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# A Golden Ratio Control Policy for a Multiple-Access Channel

ALON ITAI AND ZVI ROSBERG

*Abstract* —Consider $n$ stations sharing a single communications channel. Each station has a buffer of length one. If the arrival rate of station $i$ is $r_i$, then $1-\prod_i(1-r_i)$ is shown to be an upper bound (over all policies) on the throughput of the channel. Moreover, an optimal policy always exists and is stationary and periodic.

The throughput of two policies, the *random-policy* and the *golden-ratio policy*, are analyzed for a finite and infinite number of stations. The latter is shown to approach a limit which is within at least 98.4 percent of the upper bound.

## I. Introduction

Consider $n$ transmission stations sharing a single communication channel. Each station contains a buffer capable of storing a single packet; a message is *lost* if it arrives at a station whose buffer is full. The channel is assumed to be slotted, i.e., the channel time is divided into equal segments called slots. At each slot, several stations are given permission to transmit. When given permission, a station transmits a packet within the slot, if its buffer is not empty; otherwise, no message is sent. If a collision occurs (i.e., more than one station whose buffer is not empty is given permission to transmit at the same slot) all the messages transmitted are lost or stored separately for later retransmission. (Thus, our model does not take advantage of any information obtained from a collision.) A policy $\pi$ allocates to each slot a station; $\pi$ depends on the slot number $t$ and the state of the buffers.

Suppose that there is probability $r_i$ that a packet arrives at station $i$ during a time slot. Without loss of generality, $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_n > 0$.

Let $\boldsymbol{r}=(r_1, r_2,\cdots, r_n)$ and $V_T(\boldsymbol{r},\pi)$ be the expected number of packets successfully transmitted during the first $T$ slots using policy $\pi$. Define the *throughput of the channel* (under policy $\pi$)

$$\overline{V}(\boldsymbol{r},\pi)=\liminf_{T\to\infty} V_T(\boldsymbol{r},\pi)/T. \qquad (1.1)$$

Finally, let

$$\overline{V}(\boldsymbol{r})=\sup_{\pi}\overline{V}(\boldsymbol{r},\pi).$$

$\overline{V}$ is the *value function*. A control policy $\pi^*$ is optimal for $\boldsymbol{r}$ if $\overline{V}(\boldsymbol{r},\pi^*)=\overline{V}(\boldsymbol{r})$.

This study is a continuation of Rosberg [7], where $n=2$. There it was shown that the optimal policy does not contain conflicts (permission to more than one station to transmit in a slot). Moreover, the optimal policy permits one station to transmit every $k$ slots while the other station gets permission the rest of the time.

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The authors are with the Department of Computer Science, Technion —Israel Institute of Technology, Haifa, Israel.

Finding an optimal policy for $n>2$ turned out to be a difficult combinatorial problem. Here we shall discuss only conflict-free policies (i.e., each slot is allocated to at most one station) and conjecture that for every $\boldsymbol{r}$ there exists an optimal policy which does not contain conflicts. This conjecture is motivated from the fact that the model does not explicitly incorporate retransmitted packets. Also, it is partially supported by the results of Rosberg [7] and the following reasoning: one would benefit most from conflicts when the $r_i$'s are very small. In which case, we should use $\pi_{AT}$, the policy which gives permission to all the stations to transmit all the time, since the probability that more than one station will actually transmit is negligible. However, for equal $r_i$ the following calculation shows that $\pi_{AT}$ is inferior to $\pi_{RR}$, the round robin policy (station $i$ is permitted to transmit at slot $t$ if $t=i \bmod n$):

$$\overline{V}(\boldsymbol{r},\pi_{RR})=1-(1-r_1)^n > nr_1(1-r_1)^{n-1}=\overline{V}(\boldsymbol{r},\pi_{AT}).$$

Regardless of the conjecture, conflict-free policies are important for some communication networks, e.g., the loop circuit presented and analyzed by Konheim and Meister [5].

Our model is similar, but not identical, to that of Schoute [8] and Varaiya and Walrand [10]. The main differences are the following.

1) Our $r_i$'s are arbitrary, while theirs are all equal.

2) In their model, after a fixed delay, all stations know the buffer contents of all the stations, i.e., a delayed sharing type of information. In our model, no station has any information on the buffer contents of the other stations.

3) In our model, there is no cost for collisions, while their's has.

This conflict-free model is mainly applicable to data communication systems which use a satellite communication channel (see [8]), terrestrial loop circuit (see [5]), or local area networks of computers (see [2]). Also, even when collision detection and resolution is cost effective and reliable, it is not worthwhile to allow conflicts when the $r_i$'s are large.

In Section III we show that attention can be restricted to *loop policies*, policies for which there exists an $N$ such that for all $t$, the station allocated to slot $t$ is also allocated to slot $t+N$. For such policies the mean buffer length and the mean packet delay in equilibrium under a given policy were studied by Kosovych [6], using unjustified simplifications.

Using dynamic-programming formulation, we show in Section II that the throughput is maximized by a nonrandom time-division multiplexing policy (TDM policy). Although this result is intuitive, it is nontrivial to show that there exists an optimal policy which is periodic. The goal of this paper is to estimate $\overline{V}(\boldsymbol{r})$, investigate $\pi^*$, and give policies whose values $\overline{V}(\boldsymbol{r},\pi)$ approach $\overline{V}(\boldsymbol{r})$. In Section II our problem is formulated as a dynamic programming problem. Section III gives an upper bound to the value function. The next two sections discuss particular policies. Section IV analyzes a random control policy. In Section V we discuss a specific nonrandom TDM policy, the golden ratio policy, which approaches a limit (as the number of stations tends to infinity) greater than 98.4 percent of the upper bound.

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## II. Dynamic Programming Formulation

For every station $i$ define the following r.v.'s whose values are 0,1.

$X_i(t)=1$ iff the buffer is full at slot $t$.

$V_i(t)=1$ iff a new packet arrived at slot $t$.

$u_i(t)=1$ iff the station had permission to transmit at slot $t$.

Recall that only conflict-free policies are considered, i.e., at any slot $t$, at most one station has permission to transmit. Thus, $\Sigma_{i=1}^{n} u_i(t)$ equals 0 or 1.

From the definitions:

$$X_i(t+1)=V_i(t)+X_i(t)(1-V_i(t))(1-u_i(t)), \qquad 1\leqslant i\leqslant n, \tag{2.1}$$

where $\{X_i(0), V_i(t)\colon 1\leqslant i\leqslant n,\ t=0,1,\cdots\}$ are Bernoulli independent r.v.'s and $P(V_i(t)=1)=P(X_i(0)=1)=r_i$.

Let the *immediate reward* at slot $t, w(t)$, be the number of packets successfully transmitted during the slot

$$w(t)=w(X(t),u(t))=\sum_i X_i(t)u_i(t) \tag{2.2}$$

where $X(t)=(X_1(t),\cdots,X_n(t))$ and $u(t)=(u_1(t),\cdots,u_n(t))$. The total expected number of packets successfully transmitted during the first $T$ slots using policy $\pi$ is

$$V_T(r,\pi)=E_\pi\sum_{i=1}^{T} w(t)=\sum_{t=1}^{T}E_\pi(w(t)), \tag{2.3}$$

where $E_\pi(w(t))$ is the expected immediate reward at slot $t$ using policy $\pi$.

As in Berman [1] and Rosberg [7], we use the following sufficient statistics (for sufficient statistics in optimum control of stochastic systems, see, e.g., Striebel [9]). Let $k^{(i)}(t)$ $(i=1,2,\cdots,n)$ be the elapsed time since station $i$'s last permission to transmit. Define $k^{(i)}(0)=1$ $(i=1,\cdots,n)$. We have

$$k^{(i)}(t+1)=1+k^{(i)}(t)(1-u_i(t)), \qquad i=1,\cdots,n. \tag{2.4}$$

Let $u^{t-1}=(u(1),u(2),\cdots,u(t-1))$. Since the $V_i$'s are independent, collisions are avoided and $X_i(t)$ depends only on $u^{t-1}$ and $V_i$, we have the following.

*Lemma 2.1:*

1) Given $u^{t-1},\{X_1(t),X_2(t),\cdots,X_n(t)\}$ are mutually independent r.v.'s.

2) $P(X_i(t)=1|u^{t-1})=1-(1-r_i)^{k^{(i)}(t)}$ $\quad i=1,\cdots,n.$ ■

Let

$$p_i(k)=1-(1-r_i)^k. \tag{2.5}$$

The assumption of no conflicts, (2.2), (2.3), and Lemma 2.1 imply that

$$E_\pi(w(t)|u^{t-1})=\sum_{i=1}^{n}p_i(k^{(i)}(t))u_i(t) \tag{2.6}$$

where $u_i(t)$, $i=1,2,\cdots,n$ are the control actions taken by policy $\pi$.

A dynamic programming problem is defined by the state space $S$, the action space $A=X_{s\in S}A_s$, the law of motion $q$, and the reward function $w$.

For every $t$, $X(t)$ is a random variable whose probability distribution depends only on $u^{t-1}$. If $X(t)$ were known it could serve as a state. However, we are interested in a decentralized control policy, therefore $X(t)$ can be considered by all the stations only as a random variable whose distribution depends on $u^{t-1}$, which is common information (since all the stations know the policy).

From Lemma 2.1 the distribution of $X(t)$ is completely defined by the parameters $k(t)=(k^{(1)}(t),\cdots,k^{(n)}(t))$. Therefore, we consider the state space

$$S=\{k=(k^{(1)},\cdots,k^{(n)})|k^{(i)}=1,2,\cdots\ i=1,2,\cdots,n\}$$

and the action space at state $s\in S$

$$A_s=\{u=(u_1,u_2,\cdots,u_n)|u_i=0,1 \text{ and } u_iu_j=0 \text{ for } 0\leqslant i<j\leqslant n\}.$$

From (2.4), the law of motion (transition probabilities) becomes

$$q((k^{(1)}+1,\cdots,k^{(n)}+1)|(k^{(1)},\cdots,k^{(n)}),\ u=(0,\cdots,0))=1$$
$$q((k^{(1)}+1,\cdots,1,\cdots,k^{(n)}+1)|(k^{(1)},\cdots,k^{(n)}),\ u=(0,\cdots,1,\cdots,0))=1$$
$$q((\cdot)|(k^{(1)},\cdots,k^{(n)}),u))=0 \qquad \text{otherwise.} \tag{2.7}$$

Note that the law of motion is deterministic. Finally, from (2.6) the expected immediate reward is

$$w(k,u)=\sum_{i=1}^{n}p_i(k^{(i)})u_i. \tag{2.8}$$

Thus, from (2.3)

$$V_T(r,\pi)=\sum_{i=1}^{n}\sum_{t=1}^{T}p_i(k^{(i)}(t))u_i(t). \tag{2.9}$$

*Lemma 2.2:* Let $\pi$ be a policy in which station $i$ is permitted to transmit only a finite number of times. Then there exists a policy $\pi'$ such that

$$\overline{V}(r,\pi')>\overline{V}(r,\pi).$$

*Proof:* Let $0\leqslant s^1<s^2<s^3\leqslant 2n$. Because of the convexity of $p_j(k)$

$$f_j(s^1,s^2,s^3)\overset{\text{def}}{=}p_j(s^2-s^1)+p_j(s^3-s^2)-p_j(s^3-s^1)<1.$$

Let

$$\bar{f}\overset{\text{def}}{=}\max_{1\leqslant j\leqslant n}\ \max_{0\leqslant s^1<s^2<s^3\leqslant 2n} f_j(s^1,s^2,s^3)$$

and $\tau\geqslant 2n$ be such that $p_n(\tau)\geqslant\bar{f}$.

Let $t_0$ be the last slot policy $\pi$ permitted station $i$ to transmit. Define $t_m=t_0+m\tau$. In each interval $[t_{2(m-1)},t_{2m}]$ there is a station $j_m\neq i$ which is permitted to transmit three times: $s_m^1,s_m^2,s_m^3$.

Consider the policy $\pi'$ which is identical to $\pi$ except that at slots $s_m^2$ station $i$ is permitted to transmit instead of station $j_m$.

From (2.9) the net gain in the immediate reward for the interval $[t_{2m},t_{2m+2}]$ is

$$p_i(s_{m+1}^2-s_m^2)-(p_{j_m}(s_m^2-s_m^1)+p_{j_m}(s_m^3-s_m^2)-p_{j_m}(s_m^3-s_m^1)) > p_i(\tau)-\bar{f}>0.$$

Since this gain repeats every interval of length $2\tau$

$$\overline{V}(r,\pi')\geqslant\overline{V}(r,\pi)+\frac{p_i(\tau)-\bar{f}}{2\tau}>\overline{V}(r,\pi).$$

■

Let $d^{(i)}(\pi)=\sup_t k^{(i)}(t,\pi)$, where $k^{(i)}(t,\pi)$ is $k^{(i)}(t)$ under policy $\pi$. The following lemma is proved using techniques similar to those of Lemma 2.2.

*Lemma 2.3:* For every $\boldsymbol{r}$, there exists a $\tau$ such that for every policy $\pi$, there exists a policy $\pi'$ for which

i) $d^{(i)}(\pi')\leqslant\tau \quad i=1,\cdots,n;$

ii) $\bar{V}(\boldsymbol{r},\pi)\leqslant\bar{V}(\boldsymbol{r},\pi')$.

■

*Remark 2.1:* The weak inequality of Lemma 2.3 ii) cannot be strengthened since the long run average is not sensitive to fluctuations of zero weight.

Since the state space can be restricted to the finite space

$$S_0=\{(k^{(1)},k^{(2)},\cdots,k^{(n)})|k^{(i)}\leqslant\tau,\ i=1,2,\cdots,n\}\quad\text{for some }\tau,$$

the action space is finite and we have (see, e.g., Derman [3]).

*Theorem 2.1:* There exists a nonrandomized stationary control policy $\pi^*$ such that

$$\bar{V}(\boldsymbol{r})=\sup_\pi\bar{V}(\boldsymbol{r},\pi)=\bar{V}(\boldsymbol{r},\pi^*).$$

■

*Remark 2.2:* It should be noted that the above nonrandomized stationary policies depend only on the $k^{(i)}$'s and $\boldsymbol{r}$. If a state appears at slot $t$ and $t'$, then for every $m$ the states of slot $t+m$ and $t'+m$ are equal. Also, since the state space is finite, some state must repeat, thus the policies are ultimately periodic, i.e., they are loop policies.

*Remark 2.3:* At first glance, it may seem that loop policies are independent of the state of the system. Indeed, a distributed policy cannot be based on the physical state $X_i(t)$ $(1\leqslant i\leqslant n)$, therefore, the best one can do is to base decisions on sufficient statistics, in our case on $k^{(i)}$ $(1\leqslant i\leqslant n)$. Using these statistics the dynamic programming formulation with state space $S=\{(k^{(1)},\cdots,k^{(n)})\}$, was obtained. The loop policy depends on these states.

## III. An Upper Bound on the Value Function

An *empty action* is a slot in which no permission is given. Since for every policy which has empty actions there is a policy with no empty actions and at least the same throughput (replace every empty action by a permission to some station), we shall investigate policies with no empty actions.

Let $\pi$ be a loop policy of period $N$ and station $i$ be given $N^{(i)}$ permissions. Let $d_j^{(i)}$ $(j=1,\cdots,N^{(i)})$ be the distances between two consecutive permissions to station $i$ in the loop. Clearly, $\sum_j d_j^{(i)}=N$.

From (1.1), (2.9), the definition of $d_j^{(i)}$ and the mean ergodic theorem

$$\bar{V}(\boldsymbol{r},\pi)=\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{N^{(i)}}p_i\left(d_j^{(i)}\right)=\sum_{i=1}^{n}\frac{N^{(i)}}{N}\sum_{j=1}^{N^{(i)}}\frac{1}{N^{(i)}}p_i\left(d_j^{(i)}\right).\tag{3.1}$$

Because of the convexity of $p_i$

$$\bar{V}(\boldsymbol{r},\pi)\leqslant\sum_{i=1}^{n}\frac{N^{(i)}}{N}p_i\left(\frac{1}{N^{(i)}}\sum_{j=1}^{N^{(i)}}d_j^{(i)}\right)=\sum_{i=1}^{n}\frac{N^{(i)}}{N}p_i\left(\frac{N}{N^{(i)}}\right).\tag{3.2}$$

*Theorem 3.1:* For every policy $\pi$

$$\bar{V}(\boldsymbol{r},\pi)\leqslant1-\prod_{i=1}^{n}(1-r_i)\overset{\text{def}}{=}U(\boldsymbol{r}).$$

*Proof:* From (3.2) we have $\bar{V}(\boldsymbol{r},\pi)\leqslant U(\boldsymbol{r})$, where $U(\boldsymbol{r})$ is the solution to the following optimization problem:

$$U(\boldsymbol{r})=\max\sum_{i=1}^{n}x^{(i)}\left(1-(1-r_i)^{1/x^{(i)}}\right)$$
$$=1-\min\sum_{i=1}^{n}x^{(i)}(1-r_i)^{1/x^{(i)}}$$

subject to

$$\sum_{i=1}^{n}x^{(i)}=1\qquad\text{and } x^{(i)}\geqslant0.\tag{3.3}$$

We shall see that the optimum solution to the following problem satisfies $x^{(i)}\geqslant0$:

$$\min f(x^{(1)},\cdots,x^{(n)})=\min\sum_{i=1}^{n}x^{(i)}(1-r_i)^{1/x^{(i)}}\tag{3.4}$$

subject to

$$\sum_{i=1}^{n}x^{(i)}=1.$$

To solve (3.4) we use Lagrange multipliers. Let

$$F(x^{(1)},x^{(2)},\cdots,x^{(n)},\lambda)=\lambda\left(1-\sum_{i=1}^{n}x^{(i)}\right)+\sum_{i=1}^{n}x^{(i)}(1-r_i)^{1/x^{(i)}}.$$

A necessary and sufficient condition to the solution is

$$\frac{\partial F}{\partial x^{(i)}}=-\lambda+(1-r_i)^{1/x^{(i)}}\left(1-\ln(1-r_i)^{1/x^{(i)}}\right)=0,$$
$$\frac{\partial F}{\partial\lambda}=1-\sum_{i=1}^{n}x^{(i)}=0.$$

Therefore, the solution must satisfy

$$(1-r_i)^{1/x^{(i)}}\left(1-\ln(1-r_i)^{1/x^{(i)}}\right)=\lambda,\qquad i=1,2,\cdots,n$$

and

$$\sum_{i=1}^{n}x^{(i)}=1.\tag{3.5}$$

Since $(1-r_i)^{1/x^{(i)}}<1$ and the function $\gamma(z)=z(1-\ln(z))$ is monotonically increasing for $z<1$, it follows from (3.5) that for every $i$,

$$(1-r_i)^{1/x^{(i)}}=C\qquad\text{for some } C\tag{3.6}$$

and

$$\sum_{i=1}^{n}\frac{\ln(1-r_i)}{\ln C}=1.$$

Therefore, the minimum of (3.4) is obtained when

$$x^{(i)}=\frac{\ln(1-r_i)}{\sum_{j=1}^{n}\ln(1-r_j)}$$

and

$$C=\prod_{i=1}^{n}(1-r_i). \tag{3.7}$$

As promised, these $x^{(i)}$'s are nonnegative. From (3.3), (3.4), and (3.7) it follows that

$$U(\boldsymbol{r})=1-\prod_{i=1}^{n}(1-r_i).$$

■

*Corollary 3.1:* If $r_i=r_1$, $i=1,\cdots,n$, then the "round robbin" policy $\pi_{RR}$ is optimal.

*Proof:* A straightforward calculation shows that

$$\overline{V}(\pi_{RR})=1-(1-r_1)^n=U(\boldsymbol{r}).$$

■

If all the $r_i$'s are equal, then all the $x^{(i)}$'s are also equal and $x^{(i)}=1/n$. The interpretation is that station $i$ is permitted to transmit every $1/x^{(i)}=n$ slots. However, in general, the $x^{(i)}$'s are irrational, thus the $i$th station cannot transmit every $1/x^{(i)}$ slot. Consequently, the upper bound cannot be obtained. We shall try to approximate the optimal solution by policies which permit station $i$ to transmit $N^{(i)}\approx x^{(i)}N$ equally spaced times in a loop of size $N$ (according to Remark 2.2). Thus, we shall first calculate the $x^{(i)}$'s then consider policies, which give station $i$, $N^{(i)}$ permissions. These policies will be close to the optimal only if the permissions to station $i$ are nearly equally spaced.

Thus, we are confronted with the following *placement problem*. Given

$$N^{(1)},N^{(2)},\cdots,N^{(n)},\ \sum_{i=1}^{n}N^{(i)}=N.$$

Place the permissions of each station such that

$$\lfloor N/N^{(i)}\rfloor\leqslant \underline{d^{(i)}}\leqslant\overline{d^{(i)}}\leqslant\lceil N/N^{(i)}\rceil$$

where $\underline{d^{(i)}}$ and $\overline{d^{(i)}}$ are the minimum and maximum distance between consecutive permissions to station $i$.

This problem does not always have a solution. (For example: $n=3$, $N=6$, $N^{(1)}=3$, $N^{(2)}=2$, and $N^{(3)}=1$, ideally, station 1 should be given permission every other time slot, station 2 at time slots $t$ and $t+3$ mod 6; this is not obtainable.)

Let us consider also the asymptotic case where the number of stations $n\to\infty$ and $r_i=r_i(n)\to 0$. Suppose $\pi$ is a loop policy which permits each station to transmit at least once during a loop of size $N=N(n)$, and let $d_j^{(i)}$ be as above. From (3.1)

$$\begin{aligned}\overline{V}(\boldsymbol{r},\pi)&=\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{N^{(i)}}\left(1-(1-r_i)^{d_j^{(i)}}\right)\\&=\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{N^{(i)}}\left(d_j^{(i)}r_i+O(r_i^2)\right)\\&=r+O(r^2)\end{aligned} \tag{3.8}$$

where $r=\sum_{i=1}^{n}r_i$. However,

$$U(\boldsymbol{r})=1-\prod_{i=1}^{n}(1-r_i)=r+O(r^2).$$

Thus, if also $r\to 0$, $\overline{V}(\boldsymbol{r},\pi)\approx U(\boldsymbol{r})$ for every $\pi$.

Thus, it is important to define the asymptotic behavior nontrivially. That is, when the number of stations $n\to\infty$, the arrival rate to station $i$ (given $n$ stations) $r_i(n)\to 0$ and the total arrival rate $r=\sum_{i=1}^{n}r_i(n)$ remains fixed.

Let $x_\pi^{(i)}(n)$ be the proportion of permissions given to station $i$ by policy $\pi$ in the system $(n,\boldsymbol{r}(n))$, where

$$\boldsymbol{r}(n)=(r_1(n),\cdots,r_n(n)).$$

Also let

$$C_\pi^{(i)}(n)=\max_j\left\{\frac{r}{r_i(n)}-d_j^{(i)}(n)\right\} \tag{3.9}$$

where $d_j^{(i)}(n)$ are the distances $d_j^{(i)}$ defined above for policy $\pi$ and system $(n,\boldsymbol{r}(n))$. For a given policy $\pi$ we simplify the notation to $x^{(i)}(n)$ and $C^{(i)}(n)$.

Since attention can be restricted to loop policies, a policy $\pi$ depends only on $(n,\boldsymbol{r}(n))$. Denote by $\boldsymbol{P}$ the set of all loop policies.

*Definition 3.1:* A policy $\pi$: $\{(n,\boldsymbol{r}(n))\}\underset{n\to\infty}{\to}\boldsymbol{P}$ is asymptotically optimal if

$$\left|\overline{V}(\boldsymbol{r}(n),\pi(n,\boldsymbol{r}(n)))-\left(1-\prod_{i=1}^{n}(1-r_i(n))\right)\right|\underset{n\to\infty}{\to}0.$$

Note that under the above asymptotic conditions

$$\lim_{n\to\infty}U(\boldsymbol{r}(n))=1-\lim_{n\to\infty}\prod_{i=1}^{n}(1-r_i(n))=1-e^{-r}. \tag{3.10}$$

*Theorem 3.2:* Let $\pi$ be a loop policy, if

$$\left|x^{(i)}(n)-\frac{r_i(n)}{r}\right|\to 0$$

and

$$C^{(i)}(n)r_i(n)\to 0,$$

then $\pi$ is asymptotically optimal.

*Proof:* For any given $(n,\boldsymbol{r}(n))$ we have from (3.1)

$$\overline{V}(\boldsymbol{r}(n),\pi(n,\boldsymbol{r}(n)))=\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{N^{(i)}}p_i\left(d_j^{(i)}(n)\right)$$

where $N$, $N^{(i)}$, and $d_j^{(i)}$ are as defined earlier. Therefore, from (3.9)

$$\begin{aligned}&\overline{V}(\boldsymbol{r}(n),\pi(n,\boldsymbol{r}(n)))\\&\quad\geqslant\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{N^{(i)}}\left(1-(1-r_i(n))^{(r/r_i(n))-C^{(i)}(n)}\right)\\&\quad=\sum_{i=1}^{n}x^{(i)}(n)\left(1-(1-r_i(n))^{(r/r_i(n))-C^{(i)}(n)}\right).\end{aligned}$$

Since $\sum_{i=1}^{n}x^{(i)}(n)=1$, $(1-r_i(n))^{r/r_i(n)}\underset{n\to\infty}{\to}e^{-r}$ and $(1-r_i(n))^{-C^{(i)}(n)}\to 1$ we obtain

$$\lim_{n\to\infty}\overline{V}(\boldsymbol{r}(n),\pi(n,\boldsymbol{r}(n)))=1-e^{-r},$$

and in conjunction with (3.10) the theorem is proved. ■

*Remark 3.1:* Since the average distance between two consecutive permissions to station $i$ is $1/x^{(i)}(n)$, the condition $C^{(i)}(n)r_i(n)\to 0$ implies that $|x^{(i)}(n)-(r_i(n)/r)|\to 0$.

*Remark 3.2:* From Theorem 3.2 it follows that for a large number of stations, a policy $\pi$ might still be good, even if its

$d_j^{(i)}$'s are far away from the desired distances according to (3.7). Thus, it might not be needed to solve the placement problem.

We have not been able to prove that any policy is asymptotically optimal, even though we have a candidate (see Section VI). In the next two sections we analyze two policies.

## IV. A Random Control Policy

For a given system $(n, r(n))$, let

$$x^{(i)}(n) = \frac{\ln(1-r_i(n))}{\sum_j \ln(1-r_j(n))}. \quad (4.1)$$

*Definition 4.1:* Let $\pi_R$ be the control policy which at every slot $t$, $t=1,2,\cdots$, permits station $i$ to transmit with probability $x^{(i)}(n)$, $i=1,2,\cdots,n$.

This policy can be implemented distributively (without conflicts) if all the stations use the same random number generator.

Let $D^{(i)}$ be the number of slots between two consecutive permissions to station $i$. $D^{(i)}$ $(i=1,\cdots,n)$ are independent geometrically distributed random variables with probability of success $x^{(i)}(n)$. Therefore,

$$\bar{V}(r(n),\pi_R) = E\sum_{i=1}^{n}\left(1-(1-r_i(n))^{D^{(i)}}\right).$$

The expectation is taken over the common distribution $(I, D)$, where $I$ is the station which has been given permission to transmit at a random slot and $D$ is the distance from the last permission. Clearly,

$$\begin{aligned}\bar{V}&(r(n),\pi_R)\\ &= E_I E_{D|I}\left(1-(1-r_I)^{D^{(I)}}\right) = \sum_{i=1}^{n} x^{(i)}(n) E\left(1-(1-r_i(n))^{D^{(i)}}\right)\\ &=1-\sum_{i=1}^{n} x^{(i)}(n)\sum_{k=1}^{\infty}(1-r_i(n))^k(1-x^{(i)}(n))^{k-1}x^{(i)}(n)\\ &=1-\sum_{i=1}^{n} x^{(i)}(n)\frac{x^{(i)}(n)(1-r_i(n))}{1-(1-r_i(n))(1-x^{(i)}(n))}. \quad (4.2)\end{aligned}$$

Under the asymptotic conditions, $\ln(1-r_i(n)) \underset{r_i(n)\to 0}{\approx} r_i(n)$, and from L'Hopital's rule

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{x^{(i)}(n)(1-r_i(n))}{1-(1-r_i(n))(1-x^{(i)}(n))} = \frac{1}{1+r}. \quad (4.3)$$

Now since $\sum_{i=1}^{n} x^{(i)}(n) = 1$ for every $n$ we have from (4.2) and (4.3)

$$\lim_{n\to\infty}\bar{V}(r(n),\pi_R(n,r(n))) = 1-\frac{1}{1+r}.$$

Note that for small and large $r$, the random policy $\pi_R$ is asymptotically close to the upper bound $U(r)=1-e^{-r}$. However, for the intermediate $r$'s $\pi_R$ is not recommended since the $D^{(i)}$'s tend to deviate largely from the $1/x^{(i)}$'s (this property is typical of the geometrical distribution).

## V. The Golden Ratio Control Policy

Let $(n, r(n))$ be a given system and $x^{(i)}(n) > 0$, $i=1,2,\cdots,n$, $\sum_{i=1}^{n} x^{(i)}(n)=1$ be the desirable proportions of permissions to each of the stations. (When no confusion arises the argument $n$ will be omitted.) Also, let $F_k$ be the $k$th Fibonacci number and $N_k^{(i)}$, $i=1,2,\cdots,n$ be integers such that

$$\lfloor x^{(i)}F_k\rfloor \leqslant N_k^{(i)} \leqslant \lceil x^{(i)}F_k\rceil$$

and

$$\sum_{i=1}^{n} N_k^{(i)} = F_k. \quad (5.1)$$

Thus

$$\lim_{k\to\infty}\frac{N_k^{(i)}}{F_k} = x^{(i)}. \quad (5.2)$$

For each $k$, the golden ratio policy assigns $N_k^{(i)}$ slots to station $i$ and attempts to distribute the permissions uniformly over a loop of size $F_k$. (The analysis of Section III implies that it is desirable to distribute the permissions uniformly.)

Open address hashing confronts a similar problem: to distribute keys uniformly over a hash table. The uniformity of the distribution depends on the hash function. It has been shown that multiplicative hashing with the golden ratio multiplicand, $\varphi^{-1} = (\sqrt{5}-1)/2 \approx 0.6180339887$, distributes the keys most uniformly [4, vol. 1]. The golden ratio policy applies some of these results. Fibonacci numbers are related to the golden ratio $\varphi^{-1}$ by the equation

$$F_k = \frac{\varphi^k-(1-\varphi)^k}{\sqrt{5}}.$$

Let $frac(y) = y - \lfloor y\rfloor$, $a_j = frac(j\varphi^{-1})$, and $A_N = \{a_j | j = 0,\cdots,N-1\}$. The $t$th smallest point of $A_{F_k}$ is identified with the $t$th slot of the loop.

*Definition 5.1:* The golden ratio policy, $\pi_{GR(k)}$, is the policy which assigns to station $i$ the slots corresponding to the points

$$\left\{a_j \middle| \sum_{m=1}^{i-1} N_k^{(m)} \leqslant j < \sum_{m=1}^{i} N_k^{(m)}\right\}.$$

It will be convenient to identify the points 0 and 1, and thus the points $a_j$ are distributed over a circle $C$.

*Example 5.1:* Suppose $n=3$, $x^{(1)}=\frac{1}{2}\pm\epsilon_1$, $x^{(2)}=\frac{3}{8}\pm\epsilon_2$, $x^{(3)}=\frac{1}{8}\pm\epsilon_3$, where $\epsilon_i>0$ are arbitrarily small and $x^{(1)}+x^{(2)}+x^{(3)}=1$. Taking $F_6=8$, $N_6^{(1)}=4$, $N_6^{(2)}=3$, and $N_6^{(3)}=1$, $\pi_{GR(6)}$ assigns to station 1 the slots corresponding to 0, $\varphi^{-1}$, $frac(2\varphi^{-1})$ and $frac(3\varphi^{-1})$; to station 2 the slots corresponding to $frac(4\varphi^{-1})$, $frac(5\varphi^{-1})$, and $frac(6\varphi^{-1})$; and to station 3 the point corresponding to $frac(7\varphi^{-1})$. Thus, the loop policy keeps giving permission to the stations in the following cyclic order: "1,2,1,3,2,1,2,1."

Let $l_m = frac((-1)^m F_{m-1}\varphi^{-1})$. From Knuth [4, vol. 3, pp. 506–549] we can deduce the following.

*Theorem 5.1:* Let $N = F_m + s$ $(0 \leqslant s < F_{m-1})$:

1) $A_N$ partitions $C$ into
   - $s$ intervals of length $l_{m+1}$;
   - $F_{m-2}+s$ intervals of length $l_m$;
   - $F_{m-1}-s$ intervals of length $l_{m-1}$.

2) An additional point, $a_N$ $(N \leqslant F_{m+1})$, breaks an interval of length $l_{m-1}$ into one of length $l_m$ and one of length $l_{m+1}$.

3) The lengths of the subintervals decreases, moreover $l_{m+1} = \varphi^{-1}l_m$. ∎

*Corollary 5.1:* For each station $i$ (with $N_k^{(i)} = s_k^{(i)} + F_{k_i}$) there are at most three types of intervals:
- $s_k^{(i)}$ of length $l_{k_i+1}$;
- $F_{k_i-2}+s_k^{(i)}$ of length $l_{k_i}$;
- $F_{k_i-1}-s_k^{(i)}$ of length $l_{k_i-1}$.

*Proof:* Theorem 5.1 considered starting at $a_0=0$. However, the same result holds if we make a circular shift and start at $a_q$ where $q=\sum_{m=1}^{i} N_k^{(m)}$. ∎

In order to compute $\bar{V}(r,\pi_{GR(k)})$, we shall find the number of slots between two consecutive permissions to station $i$. Thus, we consider the number of points of $A_{F_k}$ in intervals of lengths $l_{k_i-1}$, $l_{k_i}$, and $l_{k_i+1}$.

Consider the partitioning of $C$ by $A_{F_k}$ into subintervals. A subinterval is *atomic* if it is not partitioned into any smaller subinterval. Let $X^1$ and $X^2$ be two intervals of length $|X^j| = l_m$ and whose endpoints belong to $A_{F_k}$. Let $g_k(X^j)$ denote the number of atomic subintervals in $X^j$.

From Theorem 5.1 the atomic subintervals have lengths $l_k$ and $l_{k-1}$, thus

$$l_k g_k(X^j) + l_{k-1} g_{k-1}(X^j) = l_m, \qquad j = 1,2.$$

From Theorem 5.1 (3)

$$\varphi^{-1} g_k(X^1) + g_{k-1}(X^1) = \varphi^{-1} g_k(X^2) + g_{k-1}(X^2),$$

and

$$\varphi^{-1}\big(g_k(X^1) - g_k(X^2)\big) = g_{k-1}(X^2) - g_{k-1}(X^1).$$

Since $g_k(X^j)$ are integers and $\varphi^{-1}$ is irrational:

$$g_{k-1}(X^1) = g_{k-1}(X^2)$$
$$g_k(X^1) = g_k(X^2).$$

Thus, $g_m$ is independent of the particular choice of $X^j$, and $d_k(l_m)$, the number of atomic subintervals in an interval of length $l_m$, given the partitioning $A_{F_k}$, satisfies

$$d_k(l_m) = g_k(X^1) + g_{k-1}(X^1).$$

To compute $d_k(l_m)$, notice that

$$d_k(l_k) = d_k(l_{k-1}) = 1.$$

From Theorem 5.1 each interval of size $l_m$ is partitioned into an interval of size $l_{m+1}$ and one of size $l_{m+2}$. Thus,

$$d_k(l_m) = d_k(l_{m+1}) + d_k(l_{m+2}).$$

Comparing this to Fibonacci's recursive formula yields the following.

*Lemma 5.1:*

$$d_k(l_m) = F_{k+1-m}.$$

■

*Remark 5.1* Note that $d_k(l_{k_i-1})$, $d_k(l_{k_i})$, and $d_k(l_{k_i+1})$ are the number of slots between two consecutive permissions to station $i$.

*Lemma 5.2:* Let $j_i$ satisfy $\varphi^{-j_i} \leqslant x^{(i)} < \varphi^{-j_i+1}$, if $x^{(i)} \neq \varphi^{-j_i}$ then for almost all $k$

$$k_i = k - j_i.$$

*Proof:* From (5.2) and the definition of $N_k^{(i)}$ and $s_k^{(i)}$

$$\lim_{k \to \infty} \frac{F_{k_i} + s^{(i)}}{F_k} = x^{(i)}.$$

Consequently, for almost all $k$

$$\varphi^{-j_i} < \frac{F_{k_i} + s_k^{(i)}}{F_k} < \varphi^{-j_i+1}.$$

$$F_k\varphi^{-j_i} < F_{k_i} + s_k^{(i)} < F_k\varphi^{-j_i+1}. \tag{5.3}$$

From Knuth [4]

$$F_k = \frac{\varphi^k - (1-\varphi)^k}{\sqrt{5}}.$$

Since $|1-\varphi| < 1$

$$\lim_{m \to \infty} F_m - \frac{\varphi^m}{\sqrt{5}} = 0.$$

Thus,

$$\lim_{k \to \infty} F_k\varphi^{-j_i} - F_{k-j_i} = 0.$$

Thus, from (5.3) for almost all $k$

$$F_{k-j_i} \leqslant F_{k_i} + s_k^{(i)} < F_{k-j_i+1}.$$

The result follows since $S_k^{(i)} < F_{k_i-1}$. ■

*Remark 5.2:* The above proof is not valid for $x^{(i)} = \varphi^{-j_i}$. However, in this case we may choose $N_k^{(i)} = F_{k-j_i}$.

*Corollary 5.2:*

$$\lim_{k \to \infty} \frac{F_{k_i-q}}{F_k} = \frac{F_{k-j_i-q}}{F_k} = \varphi^{-(j_i+q)}$$
$$d_k\big(l_{k_i-1+q}\big) = F_{j_i+2-q} \qquad q = 0,1,2$$
$$\lim_{k \to \infty} \frac{s_k^{(i)}}{F_k} = x^{(i)} - \varphi^{-j_i}.$$

*Theorem 5.2:*

$$\overline{V}(\boldsymbol{r}, \pi_{GR}) \overset{\text{def}}{=} \lim_{k \to \infty} \overline{V}(\boldsymbol{r}, \pi_{GR(k)})$$
$$= 1 - \sum_{i=1}^{n} \Big[\big(x^{(i)} - \varphi^{-j_i}\big)(1-r_i)^{F_{j_i}}$$
$$+ \big(x^{(i)} - \varphi^{-j_i-1}\big)(1-r_i)^{F_{j_i+1}} + \big(\varphi^{-j_i+1} - x^{(i)}\big)(1-r_i)^{F_{j_i+2}}\Big].$$

*Proof:* From Corollary 5.1, Remark 5.1, and (3.1)

$$\overline{V}(\boldsymbol{r}, \pi_{GR(k)}) = \frac{1}{F_k} \sum_{i=1}^{n} \Big[s_k^{(i)}\big(1 - (1-r_i)^{d_k(l_{k_i+1})}\big)$$
$$+ \big(F_{k_i-2} + s_k^{(i)}\big)\big(1 - (1-r_i)^{d_k(l_{k_i})}\big)$$
$$+ \big(F_{k_i-1} - s_k^{(i)}\big)\big(1 - (1-r_i)^{d_k(l_{k_i-1})}\big)\Big].$$

The theorem follows from Corollary 5.2. ■

We now investigate the asymptotic behavior of the throughput for increasing $n$ and fixed $r$.

*Theorem 5.3:* If $r_i(n) \underset{n \to \infty}{\to} 0$ and $\sum_{i=1}^{n} r_i(n) = r$, then

$$\lim_{n \to \infty} \inf \overline{V}(\boldsymbol{r}(n), \pi_{GR}) \geqslant 1 - (1-\varphi^{-1})e^{-r\varphi/\sqrt{5}} - \varphi^{-1}e^{-r\varphi^2/\sqrt{5}}.$$

*Proof:* [1]From Lemma 5.2

$$j_i = -\log_\varphi x^{(i)} + a_i$$

where $a_i = frac(\log_\varphi x^{(i)})$. Thus,

$$\varphi^{-j_i} = x^{(i)}\varphi^{-a_i}.$$

Therefore, by Theorem 5.2,

$$\overline{V}(\boldsymbol{r}(n), \pi_{GR}) = 1 - \sum_{i=1}^{n} x^{(i)}(n)h^{(i)}(n) \tag{5.4}$$

where

[1]In the proof we omit the argument $n$, whenever it is understood.

$$h^{(i)}(n)=(1-\varphi^{-a_i})(1-r_i(n))^{F_{j_i}}+(1-\varphi^{-a_i-1})(1-r_i(n))^{F_{j_i+1}}+(\varphi^{-a_i+1}-1)(1-r_i(n))^{F_{j_i+2}}.$$

We show that the right-hand side of (5.4) attains its minimum for $0\leqslant a_i\leqslant 1$, for $i=1,2,\cdots,n$, when $a_i=1$.

$r_i(n)/r\to 0$ implies that $x^{(i)}(n)\to 0$, hence, $\lim_{n\to\infty}F_{j_i}=\infty$. Therefore,

$$F_{j_i+q}=\frac{\varphi^{j_i+q}-(1-\varphi)^{j_i+q}}{\sqrt{5}}=\frac{\varphi^{a_i+q}}{x^{(i)}\sqrt{5}}+o(1), \qquad q=0,1,2.$$

Let $c_i=(1-r_i)^{1/x^{(i)}(n)\sqrt{5}}$. Clearly, $0\leqslant c_i<1$.

Thus, the $h^{(i)}(n)$'s are approximately[2]

$$(1-\varphi^{-a_i})c_i^{\varphi^{a_i}}+(1-\varphi^{-a_i-1})c_i^{\varphi^{a_i+1}}+(\varphi^{-a_i+1}-1)c_i^{\varphi^{a_i+2}}. \tag{5.5}$$

Using

$$\varphi^{a+2}=\varphi^a+\varphi^{a+1}$$

and

$$c^{\varphi^{a_i+1}}>0,$$

it is sufficient to show that

$$g(a)=(1-\varphi^{-a})c^{-\varphi^{a-1}}+(1-\varphi^{-a-1})+(\varphi^{-a+1}-1)c^{\varphi}$$

attains its maximum when $a=1$.

Since

$$(1-\varphi^{-a})+(1-\varphi^{-a-1})+(\varphi^{-a+1}-1)=0$$

it follows that

$$g(a)=(1-\varphi^{-a})(c^{-\varphi^{a-1}}-c^{\varphi^{a}})+\varphi^{-a-1}(c^{\varphi^{a}}-1)+1.$$

Clearly, the first summand of $g(a)$ increases in $a$. By differentiation and using the fact that $z(1-\ln z)$ increases for $0<z\leqslant 1$ it can be shown that the second summand also increases in $a$.

Thus, for almost all $n$, (5.4) and (5.5) imply

$$\bar{V}(r(n),\pi_{GR})\geqslant 1-\sum_{i=1}^{n}x^{(i)}(n)\left[(1-\varphi^{-1})c_i^{\varphi}+\varphi^{-1}c_i^{\varphi^2}\right].$$

To finish the proof note that $c_i \underset{n\to\infty}{\to} e^{-r/\sqrt{5}}$ and $\Sigma_{i=1}^{n}x^{(i)}(n)=1$. ■

Finding the minimum over $r$ of the ratio between $\liminf \bar{V}(r(n),\pi_{GR})$ and $\lim U(r(n))$ yields the following.

*Corollary 5.3:* Under our asymptotic conditions, for almost all $n$

$$\frac{\bar{V}(r(n),\pi_{GR})}{U(r(n))}>0.984.$$

## VI. Conclusions

Theorem 2.1 implies that there always exists a stationary periodic optimal policy, hence, there is always an asymptotically optimal policy. Even though Theorem 3.2 gives sufficient conditions for a policy to be asymptotically optimal, we have not been able to demonstrate one. The golden-ratio policy of Section V is shown to be very close to being asymptotically optimal (see Corollary 5.3). These results depend on $k$th Fibonacci number being sufficiently large. For practical implementation $k$ must be finite. We believe that when $F_k\gg 1/x^{(i)}$ the throughput is sufficiently close to the limit [see (3.7)].

The above results do not imply that we should always use the golden ratio policy. For a specific $r$ the placement problem may be easily solvable. For example: when the $r_i$'s are equal, then $x^{(i)}=1/n$ implying that the round-robin policy is optimal.

We conjecture that the following policy is asymptotically optimal. Let

$\alpha_j(t)$ be the number of slots station $i$ was permitted to transmit until time $t$;

$u_i(t)=1$ iff $i$ is the station for which

$$\min_j\left\{\frac{\alpha_j(t)}{t}-x^{(i)}\right\}$$

is obtained.

Numerical calculations indicate that this policy is promising.

[2] The correct $h^{(i)}(n)$ is obtained by replacing $c_i$ by $c_i(1+o(1))$, however this approximation is justified because $h^{(i)}(n)$ is continuous, and for large $n$ the error approaches zero.

## References

[1] Z. Berman, "Maximal traffic in radio networks with two transmitters," Masters thesis, Dep. Elec. Eng., Technion—Israel Inst. Technol., 1981 (in Hebrew).

[2] I. Chlamtach, W. R. Franta, P. C. Patton, and B. Wells, "Performance issues in back-end storage networks," *Comput.*, vol. 13, no. 2, pp. 18–31, 1980.

[3] C. Derman, *Finite State Markovian Decision Processes*. New York: Academic, 1970.

[4] D. E. Knuth, *The Art of Computer Programming*, vol. 1-3. New York: Addison-Wesley, 1973.

[5] A. G. Konheim and B. Meister, "Service in a loop system," *J. Ass. Comput. Mach.*, vol. 19, pp. 92–108, 1972.

[6] O. S. Kosovych, "Fixed assignment access technique," *IEEE Trans. Commun.*, vol. COM-26, pp. 1370–1375, Sept. 1978.

[7] Z. Rosberg, "Optimal decentralized control in a multi-access channel with partial information," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 187–193, Feb. 1983.

[8] F. C. Schoute, "Decentralized control in packet switched satellite communication," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 362–371, Apr. 1978.

[9] C. Striebel, "Sufficient statistics in the optimum control of stochastic systems," *J. Math. Anal. Appl.*, vol. 12, pp. 576–592, 1965.

[10] P. Varaiya and J. Walrand, "Decentralized control in packet switched satellite communication," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 794–796, Oct. 1979.

**Alon Itai** received the B.Sc. degree in mathematics from the Hebrew University, Jerusalem, Israel, in 1969, the M.Sc. degree in 1971, and the Ph.D. degree in computer science from the Weizmann Institute of Science, Rehovot, Israel.

Since 1976 he has been with the Department of Computer Science, Technion—Israel Institute of Technology, Haifa, Israel, where he currently holds the position of Senior Lecturer.

**Zvi Rosberg** was born in Germany on July 25, 1947. He received the B.Sc., M.A., and Ph.D. degrees from the Hebrew University, Jerusalem, Israel, in 1971, 1974, and 1978, respectively.

From 1972 to 1978 he was a Senior System Analyst in the General Computer Bureau of the Israeli Government. From 1978 to 1979 he held a research fellowship at the Center of Operation Research and Econometric (CORE), Catholic University of Louvain, Belgium. From 1979 to 1980 he was a Visiting Assistant Professor at the University of Illinois, Urbana. Since 1980 he has been with the Faculty of Computer Science, Technion—Israel Institute of Technology, Haifa, where he currently holds the position of Senior Lecturer.