A Goodness-of-Fit Test for Multivariate Multiparameter Copulas Based on Multiplier Central Limit Theorems

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Abstract

Recent large scale simulations indicate that a powerful goodness-of-fit test for copulas can be obtained from the process comparing the empirical copula with a parametric estimate of the copula derived under the null hypothesis. A first way to compute approximate p-values for statistics derived from this process consists of using the parametric bootstrap procedure recently thoroughly revisited by Genest and Rémillard. Because it heavily relies on random number generation and estimation, the resulting goodness-of-fit test has a very high computational cost that can be regarded as an obstacle to its application as the sample size increases. An alternative approach proposed by the authors consists of using a *multiplier* procedure. The study of the finite-sample performance of the multiplier version of the goodness-of-fit test for bivariate one-parameter copulas showed that it provides a valid alternative to the parametric bootstrap-based test while being orders of magnitude faster. The aim of this work is to extend the multiplier approach to multivariate multiparameter copulas and study the finite-sample performance of the resulting test. Particular emphasis is put on elliptical copulas such as the normal and the t as these are flexible models in a multivariate setting. The implementation of the procedure for the latter copulas proves challenging and requires the extension of the *Plackett formula* for the t distribution to arbitrary dimension. Extensive Monte Carlo experiments, which could be carried out only because of the good computational properties of the multiplier approach, confirm in the multivariate multiparameter context the satisfactory behavior of the goodness-of-fit test.

KEY WORDS: Normal copula; Plackett formula; Pseudo-likelihood; Pseudo-observation; Rank; t copula.

1 Introduction

Let $\mathbf{X} = (X_1, \dots, X_d)^\top$ be a random vector with continuous marginal cumulative distribution functions (c.d.f.s) $F_1, \dots, F_d$. From the work of Sklar (1959), it is well-known that the c.d.f. H of $\mathbf{X}$ can be expressed in a unique way as

$$H(\mathbf{x}) = C\{F_{\mathbf{1}}(x_1), \dots, F_d(x_d)\}, \qquad \mathbf{x} \in \mathbb{R}^d, \tag{1}$$

where $C : [0,1]^d \to [0,1]$, called a *copula*, is a d-dimensional c.d.f. with standard uniform margins. The above representation is at the origin of the increasingly frequent use of copulas for the modeling of multivariate continuous distributions. Indeed, given a random sample $\mathbf{X}_1, \dots, \mathbf{X}_n$ from c.d.f. H, representation (1) suggests breaking the construction of a multivariate model for $\mathbf{X}$ into two independent parts: the estimation of the marginal c.d.f.s and the fitting of an appropriate parametric copula (see e.g. Joe, 1997; Nelsen, 2006, for an extensive review of parametric copula families). Applications of this modeling approach are found in finance (Cherubini et al., 2004; McNeil et al., 2005) and increasingly in other fields such as hydrology (Genest and Favre, 2007; Genest et al., 2007), public health (Cui and Sun, 2004) or actuarial sciences (Frees and Valdez, 1998).

Assuming that the unknown copula C belongs to a parametric copula family $\mathcal{C}_0 = \{C_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \mathcal{O}\}$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^q$ for some integer $q \geq 1$, the next step consists of estimating the vector of parameters $\boldsymbol{\theta} = (\theta_1, \dots, \theta_q)^\top$ from the random sample $\mathbf{X}_1, \dots, \mathbf{X}_n$. When dealing with one-parameter bivariate copulas, a popular approach consists of using the simple method of moments based on the inversion of Spearman's rho or Kendall's tau. In the more general multivariate multiparameter case, such rank correlation-based methods become less natural except maybe for elliptical copulas (Genest et al., 2007). Still, as discussed e.g. in Demarta and McNeil (2005, §4.2) such approaches may lead to inconsistencies. The most natural estimation method in the multivariate multiparameter case is therefore probably the *pseudo-likelihood* approach studied in Genest et al. (1995), Shih and Louis (1995) and Genest and Werker (2002) (see also Kim et al., 2007, for empirical arguments in favor of this approach). It consists of maximizing the log pseudo-likelihood

$$\log L(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log c_{\boldsymbol{\theta}}(\hat{\mathbf{U}}_i), \tag{2}$$

where $c_{\boldsymbol{\theta}}$ denotes the density of a copula $C_{\boldsymbol{\theta}} \in \mathcal{C}_0$ assuming it exists, and where the $\hat{\mathbf{U}}_i = (\hat{U}_{i1}, \dots, \hat{U}_{id})^\top$ are the *pseudo-observations* computed from the $\mathbf{X}_i = (X_{i1}, \dots, X_{id})^\top$ by

$\hat{U}_{ij} = R_{ij}/(n+1)$, with R_{ij} being the rank of X_{ij} among $X_{1j}, \ldots, X_{nj}$. Note that, equivalently, $\hat{U}_{ij} = n\hat{F}_j(X_{ij})/(n+1)$, where $\hat{F}_j$ is the empirical c.d.f. computed from $X_{1j}, \ldots, X_{nj}$. The scaling factor $n/(n+1)$ is classically introduced to avoid problems at the boundary of $[0,1]^d$.

A very important issue that is currently being actively investigated is whether the assumption $C \in \mathcal{C}_0$ on which the previously discussed estimation step is based is actually valid or not. More formally, one wants to test

$$H_0 : C \in \mathcal{C}_0 \qquad \text{against} \qquad H_1 : C \notin \mathcal{C}_0.$$

Among the relatively large number of procedures proposed in the literature (see Charpentier et al., 2007; Genest et al., 2009, for extensive reviews), recent large scale simulations (Berg, 2009; Genest et al., 2009) indicate that powerful goodness-of-fit tests can be obtained from the process

$$\sqrt{n}\{C_n(\mathbf{u}) - C_{\boldsymbol{\theta}_n}(\mathbf{u})\}, \qquad \mathbf{u} \in [0,1]^d, \tag{3}$$

where C_n is the *empirical copula* (Deheuvels, 1981) of the data $\mathbf{X}_1, \ldots, \mathbf{X}_n$ defined by

$$C_n(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(\hat{\mathbf{U}}_i \leq \mathbf{u}), \qquad \mathbf{u} \in [0,1]^d, \tag{4}$$

and $C_{\boldsymbol{\theta}_n}$ is an estimation of C obtained assuming that $H_0 : C \in \mathcal{C}_0$ holds. The latter estimation is based on an estimator $\boldsymbol{\theta}_n$ of $\boldsymbol{\theta}$ such as the maximum pseudo-likelihood estimator discussed earlier. In Genest et al. (2009), the following Cramér-von Mises statistic

$$S_n = \int_{[0,1]^d} n\{C_n(\mathbf{u}) - C_{\boldsymbol{\theta}_n}(\mathbf{u})\}^2 \mathrm{d}C_n(\mathbf{u}) = \sum_{i=1}^{n}\{C_n(\hat{\mathbf{U}}_i) - C_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_i)\}^2$$

was found to yield the best results overall. To obtain approximate p-values for tests based on statistics derived from the goodness-of-fit process (3), such as S_n, Genest et al. (2009) used the parametric bootstrap procedure initially suggested in the univariate case by Stute et al. (1993) and showed its asymptotic validity in the rank-based context under consideration (Genest and Rémillard, 2008). The main inconvenience of this approach is its very high computational cost as each bootstrap iteration requires both random number generation from the hypothesized copula and estimation of the dependence parameters. In practice, as the sample size increases, this very high computational complexity tends to become an obstacle to the application of the parametric bootstrap, especially for so-called *implicit* copulas such as the normal or the t. Inspired by the seminal work of Scaillet (2005) and Rémillard and Scaillet (2009), a valid and much faster alternative approach was recently proposed in Kojadinovic et al. (2011). The illustration presented in the latter paper for $d = 2$ and $q = 1$ shows that for $n \approx 1500$ the use of the multiplier approach instead of the parametric bootstrap leads to a reduction in the computing time from about a day to a few minutes.

The aim of this paper is to extend the multiplier approach to multivariate multiparameter copulas and study the finite-sample performance of the resulting goodness-of-fit test.

Although for $d > 2$ the implementation of the multiplier goodness-of-fit procedure can be regarded as rather straightforward in the case of copulas with explicit c.d.f.s, for popular elliptical copulas such as the normal or the t it is much more challenging. For instance, for the t copula, the implementation of the test required the extension of the so-called *Plackett formula* for the t distribution (Genz, 2004) to the situation $d \geq 3$. From a practical perspective, the case of the elliptical copulas is also probably the most interesting as these, not necessarily satisfying the *exchangeability* property, are likely to better model multivariate data than classical one-parameter exchangeable Archimedean copulas (see e.g. Genest et al., 2007).

To the best of our knowledge, this is the first study of the finite-sample performance of a goodness-of-fit test for multiparameter normal or t copulas. It is important to stress out that such a study is made possible thanks to the computational efficiency of the multiplier approach.

In the second section, we elaborate on the asymptotic representation of the pseudo-likelihood estimator discussed earlier as it is probably the most natural way of estimating the vector of dependence parameters $\boldsymbol{\theta}$ when $q > 1$. The resulting asymptotic representation is used in Section 3 in combination with the multiplier approach proposed by Rémillard and Scaillet (2009), which yields a fast goodness-of-fit procedure. The fourth section is devoted to the main implementation issues, while more technical details necessary for the implementation of the test when the hypothesized copula family is the normal or the t are relegated to the Appendices. The fifth section presents extensive simulation results for $n = 100$, 300 and 500 and $d = 2$, 3 and 4, for five different copula families, *viz.* the Clayton, Gumbel, Frank, normal and t copulas. For the latter two families, both one-parameter exchangeable and multiparameter non-exchangeable dependence structures are considered. The last section is devoted to a discussion and concluding remarks.

Finally, note that the proposed test is implemented in the `copula` R package (Yan and Kojadinovic, 2010) available on the Comprehensive R Archive Network.

2 Maximum Pseudo-Likelihood Estimator

Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be a random sample from c.d.f. $C_{\boldsymbol{\theta}}\{F_1(x_1), \dots, F_d(x_d)\}$, where $F_1, \dots, F_d$ are continuous c.d.f.s and $C_{\boldsymbol{\theta}} \in \mathcal{C}_0$ is an absolutely continuous copula. For any $i \in \{1, \dots, n\}$, let $\mathbf{U}_i = (F_1(X_{i1}), \dots, F_d(X_{id}))^\top$. Furthermore, let $\boldsymbol{\theta}_n$ be the maximum pseudo-likelihood estimator of $\boldsymbol{\theta} = (\theta_1, \dots, \theta_q)^\top$ computed from the pseudo-observations $\hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_n$ by maximizing (2), and let $\boldsymbol{\Theta}_n = \sqrt{n}(\boldsymbol{\theta}_n - \boldsymbol{\theta})$. Also, let

$$\dot{c}_{\boldsymbol{\theta}}(\mathbf{u}) = \left(\frac{\partial c_{\boldsymbol{\theta}}(\mathbf{u})}{\partial \theta_1}, \dots, \frac{\partial c_{\boldsymbol{\theta}}(\mathbf{u})}{\partial \theta_q} \right)^\top, \qquad \mathbf{u} \in [0,1]^d.$$

Assuming similar regularity conditions as for maximum likelihood estimation, from the work of Genest et al. (1995), this estimator admits the following asymptotic representation:

$$\boldsymbol{\Theta}_n = \frac{1}{\sqrt{n}} \left[\mathbb{E}_{C_{\boldsymbol{\theta}}} \left\{ \frac{\dot{c}_{\boldsymbol{\theta}}(\mathbf{U}) \dot{c}_{\boldsymbol{\theta}}^{\top}(\mathbf{U})}{c_{\boldsymbol{\theta}}(\mathbf{U})^2} \right\} \right]^{-1} \sum_{i=1}^{n} \frac{\dot{c}_{\boldsymbol{\theta}}(\hat{\mathbf{U}}_i)}{c_{\boldsymbol{\theta}}(\hat{\mathbf{U}}_i)} + o_P(1).$$

From the work of Ruymgaart et al. (1972, §3) and Genest et al. (1995, §4 and Prop. A1), this representation can be rewritten in terms of the $\mathbf{U}_i$'s as

$$\boldsymbol{\Theta}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{J}_{\boldsymbol{\theta}}(\mathbf{U}_i) + o_P(1), \tag{5}$$

where

$$\mathbf{J}_{\boldsymbol{\theta}}(\mathbf{U}_i) = \left[\mathbb{E}_{C_{\boldsymbol{\theta}}} \left\{ \frac{\dot{c}_{\boldsymbol{\theta}}(\mathbf{U}) \dot{c}_{\boldsymbol{\theta}}^{\top}(\mathbf{U})}{c_{\boldsymbol{\theta}}(\mathbf{U})^2} \right\} \right]^{-1} \mathbf{K}_{\boldsymbol{\theta}}(\mathbf{U}_i),$$

and $\mathbf{K}_{\boldsymbol{\theta}}(\mathbf{U}_i)$ is a q-dimensional random vector whose kth component is

$$\frac{\partial \log c_{\boldsymbol{\theta}}(\mathbf{U}_i)}{\partial \theta_k} + \sum_{j=1}^{d} \int_{[0,1]^d} \{\mathbf{1}(U_{ij} \leq u_j) - u_j\} \frac{\partial^2 \log c_{\boldsymbol{\theta}}(\mathbf{U}_i)}{\partial \theta_k \partial u_j} \mathrm{d}C_{\boldsymbol{\theta}}(\mathbf{u}).$$

Upon integrating by parts with respect to u_j, the integral in the summation above can be expressed as

$$-\int_{[0,1]^d} \{1(U_{ij} \leq u_j) - u_j\} \frac{c_{\boldsymbol{\theta}}^{(j)}(\mathbf{u})}{c_{\boldsymbol{\theta}}(\mathbf{u})} \frac{\partial c_{\boldsymbol{\theta}}(\mathbf{u})}{\partial \theta_k} \mathrm{d}\mathbf{u},$$

where $c_{\boldsymbol{\theta}}^{(j)}(\mathbf{u}) = \partial c_{\boldsymbol{\theta}}(\mathbf{u}) / \partial u_j$. Hence, the $\mathbf{J}_{\boldsymbol{\theta}}(\mathbf{U}_i)$'s can be rewritten as

$$\left[\mathbb{E}_{C_{\boldsymbol{\theta}}} \left\{ \frac{\dot{c}_{\boldsymbol{\theta}}(\mathbf{U}) \dot{c}_{\boldsymbol{\theta}}^{\top}(\mathbf{U})}{c_{\boldsymbol{\theta}}(\mathbf{U})^2} \right\} \right]^{-1} \left[\frac{\dot{c}_{\boldsymbol{\theta}}(\mathbf{U}_i)}{c_{\boldsymbol{\theta}}(\mathbf{U}_i)} - \sum_{j=1}^{d} \int_{[0,1]^d} \{\mathbf{1}(U_{ij} \leq u_j) - u_j\} \frac{c_{\boldsymbol{\theta}}^{(j)}(\mathbf{u})}{c_{\boldsymbol{\theta}}(\mathbf{u})} \frac{\dot{c}_{\boldsymbol{\theta}}(\mathbf{u})}{c_{\boldsymbol{\theta}}(\mathbf{u})} \mathrm{d}C_{\boldsymbol{\theta}}(\mathbf{u}) \right]. \tag{6}$$

It is important to notice that the $\mathbf{J}_{\boldsymbol{\theta}}(\mathbf{U}_i)$'s are i.i.d. with expectation $\mathbf{0}$ and finite covariance.

As we shall see, the asymptotic representation (5) is a key element in the goodness-of-fit test procedure to be presented in the next section.

3 Goodness-of-Fit Test Procedure

Let us assume that the unknown copula C appearing in (1) belongs to the family $\mathcal{C}_0$. Also, suppose that the members $C_{\boldsymbol{\theta}}$ of $\mathcal{C}_0$ have continuous partial derivatives with respect to each u_j, which will be denoted by $C_{\boldsymbol{\theta}}^{(j)}$, and that $\boldsymbol{\theta}$ is estimated by the maximum pseudo-likelihood estimator $\boldsymbol{\theta}_n$ considered in the previous section. Furthermore, let $\boldsymbol{\Theta}$ denote the weak limit of $\boldsymbol{\Theta}_n = \sqrt{n}(\boldsymbol{\theta}_n - \boldsymbol{\theta})$, and let

$$\dot{C}_{\boldsymbol{\theta}}(\mathbf{u}) = \left(\frac{\partial C_{\boldsymbol{\theta}}(\mathbf{u})}{\partial \theta_1}, \dots, \frac{\partial C_{\boldsymbol{\theta}}(\mathbf{u})}{\partial \theta_q} \right)^\top, \qquad \mathbf{u} \in [0,1]^d,$$

Then, from the work of Quessy (2005) (see also Berg and Quessy, 2009; Genest and Rémillard, 2008, and the references therein), we have that, under mild regularity conditions, the goodness-of-fit process $\sqrt{n}(C_n - C_{\boldsymbol{\theta}_n})$ converges weakly in $\ell^\infty([0,1]^d)$ to the tight centered Gaussian process

$$\mathbb{W}_{\boldsymbol{\theta}}(\mathbf{u}) = \mathbb{C}_{\boldsymbol{\theta}}(\mathbf{u}) - \dot{C}_{\boldsymbol{\theta}}^\top(\mathbf{u})\boldsymbol{\Theta}, \qquad \mathbf{u} \in [0,1]^d, \tag{7}$$

where

$$\mathbb{C}_{\boldsymbol{\theta}}(\mathbf{u}) = \alpha_{\boldsymbol{\theta}}(\mathbf{u}) - \sum_{j=1}^d C_{\boldsymbol{\theta}}^{(j)}(\mathbf{u}) \alpha_{\boldsymbol{\theta}}(1, \dots, 1, u_j, 1, \dots, 1),$$

and where $\alpha_{\boldsymbol{\theta}}$ is a tight centered Gaussian process on $[0,1]^d$ with covariance function

$$E[\alpha_{\boldsymbol{\theta}}(\mathbf{u})\alpha_{\boldsymbol{\theta}}(\mathbf{v})] = C_{\boldsymbol{\theta}}(\mathbf{u} \wedge \mathbf{v}) - C_{\boldsymbol{\theta}}(\mathbf{u})C_{\boldsymbol{\theta}}(\mathbf{v}), \qquad \mathbf{u}, \mathbf{v} \in [0,1]^d.$$

In order to obtain approximate p-values for goodness-of-fit tests based on the empirical process $\sqrt{n}(C_n - C_{\boldsymbol{\theta}_n})$, Kojadinovic et al. (2011) have combined the approach proposed in Rémillard and Scaillet (2009) for simulating $\mathbb{C}_{\boldsymbol{\theta}}$ with the asymptotic representation (5). Before describing the resulting fast goodness-of-fit procedure, we recall the key results justifying the suggested approach.

Let N be a large integer and let $Z_i^{(k)}$, $i \in \{1, \dots, n\}$, $k \in \{1, \dots, N\}$, be i.i.d. random variables with mean 0 and variance 1 independent of the data $\mathbf{X}_1, \dots, \mathbf{X}_n$. For any $k \in \{1, \dots, N\}$, let

$$\alpha_n^{(k)}(\mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i^{(k)} \left\{ \mathbf{1}(\hat{\mathbf{U}}_\mathbf{i} \le \mathbf{u}) - C_n(\mathbf{u}) \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Z_i^{(k)} - \bar{Z}^{(k)}) \mathbf{1}(\hat{\mathbf{U}}_\mathbf{i} \le \mathbf{u}), \qquad \mathbf{u} \in [0,1]^d.$$

Furthermore, let $\ell^\infty([0,1]^d)$ be the space of bounded, real-valued functions on $[0,1]^d$ and let the arrow $\rightsquigarrow$ denote weak convergence. From Lemma A.1 in Rémillard and Scaillet (2009), one has that

$$\left(\sqrt{n}(H_n - C_{\boldsymbol{\theta}}), \alpha_n^{(1)}, \dots, \alpha_n^{(N)}\right) \rightsquigarrow \left(\alpha_{\boldsymbol{\theta}}, \alpha_{\boldsymbol{\theta}}^{(1)}, \dots, \alpha_{\boldsymbol{\theta}}^{(N)}\right)$$

in $\ell^\infty([0,1]^d)^{\otimes(N+1)}$, where H_n is the empirical c.d.f. computed from the unobservable random sample $\mathbf{U}_1, \dots, \mathbf{U}_n$, and where $\alpha_{\boldsymbol{\theta}}^{(1)}, \dots, \alpha_{\boldsymbol{\theta}}^{(N)}$ are independent copies of $\alpha_{\boldsymbol{\theta}}$. As a consistent estimator of the jth partial derivative $C_{\boldsymbol{\theta}}^{(j)}$ of $C_{\boldsymbol{\theta}}$, Rémillard and Scaillet (2009, Prop. A.2) suggested using

$$C_n^{(j)}(\mathbf{u}) = \frac{1}{2n^{-1/2}} \left\{ C_n(u_1, \dots, u_{j-1}, u_j + n^{-1/2}, u_{j+1}, \dots, u_d) \right.$$
$$\left. - C_n(u_1, \dots, u_{j-1}, u_j - n^{-1/2}, u_{j+1}, \dots, u_d) \right\}.$$

Now, for any $k \in \{1, \ldots, N\}$, let

$$\mathbb{C}_n^{(k)}(\mathbf{u}) = \alpha_n^{(k)}(\mathbf{u}) - \sum_{j=1}^d C_n^{(j)}(\mathbf{u}) \alpha_n^{(k)}(1, \ldots, 1, u_j, 1, \ldots, 1), \qquad \mathbf{u} \in [0,1]^d.$$

From the proof of Theorem 2.1 in Rémillard and Scaillet (2009), one obtains that

$$\left(\sqrt{n}(C_n - C_{\boldsymbol{\theta}}), \mathbb{C}_n^{(1)}, \ldots, \mathbb{C}_n^{(N)}\right) \rightsquigarrow \left(\mathbb{C}_{\boldsymbol{\theta}}, \mathbb{C}_{\boldsymbol{\theta}}^{(1)}, \ldots, \mathbb{C}_{\boldsymbol{\theta}}^{(N)}\right)$$

in $\ell^\infty([0,1]^d)^{\otimes(N+1)}$, where $\mathbb{C}_{\boldsymbol{\theta}}^{(1)}, \ldots, \mathbb{C}_{\boldsymbol{\theta}}^{(N)}$ are independent copies of $\mathbb{C}_{\boldsymbol{\theta}}$. Next, by analogy with the asymptotic representation (5), for any $k \in \{1, \ldots, N\}$, let

$$\boldsymbol{\Theta}_n^{(k)} = \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i^{(k)} \mathbf{J}_{\boldsymbol{\theta}}(\mathbf{U}_i). \tag{8}$$

From the multivariate central limit theorem, we immediately have that

$$(\boldsymbol{\Theta}_n, \boldsymbol{\Theta}_n^{(1)}, \ldots, \boldsymbol{\Theta}_n^{(N)}) \rightsquigarrow (\boldsymbol{\Theta}, \boldsymbol{\Theta}^{(1)}, \ldots, \boldsymbol{\Theta}^{(N)})$$

in $\mathbb{R}^{\otimes N+1}$, where $\boldsymbol{\Theta}^{(1)}, \ldots, \boldsymbol{\Theta}^{(N)}$ are independent copies of $\boldsymbol{\Theta}$, the weak limit of $\boldsymbol{\Theta}_n$. Then, under mild regularity conditions, one has that

$$\left(\sqrt{n}(C_n - C_{\boldsymbol{\theta}_n}), \mathbb{C}_n^{(1)} - \dot{C}_{\boldsymbol{\theta}_n}^\top \boldsymbol{\Theta}_n^{(1)}, \ldots, \mathbb{C}_n^{(N)} - \dot{C}_{\boldsymbol{\theta}_n}^\top \boldsymbol{\Theta}_n^{(N)}\right)$$

converges weakly to $(\mathbb{W}_{\boldsymbol{\theta}}, \mathbb{W}_{\boldsymbol{\theta}}^{(1)}, \ldots, \mathbb{W}_{\boldsymbol{\theta}}^{(N)})$ in $\ell^\infty([0,1]^d)^{\otimes(N+1)}$, where $\mathbb{W}_{\boldsymbol{\theta}}^{(1)}, \ldots, \mathbb{W}_{\boldsymbol{\theta}}^{(N)}$ are independent copies of the process $\mathbb{W}_{\boldsymbol{\theta}}$ defined in (7).

As the n random vectors $\mathbf{U}_1, \ldots, \mathbf{U}_n$ cannot be observed and $C_{\boldsymbol{\theta}}$ is obviously unknown, to compute the $\mathbf{J}_{\boldsymbol{\theta}}(\mathbf{U}_i)$'s appearing in (8), one can proceed as Genest et al. (1995, §3) for the asymptotic variance of the maximum pseudo-likelihood estimator: the $\mathbf{U}_i$'s can be replaced by the available pseudo-observations $\hat{\mathbf{U}}_i$, $\boldsymbol{\theta}$ by $\boldsymbol{\theta}_n$ and the integrals appearing in (6) can be computed with respect to $\mathrm{d}C_n$ instead of $\mathrm{d}C_{\boldsymbol{\theta}}$. This yields

$$\hat{\mathbf{J}}_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_i) = \boldsymbol{\Sigma}_n^{-1} \left[\frac{\dot{c}_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_i)}{c_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_i)} - \frac{1}{n} \sum_{j=1}^d \sum_{k=1}^n \left\{ \mathbf{1}(\hat{U}_{ij} \leq \hat{U}_{kj}) - \hat{U}_{kj} \right\} \frac{c_{\boldsymbol{\theta}_n}^{(j)}(\hat{\mathbf{U}}_k)}{c_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_k)} \frac{\dot{c}_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_k)}{c_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_k)} \right],$$

where $\boldsymbol{\Sigma}_n$ is the sample covariance matrix of $\dot{c}_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_1)/c_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_1), \ldots, \dot{c}_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_n)/c_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_n)$. Now, for any $k \in \{1, \ldots, N\}$, let

$$\hat{\boldsymbol{\Theta}}_n^{(k)} = \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i^{(k)} \hat{\mathbf{J}}_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_i).$$

Then, under additional conditions, one has that

$$\left(\sqrt{n}(C_n - C_{\boldsymbol{\theta}_n}), \mathbb{C}_n^{(1)} - \dot{C}_{\boldsymbol{\theta}_n}^\top \hat{\boldsymbol{\Theta}}_n^{(1)}, \ldots, \mathbb{C}_n^{(N)} - \dot{C}_{\boldsymbol{\theta}_n}^\top \hat{\boldsymbol{\Theta}}_n^{(N)}\right)$$

converges weakly to $(\mathbb{W}_{\boldsymbol{\theta}}, \mathbb{W}_{\boldsymbol{\theta}}^{(1)}, \ldots, \mathbb{W}_{\boldsymbol{\theta}}^{(N)})$. The previous developments thus suggest adopting the following fast goodness-of-fit procedure:

1. Compute C_n from $\hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_n$ using (4), and estimate $\boldsymbol{\theta} = (\theta_1, \dots, \theta_q)^\top$ using the maximum pseudo-likelihood estimator $\boldsymbol{\theta}_n$.

2. Compute the Cramér-von Mises statistic

$$S_n = \int_{[0,1]^d} n\{C_n(\mathbf{u}) - C_{\boldsymbol{\theta}_n}(\mathbf{u})\}^2 \mathrm{d}C_n(\mathbf{u}) = \sum_{i=1}^n \{C_n(\hat{\mathbf{U}}_i) - C_{\boldsymbol{\theta}_n}(\hat{\mathbf{U}}_i)\}^2.$$

3. Then, for some large integer N, repeat the following steps for every $k \in \{1, \dots, N\}$:

   (a) Generate n i.i.d. random variates $Z_1^{(k)}, \dots, Z_n^{(k)}$ with expectation 0 and variance 1.

   (b) Form an approximate realization of the test statistic under H_0 by

$$S_n^{(k)} = \int_{[0,1]^d} \left\{\mathbb{C}_n^{(k)}(\mathbf{u}) - \dot{C}_{\boldsymbol{\theta}_n}^\top(\mathbf{u})\hat{\boldsymbol{\Theta}}_n^{(k)}\right\}^2 \mathrm{d}C_n(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^n \left\{\mathbb{C}_n^{(k)}(\hat{\mathbf{U}}_i) - \dot{C}_{\boldsymbol{\theta}_n}^\top(\hat{\mathbf{U}}_i)\hat{\boldsymbol{\Theta}}_n^{(k)}\right\}^2.$$

4. An approximate p-value for the test is then given by $N^{-1}\sum_{k=1}^N \mathbf{1}(S_n^{(k)} \geq S_n)$.

The computational efficiency of the procedure is due to the fact that Step 3 (b) only involves simple arithmetic operations between the $Z_i^{(k)}$'s and terms that need to be computed only once. Notice that, when the hypothesis $H_0 : C \in \mathcal{C}_0$ is not true, the $S_n^{(k)}$, $k \in \{1, \dots, N\}$, cannot be regarded anymore as approximate independent copies of S_n under H_0 because $\mathbf{X}_1, \dots, \mathbf{X}_n$ is not anymore a random sample from a c.d.f. $C_{\boldsymbol{\theta}}\{F_1(x_1), \dots, F_d(x_d)\}$, where $C_{\boldsymbol{\theta}} \in \mathcal{C}_0$. This does not however affect the consistency of the procedure as S_n will tend to infinity in probability if H_0 is false.

4 Implementation Issues

To implement the previous procedure for a given copula family $\mathcal{C}_0$, one needs to be able to estimate the vector of dependence parameters $\boldsymbol{\theta}$ from the available data. The corresponding pseudo-likelihood is maximized using the well-designed general-purpose R `optim` routine (R Development Core Team, 2011). Also, for every $C_{\boldsymbol{\theta}} \in \mathcal{C}_0$, one needs to be able to compute, the quantities $\dot{C}_{\boldsymbol{\theta}}$, $c_{\boldsymbol{\theta}}^{(j)}/c_{\boldsymbol{\theta}}$ and $\dot{c}_{\boldsymbol{\theta}}/c_{\boldsymbol{\theta}}$, the latter two appearing in the expression of the function $\mathbf{J}_\theta$ defined by (6). For copula families whose c.d.f. can be explicitly written, these expressions can be obtained by differentiation. In this work, we considered three such explicit one-parameter copulas from the Archimedean family, namely the Clayton, Gumbel and Frank copulas. For each copula, expressions for $\dot{C}_{\boldsymbol{\theta}}$, $c_{\boldsymbol{\theta}}^{(j)}/c_{\boldsymbol{\theta}}$ and $\dot{c}_{\boldsymbol{\theta}}/c_{\boldsymbol{\theta}}$ were obtained using symbolic computation software and were stored for future access. For the Clayton and Gumbel copulas, calculations were performed up to dimension 10, whereas for the Frank copula they were carried out only up to dimension 6 because of the complexity of the resulting expressions.

Other popular copulas such as the elliptical ones do not have explicit c.d.f. expressions. In this study, two elliptical copulas were considered: the normal and the t copula with ν degrees of freedom. The normal copula with correlation matrix $\mathbf{\Sigma}$ is defined as

$$C^N_{\mathbf{\Sigma}}(\mathbf{u}) = \Phi_{\mathbf{\Sigma}}\{\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)\}, \qquad \mathbf{u} \in [0,1]^d,$$

where $\Phi_{\mathbf{\Sigma}}$ is the multivariate standard normal c.d.f. with correlation matrix $\mathbf{\Sigma}$ and where Φ is the univariate standard normal c.d.f. Recall that the p.d.f. of the centered multivariate normal with covariance matrix $\mathbf{\Sigma}$ is given by

$$\phi_{\mathbf{\Sigma}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}|\mathbf{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{x}^\top\mathbf{\Sigma}^{-1}\mathbf{x}\right), \qquad \mathbf{x} \in \mathbb{R}^d. \tag{9}$$

Similarly, the t copula with correlation matrix $\mathbf{\Sigma}$ and ν degrees of freedom is defined as

$$C^t_{\nu,\mathbf{\Sigma}}(\mathbf{u}) = T_{\nu,\mathbf{\Sigma}}\{T_\nu^{-1}(u_1), \dots, T_\nu^{-1}(u_d)\}, \qquad \mathbf{u} \in [0,1]^d,$$

where $T_{\nu,\mathbf{\Sigma}}$ is the multivariate standard t c.d.f. with correlation matrix $\mathbf{\Sigma}$ and ν degrees of freedom, and T_ν is the univariate standard t c.d.f. with ν degrees of freedom. The p.d.f. of the centered multivariate t with covariance matrix $\mathbf{\Sigma}$ and ν degrees of freedom is given by

$$t_{\nu,\mathbf{\Sigma}}(\mathbf{x}) = \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{(\pi\nu)^{\frac{d}{2}}\Gamma\left(\frac{\nu}{2}\right)|\mathbf{\Sigma}|^{\frac{1}{2}}} \left(1 + \frac{1}{\nu}\mathbf{x}^\top\mathbf{\Sigma}^{-1}\mathbf{x}\right)^{-\frac{\nu+d}{2}}. \tag{10}$$

In this work, following Genest et al. (2009) and Berg (2009), the number of degrees of freedom ν is not regarded as a parameter to be estimated from the data. The more complex situation where ν needs to be estimated (see e.g. Demarta and McNeil, 2005) will be investigated in the future.

In our implementation, we considered four different ways of parameterizing the correlation matrix $\mathbf{\Sigma}$ defining the dependence structure of the normal or the t copula:

- For a given θ in $[-1,1]$, the *exchangeable* dependence structure is obtained by defining elements of $\mathbf{\Sigma}$ by $\rho_{i,j} = \theta$ for $i \neq j$. The resulting copulas thus depend only on one parameter and satisfy the so-called *exchangeability* property.
- For a given θ in $[-1,1]$, the *AR1* dependence structure is obtained by setting $\rho_{i,j} = \theta^{|i-j|}$ for $i \neq j$. The resulting copulas are again one-parameter copulas.
- For a given vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{d-1})$ in $[-1,1]^{d-1}$, the *Toeplitz* dependence structure is obtained by defining elements of $\mathbf{\Sigma}$ by $\rho_{i,j} = \theta_{|i-j|}$ for $i \neq j$. The resulting copulas have therefore $d-1$ parameters.
- Finally, the *unstructured* case corresponds to copulas defined by $d(d-1)/2$ parameters.

As we continue, we shall therefore regard $\mathbf{\Sigma}$ as parameterized by q reals $\theta_1, \dots, \theta_q$ where $q = 1$, $d-1$ or $d(d-1)/2$ depending on the underlying dependence structure.

The derivation of the quantities $\dot{C}_{\boldsymbol{\theta}}$, $c_{\boldsymbol{\theta}}^{(j)}/c_{\boldsymbol{\theta}}$ and $\dot{c}_{\boldsymbol{\theta}}/c_{\boldsymbol{\theta}}$ for the normal and t copulas is relegated to Appendix A. As we shall see, it is much more challenging than for explicit copulas. For instance, the computation of $\dot{C}^t_{\nu,\mathbf{\Sigma}}$ required the extension of the *Plackett formula* for the t distribution (Genz, 2004) to the situation $d \geq 3$. The proof of that very useful result is given in Appendix B.

5 Finite-Sample Performance

The finite-sample performance of the goodness-of-fit procedure given in Section 3 is investigated in two simulation studies. The first one involves only one-parameter multivariate copulas satisfying the exchangeability property, while the second one additionally involves non-exchangeable, possibly multiparameter, multivariate copulas. In both studies, two characteristics of interests are the empirical levels and the empirical powers of the proposed testing procedure.

5.1 Exchangeable Copula Families

Five copula families are considered, namely the Clayton, Gumbel, Frank, normal, and t with $\nu = 4$ degrees of freedom. They will be abbreviated by the letters C, G, F, N and t respectively in the forthcoming tables. Each copula family is used both as hypothesized family and as data generating family. For generating data, four levels of dependence are considered corresponding respectively to a Kendall's tau of 0.2, 0.4, 0.6 and 0.8 in the bivariate case. The dimension d is set to either 2, 3 or 4. In order to gain an idea on how large a sample needs to be for the goodness-of-fit procedure discussed in Section 3 to work well, three values for n are considered, *viz.* 100, 300 and 500. The number of multiplier iterations N is set to 1000. For each combination of copula family, Kendall's tau, dimension and sample size, 10 000 samples are generated and are then used to estimate the rejection percentages of the five null hypotheses under consideration. These empirical rejection percentages are given in Tables 1, 2 and 3 for sample size 100, 300 and 500 respectively. The empirical levels are italicized.

[Table 1 about here.]

[Table 2 about here.]

[Table 3 about here.]

By comparing the empirical levels to the 5% nominal level, one can see that the proposed procedure globally appears to improve as the sample size n increases from 100 to 300 and then to 500, and as the dimension d decreases from 4 to 3 and 2. For $n = 100$, the range of the empirical levels is 2.9–9.0% for $d = 2$, 0.2–7.2% for $d = 3$, and 0.0–7.4% for $d = 4$, respectively. The levels are above 5% when the true copula is the Clayton or Frank for $d = 2$. They are smaller than 5% in most other scenarios, especially for higher dimension and strong dependence. For $n = 300$, the range of the empirical levels is 3.0–6.3% for $d = 2$, 1.3–5.6% for $d = 3$, and 0.8–5.9% for $d = 4$. For $n = 500$, the ranges become 2.8–5.8% for $d = 2$, 1.7–5.5% for $d = 3$, and 1.1—5.3% for $d = 4$. Although the agreement with the 5% nominal level appears to improve as the sample size increases, the procedure remains disappointingly conservative for $\tau = 0.8$ and $d = 3$ and 4.

Let us now comment on the power of the proposed testing procedure. As expected, the empirical powers appear to increase as the sample size increases. As the dimension increases, the empirical powers increase in most scenarios. Noticeable exceptions include the cases where the null hypothesis involves the normal copula and the data are generated from a Frank copula or a t copula. For $n = 300$ for example, when the data are generated from the Frank copula with $\tau = 0.4$, the rejection percentage of the normal family is 58.1 for $d = 2$, 41.0 for $d = 3$, and 23.7 for $d = 4$. When the data are generated from the t copula with $\tau = 0.6$, the rejection percentage of the normal family are is 7.9 for $d = 2$, 7.0 for $d = 3$, and 6.1 for $d = 4$. These exceptions are not surprising because the Frank copula with moderate dependence and the t copula with high dependence are known to be similar to normal copulas. Finally, notice that in some scenarios, as the dependence gets stronger, the power increases first and then decreases, which may be explained by the fact that the difference between the true copula and the hypothesized copula increases first and then decreases.

5.2 Non-exchangeable Copulas

In the second study, copulas not satisfying the exchangeability property are considered in addition to the previously used exchangeable copulas. The non-exchangeable copulas under consideration are the normal and t with AR1, Toeplitz and unstructured correlation matrix defined in Section 4. As they are very useful in practical multivariate applications, it is important to assess the performance of the goodness-of-fit test for them. The four possible dependence structures for the normal and t copulas will be abbreviated as ex, ar1, tp, and un as we continue. Hence, overall, eleven copula families are used both as hypothesized and as data generating families. For data generation, the value of the dependence parameter θ for the exchangeable copulas is set to correspond to a τ of 0.4 in the bivariate case. For the elliptical copulas having AR1 correlation matrix structure, the value of θ is set such that Kendall's tau for pairs of variables $\{X_i, X_{i+1}\}$, $i \in \{1, \dots, d-1\}$ is 0.4 as well. For the Toeplitz dependence structure, the values of $\theta_1, \dots, \theta_{d-1}$ parameterizing the correlation matrix $\mathbf{\Sigma}$ are set to correspond to τ's equally spaced between 0.2 and 0.6 so that their mean is 0.4. Finally, in the unstructured case, the values of the $d(d-1)/2$ elements $\rho_{2,1}, \rho_{3,1}, \dots, \rho_{d,d-1}$ of $\mathbf{\Sigma}$ are also set to correspond to τ's equally spaced between 0.2 and 0.6. Note that the dependence structures ex and ar1 are particular cases of both tp and un, and that tp is a special case of un. The dimension d is set to either 3 or 4 and sample sizes $n = 100$ and $n = 300$ are considered. As in the previous study, 10 000 repetitions are used to obtain empirical rejection percentages and N is set to 1000. The results are presented in Table 4. As previously, the empirical levels are italicized.

[Table 4 about here.]

For the dimensions under consideration, the empirical levels overall appear to be quite lower than the 5% nominal level for $n = 100$, but as expected, the agreement gets better for $n = 300$. For every hypothesized copula family, the empirical powers of the goodness-of-fit test increase with the sample size but not necessarily with the dimension. For example, for $n = 300$, when the true copula is Gumbel, the rejection percentage of N-ex increases from 70.7 for $d = 3$ to 84.1 for $d = 4$, while the rejection percentage of t-ex decreases from 51.6 for $d = 3$ to 33.8 for $d = 4$. The test seems to have quite good power when the true copula and the hypothesized copula are dissimilar. It has no power when the true copula is a special case of the hypothesized copula family, which is reassuring. For instance, when N-tp is considered as a model for data generated from an N-ar1 copula, the empirical power, which is also an empirical level, is below 5%. On the other hand, when the hypothesized copula family is strictly contained in the true copula family, the test has substantial power. For example, for $n = 100$ and $d = 4$, the empirical rejection percentage is 99.7 when N-un is the true copula and the hypothesized family is N-ar1. When the hypothesized copula and the true copula are not nested but belong to the same broader family, the test has also good power. For instance, for $n = 100$ and $d = 4$, when N-ex is the true copula and N-ar1 is the hypothesized family, the empirical rejection percentage is 98.1.

6 Discussion

From the extensive simulation results presented in the previous section, it seems sensible to conclude that as soon as the sample size reaches 300, the multiplier approach can be safely used in all circumstances. Its performance remains satisfactory even for non-exchangeable multiparameter copulas such as the normal or the t. The fact that the proposed goodness-of-fit procedure appears to be very conservative for very strongly dependent data sets may not be of great practical importance as such situations seem rather rare in real world problems. The results of the Monte Carlo experiments actually suggest that the proposed procedure can be safely used even in the case of samples of size as small as 100, as long as the Clayton dependence structure is not being tested.

The results of the experiments carried out in Kojadinovic et al. (2011) in the bivariate one-parameter case for $n = 150$ suggest that the proposed multiplier procedure provides a valid alternative to the corresponding parametric bootstrap-based procedure. The illustration presented in the latter paper also demonstrates that the multiplier procedure is orders of magnitude faster. The procedure actually remains extremely fast even for higher dimensions and larger sample size. For instance, for $n = 500$ and $d = 4$, to test whether N-un (resp. the Clayton copula) may be considered as an appropriate model for the data, the procedure of Section 3 takes approximately 1.73 seconds (resp. 0.20 seconds) on one Pentium M 2.2 GHz processor with $N = 1000$. The computational efficiency of the multiplier approach does not however prevent it from being rapidly affected by the increasing dimensionality of the data for certain models. Indeed, if the dimension d is increased to 5 while all other parameters remain unchanged, the procedure takes approximately 0.22 seconds if the Clayton copula is hypothesized but 4.28 seconds if N-un is hypothesized. If d is increased to 6, the approximate execution times become 0.26 and 9.34 seconds respectively. Note however that the previous measures are based on our mixed R and C implementation which is not optimal in terms of speed, especially when the hypothesized family is the normal or the t.

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A Expressions for normal and t copulas

A.1 Expressions of $\dot{c}_{\mathbf{\Sigma}}/c_{\mathbf{\Sigma}}$

Let $f_{\mathbf{\Sigma}}$ stand for $\phi_{\mathbf{\Sigma}}$ or $t_{\nu,\mathbf{\Sigma}}$, let F stand for Φ or T_ν, and let f stand for the univariate standard normal p.d.f. ϕ or for t_ν, the univariate standard t p.d.f. with ν degrees of freedom. Then, it can be checked that

$$c_{\mathbf{\Sigma}}(\mathbf{u}) = \frac{f_{\mathbf{\Sigma}}\left\{F^{-1}(u_1), \ldots, F^{-1}(u_d)\right\}}{f\left\{F^{-1}(u_1)\right\} \ldots f\left\{F^{-1}(u_d)\right\}}, \qquad \mathbf{u} \in [0,1]^d,$$

where $c_{\mathbf{\Sigma}}$ stands for the p.d.f. obtained from $C^N_{\mathbf{\Sigma}}$ or $C^t_{\nu,\mathbf{\Sigma}}$. Furthermore, let

$$\dot{f}_{\mathbf{\Sigma}}(\mathbf{u}) = \left(\frac{\partial f_{\mathbf{\Sigma}}(\mathbf{u})}{\partial \theta_1}, \ldots, \frac{\partial f_{\mathbf{\Sigma}}(\mathbf{u})}{\partial \theta_q}\right)^\top, \qquad \mathbf{u} \in [0,1]^d,$$

and define $\dot{c}_{\mathbf{\Sigma}}$ correspondingly. It follows that

$$\dot{c}_{\mathbf{\Sigma}}(\mathbf{u}) = \frac{\dot{f}_{\mathbf{\Sigma}}\left\{F^{-1}(u_1), \ldots, F^{-1}(u_d)\right\}}{f\left\{F^{-1}(u_1)\right\} \ldots f\left\{F^{-1}(u_d)\right\}}, \qquad \mathbf{u} \in [0,1]^d,$$

and therefore that

$$\frac{\dot{c}_{\mathbf{\Sigma}}(\mathbf{u})}{c_{\mathbf{\Sigma}}(\mathbf{u})} = \frac{\dot{f}_{\mathbf{\Sigma}}\left\{F^{-1}(u_1), \ldots, F^{-1}(u_d)\right\}}{f_{\mathbf{\Sigma}}\left\{F^{-1}(u_1), \ldots, F^{-1}(u_d)\right\}}, \qquad \mathbf{u} \in [0,1]^d. \tag{11}$$

In the normal case, starting from (9), we obtain

$$\frac{\partial \phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \theta_i} = -\frac{1}{2}\phi_{\mathbf{\Sigma}}(\mathbf{x})\left(\frac{1}{|\mathbf{\Sigma}|}\frac{\partial |\mathbf{\Sigma}|}{\partial \theta_i} + \mathbf{x}^\top \frac{\partial \mathbf{\Sigma}^{-1}}{\partial \theta_i}\mathbf{x}\right), \qquad \mathbf{x} \in \mathbb{R}^d. \tag{12}$$

Similarly, in the t case, starting from (10), we get

$$\frac{\partial t_{\nu,\mathbf{\Sigma}}(\mathbf{x})}{\partial \theta_i} = -\frac{1}{2}t_{\nu,\mathbf{\Sigma}}(\mathbf{x})\left\{\frac{1}{|\mathbf{\Sigma}|}\frac{\partial |\mathbf{\Sigma}|}{\partial \theta_i} + \frac{(\nu+d)\mathbf{x}^\top \frac{\partial \mathbf{\Sigma}^{-1}}{\partial \theta_i}\mathbf{x}}{\nu + \mathbf{x}^\top \mathbf{\Sigma}^{-1}\mathbf{x}}\right\}, \qquad \mathbf{x} \in \mathbb{R}^d. \tag{13}$$

From e.g. Seber (2008, Chap. 17), we have that

$$\frac{\partial |\mathbf{\Sigma}|}{\partial \theta_i} = |\mathbf{\Sigma}| \operatorname{trace}\left(\mathbf{\Sigma}^{-1}\frac{\partial \mathbf{\Sigma}}{\partial \theta_i}\right) \qquad \text{and that} \qquad \frac{\partial \mathbf{\Sigma}^{-1}}{\partial \theta_i} = -\mathbf{\Sigma}^{-1}\frac{\partial \mathbf{\Sigma}}{\partial \theta_i}\mathbf{\Sigma}^{-1}.$$

We used the two latter expressions for the exchangeable, AR1 and Toeplitz dependence structures. In the unstructured case, it is faster to use the fact that

$$\frac{\partial |\mathbf{\Sigma}|}{\partial \rho_{i,j}} = 2K_{ij} \qquad \text{and that} \qquad \frac{\partial \mathbf{\Sigma}^{-1}}{\partial \rho_{i,j}} = -r_i r_j^\top - r_j r_i^\top,$$

where K_{ij} is the cofactor of $\rho_{i,j}$, and where r_i is the i-th column of $\mathbf{\Sigma}^{-1}$. Finally, combining (11) with (12) (resp. (13)), we obtain the desired expression for the normal (resp. t) copula.

A.2 Expressions of $c_{\mathbf{\Sigma}}^{(j)}/c_{\mathbf{\Sigma}}$

Using the same generic notation as in the previous subsection, it can be checked that

$$\frac{c_{\mathbf{\Sigma}}^{(j)}(\mathbf{u})}{c_{\mathbf{\Sigma}}(\mathbf{u})} = \frac{f_{\mathbf{\Sigma}}^{(j)}\{F^{-1}(u_1),\dots,F^{-1}(u_d)\}}{f\{F^{-1}(u_j)\}f_{\mathbf{\Sigma}}\{F^{-1}(u_1),\dots,F^{-1}(u_d)\}} - \frac{f'\{F^{-1}(u_j)\}}{f\{F^{-1}(u_j)\}^2}, \qquad j \in \{1,\dots,d\}, \mathbf{u} \in [0,1]^d.$$

In the normal case, starting from (9), we obtain

$$\phi_{\mathbf{\Sigma}}^{(j)}(\mathbf{x}) = -\frac{\mathbf{x}^\top \mathbf{\Sigma}^{-1}\mathbf{e}_j}{(2\pi)^{\frac{d}{2}}|\mathbf{\Sigma}|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}\mathbf{x}^\top \mathbf{\Sigma}^{-1}\mathbf{x}\right) = -\mathbf{x}^\top \mathbf{\Sigma}^{-1}\mathbf{e}_j \phi_{\mathbf{\Sigma}}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d,$$

where $\mathbf{e}_j$ is the unit vector of $\mathbb{R}^d$ whose ith component is 1 if $i = j$ and 0 otherwise. Hence,

$$\frac{c_{\mathbf{\Sigma}}^{N^{(j)}}(\mathbf{u})}{c_{\mathbf{\Sigma}}^{N}(\mathbf{u})} = \frac{-\left(\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_d)\right)\mathbf{\Sigma}^{-1}\mathbf{e}_j}{\phi\{\Phi^{-1}(u_j)\}} + \frac{\Phi^{-1}(u_j)}{\phi\{\Phi^{-1}(u_j)\}}, \qquad \mathbf{u} \in [0,1]^d.$$

Similarly, in the t case, starting from (10), we get

$$t_{\nu,\mathbf{\Sigma}}^{(j)}(\mathbf{x}) = -\frac{(\nu+d)\mathbf{x}^\top \mathbf{\Sigma}^{-1}\mathbf{e}_j}{\nu + \mathbf{x}^\top \mathbf{\Sigma}^{-1}\mathbf{x}}t_{\nu,\mathbf{\Sigma}}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d,$$

which, in the univariate case, gives

$$t'_\nu(x) = -\frac{(\nu+1)x}{\nu+x^2} t_\nu(x), \qquad x \in \mathbb{R}.$$

Finally, we obtain that

$$\frac{c^{t(j)}_{\nu,\mathbf{\Sigma}}(\mathbf{u})}{c^{t}_{\nu,\mathbf{\Sigma}}(\mathbf{u})} = -\frac{(\nu+d)\mathbf{x}^\top \mathbf{\Sigma}^{-1}\mathbf{e}_j}{(\nu+\mathbf{x}^\top\mathbf{\Sigma}^{-1}\mathbf{x})t_\nu\{T_\nu^{-1}(u_j)\}} + \frac{(\nu+1)T_\nu^{-1}(u_j)}{\{\nu+T_\nu^{-1}(u_j)^2\}t_\nu\{T_\nu^{-1}(u_j)\}}, \qquad \mathbf{u} \in [0,1]^d,$$

where $\mathbf{x} = (T_\nu^{-1}(u_1), \dots, T_\nu^{-1}(u_d))^\top$.

A.3 Expressions of $C^{(j)}_{\mathbf{\Sigma}}$

Let $F_{\mathbf{\Sigma}}$ stand for $\Phi_{\mathbf{\Sigma}}$ or $T_{\nu,\mathbf{\Sigma}}$ and let $\mathbf{X}$ be a random vector with c.d.f. $F_{\mathbf{\Sigma}}$. It can then be checked that

$$\frac{\partial F_{\mathbf{\Sigma}}(F^{-1}(u_1),\dots,F^{-1}(u_d))}{\partial u_j} = F_{\mathbf{\Sigma}|X_j=F^{-1}(u_j)}\left(F^{-1}(u_1),\dots,F^{-1}(u_{j-1}),F^{-1}(u_{j+1}),\dots,F^{-1}(u_d)\right),$$

where $F_{\mathbf{\Sigma}|X_j=x_j}$ is the c.d.f. of $\mathbf{X}_{-j} = (X_1,\dots,X_{j-1},X_{j+1},\dots,X_d)^\top$ given $X_j = x_j$.

If $\mathbf{X}$ is multivariate standard normal with correlation matrix $\mathbf{\Sigma}$, then, conditionally on $X_j = x_j$, it is well-known (see e.g. Fang et al., 1990) that the random vector $\mathbf{X}_{-j}$ is multivariate normal with expectation $x_j\mathbf{\Sigma}_{-j,j}$ and covariance matrix

$$\mathbf{\Lambda}_j = \mathbf{\Sigma}_{-j,-j} - \mathbf{\Sigma}_{-j,j}\mathbf{\Sigma}_{j,-j}.$$

In the previous expression, $\mathbf{\Sigma}_{-j,-j}$ is a $(d-1)\times(d-1)$ matrix obtained by removing the jth row and the jth column of $\mathbf{\Sigma}$, $\mathbf{\Sigma}_{-j,j}$ is a $(d-1)\times 1$ matrix obtained by removing the jth row and keeping only the jth column of $\mathbf{\Sigma}$, and $\mathbf{\Sigma}_{j,-j} = \mathbf{\Sigma}_{-j,j}^\top$. Hence,

$$C^{N(j)}_{\mathbf{\Sigma}}(\mathbf{u}) = \Phi_{\mathbf{\Lambda}_j}(\mathbf{x}_{-j} - x_j\mathbf{\Sigma}_{-j,j}), \qquad \mathbf{u} \in [0,1]^d, \tag{14}$$

where $\mathbf{x} = (\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_d))^\top$.

Similarly, from Nadarajah and Kotz (2005, p 66), if $\mathbf{X}$ is standard multivariate t with ν degrees of freedom and parameter correlation matrix $\mathbf{\Sigma}$, then, conditionally on $X_j = x_j$, we have that

$$\sqrt{\frac{\nu+1}{\nu+x_j^2}}\,(\mathbf{X}_{-j} - x_j\mathbf{\Sigma}_{-j,j})$$

is multivariate standard t with $\nu+1$ degrees of freedom and parameter covariance matrix $\mathbf{\Lambda}_j = \mathbf{\Sigma}_{-j,-j} - \mathbf{\Sigma}_{-j,j}\mathbf{\Sigma}_{j,-j}$. Hence,

$$C^{t(j)}_{\nu,\mathbf{\Sigma}}(\mathbf{u}) = T_{\nu+1,\mathbf{\Lambda}_j}\left(\sqrt{\frac{\nu+1}{\nu+x_j^2}}(\mathbf{x}_{-j} - x_j\mathbf{\Sigma}_{-j,j})\right), \qquad \mathbf{u} \in [0,1]^d, \tag{15}$$

where $\mathbf{x} = (T_\nu^{-1}(u_1),\dots,T_\nu^{-1}(u_d))^\top$.

A.4 Expressions of $\dot{C}_{\mathbf{\Sigma}}$

For the normal copula, the expression of $\dot{C}^N_{\mathbf{\Sigma}}$ can be obtained from the so-called *Plackett formula* (Plackett, 1954). In the bivariate case, it is given by

$$\frac{\partial \Phi_{\mathbf{\Sigma}}(x_1, x_2)}{\partial \rho} = \frac{\exp\left\{-\frac{x_1^2+x_2^2-2\rho x_1 x_2}{2(1-\rho^2)}\right\}}{2\pi\sqrt{1-\rho^2}}, \qquad (x_1, x_2) \in \mathbb{R}^2.$$

For $d > 2$, the formula is

$$\frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} = \frac{\exp\left\{-\frac{x_i^2-2\rho_{i,j}x_i x_j+x_j^2}{2(1-\rho_{i,j}^2)}\right\}}{2\pi\sqrt{1-\rho_{i,j}^2}} \Phi_{\mathbf{S}_{ij}}\left(\mathbf{x}_{-ij} - \mathbf{M}_{ij}\mathbf{x}_{ij}\right), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{16}$$

where $\mathbf{M}_{ij} = \mathbf{\Sigma}_{-ij,ij}\mathbf{\Sigma}_{ij,ij}^{-1}$ and $\mathbf{S}_{ij} = \mathbf{\Sigma}_{-ij,-ij} - \mathbf{\Sigma}_{-ij,ij}\mathbf{\Sigma}_{ij,ij}^{-1}\mathbf{\Sigma}_{ij,-ij}$. The matrix $\mathbf{\Sigma}_{-ij,ij}$ in the previous expressions is a $(d-2) \times 2$ matrix obtained from $\mathbf{\Sigma}$ by removing rows i and j and by keeping only columns i and j. Similarly, the matrix $\mathbf{\Sigma}_{ij,ij}$ is 2 by 2 and is obtained from $\mathbf{\Sigma}$ by keeping only rows i and j and columns i and j, and $\mathbf{\Sigma}_{ij,-ij} = \mathbf{\Sigma}_{-ij,ij}^\top$. Note that the 2 by 2 matrix $\mathbf{\Sigma}_{ij,ij}^{-1}$ has the following simple form:

$$\frac{1}{1-\rho_{i,j}^2}\begin{bmatrix} 1 & -\rho_{i,j} \\ -\rho_{i,j} & 1 \end{bmatrix}.$$

In the unstructured case, for any $\mathbf{u} \in [0,1]^d$, $\dot{C}^N_{\mathbf{\Sigma}}(\mathbf{u})$ is the $d(d-1)/2$-dimensional vector defined by

$$\left(\frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{2,1}}, \ldots, \frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{d,d-1}}\right)^\top,$$

where $\mathbf{x} = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))^\top$. Its elements can be computed using (16). In case of the exchangeable or AR1 dependence structure, the copula depends only on one parameter, θ_1, and therefore

$$\dot{C}^N_{\mathbf{\Sigma}}(\mathbf{u}) = \frac{\partial \Phi_{\mathbf{\Sigma}}\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\}}{\partial \theta_1}, \qquad \mathbf{u} \in [0,1]^d.$$

The result then follows from the chain rule. In the exchangeable case (resp. in the AR1 case), we have

$$\frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \theta_1} = \sum_{i>j} \frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} \qquad \left(\text{resp. } \frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \theta_1} = \sum_{i>j}(i-j)\theta_1^{i-j-1}\frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}}\right), \qquad \mathbf{x} \in \mathbb{R}^d,$$

Finally, for the Toeplitz dependence structure,

$$\dot{C}^N_{\mathbf{\Sigma}}(\mathbf{u}) = \left(\frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \theta_1}, \ldots, \frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \theta_{d-1}}\right)^\top, \qquad \mathbf{u} \in [0,1]^d,$$

where $\mathbf{x} = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))^\top$, and from the chain rule,

$$\frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \theta_k} = \sum_{\substack{i>j \\ i-j=k}} \frac{\partial \Phi_{\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}}, \qquad k \in \{1, \ldots, d-1\}, \qquad \mathbf{x} \in \mathbb{R}^d.$$

For the t distribution, the analog of the Plackett formula has been obtained by Genz (2004) in the bivariate and the trivariate case. For $d = 2$, it is given by

$$\frac{\partial T_{\nu,\mathbf{\Sigma}}(x_1, x_2)}{\partial \rho} = \frac{\left\{1 + \frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{\nu(1-\rho^2)}\right\}^{-\frac{\nu}{2}}}{2\pi\sqrt{1-\rho^2}}, \qquad (x_1, x_2) \in \mathbb{R}^2.$$

The following result, whose proof is given in Appendix B, extends the formulas obtained by Genz (2004) to the situation $d \geq 3$.

Proposition 1. *For $d \geq 3$ and any $\mathbf{x} \in \mathbb{R}^d$,*

$$\frac{\partial T_{\nu,\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} = \frac{\left\{1 + \frac{x_i^2 + x_j^2 - 2\rho_{i,j} x_i x_j}{\nu(1-\rho_{i,j}^2)}\right\}^{-\frac{\nu}{2}}}{2\pi\sqrt{1-\rho_{i,j}^2}} T_{\nu,\mathbf{S}_{ij}} \left[\left\{1 + \frac{x_i^2 + x_j^2 - 2\rho_{i,j} x_i x_j}{\nu(1-\rho_{i,j}^2)}\right\}^{-\frac{1}{2}} (\mathbf{x}_{-ij} - \mathbf{M}_{ij}\mathbf{x}_{ij})\right].$$

where $\mathbf{M}_{ij} = \mathbf{\Sigma}_{-ij,ij}\mathbf{\Sigma}_{ij,ij}^{-1}$ and $\mathbf{S}_{ij} = \mathbf{\Sigma}_{-ij,-ij} - \mathbf{\Sigma}_{-ij,ij}\mathbf{\Sigma}_{ij,ij}^{-1}\mathbf{\Sigma}_{ij,-ij}$.

The expression of $\dot{C}^t_{\mathbf{\Sigma}}(\mathbf{u})$, $\mathbf{u} \in [0,1]^d$, for the different dependence structures can then be obtained by proceeding as in the normal case.

To compute the normal and t c.d.f.s involved in the above expressions, we used the algorithms proposed in Genz (1992, 1993) and Genz and Bretz (1999, 2002), and implemented in the R package `mvtnorm` (Genz et al., 2011).

B Proof of Proposition 1

Proof. The proof generalizes the approach adopted in Genz (2004) for the trivariate case. From the work of Cornish (1954), the c.d.f. of the standard multivariate t with correlation matrix $\mathbf{\Sigma}$ can be alternatively expressed as

$$T_{\nu,\mathbf{\Sigma}}(\mathbf{x}) = \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty s^{\nu-1} \exp\left(-\frac{s^2}{2}\right) \Phi_{\mathbf{\Sigma}}\left(\frac{s}{\sqrt{\nu}}\mathbf{x}\right) \mathrm{d}s.$$

Then, from Leibniz' integral rule, we can write

$$\frac{\partial T_{\nu,\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} = \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty s^{\nu-1} \exp\left(-\frac{s^2}{2}\right) \frac{\partial \Phi_{\mathbf{\Sigma}}\left(\frac{s}{\sqrt{\nu}}\mathbf{x}\right)}{\partial \rho_{i,j}} \mathrm{d}s.$$

As we continue, for any $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{x}_{ij}$ will designate the vector of $\mathbb{R}^2$ whose components are x_i and x_j while $\mathbf{x}_{-ij}$ will designate the vector of $\mathbb{R}^{d-2}$ obtained from $\mathbf{x}$ by removing x_i and x_j. Using the Plackett formula (16), we obtain

$$\frac{\partial T_{\nu,\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} = \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty s^{\nu-1} \exp\left(-\frac{s^2}{2}\right) \frac{\exp\left(-\frac{s^2}{\nu}\frac{\mathbf{x}_{ij}^\top \mathbf{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}}{2}\right)}{2\pi(1-\rho_{i,j}^2)^{\frac{1}{2}}} \Phi_{\mathbf{S}_{ij}}\left(\frac{s}{\sqrt{\nu}}\mathbf{y}_{-ij}\right) \mathrm{d}s. \tag{17}$$

where $\mathbf{y}_{-ij} = \mathbf{x}_{-ij} - \mathbf{M}_{ij}\mathbf{x}_{ij}$. Now,

$$\Phi_{\mathbf{S}_{ij}}\left(\frac{s}{\sqrt{\nu}}\mathbf{y}_{-ij}\right) = \int_{\mathbf{u}_{-ij} \le \frac{s}{\sqrt{\nu}}\mathbf{y}_{-ij}} \frac{1}{(2\pi)^{\frac{d-2}{2}} |\mathbf{S}_{ij}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{u}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{u}_{-ij}\right) \mathrm{d}\mathbf{u}_{-ij}.$$

Consider the change of variable $\mathbf{u}_{-ij} = \frac{s}{\sqrt{\nu}}\mathbf{v}_{-ij}$. Then,

$$\Phi_{\mathbf{S}_{ij}}\left(\frac{s}{\sqrt{\nu}}\mathbf{y}_{-ij}\right) = \int_{\mathbf{v}_{-ij} \le \mathbf{y}_{-ij}} \frac{s^{d-2}}{(2\pi\nu)^{\frac{d-2}{2}} |\mathbf{S}_{ij}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\frac{s^2}{\nu}\mathbf{v}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{v}_{-ij}\right) \mathrm{d}\mathbf{v}_{-ij}.$$

Combining the previous equation with (17), we obtain

$$\begin{aligned} \frac{\partial T_{\nu,\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} = \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty s^{\nu-1} \exp\left(-\frac{s^2}{2}\right) \frac{\exp\left(-\frac{s^2}{\nu}\frac{\mathbf{x}_{ij}^\top \mathbf{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}}{2}\right)}{2\pi(1-\rho_{i,j}^2)^{\frac{1}{2}}} \\ \times \int_{\mathbf{v}_{-ij} \le \mathbf{y}_{-ij}} \frac{s^{d-2}}{(2\pi\nu)^{\frac{d-2}{2}} |\mathbf{S}_{ij}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\frac{s^2}{\nu}\mathbf{v}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{v}_{-ij}\right) \mathrm{d}\mathbf{v}_{-ij}, \end{aligned}$$

that is,

$$\begin{aligned} \frac{\partial T_{\nu,\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} = \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty \int_{\mathbf{v}_{-ij} \le \mathbf{y}_{-ij}} \frac{s^{\nu+d-3}}{(2\pi\nu)^{\frac{d-2}{2}} |\mathbf{S}_{ij}|^{\frac{1}{2}} 2\pi(1-\rho_{i,j}^2)^{\frac{1}{2}}} \\ \times \exp\left\{-\frac{s^2}{2}\left(1 + \frac{\mathbf{x}_{ij}^\top \mathbf{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij} + \mathbf{v}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{v}_{-ij}}{\nu}\right)\right\} \mathrm{d}\mathbf{v}_{-ij}\mathrm{d}s. \end{aligned}$$

Next, consider the change of variable $r = s\left(1 + \frac{\mathbf{x}_{ij}^\top \mathbf{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij} + \mathbf{v}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{v}_{-ij}}{\nu}\right)^{\frac{1}{2}}$. We then obtain

$$\frac{\partial T_{\nu,\mathbf{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} = \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_{\mathbf{v}_{-ij} \le \mathbf{y}_{-ij}} \frac{\left(1 + \frac{\mathbf{x}_{ij}^\top \mathbf{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij} + \mathbf{v}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{v}_{-ij}}{\nu}\right)^{-\frac{\nu+d-2}{2}}}{(2\pi\nu)^{\frac{d-2}{2}} |\mathbf{S}_{ij}|^{\frac{1}{2}} 2\pi(1-\rho_{i,j}^2)^{\frac{1}{2}}} \int_0^\infty r^{\nu+d-3} \exp\left(-\frac{r^2}{2}\right) \mathrm{d}r\mathrm{d}\mathbf{v}_{-ij}.$$

Using the fact that

$$\int_0^\infty r^{\nu+d-3} \exp\left(-\frac{r^2}{2}\right) \mathrm{d}r = \Gamma\left(\frac{\nu+d-2}{2}\right) 2^{\frac{\nu+d-2}{2}-1},$$

we obtain

$$\frac{\partial T_{\nu,\boldsymbol{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} = \frac{\Gamma\left(\frac{\nu+d-2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{\frac{d-2}{2}}|\mathbf{S}_{ij}|^{\frac{1}{2}}} \int_{\mathbf{v}_{-ij}\leq \mathbf{y}_{-ij}} \frac{\left(1+\frac{\mathbf{x}_{ij}^\top \boldsymbol{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}+\mathbf{v}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{v}_{-ij}}{\nu}\right)^{-\frac{\nu+d-2}{2}}}{2\pi(1-\rho_{i,j}^2)^{\frac{1}{2}}} \mathrm{d}\mathbf{v}_{-ij}.$$

Next, it can be checked that

$$\left(1+\frac{\mathbf{x}_{ij}^\top \boldsymbol{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}+\mathbf{v}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{v}_{-ij}}{\nu}\right) = \left(1+\frac{\mathbf{x}_{ij}^\top \boldsymbol{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}}{\nu}\right)\left(1+\frac{\mathbf{v}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{v}_{-ij}}{\nu+\mathbf{x}_{ij}^\top \boldsymbol{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}}\right).$$

Then, we obtain

$$\frac{\partial T_{\nu,\boldsymbol{\Sigma}}(\mathbf{x})}{\partial \rho_{i,j}} = \frac{\Gamma\left(\frac{\nu+d-2}{2}\right)\left(1+\frac{\mathbf{x}_{ij}^\top \boldsymbol{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}}{\nu}\right)^{-\frac{\nu+d-2}{2}}}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{\frac{d-2}{2}}2\pi(1-\rho_{i,j}^2)^{\frac{1}{2}}} \int_{\mathbf{v}_{-ij}\leq \mathbf{y}_{-ij}} \left(1+\frac{\mathbf{v}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{v}_{-ij}}{\nu+\mathbf{x}_{ij}^\top \boldsymbol{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}}\right)^{-\frac{\nu+d-2}{2}} \mathrm{d}\mathbf{v}_{-ij}.$$

Finally, consider the change of variable $\mathbf{w}_{-ij} = \left(1+\frac{\mathbf{x}_{ij}^\top \boldsymbol{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}}{\nu}\right)^{-\frac{1}{2}}\mathbf{v}_{-ij}$. The quantity $\partial T_{\nu,\boldsymbol{\Sigma}}(\mathbf{x})/\partial\rho_{i,j}$ is then equal to

$$\frac{\Gamma\left(\frac{\nu+d-2}{2}\right)\left(1+\frac{\mathbf{x}_{ij}^\top \boldsymbol{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}}{\nu}\right)^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{\frac{d-2}{2}}2\pi(1-\rho_{i,j}^2)^{\frac{1}{2}}} \int_{\mathbf{w}_{-ij}\leq \left(1+\frac{\mathbf{x}_{ij}^\top \boldsymbol{\Sigma}_{ij,ij}^{-1}\mathbf{x}_{ij}}{\nu}\right)^{-\frac{1}{2}}\mathbf{y}_{-ij}} \left(1+\frac{\mathbf{w}_{-ij}^\top \mathbf{S}_{ij}^{-1}\mathbf{w}_{-ij}}{\nu}\right)^{-\frac{\nu+d-2}{2}} \mathrm{d}\mathbf{w}_{-ij},$$

and the desired result follows from (10). □

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Table 1: Percentage of rejection of the null hypothesis for exchangeable copulas and $n = 100$ obtained from 10 000 repetitions of the procedure given in Section 3 with $N = 1000$.

True	τ	$d=2$					$d=3$					$d=4$				
copula		C	G	F	N	t	C	G	F	N	t	C	G	F	N	t
C	0.2	*7.8*	48.7	19.9	12.6	16.6	*6.2*	63.7	33.7	17.4	32.5	*4.2*	65.9	40.6	21.1	38.6
	0.4	*7.8*	91.7	58.7	41.5	45.1	*7.2*	97.0	74.6	53.1	60.0	*5.4*	97.3	77.5	57.8	57.6
	0.6	*8.3*	99.1	77.5	70.7	66.9	*5.7*	99.6	83.7	73.2	65.5	*4.1*	99.5	79.6	68.0	47.5
	0.8	*9.0*	99.8	60.0	85.9	67.4	*2.0*	99.6	20.6	63.3	13.9	*0.6*	98.8	4.2	28.6	0.8
G	0.2	67.3	*4.1*	20.3	12.1	15.5	69.4	*4.0*	19.4	9.0	6.1	69.2	*3.5*	22.4	10.6	2.3
	0.4	97.9	*3.8*	42.9	22.1	26.8	99.0	*3.2*	49.7	21.7	17.8	99.2	*2.2*	52.5	25.3	10.2
	0.6	100.0	*2.9*	62.3	29.9	33.5	100.0	*1.5*	67.3	22.3	22.5	100.0	*0.7*	67.7	18.2	14.8
	0.8	100.0	*3.1*	77.2	39.5	41.1	100.0	*0.2*	50.1	9.2	8.7	100.0	*0.0*	26.3	1.3	1.2
F	0.2	52.3	20.8	*6.0*	9.3	13.7	54.6	19.3	*7.0*	3.3	11.6	52.8	15.3	*7.4*	1.0	7.3
	0.4	92.9	43.0	*5.2*	17.4	35.2	96.7	41.5	*4.8*	5.5	32.4	97.6	34.1	*4.6*	1.8	27.1
	0.6	99.6	63.4	*5.7*	40.6	61.8	99.9	54.7	*3.5*	22.4	53.2	99.9	41.0	*2.2*	10.5	37.0
	0.8	99.9	79.8	*6.3*	77.0	85.4	100.0	54.3	*0.8*	46.5	44.8	100.0	32.5	*0.2*	19.3	8.1
N	0.2	42.3	13.6	7.7	*4.6*	10.0	61.4	39.9	27.3	*2.6*	18.1	74.1	59.8	52.4	*1.5*	26.0
	0.4	83.0	19.9	13.7	*3.8*	11.9	96.5	57.9	39.0	*2.2*	20.9	99.1	79.4	70.4	*1.5*	28.4
	0.6	97.6	17.3	27.8	*3.4*	10.8	99.7	44.4	50.8	*1.6*	12.2	100.0	61.0	70.3	*0.9*	11.5
	0.8	99.8	10.0	39.2	*4.7*	11.5	99.9	8.0	33.7	*0.3*	1.5	99.9	5.7	23.5	*0.0*	0.2
t	0.2	36.9	9.4	10.9	8.7	*5.0*	48.9	29.5	28.0	7.2	*3.0*	56.6	47.8	46.1	5.1	*2.3*
	0.4	77.3	13.7	19.5	5.7	*4.2*	92.5	45.3	41.4	3.8	*2.5*	97.0	68.4	63.1	2.5	*1.5*
	0.6	96.9	13.7	35.7	6.0	*4.2*	99.7	42.7	58.4	3.1	*2.3*	99.9	63.7	75.3	1.6	*1.0*
	0.8	99.7	9.7	46.9	8.0	*5.2*	99.9	11.7	39.9	0.8	*0.3*	99.9	10.4	28.1	0.1	*0.0*

Table 2: Percentage of rejection of the null hypothesis for exchangeable copulas and $n = 300$ obtained from 10 000 repetitions of the procedure given in Section 3 with $N = 1000$.

True	τ	$d = 2$					$d = 3$					$d = 4$				
copula		C	G	F	N	t	C	G	F	N	t	C	G	F	N	t
C	0.2	*5.3*	93.4	60.1	44.3	58.4	*4.9*	99.2	86.0	71.8	89.9	*5.0*	99.6	93.1	86.3	96.3
	0.4	*5.9*	100.0	99.0	96.2	97.5	*5.2*	100.0	100.0	99.5	99.8	*5.0*	100.0	100.0	99.9	100.0
	0.6	*6.0*	100.0	100.0	100.0	100.0	*5.1*	100.0	100.0	100.0	100.0	*3.9*	100.0	100.0	100.0	100.0
	0.8	*6.3*	100.0	100.0	100.0	100.0	*3.3*	100.0	100.0	100.0	100.0	*1.7*	100.0	100.0	100.0	99.9
G	0.2	95.7	*4.0*	41.4	24.8	33.7	98.6	*3.9*	47.1	36.4	18.7	99.4	*3.8*	54.2	55.5	6.8
	0.4	100.0	*4.0*	84.2	53.9	59.8	100.0	*3.9*	93.3	69.8	51.0	100.0	*3.1*	96.2	84.3	32.8
	0.6	100.0	*3.3*	97.5	67.0	69.7	100.0	*3.1*	99.2	73.3	65.8	100.0	*2.8*	99.6	80.5	55.4
	0.8	100.0	*3.2*	99.7	71.3	71.2	100.0	*1.3*	100.0	60.3	56.5	100.0	*0.8*	100.0	50.4	43.9
F	0.2	86.8	58.2	*4.7*	19.0	45.6	93.8	58.9	*5.6*	8.2	47.6	95.3	52.1	*5.9*	3.5	36.8
	0.4	100.0	95.3	*4.8*	58.1	89.8	100.0	97.1	*4.9*	41.0	93.1	100.0	97.1	*5.0*	23.7	92.2
	0.6	100.0	99.8	*4.5*	93.8	99.4	100.0	99.9	*4.0*	90.7	99.6	100.0	99.7	*3.6*	83.2	99.3
	0.8	100.0	100.0	*3.2*	99.9	100.0	100.0	100.0	*2.0*	99.9	100.0	100.0	100.0	*1.3*	99.5	99.7
N	0.2	72.5	31.7	9.1	*4.0*	24.6	95.7	84.7	45.4	*3.4*	64.4	99.4	98.4	88.9	*2.8*	86.0
	0.4	99.6	59.3	28.7	*3.9*	35.1	100.0	99.0	83.4	*3.6*	76.9	100.0	100.0	99.8	*2.7*	92.6
	0.6	100.0	64.8	64.9	*3.8*	28.5	100.0	99.0	97.3	*3.4*	60.0	100.0	100.0	100.0	*2.6*	78.6
	0.8	100.0	54.0	92.3	*3.0*	20.4	100.0	95.1	99.8	*1.7*	28.9	100.0	99.2	100.0	*0.8*	28.3
t	0.2	64.4	19.7	26.0	11.9	*4.2*	89.2	62.8	52.8	12.6	*3.5*	96.0	88.6	82.3	13.7	*3.5*
	0.4	99.1	41.9	58.2	9.1	*4.3*	100.0	91.3	88.7	8.6	*3.6*	100.0	99.5	99.0	8.1	*3.0*
	0.6	100.0	51.4	86.0	7.9	*4.0*	100.0	95.4	99.3	7.0	*3.3*	100.0	99.8	100.0	6.1	*2.5*
	0.8	100.0	42.7	97.2	8.3	*3.7*	100.0	90.8	100.0	4.9	*1.8*	100.0	99.0	100.0	2.8	*0.9*

Table 3: Percentage of rejection of the null hypothesis for exchangeable copulas and $n = 500$ obtained from 10 000 repetitions of the procedure given in Section 3 with $N = 1000$.

True	τ	$d=2$					$d=3$					$d=4$				
copula		C	G	F	N	t	C	G	F	N	t	C	G	F	N	t
C	0.2	*5.0*	99.5	85.0	71.6	86.1	*5.0*	100.0	98.4	94.1	99.4	*4.4*	100.0	99.6	98.9	100.0
	0.4	*5.8*	100.0	100.0	99.9	99.9	*5.5*	100.0	100.0	100.0	100.0	*5.1*	100.0	100.0	100.0	100.0
	0.6	*5.3*	100.0	100.0	100.0	100.0	*4.8*	100.0	100.0	100.0	100.0	*4.2*	100.0	100.0	100.0	100.0
	0.8	*5.7*	100.0	100.0	100.0	100.0	*3.1*	100.0	100.0	100.0	100.0	*1.8*	100.0	100.0	100.0	100.0
G	0.2	99.7	*4.0*	60.9	37.7	52.3	100.0	*4.1*	70.9	63.2	36.6	100.0	*3.7*	79.3	87.1	17.2
	0.4	100.0	*4.2*	97.0	75.3	81.7	100.0	*4.1*	99.4	92.2	77.7	100.0	*3.5*	99.8	98.5	62.6
	0.6	100.0	*3.8*	99.9	87.4	89.8	100.0	*3.4*	100.0	93.5	89.3	100.0	*2.7*	100.0	97.2	84.6
	0.8	100.0	*2.8*	100.0	89.7	89.5	100.0	*1.8*	100.0	87.6	85.5	100.0	*1.1*	100.0	85.4	79.9
F	0.2	97.3	82.3	*4.7*	28.9	74.5	99.3	83.1	*5.1*	12.8	78.6	99.8	79.7	*5.3*	5.8	73.2
	0.4	100.0	99.8	*4.3*	83.9	99.3	100.0	99.9	*4.3*	71.2	99.6	100.0	100.0	*4.4*	50.7	99.6
	0.6	100.0	100.0	*4.2*	99.7	100.0	100.0	100.0	*3.8*	99.4	100.0	100.0	100.0	*3.4*	98.4	100.0
	0.8	100.0	100.0	*2.9*	100.0	100.0	100.0	100.0	*2.2*	100.0	100.0	100.0	100.0	*1.5*	100.0	100.0
N	0.2	88.9	50.3	11.8	*4.3*	46.5	99.7	97.4	62.2	*3.6*	91.3	100.0	100.0	97.8	*3.1*	98.8
	0.4	100.0	82.6	47.3	*4.3*	59.2	100.0	100.0	97.5	*3.6*	96.3	100.0	100.0	100.0	*3.3*	99.7
	0.6	100.0	90.0	88.5	*3.9*	48.8	100.0	100.0	99.9	*3.1*	87.8	100.0	100.0	100.0	*2.9*	97.9
	0.8	100.0	85.3	99.6	*2.9*	32.8	100.0	100.0	100.0	*1.7*	57.2	100.0	100.0	100.0	*1.2*	70.6
t	0.2	83.3	33.0	45.4	16.7	*4.4*	98.4	84.9	77.4	20.8	*3.9*	99.8	98.2	95.9	23.0	*3.4*
	0.4	100.0	66.2	84.3	12.6	*4.5*	100.0	99.2	99.1	13.3	*4.0*	100.0	100.0	100.0	13.5	*3.1*
	0.6	100.0	78.2	98.6	10.7	*4.2*	100.0	99.8	100.0	10.4	*3.4*	100.0	100.0	100.0	9.9	*3.2*
	0.8	100.0	73.2	99.9	9.5	*3.6*	100.0	99.6	100.0	6.8	*2.2*	100.0	100.0	100.0	4.8	*1.2*

Table 4: Percentage of rejection of the null hypothesis for exchangeable and non-exchangeable copulas obtained from 10 000 repetitions of the procedure given in Section 3 with $N = 1000$.

n	d	True	C	G	F	N-ex	N-ar1	N-tp	N-un	t-ex	t-ar1	t-tp	t-un
100	3	C	*6.7*	96.9	74.5	52.5	88.9	52.8	52.9	59.9	93.7	59.6	59.0
		G	98.9	*2.9*	48.8	22.1	62.9	22.4	22.9	17.2	66.2	17.1	17.1
		F	96.6	40.9	*5.0*	5.3	75.8	5.4	5.5	31.8	90.7	31.3	30.7
		N-ex	96.5	58.1	39.2	*2.3*	61.0	*2.3*	*2.3*	22.0	90.0	21.5	21.1
		N-ar1	95.1	69.7	55.6	8.2	*3.0*	*2.5*	*2.4*	34.7	20.2	21.0	20.0
		N-tp	99.4	96.4	95.2	77.3	100.0	*2.1*	*2.0*	83.9	100.0	13.7	13.9
		N-un	99.7	95.1	93.5	54.6	99.5	56.3	*1.9*	67.3	99.4	68.1	12.4
		t-ex	92.8	45.8	41.8	4.0	63.9	4.0	4.3	*2.5*	50.7	*2.4*	*2.3*
		t-ar1	88.2	53.2	50.9	10.6	6.6	5.6	5.8	6.7	*3.8*	*3.0*	*2.9*
		t-tp	97.1	85.5	87.8	65.8	100.0	5.2	5.6	47.7	100.0	*2.3*	*2.3*
		t-un	98.5	82.1	85.5	44.6	98.3	46.3	5.2	26.2	84.2	27.4	*2.5*
	4	C	*5.5*	97.5	77.5	57.8	99.3	57.4	56.9	57.9	99.8	56.6	54.8
		G	99.3	*2.2*	52.8	24.3	91.6	24.4	24.6	9.8	86.0	9.2	8.8
		F	97.4	33.2	*4.4*	2.1	97.7	2.0	2.2	26.2	99.4	25.3	24.6
		N-ex	99.1	79.7	70.0	*1.5*	98.1	*1.3*	*1.3*	28.4	99.8	26.9	25.6
		N-ar1	97.8	89.8	84.9	11.9	*2.4*	*1.4*	*1.2*	52.5	22.6	24.4	23.6
		N-tp	99.9	99.3	98.9	59.5	100.0	*0.8*	*0.8*	82.3	100.0	16.5	15.6
		N-un	99.8	94.8	92.6	27.2	99.7	25.2	*1.1*	52.3	99.8	50.1	16.8
		t-ex	97.1	68.8	63.7	2.7	97.4	2.8	3.0	*1.6*	91.3	*1.6*	*1.5*
		t-ar1	90.4	74.9	71.1	13.4	5.9	4.1	4.5	6.2	*3.1*	*1.8*	*1.5*
		t-tp	98.8	92.7	92.9	47.0	100.0	3.7	3.8	22.8	100.0	*1.0*	*1.1*
		t-un	98.4	84.9	82.5	21.3	98.8	20.7	3.2	12.1	89.8	11.2	*1.0*
300	3	C	*5.4*	100.0	100.0	99.5	100.0	99.6	99.7	99.9	100.0	99.9	99.9
		G	100.0	*4.0*	93.5	70.7	98.9	70.9	71.2	51.6	98.8	51.6	51.5
		F	100.0	97.2	*4.6*	40.9	100.0	41.0	41.1	93.0	100.0	92.9	93.2
		N-ex	100.0	99.0	83.8	*3.4*	98.4	*3.3*	*3.2*	76.7	100.0	76.8	76.9
		N-ar1	100.0	99.8	97.7	43.2	*3.8*	*3.4*	*3.3*	95.3	64.2	73.6	74.3
		N-tp	100.0	100.0	100.0	100.0	100.0	*3.1*	*3.2*	100.0	100.0	63.1	63.2
		N-un	100.0	100.0	100.0	100.0	100.0	100.0	*3.4*	100.0	100.0	100.0	58.0
		t-ex	100.0	91.2	89.0	8.4	99.0	8.8	8.8	*3.8*	93.0	*3.8*	*3.7*
		t-ar1	100.0	95.8	96.6	43.8	10.0	10.1	10.5	23.4	*3.8*	*3.5*	*3.5*
		t-tp	100.0	100.0	100.0	99.9	100.0	10.7	10.7	99.3	100.0	*3.2*	*3.3*
		t-un	100.0	100.0	100.0	99.7	100.0	99.7	9.8	97.4	100.0	97.7	*3.4*
	4	C	*5.1*	100.0	100.0	99.9	100.0	99.9	99.9	100.0	100.0	100.0	100.0
		G	100.0	*3.5*	95.8	84.1	100.0	84.0	84.2	33.8	100.0	33.5	32.8
		F	100.0	96.9	*5.3*	23.4	100.0	23.5	23.8	91.7	100.0	91.4	91.4
		N-ex	100.0	100.0	99.7	*3.0*	100.0	*2.9*	*2.8*	92.2	100.0	92.0	92.1
		N-ar1	100.0	100.0	100.0	81.5	*3.6*	*2.9*	*2.8*	99.8	71.4	88.7	88.8
		N-tp	100.0	100.0	100.0	100.0	100.0	*2.8*	*2.9*	100.0	100.0	83.7	84.1
		N-un	100.0	100.0	100.0	99.6	100.0	99.6	*3.1*	100.0	100.0	100.0	83.8
		t-ex	100.0	99.5	99.2	7.9	100.0	8.3	9.0	*3.3*	100.0	*3.2*	*3.2*
		t-ar1	100.0	99.8	99.8	73.5	11.8	10.6	11.3	45.8	*3.8*	*3.0*	*2.9*
		t-tp	100.0	100.0	100.0	99.9	100.0	9.9	10.2	96.1	100.0	*2.5*	*2.5*
		t-un	100.0	100.0	100.0	96.5	100.0	96.9	9.8	92.6	100.0	92.2	*2.5*