering the second variation of J, and showing that J does not have a constrained local maximum on the sphere $S(\eta)$ at ψ. Let

$$u(\beta) = \psi\cos\beta + \eta^{1/2}v\sin\beta/\|v\|,$$

where
$$\langle v, \psi\rangle = 0, \; v\in H(D)\cap C(\bar{D}) \quad \text{and} \quad \|v\| > 0,$$

then $u(\beta)\in S(\eta)$ for all β, and $u(0) = \psi$. Expanding $J(u(\beta))$ for small values of β, and using (3.1) and (3.7), one obtains

$$J(u(\beta)) - J(\psi) = \frac{1}{2}\frac{\beta^2\eta}{\|v\|^2}\left\{\iint_D v^2 f'(\Psi) - \frac{\|v\|^2}{\lambda}\right\} + o(\beta^2),$$

where the o-term results from the uniform continuity of f' on $[0, \max u(\beta, x)]$, the maximum being over $[-\pi/2, \pi/2]\times\bar{D}$. We shall have the desired contradiction (for β sufficiently small) if the term in curly brackets is positive for some choice of v.

Define φ_j, $j = 1, 2$, by: $\varphi_j(x) = \Psi(x)$ in E_j, and $\varphi_j(x) = 0$ elsewhere; then, by (3.11) and (3.1),

$$\|\varphi_j\|^2 = \langle\varphi_j, \psi\rangle = \lambda\iint_D \varphi_j f(\Psi) = \lambda\iint_{E_j}\Psi f(\Psi), \tag{3.15}$$

and $\langle\varphi_1, \varphi_2\rangle = 0$. Now choose $v = c_1\varphi_1 - c_2\varphi_2$, where c_1 and c_2 are strictly positive constants such that

$$\langle v, \psi \rangle = c_1 \langle \varphi_1, \psi \rangle - c_2 \langle \varphi_2, \psi \rangle = c_1 \|\varphi_1\|^2 - c_2 \|\varphi_2\|^2 = 0.$$

Using (3.15) for $\|v\|^2 = c_1^2 \|\varphi_1\|^2 + c_2^2 \|\varphi_2\|^2$, and finally applying (3.14), we obtain

$$\iint_D v^2 f'(\Psi) - \frac{\|v\|^2}{\lambda} = \sum_{j=1}^{2} c_j^2 \iint_{E_j} \{\Psi^2 f'(\Psi) - \Psi f(\Psi)\} > 0.$$

4. The case of discontinuous vorticity

Let the vorticity function $f(t)$ have a simple discontinuity at $t=0$, corresponding to a jump in vorticity at the boundary ∂A_ψ of the cross-section of the vortex ring. The only change in (2.7) is that we weaken the qualification in (2.7c) to $t \geqslant s > 0$; then the limit $f(0+)$ exists and is positive, and we assume it to be such that (2.8) still holds.

We form continuous approximations f_j to f by introducing a non-decreasing function $\chi \in C^\infty(\mathbf{R})$ such that $\chi(s) = 0$ for $s \leqslant 0$ and $\chi(s) = 1$ for $s \geqslant 1$, and by defining

$$f_j(t) = f(t) \chi(jMt), \quad j = 1, 2, \ldots \tag{4.1}$$

where M is the constant in (2.8). Then $f_j(t)$ is non-decreasing with respect to both j and t. The corresponding solutions are denoted by (ψ_j, λ_j), and we have $0 < c < \lambda_j < C$ with c and C independent of j if we use f_1 in (3.8).

LEMMA 4A. *For any $v \in (0, 1)$ there exist a function $\psi \in C^{1+\nu}(\bar{D})$, a number $\lambda \in [c, C]$, and a subsequence $\{(\psi_{j_n}, \lambda_{j_n})\}$ which we relabel $\{(\psi_n, \lambda_n)\}$ such that*

$$(\psi_n, \lambda_n) \to (\psi, \lambda) \quad \text{in} \quad C^{1+\nu}(\bar{D}) \times \mathbf{R} \quad \text{as} \quad n \to \infty;$$

also, $\psi = O(r^2)$ and $\psi_r = O(r)$ for $r \to 0$.

Proof. One needs no continuity of f_j to show that the sequence $\{\boldsymbol{\alpha}_j\}$ of vector potentials is bounded (independently of j) in $\mathbf{C}^{1+\varkappa}(\bar{V})$ for each $\varkappa \in (0, 1)$; the bound (2.8) for f_j and the representation of $\boldsymbol{\alpha}_j$ in terms of the Green function are sufficient (cf. Appendix II: one uses (II.2) and estimates as in (II.7)). Since the sequence $\{(\boldsymbol{\alpha}_j, \lambda_j)\}$ is bounded in $\mathbf{C}^{1+\varkappa}(\bar{V}) \times \mathbf{R}$ for $\nu < \varkappa < 1$, it is relatively compact in $\mathbf{C}^{1+\nu}(\bar{V}) \times \mathbf{R}$; we extract a convergent subsequence and use the transformation (2.9b) from $\boldsymbol{\alpha}$ to ψ.

THEOREM 4B. *The limit function ψ (in Lemma 4A) maximizes the functional $J(u)$ over the sphere $S(\eta)$.*

Proof. That $\|\psi\|^2 = \eta$ follows from (2.11) and the proof of Lemma 4A. Denote by J_n the functional (3.2) when f is replaced by $f_{j_n} = f_n$ (in the notation of Lemma 4A) and let

$\sup J_n \equiv \sup_{u \in S(\eta)} J_n(u)$. We show that (a) $\sup J_n \to J(\psi)$, and (b) $\sup J_n \to \sup J$, which proves the theorem.

(a) Since $f_n(t)$ differs from $f(t)$ only for $0 < t < 1/j_n M$, we have

$$0 \leqslant F(t) - F_n(t) \leqslant 2/j_n M \quad \text{for all } t \in \mathbf{R},$$

and
$$0 \leqslant J(u) - J_n(u) \leqslant 2 |D| /j_n M \quad \text{for all } u \in H(D), \tag{4.2}$$

where $|D| = a^2 b$ is the τ-measure of D. Also the functions F_n are equicontinuous on any compact subset of $\mathbf{R}$. Accordingly

$$\left| J(\psi) - \sup J_n \right| = \left| J(\psi) - J_n(\psi_n) \right| \leqslant \left| J(\psi) - J_n(\psi) \right| + \left| J_n(\psi) - J_n(\psi_n) \right| \to 0.$$

(b) Again using (4.2), we have

$$0 \leqslant \sup J - \sup J_n = \sup \left\{ (J - J_n) + J_n \right\} - \sup J_n \leqslant \sup (J - J_n) \to 0.$$

LEMMA 4 C. $\psi(r, -z) = \psi(r, z)$ on $\bar{D}$; $\psi_z < 0$ in D_+; and $\psi > 0$ in D.

Proof. Theorem 3D and Lemma 4A show that ψ is even in z and that $\psi_z \leqslant 0$ on $\bar{D}_+$. Again we sharpen this latter to $\psi_z < 0$ in D_+: with V' as in the proof of Theorem 3D,

$$\int_{V'} \frac{\partial \alpha_2}{\partial z} \Delta \Phi \, dX = \lim_{n \to \infty} \int_{V'} \frac{\partial \alpha_{n,2}}{\partial z} \Delta \Phi \, dX \geqslant 0$$

for every non-negative $\Phi \in C_0^\infty(V')$; and since $\partial \alpha_2 / \partial z$ is still continuous on $\bar{V}$, we can reason as before.

Remark. It is easily verified that a subset of D had τ-measure zero if and only if it has plane Lebesgue measure zero; hence 'measure zero' can have either meaning in what follows.

LEMMA 4D. *The set* $\Psi^{-1}(0) = \{x \in \bar{D} \,|\, \Psi(x) = 0\}$ *has measure zero.*

Proof. Since Lemma 4C shows that $\Psi_z < 0$ in D_+, we can construct a function h just like that in Lemma 3E, except that now h is only $C^{1+\nu}$ in each open interval (p_n, q_n). For $k > 0$, we have $h \in C[0, a]$ and $\Psi^{-1}(0)$ is a proper subset of $\{z = \pm h(r)\}$; for $k = 0$, we have $h \in C(0, a]$ and $\Psi^{-1}(0)$ is a proper subset of $\{z = \pm h(r)\} \cup \{r = 0\}$; in either case, $\Psi^{-1}(0)$ has measure zero.

LEMMA 4E. $f_n(\Psi_n(x)) \to f(\Psi(x))$ *almost everywhere on* $\bar{D}$, *as* $n \to \infty$.

Proof. Let x_0 be fixed and such that $\Psi(x_0) > 0$; there exists a number $N = N(x_0)$ such that $\Psi_n(x_0) > 1/j_n M$ for $n > N$. Accordingly, $f_n(\Psi_n(x_0)) = f(\Psi_n(x_0))$ for $n > N$, and $f(\Psi_n(x_0)) \to$

$f(\Psi(x_0))$ because $f(t)$ is continuous for $t > 0$ and $\Psi_n \to \Psi$ in $C^{1+\nu}(\bar{D})$. For x_0 such that $\Psi(x_0) < 0$, the argument is similar but simpler. The result now follows from Lemma 4 D.

THEOREM 4 F. *Let (ψ, λ) be as in Lemma 4 A, so that $\psi \in C^{1+\nu}(\bar{D})$ for any $\nu \in (0, 1)$. Then ψ is $C^{2+\mu}$ in $D - \Psi^{-1}(0)$, and, apart from failing to satisfy the differential equation (2.6 a) on the set $\Psi^{-1}(0)$ (of measure zero), the pair (ψ, λ) is a pointwise solution of the problem (2.6).*

Proof. Each approximating vector potential has the representation

$$\alpha_n(X^0) = \lambda_n \int_V \Gamma(X^0, X)\, [-X_2, X_1, 0]\, f_n(\Psi_n(x))\, dX.$$

We can pass to the limit by using Lemma 4 A for the left member and for λ_n, and by using Lemma 4 E and the Lebesgue dominated convergence theorem for the integral. The resulting representation of α implies the assertions above.

5. The unbounded flow field

5.1. Preliminary description of the limiting process

Henceforth we allow, but do not insist on, a simple discontinuity of $f(t)$ at $t = 0$. Our principal task is now to show that, for large values of the lengths a and b characterizing D, the cross-section A_ψ of the vortex ring remains in a certain bounded domain Ω in the rz-plane; Ω depends on f, W, k and η, but is independent of a and b. We begin by establishing certain geometrical restrictions on the set A_u defined by (3.3) for any function u in $H(D) \cap C(\bar{D})$; then we use the special properties of ψ to show that A_ψ lies between two curves $z = \pm Z(r)$ such that $Z(r) \to 0$ as $r \to \infty$; and we complete the specification of Ω by means of the representation formula

$$\psi(x_0) = \lambda \iint_D G(x_0, x)\, f(\Psi), \quad \Psi = \Psi(x). \tag{5.1}$$

Here G is the Green function of the Dirichlet problem for the operator L in D, and is related to the corresponding Green function $\tilde{\Gamma}(x_0, x, \theta - \theta_0)$ of the Laplace operator in V by

$$G(x_0, x) = P(x_0, x) - g(x_0, x) = r_0 r \int_{-\pi}^{\pi} \cos \beta\, \tilde{\Gamma}(x_0, x, \beta)\, d\beta, \tag{5.2 a}$$

where P is the fundamental solution of L,

$$P(x_0, x) = \frac{r_0 r}{4\pi} \int_{-\pi}^{\pi} \frac{\cos \beta\, d\beta}{\{r_0^2 + r^2 - 2r_0 r \cos \beta + (z_0 - z)^2\}^{\frac{1}{2}}}. \tag{5.2 b}$$

By an application of the classical maximum principle [24] and known bounds for P ([21], p. 278; our function P is denoted by $G/2\pi r$ there), we have

$$0 \leqslant G(x_0, x) \leqslant P(x_0, x) \leqslant \begin{cases} \dfrac{r_0^2 r^2}{\pi |x_0 - x|^3}, & (5.3\,\mathrm{a}) \\[2ex] \dfrac{(r_0 r)^{\frac{1}{2}}}{2\pi} \sinh^{-1} \dfrac{(2 r_0 r)^{\frac{1}{2}}}{|x_0 - x|}, & (5.3\,\mathrm{b}) \end{cases}$$

where $|x| = (r^2 + z^2)^{\frac{1}{2}}$; these two distinct estimates are useful for large and for small values of $|x_0 - x| / (r_0 r)^{\frac{1}{2}}$, respectively. We also note that

$$\iint_{0 < r < c} P(x_0, x) = \begin{cases} \frac{1}{8} r_0^2 (2c^2 - r_0^2), & 0 \leqslant r_0 \leqslant c, \\[1ex] \frac{1}{8} c^4, & r_0 \geqslant c. \end{cases} \tag{5.4}$$

5.2. Estimates of A_ψ in large domains D

Recalling the definition (3.3) of A_u for any $u \in H(D)$, we now define B_u to be the set of points in D that are in A_u or to the left of it (when r increases to the right):

$$B_u = \{(r, z) \,|\, \exists\, (\varrho, \zeta) \in A_u \quad \text{with} \quad \varrho \geqslant r > 0, \quad \zeta = z\}.$$

Also, $|A_u|$ and $|B_u|$ denote the τ-measures of these sets ($d\tau = r\,dr\,dz$), and l_u is the projection of A_u (or B_u) onto the z-axis:

$$l_u = \{z \,|\, (r, z) \in A_u \text{ for some } r\}, \quad |l_u| = \int_{l_u} dz.$$

We continue to use the notation (2.12).

LEMMA 5A. Let $u \in H(D) \cap C(\bar{D})$ and define $U(x) = u(x) - \frac{1}{2} W r^2 - k$. Then

$$W^2 |A_u| = \|u\|_{A_u}^2 - \|U\|_{A_u}^2, \tag{5.5}$$

and

$$W^2 |B_u| + 2 W k |l_u| = \|u\|_{B_u}^2 - \|U\|_{B_u}^2. \tag{5.6}$$

Proof. Abbreviate the symbols A_u, B_u and l_u to A, B and l. To prove (5.5) we again use the fact that the non-negative part U^+ of U belongs to the Sobolev space $\overset{\circ}{W}_{1,2}(D)$ ([14], p. 50); then

$$\|u\|_A^2 - \|U\|_A^2 = \iint_A \frac{1}{r^2} \{u_r^2 - (u_r - Wr)^2\} = W \iint_A \left(\frac{2}{r} u_r - W\right) = 2W \iint_D U_r^+ \, dr\,dz + W^2 |A|,$$

and the integral of U_r^+ is zero because U^+ vanishes on ∂D.

To prove (5.6) we first assume that the boundary ∂B is a finite union of disjoint closed curves, each of which is piecewise of class C^1. Then

$$\|u\|_B^2 - \|U\|_B^2 = 2W \iint_B u_r \, dr \, dz - W^2|B| = 2W \int_{\partial B} u \, dz - W^2|B|,$$

and ∂B consists of (a) a part of the axis $r=0$, on which $u=0$; (b) straight line segments $z=$const.; and (c) the set $\partial B \cap \partial A$, on which $u=\frac{1}{2}Wr^2+k$. Hence

$$2W \int_{\partial B} u \, dz = 2W \int_{\partial B \cap \partial A} (\tfrac{1}{2}Wr^2 + k) \, dz = 2W^2|B| + 2Wk|l|,$$

and (5.6) follows.

To remove the assumption about ∂B, we note that the open set A can be approximated by a finite union A_n of squares with disjoint interiors and with sides of length 2^{-n}, such that $A_n \nearrow A$ as $n \to \infty$ and for all $x \in \partial A_n$ the distance $d(x, \partial A) < 2^{-n+1}$. Let B_n and l_n be related to A_n as B and l are to A. Since u is uniformly continuous on $\bar{D}$, and $d(x, \partial A) \to 0$ uniformly for $x \in \partial A_n$,

$$\sup_{x \in \partial A_n} |u(x) - \tfrac{1}{2}Wr^2 - k| \to 0 \quad \text{as } n \to \infty,$$

and since $\int_{\partial B \cap \partial A} |dz| < 2b$, we have

$$\iint_{B_n} u_r \, dr \, dz - W|B_n| - k|l_n| \to 0 \quad \text{as } n \to \infty.$$

LEMMA 5B. *The cross-section A_ψ of the vortex ring lies in the region between the curves*

$$z = \pm \frac{\eta}{W(Wr^2 + 4k)}, \quad r > 0.$$

Proof. Let $(r_0, z_0) \in \partial A_\psi$, with $r_0 > 0$ and $z_0 > 0$. Then all points (r_0, z) with $|z| < z_0$ belong to A_ψ (because $\Psi_z < 0$ in D_+ and Ψ is an even function of z), the open rectangle $(0, r_0) \times (-z_0, z_0)$ is a subset of B_ψ, and (5.6) implies the estimate

$$W^2 r_0^2 z_0 + 4Wk z_0 < \eta.$$

LEMMA 5C. *Recall that $D = (0, a) \times (-b, b)$ and let $b \geqslant b_0 > 0$. Then there exists a number r_*, independent of a and b, such that $r < r_*$ if $(r, z) \in A_\psi$.*

Proof. (i) We adopt the notation $A = A_\psi$, $r_0 = \sup_A r$, $x_0 = (r_0, 0)$ and $s = |x - x_0|$. The supremum r_0 exists for each finite D, with $\Psi(r_0, 0) = 0$ and $\Psi(r_0, z) < 0$ for $z \neq 0$. We shall use the Green-function representation (5.1) to show that $\psi(x_0)$ cannot equal $\frac{1}{2}Wr_0^2 + k$ if r_0 is sufficiently large. To this end, we partition the set A, to which the integration in (5.1) may be restricted, by defining

$$\mathcal{B}_\varepsilon = \{x \mid s = |x - x_0| < \varepsilon r_0\}, \quad 0 < \varepsilon \leqslant \tfrac{1}{2},$$

$$A_1 = A \cap \{r < \tfrac{1}{2}r_0\}, \quad A_2 = A \cap \{r \geqslant \tfrac{1}{2}r_0\} - \mathcal{B}_\varepsilon, \quad A_3 = A \cap \mathcal{B}_\varepsilon,$$

where ε will be chosen later. Writing

$$\psi_j(x_0) = \lambda \iint_{A_j} G(x_0, x) f(\Psi'), \quad (j = 1, 2, 3),$$

we now estimate $\psi_1(x_0)$ by using (5.3a) for G, and $\psi_2(x_0)$, $\psi_3(x_0)$ by using (5.3b); in estimating $\psi_3(x_0)$ we also introduce the inequality $\sinh^{-1} t < \log 2t + \tfrac{1}{4} t^{-2}$ for $t > 0$. Accordingly,

$$\psi_1(x_0) \leqslant \frac{8\lambda}{\pi r_0} \iint_{A_1} r^2 f(\Psi'),$$

$$\psi_2(x_0) \leqslant \frac{2^{\frac{1}{2}} \lambda}{\pi r_0} \sinh^{-1} \left(\frac{2^{\frac{1}{2}}}{\varepsilon} \right) \iint_{A_2} r^2 f(\Psi'),$$

$$\psi_{31}(x_0) \equiv \psi_3(x_0) - \psi_{32}(x_0) \leqslant \frac{\lambda}{2\pi r_0 (1-\varepsilon)^{\frac{3}{2}}} \left(\log \frac{2^{\frac{3}{2}}}{\varepsilon} + \tfrac{1}{8} \varepsilon^2 \right) \iint_{A_3} r^2 f(\Psi'),$$

where $\quad \psi_{32}(x_0) = \dfrac{\lambda r_0}{2\pi} \displaystyle\iint_{A_3} \log \frac{1}{\sigma} f(\Psi'), \quad \sigma = \dfrac{s}{\varepsilon r_0},$

and where we have combined (5.3b) with estimates like $r^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}} r^2 / r_0^{\frac{3}{2}}$ on A_2, in order to keep the integrand $r^2 f(\Psi')$. For since $\psi > \tfrac{1}{2} W r^2$ on A, we have

$$\lambda = \eta \bigg/ \iint_A \psi f(\Psi') < 2\eta \bigg/ W \iint_A r^2 f(\Psi'). \tag{5.7}$$

Therefore $\qquad\qquad \psi_1(x_0) + \psi_2(x_0) + \psi_{31}(x_0) < \dfrac{\eta K(\varepsilon)}{\pi W} \dfrac{1}{r_0}, \tag{5.8a}$

where $\qquad\qquad K(\varepsilon) = \max \left\{ 16, 2^{\frac{3}{2}} \left(\log \frac{2^{\frac{3}{2}}}{\varepsilon} + \tfrac{1}{8} \varepsilon^2 \right) \right\}, \tag{5.8b}$

the maximum being taken at each fixed ε.

(ii) The next step is to estimate $\psi_{32}(x_0)$. Choosing a positive number a_0, we assume that $a \geqslant a_0$ (for otherwise $r_0 < a_0$). Then the upper bound C for λ, established in Theorem 3 A and Lemma 4 A, can be chosen to be independent of a and b. By the bound (2.8) for f, and the Schwarz inequality,

$$\psi_{32}(x_0) \leqslant \frac{C r_0}{2\pi} \left\{ \left\| \log \frac{1}{\sigma} \right\|_{1, A_3} + M^m \left\| \log \frac{1}{\sigma} \right\|_{2, A_3} \|\Psi^+\|_{2m, A_3}^m \right\},$$

where Ψ^+ is the non-negative part of Ψ'. Now A_3 lies in the left half of the ball $\mathcal{B}_\varepsilon$, so that

$$\iint_{A_3} \left(\log \frac{1}{\sigma} \right)^p \leqslant \pi \varepsilon^2 r_0^3 \int_0^1 \left(\log \frac{1}{\sigma} \right)^p \sigma d\sigma = \pi \varepsilon^2 r_0^3 2^{-p-1} p!, \quad (p = 1, 2),$$

and Ψ^+ vanishes on $r=r_0$, so that $\|\Psi^+\|_{2m, A_3}$ can be estimated in terms of ε, r_0 and $\|\Psi^+\|$ by means of (2.14), applied to the rectangle $R'=(r_0-\varepsilon r_0, r_0)\times(-\varepsilon r_0, \varepsilon r_0)$. Since $\|\Psi^+\| = \|\Psi\|_A < \eta^{1/2}$ by (5.5), we obtain

$$\psi_{32}(x_0) < \tfrac{1}{8}C\varepsilon^2 r_0^4\{1+M^m c_m' r_0^{m/2}\eta^{m/2}\}, \tag{5.9}$$

where c_m' depends only on m.

(iii) Now let $\varepsilon_1(r_0)$ be the value of ε that makes the right-hand member of (5.9) equal to $\eta/\pi W r_0$, and choose $\varepsilon=\varepsilon_0(r_0)=\min\{\tfrac{1}{2}, \varepsilon_1(r_0)\}$. Then in (5.8) we have $K(\varepsilon)=K_0(r_0)$, say, and hence the estimate

$$\psi(r_0, 0)=\tfrac{1}{2}Wr_0^2+k < \frac{\eta}{\pi W}\{K_0(r_0)+1\}\frac{1}{r_0}, \tag{5.10}$$

where K_0 is a known, non-decreasing function defined for $r_0 > 0$ and such that

$$K_0(r_0) \sim 2^{1/2}(5+\tfrac{1}{2}m)\log r_0 \quad \text{as} \quad r_0 \to \infty.$$

(5.10) implies an upper bound, say r_1, for r_0 when $a \geqslant a_0$, and we define r_* to be the greater of a_0 and r_1.

Remark. The following lemma is necessary because Lemma 5B fails to bound z on A_ψ if $k=0$ and $r \to 0$.

LEMMA 5D. *Let $a \geqslant a_0 > 0$. Then there exists a number z_*, independent of a, b and of the flux constant k, such that $|z| < z_*$ if $(r, z) \in A_\psi$.*

Proof. (i) Choosing positive numbers b_0 and k_0, we may suppose that $b \geqslant b_0$ (for otherwise $|z| < b_0$ on A_ψ) and that $k \leqslant k_0$ (for otherwise $|z| < \eta/4Wk_0$ on A_ψ, by Lemma 5B). We also choose a number $\varrho > 0$, and first bound $\psi(x_0)$ independently of x_0, a, b and k for $r_0 \leqslant \varrho$.

Let R be the square $(0, 2\varrho)\times(z_0-\varrho, z_0+\varrho)$ and abbreviate A_ψ to A. By (5.3a) and (5.7)

$$\lambda\iint_{A-R} G(x_0, x)f(\Psi) \leqslant \frac{\lambda r_0^2}{\pi\varrho^3}\iint_{A-R} r^2 f(\Psi) \leqslant \frac{2\eta}{\pi W\varrho}.$$

Also, $\quad \lambda\iint_R G(x_0, x)f(\Psi) \leqslant \frac{\lambda\varrho}{2^{\frac{1}{2}}\pi}\iint_R \sinh^{-1}\left|\frac{2\varrho}{x_0-x}\right|f(\Psi),$

and this can be bounded as $\psi_{32}(x_0)$ was bounded in the proof of Lemma 5C, since $\Psi^+(x)=0$ on $r=0$. Hence there exists a number β (independent of $r_0 \leqslant \varrho$, z_0, $a \geqslant a_0$, $b \geqslant b_0$ and $k \leqslant k_0$) such that $\psi(x_0) \leqslant \beta$, and then

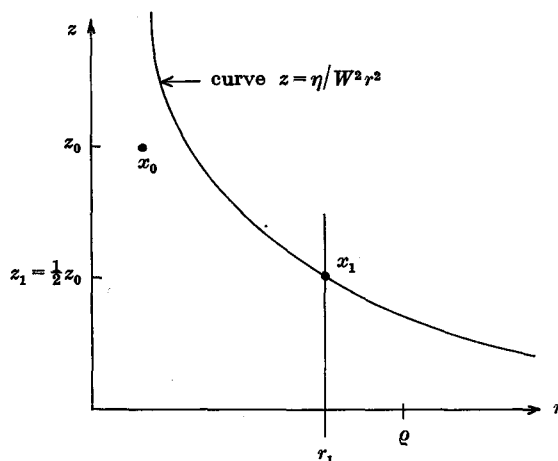$$f(\Psi(x)) < 1+(M\beta)^m = M'(\text{say}) \quad \text{for} \quad r \leqslant \varrho. \tag{5.11}$$

Fig. 2. Notation used in the proof of Lemma 5 D.

(ii) Now let $x_0 \in A$, so that $z_0 < \eta/W^2 r_0^2$ by Lemma 5 B, and suppose that $z_0 \geqslant 2\eta/W^2 \varrho^2$. Let x_1 be the point on the curve $z = \eta/W^2 r^2$ such that $z_1 = \frac{1}{2} z_0$ (Figure 2). Estimating as before by means of (5.3 a) and (5.7), we obtain

$$\lambda \iint_{r > r_1} G(x_0, x) f(\Psi) < \frac{16\eta}{\pi W} \frac{r_0^2}{z_0^3}.$$

With $\lambda \leqslant C$ (where C is independent of $a \geqslant a_0$, $b \geqslant b_0$ and $k \leqslant k_0$) and with $r_1^2 = 2\eta/W^2 z_0 \leqslant \varrho^2$, we have from (5.4) and (5.11)

$$\lambda \iint_{r < r_1} G(x_0, x) f(\Psi) < CM' \tfrac{1}{8} r_0^2 (2r_1^2 - r_0^2) < CM' \tfrac{1}{4} r_0^2 r_1^2 = \frac{CM'\eta}{2W^2} \frac{r_0^2}{z_0}.$$

But $\psi(x_0) > \frac{1}{2} W r_0^2$ because $x_0 \in A$; hence

$$\frac{16\eta}{\pi W} \frac{1}{z_0^3} + \frac{CM'\eta}{2W^2} \frac{1}{z_0} > \frac{\psi(x_0)}{r_0^2} > \tfrac{1}{2} W,$$

and this implies an upper bound, say z_2, for z_0, under the assumed conditions. Finally, we define z_* to be the greatest of b_0, $\eta/4 W k_0$, $2\eta/W^2 \varrho^2$ and z_2.

THEOREM 5E. *If $a \geqslant a_0 > 0$ and $b \geqslant b_0 > 0$, then A_ψ is a subset of*

$$\Omega = \left\{ (r, z) \, \middle| \, 0 < r < r_*, \, |z| < z_*, \, |z| < \frac{\eta}{W(Wr^2 + 4k)} \right\}, \tag{5.12}$$

where r_* is independent of a and b, and z_* is independent of a, b and k.

Proof. The result follows from Lemmas 5 B to 5 D.

5.3. The solution for the half-plane Π

Let Π denote the half-plane $\{x \,|\, r > 0\}$. By *the problem* (2.6, Π) we mean the problem that results from

(i) replacing D by Π in (2.6),

(ii) adding the condition: $\psi \to 0$ and $|\nabla \psi| \to 0$ as $r^2 + z^2 \to \infty$.

We now solve this problem by considering a sequence $\{D_j\}$, $j = 1, 2, \ldots$, of domains such that (a) $\Omega \subset D_j \subset D_{j+1}$ for each j, (b) $a_j \to \infty$ as $j \to \infty$ and (c) b_j/a_j is bounded away from zero and infinity. The corresponding solutions are denoted by (ψ_j, λ_j); we have $c \leqslant \lambda_j \leqslant C$ with c, C independent of j because the integral defining λ in (3.7) need only be taken over Ω. Initially, to retain compactness properties, we restrict attention to the bounded domain Ω; once we have established a limit function ψ in $C^{1+\nu}(\overline{\Omega})$, we can use the limiting form of (5.1) to extend ψ to the whole half-plane.

LEMMA 5F. (a) *For any* $\nu \in (0, 1)$ *there exist a function* $\psi \in C^{1+\nu}(\overline{\Omega})$, *a number* $\lambda \in [c, C]$, *and a subsequence* $(\psi_{j_n}, \lambda_{j_n})$ *which we relabel* (ψ_n, λ_n) *such that*

$$(\psi_n, \lambda_n) \to (\psi, \lambda) \quad \text{in} \quad C^{1+\nu}(\overline{\Omega}) \times \mathbf{R} \quad \text{as} \quad n \to \infty;$$

also, $\psi = O(r^2)$ *and* $\psi_r = O(r)$ *for* $r \to 0$.

(b) $\psi(r, -z) = \psi(r, z)$ *on* $\overline{\Omega}$; $\psi_z < 0$ *in* $\Omega_+ = \{x \in \Omega \,|\, z > 0\}$; *and* $\psi > 0$ *in* Ω.

(c) *The set* $\Psi^{-1}(0)$ *has measure zero.*

(d) $f(\Psi_n(x)) \to f(\Psi(x))$ *almost everywhere on* $\overline{\Omega}$, *as* $n \to \infty$.

Proof. This proceeds just as for Lemmas 4A, 4C, 4D and 4E. It is not difficult, at the first stage, to bound the Green function Γ_j of the cylinder V_j, and its derivatives up to the second order, independently of j.

LEMMA 5G. (a) *The limit function* ψ *in Lemma* 5F *has the representation*

$$\psi(x_0) = \lambda \iint_\Omega P(x_0, x) f(\Psi), \quad \Psi = \Psi(x), \tag{5.13}$$

for $x_0 \in \overline{\Omega}$; *here* P *is the fundamental solution* (5.2 b).

(b) *Extend* ψ_n *to* $\overline{\Pi}$ (*as a piecewise* C^1 *function*) *by defining it to be zero outside* $\overline{D}_n$. *Extend* ψ *to* $\overline{\Pi}$ *by means of* (5.13). *Then* $\psi_n \to \psi$ *and* grad $\psi_n \to$ grad ψ *uniformly on* $\overline{\Pi}$.

(c) *The properties* (b) *in Lemma* 5F *remain true in* Π, *and* $\Psi < 0$ *in* $\Pi - \overline{\Omega}$.

Proof. Let G_n be the Green function of the domain D_n, and define it to be zero outside $\overline{D}_n \times \overline{D}_n$. Let $\partial_0^\beta = (\partial/\partial r_0)^{\beta_1} (\partial/\partial z_0)^{\beta_2}$, where β is a double index. We can prove both (a) and (b) by showing that

$$\sup_{x_0 \in \Pi} \left| \lambda \int\!\!\int_\Omega \partial_0^\beta P(x_0, x)\, f(\Psi) - \lambda_n \int\!\!\int_\Omega \partial_0^\beta G_n(x_0, x)\, f(\Psi_n) \right| \to 0 \quad \text{for } |\beta| \leqslant 1 \qquad (5.14)$$

as $n \to \infty$, since the second integral is known to be $\partial_0^\beta \psi_n(x_0)$, which tends to $\partial_0^\beta \psi(x_0)$ on $\overline{\Omega}$.

Now we know from Lemma 5F that $\lambda_n \to \lambda$ and that $f(\Psi_n) \to f(\Psi)$ almost everywhere in Ω. To bound $P - G_n$ and its first derivatives, we note that

$$P(x_0, x) - G_n(x_0, x) = r_0\, r \int_{-\pi}^{\pi} \cos \omega\, \tilde{\gamma}_n(x_0, x, \omega)\, d\omega, \qquad (5.15)$$

where $\tilde{\gamma}_n(x_0, x, \theta_0 - \theta) = \gamma_n(X_0, X)$ is the difference between the fundamental solution and the Green function of the Laplace operator in the cylinder V_n corresponding to D_n; it may be written $\gamma_{n,0} + \gamma_{n,1}$ where, for fixed X such that $x \in \Omega$,

$$\Delta_0 \gamma_{n,i} = 0 \quad \text{for } X_0 \in V_n \quad \text{and } i = 0, 1,$$

$$\gamma_{n,0} = \frac{1}{4\pi |X_0|} \quad \text{and} \quad \gamma_{n,1} = \frac{1}{4\pi} \left\{ \frac{1}{|X_0 - X|} - \frac{1}{|X_0|} \right\} \quad \text{for } X_0 \in \partial V_n.$$

Then $\gamma_{n,0}$ is independent of $\theta_0 - \theta$ and does not contribute to the integral in (5.15), while for $x \in \Omega$ we have $|\gamma_{n,1}| \leqslant \text{const.}/a_n^2$ by the maximum principle, for sufficiently large values of the radius a_n of V_n. The estimate (II.3) in Appendix II shows that $|\nabla_0 \gamma_n| \leqslant \text{const.}/a_n^2$ for $x \in \Omega$; thus, for $|\beta| \leqslant 1$,

$$\left| \partial_0^\beta \{P(x_0, x) - G_n(x_0, x)\} \right| \leqslant \text{const.}/a_n \to 0 \quad \text{on} \quad D_n \times \Omega, \quad \text{as } n \to \infty,$$

and this estimate is easily extended to $\overline{\Pi} \times \Omega$ because $G_n = 0$ on $(\overline{\Pi} - D_n) \times \Omega$. Accordingly (5.14) follows from Lebesgue's dominated convergence theorem.

Regarding (c): the three familiar properties are proved as before. Since the sequence $\{\psi_n\}$ makes $\Psi \leqslant 0$ in $\Pi - \overline{\Omega}$ and since $L\Psi = 0$ there by (5.13), the maximum principle shows that $\Psi < 0$ there.

THEOREM 5H. *The pair (ψ, λ) defined by Lemmas 5F and 5G is a pointwise solution of the problem $(2.6, \Pi)$ in the sense of Theorem 3C if f is continuous, and of Theorem 4F otherwise.*

Proof. Multiplying (5.13) by $-\sin \theta_0/r_0$ and $\cos \theta_0/r_0$, and inserting (5.2b) for P, we obtain

$$\alpha(X_0) = \frac{\lambda}{4\pi} \int [-X_2, X_1, 0] \frac{f(\Psi)}{|X_0 - X|}\, dX,$$

and this implies the result by the method of Appendix II, but more simply.

5.4. Variational characterization of the solution for Π

The variational principle has not yet been considered in connection with the half-plane Π. Let $H(\Pi)$, $L_p(\Pi, \tau)$ and $J(u)$ be defined as in sections 2.4 and 3.1 but with Π replacing D. Sets bounded in $H(\Pi)$ need not be bounded, let alone compact, in $L_p(\Pi, \tau)$, and the methods of Theorem 3A are inadequate for discussion of the functional J on the space $H(\Pi)$. Nevertheless, it turns out that the variational characterization of ψ survives the limiting process $D \to \Pi$. In Appendix III we prove

THEOREM 5I. *The functional J is uniformly continuous on any bounded subset of $H(\Pi)$, and the solution ψ established in Theorem* 5H *maximizes $J(u)$ over the sphere $\{\|u\|^2 = \eta\}$ in $H(\Pi)$.*

Finally, we relate $J(\psi)$ to the energy $\pi \varrho \eta$ and *impulse* $\pi \varrho p$ of the vortex ring; here ϱ is the fluid density, and the impulse ([15], §§ 152, 162) is defined by

$$p = \iint_\Pi r\omega = \lambda \iint_\Pi r^2 f(\Psi).$$

Integration by parts shows that

$$\iint_\Pi f(\Psi)(r\psi_r - 2z\psi_z) = \iint_\Omega F'(\Psi)\{(r\Psi_r + Wr^2) - 2z\Psi_z\} = W\iint_\Omega r^2 F'(\Psi) = \frac{pW}{\lambda},$$

$$\iint_\Pi f(\Psi)(r\psi_r + z\psi_z) = -\frac{1}{\lambda}\iint_\Pi \frac{1}{r^2} L\psi(r\psi_r + z\psi_z) = \frac{1}{2\lambda}\|\psi\|^2;$$

subtracting these two, we therefore obtain

$$J(\psi) = \iint_\Pi F(\psi) = -\iint_\Pi zf(\Psi)\psi_z = \frac{1}{3\lambda}(pW - \tfrac{1}{2}\eta).$$

Evidently the variational principle has no obvious physical interpretation. In particular, it *cannot* be said that ψ maximizes the impulse for given energy. For, (a) the constant λ is not prescribed, but determined only after the maximization; (b) if, as the statement implies, $\pi\varrho\|u\|^2$ denotes the energy associated with any function $u \in H(\Pi)$, then, no matter whether we take $-Lu/r$ or $\lambda rf(u - \tfrac{1}{2}Wr^2 - k)$ as the vorticity associated with u, the final expression for J holds *only for a solution ψ*.

Appendix I. Steiner symmetrization of functions in $H^+(D)$

The following is an adaptation, with some extensions, of ideas found in [23] and, for example, [10], [19]. We have not seen elsewhere the joint use of symmetrization and weak convergence that characterizes Theorem 1C.

(i) Let l_ϱ denote the line $\{r=\varrho, \text{ all } z\}$. Given an open or closed set $E \subset \bar{D}$, write $l_{E,\varrho} = E \cap l_\varrho$, denote its one-dimensional Lebesgue measure by $|l_{E,\varrho}|$, and define

$$l_{E,\varrho}^* = \{(r, z)\,|\,r = \varrho, \ -\tfrac{1}{2}|l_{E,\varrho}| \prec z \prec \tfrac{1}{2}|l_{E,\varrho}|\},$$

where the symbol $\prec$ means $<$ if E is open, and $\leqslant$ if E is closed, with the understanding that $l_{E,\varrho}^*$ is empty when $l_{E,\varrho}$ is. *The Steiner symmetrization of the set E* (about the line $z = 0$) is then defined to be

$$E^* = \bigcup_{0 \leqslant \varrho \leqslant a} l_{E,\varrho}^*.$$

It is clear that $E_1 \subset E_2$ implies $E_1^* \subset E_2^*$, and that E^* has the same plane Lebesgue measure (and τ-measure) as E; one can also show that if E is open [closed] then E^* is open [closed].

(ii) Let $C_0^+(D)$ be the set of non-negative continuous functions that have compact support in D, and let $x = (r, z)$. *The Steiner symmetrization φ^* of a function $\varphi \in C_0^+(D)$* is defined by

$$\varphi^*(x) = c \quad \text{if and only if} \quad x \in \{\varphi^{-1}[c, \infty)\}^* - \{\varphi^{-1}(c, \infty)\}^*. \tag{I.1}$$

One proves the following lemma with no great difficulty; item (b) is most easily proved in the first instance for step functions (to which (I.1) is readily adapted), and (c) is proved for functions of one variable in [9], p. 278.

LEMMA IA. *If $\varphi \in C_0^+(D)$, then $\varphi^* \in C_0^+(D)$ and*

(a) φ^* *is an even function of z, non-increasing for $z \geqslant 0$;*

(b) *if $g: [0, \infty) \times [0, a] \to \mathbf{R}$ is continuous (say), then*

$$\iint_D g(\varphi^*(x), r)\,dr\,dz = \iint_D g(\varphi(x), r)\,dr\,dz;$$

(c) *if $h: (0, a) \to [0, \infty)$ is continuous, and $\chi \in C_0^+(D)$, then*

$$\iint_D \varphi^* \chi^* h(r)\,dr\,dz \geqslant \iint_D \varphi \chi h(r)\,dr\,dz.$$

(iii) To discuss integrals of Dirichlet type, we introduce the set $P_0^+(D) \subset C_0^+(D)$ of piecewise linear functions (such a function is linear on each of a finite number of closed triangles within D). For this particular class the simplest geometrical arguments in [23] are sufficient to prove

LEMMA IB. *If $\varphi \in P_0^+(D)$ and $h: (0, a) \to [0, \infty)$ is continuous, then*

$$\iint_D (\varphi_r^{*2} + \varphi_z^{*2})\,h(r)\,dr\,dz \leqslant \iint_D (\varphi_r^2 + \varphi_z^2)\,h(r)\,dr\,dz.$$

(iv) We now extend results of the foregoing kind to the space $H^+(D)$, which is the completion, in the norm $\|\cdot\|$ of $H(D)$, of the set of non-negative functions in $C_0^\infty(D)$.

THEOREM IC. (a) *For any $u \in H^+(D)$, there exists a sequence $\{\varphi_n\}$ of functions in $P_0^+(D)$ such that $\varphi_n \to u$ in $H(D)$ as $n \to \infty$.*

(b) *The sequence $\{\varphi_n^*\}$ of symmetrized functions is a Cauchy sequence in $L_2(D, \tau)$, and without ambiguity we define u^* to be the limit of $\{\varphi_n^*\}$ in $L_2(D, \tau)$.*

(c) *It follows that $u^* \in H^+(D)$, that $\varphi_n^* \to u^*$ weakly in $H^+(D)$, and hence that $\|u^*\|_p = \|u\|_p$, $(p \geqslant 1)$ and $\|u^*\| \leqslant \|u\|$.*

(d) *If $g: [0, \infty) \times [0, a] \to \mathbf{R}$ is continuous and (for all $s \geqslant 0$, $t \geqslant 0$ and $r \in [0, a]$)*

$$\left| g(s,r) - g(t,r) \right| \leqslant \text{const.} \ \{1 + s^c + t^c\} \left| s - t \right|, \quad (c \geqslant 0),$$

then
$$\iint_D g(u^*(x), r) \, dr \, dz = \iint_D g(u(x), r) \, dr \, dz.$$

Proof. (a) There exists a sequence $\{\chi_n\}$ of non-negative functions in $C_0^\infty(D)$ such that $\|u - \chi_n\| \to 0$. Taking a sequence of increasingly fine triangulations T_k of D by isosceles right-angled triangles, we construct $\varphi_{n,k} \in P_0^+(D)$ by setting $\varphi_{n,k} = \chi_n$ at each vertex. Then as $k \to \infty$ with n fixed, $\operatorname{grad} \varphi_{n,k} \to \operatorname{grad} \chi_n$ uniformly on D, and so $\varphi_{n,k} \to \chi_n$ in $H(D)$, for the weighting function $1/r$ in the norm of $H(D)$ is rendered harmless by the compact support of χ_n. Choosing $\varphi_n = \varphi_{n,k_n}$, where $\|\chi_n - \varphi_{n,k_n}\| < 1/n$, we have desired sequence.

(b) By (b) and (c) of Lemma I A,

$$\iint_D (\varphi_m^{*2} + \varphi_n^{*2}) = \iint_D (\varphi_m^2 + \varphi_n^2) \quad \text{and} \quad -2 \iint_D \varphi_m^* \varphi_n^* \leqslant -2 \iint_D \varphi_m \varphi_n,$$

so that $\|\varphi_m^* - \varphi_n^*\|_2 \leqslant \|\varphi_m - \varphi_n\|_2$, and $\{\varphi_n^*\}$ is a Cauchy sequence in $L_2(D, \tau)$. One readily verifies that, for given u, different approximating sequences $\{\varphi_n\}$ lead to the same element u^* of $L_2(D, \tau)$.

(c) Lemma I B, with $h(r) = 1/r$, shows that $\|\varphi_n^*\| \leqslant \|\varphi_n\|$. Thus $\{\varphi_n^*\}$ is bounded in $H^+(D)$, and there exists a subsequence $\{\varphi_{n_j}^*\}$ and an element $v^* \in H^+(D)$ such that, as $j \to \infty$, $\varphi_{n_j}^* \to v^*$ weakly in $H(D)$ and hence strongly in each $L_p(D, \tau)$, $p \geqslant 1$. Therefore $v^* = u^*$ and $u^* \in H^+(D)$. Because $\varphi_n^* \to u^*$ in $L_2(D, \tau)$, we have for any $\chi \in C_0^\infty(D)$

$$\langle \varphi_n^*, \chi \rangle = -\iint_D \frac{1}{r^2} \varphi_n^* L\chi \to -\iint_D \frac{1}{r^2} u^* L\chi = \langle u^*, \chi \rangle,$$

and such functions χ are dense in $H(D)$; accordingly, the whole sequence $\{\varphi_n^*\}$ converges

to u^* weakly in $H(D)$ and strongly in each $L_p(D, \tau)$. Therefore, the result $\|\varphi_n^*\|_p = \|\varphi_n\|_p$ remains true in the limit; also,

$$\|u^*\| \leqslant \lim \inf \|\varphi_n^*\| \leqslant \lim \inf \|\varphi_n\| = \|u\|,$$

since $\varphi_n \to u$ (strongly) in $H(D)$.

(d) We apply (b) of Lemma I A to φ_n^* and φ_n, and note that the conditions on g are sufficient to make the integral continuous with respect to weak convergence in $H(D)$, (cf. the proof of Theorem 3 A, step (ii)).

Appendix II. Regularity theory

Let $U = \{X \mid r < a\}$ be the infinite cylinder corresponding to V; we begin by considering the Green functions

$$\Gamma_U = \frac{1}{4\pi R} - \gamma_U(X^0, X) \quad \text{and} \quad \Gamma = \frac{1}{4\pi R} - \gamma(X^0, X), \quad (R = |X - X^0|),$$

of the Dirichlet problem for the Laplace operator in U and V respectively; Γ_U is relevant because of (II.1) below. In addition to the usual symmetry and positivity properties ($\gamma(X, X^0) = \gamma(X^0, X)$ and $0 < \gamma < 1/4\pi R$ on $V \times V$), these functions have the following behaviour.

(a) Γ_U depends only on r, r^0, $\theta - \theta^0$ and $z - z^0$; it is an even function of $\theta - \theta^0$ and $z - z^0$, and has period 2π in $\theta - \theta^0$.

(b) Let $j_{0,1} = 2.40 \ldots$ denote the smallest positive zero of the Bessel function J_0. There exists a constant K such that

$$\Gamma_U(X^0, X) \leqslant K \exp (-j_{0,1}|z - z^0|/a) \quad \text{for} \quad |z - z^0| \geqslant a,$$

and similarly for the modulus of each derivative of Γ_U.

(c) Let $U_\delta = \{X \mid r < a - \delta\}$ and $V_\delta = \{X \in V \mid r < a - \delta\}$, where $\delta > 0$. Then γ_U is (real) analytic on $\bar{U} \times \bar{U}_\delta$, and harmonic in each variable ($\Delta^0 \gamma_U = \Delta \gamma_U = 0$) there.

(d) With $(X^0, X) \in \bar{V} \times \bar{V}$ and $X^0 \neq X$, and with the same values of X_1^0, X_2^0, X_1 and X_2 implied on each side of the equation, we have

$$\Gamma(z^0, z) = \Gamma_U(z - z^0) - \Gamma_U(z + z^0 - 2b) + \Gamma_U(z - z^0 - 4b) - \Gamma_U(z + z^0 - 6b) + \ldots$$

$$- \Gamma_U(z + z^0 + 2b) + \Gamma_U(z - z^0 + 4b) - \Gamma_U(z + z^0 + 6b) + \ldots, \quad (\text{II.1})$$

for it is readily verified that this formula makes $\Gamma = 0$ on $z = \pm b$. The series converges rapidly because of (b). With the notation $X = (r, \theta, z)$, denote the primary image points of X^0 with respect to the planes $z = \pm b$ by

$$X^{(1)} = (r^0, \theta^0, 2b - z^0) \quad \text{and} \quad X^{(2)} = (r^0, \theta^0, -2b - z^0),$$

and let $R_j = |X - X^{(j)}|$. It then follows from (c) and (II.1) that

$$\Gamma(X^0, X) = \frac{1}{4\pi}\left(\frac{1}{R} - \frac{1}{R_1} - \frac{1}{R_2}\right) + A(X^0, X), \tag{II.2}$$

where A is analytic on $V \times V_\delta$, and harmonic in each variable there.

(e) We can also estimate derivatives like $\partial\gamma/\partial r^0$ on $V \times V$ by means of the maximum principle, without using (II.1). To this end, we define the pseudo-image points

$$X^{(3)} = (2a - r^0, \theta^0, z^0),\ X^{(4)} = (2a - r^0, \theta^0, 2b - z^0)\ \text{and}\ X^{(5)} = (2a - r^0, \theta^0, -2b - z^0).$$

One can show that

$$1 \leqslant \frac{R_3}{R} \leqslant \frac{3a - r^0}{a + r^0} \equiv B, \quad \text{say, for } r^0 < a, r = a, \text{ all } z - z^0,$$

$$\text{and for } r^0 < a, r \leqslant a, |z - z^0| \geqslant 2a;$$

and that, for fixed $X^0 \in U$, the function $(B^3/4\pi)\partial(1/R_3)/\partial r^0$, which is a positive harmonic function of $X \in \bar{U}$, dominates $\pm \partial\gamma_U/\partial r^0$ on ∂U and hence (by the maximum principle) on $\bar{U}$. In this way, one ultimately obtains the estimate

$$|\nabla^0\gamma(X^0, X)| \leqslant \frac{1}{4\pi}\sum_{j=1}^{2}\left|\nabla_j\frac{1}{R_j}\right| + \frac{cB^3}{4\pi}\sum_{j=3}^{5}\left|\nabla_j\frac{1}{R_j}\right| \quad \text{on} \quad V \times V, \tag{II.3}$$

where ∇_j is the gradient operator $\{\partial/\partial X_i^{(j)}\}$, $i = 1, 2, 3$, and c is a constant depending only on b/a, with $c = 1$ for $b \geqslant a$. It follows immediately (since $R_j \geqslant R$ on $V \times V$ and $B \leqslant 3$) that

$$|\nabla^0\Gamma(X^0, X)| \leqslant \text{const.}\frac{1}{R^2} \quad \text{on} \quad V \times V, \tag{II.4}$$

where the constant depends only on b/a.

LEMMA II A. *If (ψ, λ) is a generalized solution, the vector potential $\boldsymbol{\alpha} \in \overset{\circ}{W}_{1,2}(V)$ and has the representation*

$$\boldsymbol{\alpha}(X^0) = \lambda \int_V \Gamma(X^0, X)[-X_2, X_1, 0]f(\Psi)\,dX, \quad (\Psi = \Psi(x)). \tag{II.5}$$

Proof. That $\boldsymbol{\alpha} \in \overset{\circ}{W}_{1,2}(V)$ was noted after (2.11). To prove (II.5), we shall choose the test function φ in (3.1) to be a mollified Green function for L in D. Let $\mu \in C^\infty(\mathbf{R})$ be a non-decreasing function such that $\mu(t) = 0$ for $t \leqslant \frac{1}{2}$ and $\mu(t) = 1$ for $t \geqslant 1$, and define

$$\Gamma_n(X^0, X) = \tilde{\Gamma}_n(x^0, x, \theta - \theta^0) = \frac{1}{4\pi R}\, \mu(nR) - \gamma(X^0, X), \quad (n = 1, 2, \ldots), \qquad \text{(II.6a)}$$

$$G_n(x^0, x) = r^0 r \int_{-\pi}^{\pi} \cos \omega\, \tilde{\Gamma}_n(x^0, x, \omega)\, d\omega. \qquad \text{(II.6b)}$$

For the moment we restrict x^0 to that rectangular subset, say D_n, of D whose distance from ∂D is $2/n$. Then $\tilde{\Gamma}_n = \tilde{\Gamma}$ for $d(x, \partial D) < 1/n$, and one can verify that $\partial G_n/\partial r$ and $\partial G_n/\partial z$ are $0(r)$ for $r \to 0$, and that $G_n(x^0, .)$ belongs to $H(D)$. Accordingly, we choose $\varphi(x) = G_n(x^0, x)$ in (3.1) and multiply by $-\sin \theta^0/r^0$ and by $\cos \theta^0/r^0$ to obtain, after a little manipulation

$$\int_V \nabla \Gamma_n(X^0, X) \cdot \nabla[\alpha_1, \alpha_2, 0]\, dX = \lambda \int_V \Gamma_n(X^0, X)\, [-X_2, X_1, 0]\, f(\Psi)\, dX,$$

where $\nabla = \{\partial/\partial X_i\}$ and X^0 belongs to the figure of revolution V_n generated by D_n. Integrating by parts on the left-hand side, and introducing the notation, for $X^0 \in V$,

$$\alpha_n(X^0) \equiv - \int_V \Delta \Gamma_n(X^0, X)\, \alpha(X)\, dX,$$

$$\begin{bmatrix} \beta_n(X^0) \\ \beta\ (X^0) \end{bmatrix} \equiv \lambda \int_V \begin{bmatrix} \Gamma_n(X^0, X) \\ \Gamma\ (X^0, X) \end{bmatrix} [-X_2, X_1, 0]\, f(\Psi)\, dX,$$

we see that $\alpha_n(X^0) = \beta_n(X^0)$ for $X^0 \in V_n$. It is sufficient to prove that $\alpha = \beta$ in $L_2(V)$, and with $\|.\|_{2,V}$ denoting the norm of $L_2(V)$ for the moment, we have

$$\|\alpha - \beta\|_{2,V} \leqslant \|\alpha - \alpha_n\|_{2,V} + \|\alpha_n - \beta_n\|_{2,V-V_n} + \|\beta_n - \beta\|_{2,V}.$$

Now the kernel $-\Delta\Gamma_n$ in the definition of α_n is a mollifying (or 'averaging') kernel: (II.6a) implies, since $\Delta\gamma = 0$, that $-\Delta\Gamma_n$ vanishes outside the ball $\{R < 1/n\}$, and the divergence theorem shows that the integral of $-\Delta\Gamma_n$ over this ball is 1, while the integral of $|\Delta\Gamma_n|$ is easily bounded. It follows from standard theory that

$$\|\alpha - \alpha_n\|_{2,V} \to 0 \quad \text{and} \quad \|\alpha_n\|_{2,V-V_n} \leqslant \|\alpha\|_{2,V-V_n} \to 0 \quad \text{as } n \to \infty.$$

We easily bound $\sup_V |\beta_n(X^0)|$ and $\sup_V |\beta(X^0) - \beta_n(X^0)|$ by means of the Schwarz inequality, since $|\Gamma_n| \leqslant 1/4\pi R$, $\Gamma = \Gamma_n$ for $R \geqslant 1/n$, and $f(\Psi) = f(\Psi^+)$ is in $L_2(V)$ with a norm that depends only on f, V and η (cf. the proof of Theorem 3A). Then $\|\beta_n\|_{2,V-V_n} \to 0$ and $\|\beta_n - \beta\|_{2,V} \to 0$ as $n \to \infty$.

THEOREM II B. *The vector potential* $\alpha \in C^{2+\mu}(\bar{V})$ *and satisfies* (2.10a, b) *pointwise.*

Proof. The bound described in Lemma II A for $\sup_V |\beta_n(X^0)|$ now serves also for $\alpha(X^0)$, and, since $\psi = X_1\alpha_2 - X_2\alpha_1$, the functions ψ, Ψ and $f(\Psi)$ are also bounded point-

wise. It is then easy to use the estimate (II.4) to show that $\boldsymbol{\alpha}$ is uniformly Lipschitz continuous on V (indeed, this would follow if $f(\Psi)$ were merely in $L_p(V)$, $p > 3$). For consider two points X^0 and $X' = X^0 + h$ in V, and write $R' = |X - X'|$; bounding $|\Gamma(X', X) - \Gamma(X^0, X)|$ for $R \geqslant 2|h|$ by means of (II.4), we obtain

$$|\alpha_i(X^0 + h) - \alpha_i(X^0)| \leqslant \text{const.} \left\{ \int_{R < 2|h|} \left(\frac{1}{R} + \frac{1}{R'} \right) dX + \int_{V \cap \{R > 2|h|\}} \frac{|h|}{R^2} dX \right\}$$

$$\leqslant \text{const.} \; |h|, \quad (i = 1, 2), \tag{II.7}$$

where the constant depends only on the data of the problem.

This result allows us to extend $\boldsymbol{\alpha}$ to $\bar{V}$ as a Lipschitz continuous function, and since $\boldsymbol{\alpha}$ then belongs to $\overset{\circ}{\mathbf{W}}_{1,2}(V) \cap \mathbf{C}(\bar{V})$, it vanishes on ∂V. Then Ψ is Lipschitz continuous and equal to $-\frac{1}{2} W a^2 - k$ on $r = a$, and there exists a number $\delta > 0$ such that for $r \geqslant a - \delta$ we have $\Psi \leqslant 0$ and hence $f(\Psi) = 0$. Therefore we can restrict X to V_δ in (II.5) and use the form (II.2) of Γ. But this means that the component $\alpha_2(X^0)$, say, can be regarded as the sum of three Newtonian potentials

$$u_0(X^0) = \frac{\lambda}{4\pi} \int_{V_\delta} \frac{1}{R} X_1 f(\Psi) \, dX,$$

$$u_j(X^{(j)}) = -\frac{\lambda}{4\pi} \int_{V_\delta} \frac{1}{R_j} X_1 f(\Psi) \, dX, \quad (j = 1, 2),$$

and of a fourth function that is clearly analytic and harmonic on $\bar{V}$ because the kernel A is. Moreover, the density function $X_1 f(\Psi)$ of the Newtonian potentials belongs to $C^\mu(\bar{V}_\delta)$, by (2.7 c) and the Lipschitz continuity of Ψ, and is zero on ∂V_δ. Under these circumstances it is classical that $u_j \in C^{2+\mu}(\mathbf{R}^3)$, $(j = 0, 1, 2)$, that u_0 satisfies the second component of the differential equation (2.10 a), and that u_1 and u_2 are harmonic functions of $X^0 \in \bar{V}$ (since $X^{(1)}$ and $X^{(2)}$ are then outside V_δ).

THEOREM II C. *The stream function* $\psi \in C^{2+\mu}(\bar{D})$ *and satisfies* (2.6a, b) *pointwise; also,* $\psi = O(r^2)$ *and* $\psi_r = O(r)$ *for* $r \to 0$.

Proof. Since $\psi(x) = X_1 \alpha_2 - X_2 \alpha_1$ and $\boldsymbol{\alpha} \in C^{2+\mu}(\bar{V})$, it is clear that $\psi \in C^{2+\mu}(\bar{D})$ and that $\psi = 0$ on $r = 0$; since also $\boldsymbol{\alpha} = 0$ on ∂V, ψ vanishes on ∂D. To check the differential equation (2.6a) we merely transform (2.10a), recalling that $r > 0$ in D. Finally,

$$\nabla \alpha_2 = \left(\left(\frac{\psi_r}{r} - \frac{\psi}{r^2} \right) \cos \theta, \; -\frac{\psi}{r^2} \sin \theta, \frac{\psi_z}{r} \cos \theta \right),$$

and the condition $\boldsymbol{\alpha} \in C^1(\bar{V})$ is therefore sufficient to bound ψ/r^2 and ψ_r/r uniformly on D.

Appendix III. The functional J on the space $H(\Pi)$

In this Appendix we show that the functional J is continuous on $H(\Pi)$, and hence that the solution ψ for the half-plane Π maximizes J over the sphere $S(\eta)$ in $H(\Pi)$. Here $H(\Pi)$, $S(\eta)$ and $J(u)$ are defined as in sections 2.4 and 3.1, but Π replaces D; the norm $\|.\|$ is now that of $H(\Pi)$.

LEMMA III A. *Let* $\Phi(x) = \varphi(x) - \frac{1}{2}Wr^2 - k$, *where* $\varphi \in C_0^\infty(\Pi)$ *and* $\|\varphi\|^2 \leqslant \eta$. *There exist numbers* $K_p = K_p(W, k, \eta)$ *and* $K = K(W, \eta)$ *such that*

(a) $\qquad \iint_\Pi (\Phi^+)^p \leqslant K_p, \quad p \geqslant 1, \quad and \quad$ (b) $\iint_{r > \varrho} (\Phi^+)^2 \leqslant \dfrac{K}{\varrho^2}.$ $\qquad$ (III. 1 a, b)

Proof. (a) Let $f(t) = p(Mt)^{p-1}$ for $t > 0$; let ψ be the corresponding solution of (2.6) for a domain D containing the support of φ; and suppose that $\|\varphi\|^2 = \eta$. Then

$$M^{p-1} \iint_\Pi (\Phi^+)^p = J(\varphi) \leqslant J(\psi) = M^{p-1} \iint_\Omega (\Psi^+)^p,$$

where Ω is as in (5.12). The last integral is bounded in terms of r_*, z_* and $\|\Psi^+\|$ by (2.13), and $\|\Psi^+\|^2 < \eta$ by (5.5). If $\|\varphi\|^2 < \eta$, define $\tilde{\varphi} = \eta^{1/2}\varphi/\|\varphi\|$; then $J(\varphi) < J(\tilde{\varphi})$ and $\|\tilde{\varphi}\|^2 = \eta$.

(b) Again we note that $\varphi \in H(D)$ for some D, and refer to Lemma 5 A. Let $l_{\varphi,r} = \{z \mid (r, z) \in A_\varphi\}$; since the set $(0, r) \times l_{\varphi,r}$ is a subset of B_φ and has τ-measure $\frac{1}{2}r^2 |l_{\varphi,r}|$, we deduce from (5.6) that

$$\tfrac{1}{2} W^2 r^2 |l_{\varphi,r}| \leqslant \|\varphi\|^2. \qquad (\text{III.2})$$

Now $l_{\varphi,r}$ is a countable union of open disjoint intervals I_n, say, of length $|I_n|$, such that $\Phi^+ = 0$ at the end-points of each. By a well known inequality ([9], p. 185]

$$\int_{I_n} (\Phi^+)^2 \, dz \leqslant \frac{|I_n|^2}{\pi^2} \int_{I_n} (\Phi_z^+)^2 \, dz,$$

so that

$$\int_{l_{\varphi,r}} (\Phi^+)^2 \, dz \leqslant \frac{1}{\pi^2} \left\{ \sum_n |I_n| \right\}^2 \left\{ \sum_n \int_{I_n} (\Phi_z^+)^2 \, dz \right\} \leqslant \frac{4 \|\varphi\|^4}{\pi^2 W^4 r^4} \int_{l_{\varphi,r}} \varphi_z^2 \, dz,$$

by (III.2). Multiplying by r and integrating with respect to r over $\{r > \varrho\}$, we obtain (III.1 b), with $K = 4\eta^3/\pi^2 W^4$.

THEOREM III B. *The functional* J *is uniformly continuous on any bounded subset of* $H(\Pi)$.

Proof. On any bounded set $B \subset H(\Pi)$ we have $\|\varphi\|^2 \leqslant \eta$ for some η, and it suffices to prove J uniformly continuous on the set $B \cap C_0^\infty(\Pi)$, which is dense in B. Let φ and ψ be any two functions in $B \cap C_0^\infty(\Pi)$ (in this proof ψ is not necessarily a solution); then

$$|J(\varphi) - J(\psi)| \leqslant \iint_{\Pi} \{1 + (M\Phi^+)^m + (M\Psi'^+)^m\} |\Phi^+ - \Psi'^+|$$

$$\leqslant \{|A_\varphi \cup A_\psi|^{\frac{1}{2}} + M^m(\|\Phi^+\|_{2m}^m + \|\Psi'^+\|_{2m}^m)\} \|\Phi^+ - \Psi'^+\|_2, \qquad \text{(III.3)}$$

by the Schwarz inequality. Now $|A_\varphi \cup A_\psi| \leqslant 2\eta/W^2$ by (5.5), and $\|\Phi^+\|_{2m}$ and $\|\Psi'^+\|_{2m}$ are bounded by (III.1a). Since $|\Phi^+ - \Psi'^+| \leqslant |\varphi - \psi|$ at any point, we also have

$$\|\Phi^+ - \Psi'^+\|_2 \leqslant \|\varphi - \psi\|_{2, \langle r < \varrho \rangle} + \|\Phi^+\|_{2, \langle r > \varrho \rangle} + \|\Psi'^+\|_{2, \langle r > \varrho \rangle}$$

$$\leqslant 2^{-\frac{1}{2}}\varrho^2 \|\varphi - \psi\| + \frac{2K^{\frac{1}{2}}}{\varrho}, \qquad \text{(III.4)}$$

where the term in ϱ^2 comes from integrating the estimate

$$u^2(r_0, z) = \left\{\int_0^{r_0} u_r \, dr\right\}^2 \leqslant \tfrac{1}{2} r_0^2 \int_0^{r_0} r^{-1} u_r^2 \, dr,$$

while the term in $1/\varrho$ comes from (III.1b). Choose $\varrho = \text{const.} \|\varphi - \psi\|^{-1/3}$; then (III.3) and (III.4) show that

$$|J(\varphi) - J(\psi)| \leqslant \varkappa \|\varphi - \psi\|^{1/3}, \quad \text{where } \varkappa = \varkappa(f, W, k, \eta).$$

THEOREM IIIC. *The solution ψ defined by Lemma 5F maximizes $J(u)$ over the sphere $S(\eta)$ in $H(\Pi)$.*

Proof. As in section 5.3, consider an expanding sequence $\{D_j\}$ of domains tending to Π; let $S_j(\eta) = \{u \in H(D_j) \mid \|u\|^2 = \eta\}$; let ψ_j be our maximizer of J over $S_j(\eta)$; and write

$$\sup_j J(\psi_j) = \sigma, \quad \sup_{u \in S(\eta)} J(u) = s.$$

The sequence $\{J(\psi_j)\}$ is non-decreasing because ψ_j, extended to be zero outside D_j, belongs to $S_{j+1}(\eta)$; therefore $J(\psi_j) \to \sigma$ as $j \to \infty$. Since $\psi_{j_n} \to \psi$ in $C^{1+\nu}(\overline{\Omega})$, we know that $J(\psi_{j_n}) \to J(\psi)$. Accordingly, $J(\psi) = \sigma$ and $\sigma \leqslant s$ because $\psi \in S(\eta)$.

On the other hand, there exists a sequence $\{u_n\}$ in $S(\eta)$ such that $J(u_n) \to s$ as $n \to \infty$, and because of Theorem IIIB, we can approximate each u_n by a function $\varphi_n \in S(\eta) \cap C_0^\infty(\Pi)$ such that $J(\varphi_n) \to s$. But each $\varphi_n \in S_j(\eta)$ for some $j = j(n)$; therefore $J(\varphi_n) \leqslant \sigma$ for each n, and if $\sigma < s$, then $J(\varphi_n) \nrightarrow s$. Hence $\sigma = s$.

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