A Gramian-based Approach to Model Reduction for Uncertain Systems

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***Abstract*—The paper considers a problem of model reduction for a class of uncertain systems with structured norm bounded uncertainty. The paper introduces controllability and observability Gramians in terms of certain parameterized algebraic Riccati inequalities. Based on these Gramians, three model reduction approaches are investigated for the underlying uncertain systems.**

***Index Terms*—Model Reduction, Uncertain Systems, Linear Fractional Transformation, Linear Matrix Inequality**

I. Introduction

Model reduction is an important aspect of linear systems theory. One commonly applied model reduction method for linear time invariant (LTI) systems is balanced truncation [1]. By means of balancing controllability and observability Gramians, a reduced order model is constructed together with an a priori error bound; e.g., see [2], [3], [4]. In [5], it was shown that generalized controllability and observability Gramians can also be used to characterize H_∞ model reduction problems. For unstable systems, LQG balanced truncation was proposed in [6]; see also [7]. Being a *closed-loop* balancing approach, LQG balanced truncation removes a stability requirement in balanced truncation and H_∞ model reduction methods.

Uncertain systems commonly arise in robust control theory; e.g., see [8], [9]. Model reduction methods for uncertain systems are very useful in the design of practical robust control systems in which the dimension of controllers needs to be limited. In discrete-time cases, balanced truncation for uncertain systems can be traced back to [10], [11] within the framework of linear fractional transformations (LFTs). In [12], [13], balanced model reduction was extended to linear time-varying (LTV) systems. In continuous-time cases, model reduction for linear parameter-varying (LPV) systems was proposed in [14], [15]. Closely related problems, such as approximation, truncation and simplification of uncertain systems were presented in [16], [17].

In [18], [19], problems of controllability and unobservability were investigated for a class of structured uncertain systems in which the uncertainty is described by Integral Quadratic Constraints (IQCs). These results motivate the question as to whether model reduction methods, based on controllability and observability Gramians, can be obtained for uncertain systems. In this paper, we study model reduction problems for continuous-time uncertain systems modeled by an LFT representation, as a counterpart to corresponding results for discrete-time uncertain systems [10], [11]. We consider uncertain systems with norm bounded uncertainty rather than the IQC uncertainty description considered in [20], [18], [19]. This enables us to construct generalized Gramians and develop a series of model reduction methods for uncertain systems. These methods are balanced truncation and H_∞ model reduction for robustly stable uncertain systems, and LQG balanced truncation for uncertain systems which are not robustly stable.

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The main contribution of this paper is to characterize generalized observability and controllability Gramians for continuous-time uncertain systems with structured norm bounded uncertainty by means of parameterized Riccati inequalities, and to develop a systematic general framework for reducing the dimension of uncertain systems. We present a balanced truncation model reduction method for the underlying uncertain systems and derive its error bounds. These results extend existing results for LTI systems [1], [2], [3], [4] and discrete-time uncertain systems [11], [21] to this class of systems. The results also verify those in [16] in the context of IQCs when a norm bounded uncertainty setting is adopted. In particular, the second error bound developed considers different uncertainties in the original and the reduced uncertain system, and provides a Hausdorff distance between the two uncertain systems. H_∞ model reduction for uncertain systems is also investigated. Analogous to [5], a sufficient condition for the existence of a reduced order model is provided which involves the underlying Gramians together with a rank constraint. It turns out that our method, compared to the related results in [15], is less computationally demanding since [15] solves $2\nu-2$ (ν is the number of vertices of the underlying polytope) more matrix inequalities. LQG balanced truncation is proposed for uncertain systems which are not robustly stable, as a counterpart to the results in [6]. Similarly, a coprime factorization technique was used in [22] for discrete-time unstable systems. However, the results in [22] rely on an assumption that one of the system matrices is of full column rank. Our results overcome this restriction and provide a more general solution to constructing reduced-order uncertain systems. It is worth noting that the same framework has been further developed in [23] to construct contractive coprime factorizations for continuous-time uncertain systems and to derive a corresponding model reduction algorithm, without the full rank assumption.

In this paper, we also present a tutorial overview of model reduction methods for uncertain systems and aim to provide insight into these methods from a Gramian-based point of view. Note that the proposed balanced truncation and H_∞ model reduction approaches face some computational and scalability difficulties. However, the problem of overcoming these difficulties is beyond the scope of this paper, and may be a topic for future research.

Notation Let $\mathbf{L}_2^m = \mathbf{L}_2^m[0,\infty)$ be the space of square integrable functions in $\mathbf{R}^m$, and $\mathcal{L}(\mathbf{L}_2^m)$ denote the space of all linear bounded operators mapping from $\mathbf{L}_2^m$ to $\mathbf{L}_2^m$. The gain of an operator Δ in $\mathcal{L}(\mathbf{L}_2^m)$ is given by $\|\Delta\| = \sup_{z\in\mathbf{L}_2^m[0,\infty),z\neq 0} \frac{\|\Delta z\|}{\|z\|}$, and the adjoint operator of Δ is denoted as Δ^* if Δ is linear. If $\Delta=\Delta^*$, $\Delta<0$ means that $x^*\Delta x<0$ for any $x\neq 0$ in $\mathbf{R}^m$. We also use M^* to denote the complex conjugate transpose of a complex matrix M. For $z\in\mathbf{R}^m$ and a nonnegative matrix $\Lambda\in\mathbf{R}^{m\times m}$, $|z|_\Lambda^2 = z^*\Lambda z$, and Λ is omitted when it is an identity matrix. The state-space realization of a transfer matrix is denoted by $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI-A)^{-1}B+D$.

II. Problem Formulation

We consider the uncertainty structure

$$\mathbf{\Delta^c} = \{\mathrm{diag}(\Delta_1,\cdots,\Delta_k) : \Delta_i \in \mathcal{L}(\mathbf{L}_2^{h_i}), \Delta_i \text{ causal}, \|\Delta_i\|\leq 1\},$$

and the following uncertain system:

$$\mathcal{G}_\Delta : \begin{cases} \dot{x} = Ax + E\xi + Bu, \\ z = Kx + Gu, \\ y = Cx + D\xi, \\ \xi = \Delta z, \quad \Delta \in \mathbf{\Delta^c}, \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $u(t) \in \mathbf{R}^m$ is the *control input*, $z(t) \in \mathbf{R}^h$ is the *uncertainty output*, $y(t) \in \mathbf{R}^l$ is the *measured output* and $\xi(t) \in \mathbf{R}^h$ is the *uncertainty input*; here $h = h_1 + \cdots + h_k$. Note that it is assumed in (1) that ξ does not appear in z. This assumption is made only for notational simplification purposes, and the results in the paper can be readily extended to more general cases.

Let the nominal system be denoted by

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & E & B \\ \hline K & \mathbf{0}_h & G \\ C & D & \mathbf{0}_{l\times m} \end{array}\right].$$

Then, the uncertain system (1) is defined by an LFT representation as $\mathcal{F}_u(M,\Delta) := M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$, whenever $I - M_{11}\Delta$ is non-singular. Define operators

$$\begin{bmatrix} \mathcal{A}_\Delta & \mathcal{B}_\Delta \\ \mathcal{C}_\Delta & \mathcal{D}_\Delta \end{bmatrix} = \begin{bmatrix} A + E\Delta K & B + E\Delta G \\ C + D\Delta K & D\Delta G \end{bmatrix}.$$

Definition 1 (Robust Stability [24]): The uncertain system (1) is *robustly stable* if $(I - M_{11}\Delta)^{-1}$ exists in $\mathcal{L}(\mathbf{L}_2^h)$ and is causal, for all $\Delta \in \boldsymbol{\Delta^c}$.

Definition 2: The uncertain system (1) is said to be *robustly stabilizable* if there exists a static state feedback law $u = Fx$ such that the corresponding closed-loop uncertain system is robustly stable. Also, *robust detectability* can be defined similarly.

The following lemma states a necessary and sufficient condition for robust stability. This lemma is given in terms of the positive commutant set corresponding to $\boldsymbol{\Delta^c}$ defined as

$$\boldsymbol{P_\Theta} = \{\mathrm{diag}(\theta_1 I_{h_1}, \cdots, \theta_k I_{h_k}) : \theta_i > 0\}. \tag{2}$$

Lemma 3: (see [24]) The uncertain system (1) is robustly stable if and only if there exist $\Theta \in \boldsymbol{P_\Theta}$ and $X > 0$, such that

$$A^*X + XA + K^*\Theta K + XE\Theta^{-1}E^*X < 0. \tag{3}$$

III. Controllability and Observability Gramians

As is well known, the controllability and observability Gramians play very important roles in LTI balanced truncation approaches to model reduction; see [1]. In this section, we introduce generalized Gramians for the uncertain system (1), as defined below.

Definition 4: The matrices $S > 0, P > 0$ are said to be generalized controllability or observability Gramian[1], respectively, for the uncertain system (1) if the following inequalities hold,

$$\mathcal{A}_\Delta S + S\mathcal{A}_\Delta^* + \mathcal{B}_\Delta\mathcal{B}_\Delta^* < 0 \quad \forall \Delta \in \boldsymbol{\Delta^c}, \tag{4}$$

$$\mathcal{A}_\Delta^* P + P\mathcal{A}_\Delta + \mathcal{C}_\Delta^*\mathcal{C}_\Delta < 0 \quad \forall \Delta \in \boldsymbol{\Delta^c}. \tag{5}$$

In [19], [18], issues of robust controllability and unobservability for uncertain linear systems with structured uncertainty were discussed in the framework of IQCs. In these references, LTV systems with nonlinear uncertainties were studied, and parameterized Riccati differential equations were derived to characterize robust controllability and unobservability of uncertain systems. In this section, we will apply the ideas in [19], [18] to the uncertain system (1).

Consider the following algebraic Riccati inequalities (ARIs):

$$AS + SA^* + (SK^* + BG^*)(\Lambda_c^{-1} - GG^*)^{-1}(KS + GB^*) + E\Lambda_c^{-1}E^* + BB^* < 0, \tag{6}$$

$$A^*P + PA + (PE + C^*D)(\Lambda_o - D^*D)^{-1}(E^*P + D^*C) + K^*\Lambda_o K + C^*C < 0, \tag{7}$$

[1]Here generalized Gramians are defined in the sense that they satisfy underlying Lyapunov inequalities rather than equations [24]. Note that the same notion also refers to structured Gramians in the literature when uncertainty structures are involved; e.g., [11], [25].

where $S > 0, P > 0$, and $\Lambda_c \in \boldsymbol{P_\Theta}, \Lambda_o \in \boldsymbol{P_\Theta}$ are such that $\Lambda_c^{-1} - GG^* > 0, \Lambda_o - D^*D > 0$. Alternatively, (6) and (7) can be rewritten as

$$AS + SA^* + SK^*\Lambda_c KS + E\Lambda_c^{-1}E^* + (B + SK^*\Lambda_c G)(I_m - G^*\Lambda_c G)^{-1}(B^* + G^*\Lambda_c KS) < 0, \tag{8}$$

$$A^*P + PA + PE\Lambda_o^{-1}E^*P + K^*\Lambda_o K + (C^* + PE\Lambda_o^{-1}D^*)(I_l - D\Lambda_o^{-1}D^*)^{-1}(C + D\Lambda_o^{-1}E^*P) < 0. \tag{9}$$

The following example shows that S in (6) or (8) is analogous to the LTI controllability Gramians. A similar result also holds for P in (7) or (9).

Observation 5: (**Controllability Gramian**) Consider the uncertain system (1) on the interval $(-\infty, 0]$ with $x(-\infty) = 0$, and assume that ARI (8) admits a solution $S > 0$ for some $\Lambda_c \in \boldsymbol{P_\Theta}$ such that $I_m - G^*\Lambda_c G > 0$. Using $x^*(t)S^{-1}x(t)$ as a candidate Lyapunov function, we have

$$\begin{aligned}
&\int_{-\infty}^0 |u|^2 dt \\
&\geq x_0^*S^{-1}x_0 + \int_{-\infty}^0 (|z|_{\Lambda_c}^2 - |\xi|_{\Lambda_c}^2)dt + \int_{-\infty}^0 \left|\xi - \Lambda_c^{-1}E^*S^{-1}x\right|_{\Lambda_c}^2 dt \\
&\quad + \int_{-\infty}^0 \left|u - (I_m - G^*\Lambda_c G)^{-1}(B^*S^{-1} + G^*\Lambda_c K)x\right|_{(I_m - G^*\Lambda_c G)}^2 dt \\
&\geq x_0^*S^{-1}x_0.
\end{aligned}$$

Therefore, $\min_{u,\xi}\int_{-\infty}^0 |u|^2 dt \geq x_0^*S^{-1}x_0$. Recall that equality is achieved for LTI cases (without uncertainty); while for the uncertain system (1), $x_0^*S^{-1}x_0$ provides a lower bound on the minimum control energy required to drive the state from $x(-\infty) = 0$ to $x(0) = x_0$.

Solutions to (6-7) or (8-9) are closely related to generalized Gramians for the uncertain system (1). Before showing this, it is necessary to address the feasibility of the inequalities (6-7).

Theorem 6: The following statements are equivalent:

- (i) The uncertain system (1) is robustly stable.
- (ii) The Riccati inequality (6) admits a solution $S > 0$ for some $\Lambda_c \in \boldsymbol{P_\Theta}$.
- (iii) The Riccati inequality (7) admits a solution $P > 0$ for some $\Lambda_o \in \boldsymbol{P_\Theta}$.

Proof: We only prove the equivalence between (i) and (ii).

$(ii) \Rightarrow (i)$: (3) holds with $X = S^{-1}, \Theta = \Lambda_c$ by using (8). Then (i) follows using Lemma 3.

$(i) \Rightarrow (ii)$: Using Lemma 3, it follows that (3) holds. Then we can choose $\varepsilon > 0$ sufficiently small, such that $\varepsilon^{-1}I_m - G^*\Theta G > 0$ and

$$A^*X + XA + K^*\Theta K + XE\Theta^{-1}E^*X + (XB + K^*\Theta G)(\varepsilon^{-1}I_m - G^*\Theta G)^{-1}(B^*X + G^*\Theta K) < 0. \tag{10}$$

Let $X = (\varepsilon S)^{-1}, \Theta = \varepsilon^{-1}\Lambda_c$, and substitute these values into (10). From this, it is not difficult to derive (8), and thus (6) holds. ∎

The following theorem relates (6) and (7) to the generalized controllability and observability Gramians for the uncertain system (1), as defined in Definition 4.

Theorem 7: If there exist $S > 0$, $P > 0$, $\Lambda_c \in \boldsymbol{P_\Theta}$, $\Lambda_o \in \boldsymbol{P_\Theta}$ solving ARIs (6), (7), then S, P are generalized controllability and observability Gramians for the uncertain system (1).

Proof: We only prove the controllability part.

$$\begin{aligned}
&\mathcal{A}_\Delta S + S\mathcal{A}_\Delta^* + \mathcal{B}_\Delta\mathcal{B}_\Delta^* \\
&= (A + E\Delta K)S + S(A + E\Delta K)^* + (B + E\Delta G)(B + E\Delta G)^* \\
&= AS + SA^* + E\Delta\Lambda_c^{-1}\Delta^*E^* + BB^* \\
&\quad + (SK^* + BG^*)(\Lambda_c^{-1} - GG^*)^{-1}(KS + GB^*) \\
&\quad - [SK^* + BG^* - E\Delta(\Lambda_c^{-1} - GG^*)](\Lambda_c^{-1} - GG^*)^{-1} \\
&\quad \times [SK^* + BG^* - E\Delta(\Lambda_c^{-1} - GG^*)]^*.
\end{aligned}$$

Then (4) holds from $E\Delta\Lambda_c^{-1}\Delta^*E^* = E\Lambda_c^{-1/2}\Delta\Delta^*\Lambda_c^{-1/2}E^* \le E\Lambda_c^{-1}E^*$ and $\Lambda_c^{-1} - GG^* > 0$. ∎

IV. Balanced Truncation

It is shown that solutions to ARIs (6-7) are generalized Gramians for $\mathcal{G}_\Delta$ in (1). Consequently, traditional balanced truncation technique for model reduction can be applied. Firstly, we present a method to solve ARIs (6-7). By using the Schur complement twice and letting $\overline{\Lambda}_c = \Lambda_c^{-1}$, (6-7) can be transformed into Linear Matrix Inequalities (LMIs), as in the following propositions.

Proposition 8: If there exist matrices $S > 0$ and $\overline{\Lambda}_c \in \boldsymbol{P_\Theta}$ solving the following LMI

$$\begin{bmatrix} SA^* + AS + E\overline{\Lambda}_c E^* & SK^* & B \\ \star & -\overline{\Lambda}_c & G \\ \star & \star & -I_m \end{bmatrix} < 0, \tag{11}$$

then S is a generalized controllability Gramian for the uncertain system (1).

Proposition 9: If there exist matrices $P > 0$ and $\Lambda_o \in \boldsymbol{P_\Theta}$ solving the following LMI

$$\begin{bmatrix} A^*P + PA + K^*\Lambda_o K & PE & C^* \\ \star & -\Lambda_o & D^* \\ \star & \star & -I_l \end{bmatrix} < 0, \tag{12}$$

then P is a generalized observability Gramian for the uncertain system (1).

Note that solutions to LMIs (11) and (12) are not unique. A possible heuristic is, taking (11) for example, to solve the following Semi-Definite Programming (SDP) problem: minimize $trace(S)$, subject to (11); see e.g. [24]. Here the objective function is chosen such that, in the absence of uncertainty, the solution leads to the standard controllability Gramian.

Definition 10: An uncertain system of the form (1) is said to be *balanced* if it has generalized observability and controllability Gramians which are identical diagonal matrices. The diagonal entries are then referred to as generalized Hankel singular values for the uncertain system.

We summarize the proposed model reduction algorithm as follows.

Procedure 11 (Balanced Truncation):

1) Solve LMIs (11) and (12), or the associated SDP problems, to obtain generalized Gramians $S > 0, P > 0$.
2) Balance S, P by constructing a state transformation matrix T [2] such that
$$TST^* = (T^{-1})^*PT^{-1} = \Sigma = \text{diag}(\Sigma_1, \Sigma_2) = diag(\gamma_1, ..., \gamma_n), \tag{13}$$
where $\gamma_1 \ge ... \ge \gamma_d > \gamma_{d+1} \ge ... \ge \gamma_n > 0$, $\Sigma_1 = diag(\gamma_1, ..., \gamma_d)$, $\Sigma_2 = diag(\gamma_{d+1}, ..., \gamma_n)$.
3) Write the transformed nominal system of (1) as $M = \left[\begin{array}{c|cc} \overline{A} & \overline{E} & \overline{B} \\ \hline \overline{K} & \mathbf{0}_h & G \\ \overline{C} & D & \mathbf{0}_{l\times m} \end{array}\right]$, where $\overline{A} = TAT^{-1}$; $\overline{E} = TE$; $\overline{B} = TB$; $\overline{C} = CT^{-1}$; $\overline{K} = KT^{-1}$. The sub-matrices of this balanced realization of M corresponding to the matrix Σ_2 in (13) are truncated to obtain a reduced order uncertain system defined by $M_r = \left[\begin{array}{c|cc} \overline{A}_r & \overline{E}_r & \overline{B}_r \\ \hline \overline{K}_r & \mathbf{0}_h & G \\ \overline{C}_r & D & \mathbf{0}_{l\times m} \end{array}\right]$ with order d.
4) Write the reduced dimension uncertain system as $\mathcal{G}_{r\Delta} = \mathcal{F}_u(M_r, \Delta), \Delta \in \boldsymbol{\Delta^c}$.

Theorem 12: Consider a robustly stable uncertain system (1) and suppose that the reduced dimension uncertain system $\mathcal{G}_{r\Delta}$ is obtained as described in Procedure 11. Then $\mathcal{G}_{r\Delta}$ is also balanced and robustly stable. Furthermore,

$$\sup_{\Delta\in\boldsymbol{\Delta}^c} \|\mathcal{G}_\Delta(s) - \mathcal{G}_{r\Delta}(s)\|_\infty \le 2(\gamma_1^t + \cdots + \gamma_q^t), \tag{14}$$

where γ_i^t denote the distinct generalized Hankel singular values of $\gamma_{d+1}, \cdots, \gamma_n$.

Proof: It is easy to show that $\mathcal{G}_{r\Delta}$ satisfies (6) and (7) with balanced Gramian Σ_1. Therefore, $\mathcal{G}_{r\Delta}$ is balanced from Theorem 7, and robustly stable from Theorem 6. As for the bound in (14), the proof is analogous to that of Theorem 13, and thus omitted here. ∎

In the above theorem, we assume that the original system and the reduced system have identical uncertainties. If different uncertainties are allowed, the error bound will require an additional term $\overline{\theta}$ determined by $\overline{\Lambda}_c, \Lambda_o$, as to be shown below.

Theorem 13: Consider a robustly stable uncertain system (1) and suppose that the reduced dimension uncertain system $\mathcal{G}_{r\Delta}$ is obtained as described in Procedure 11. Then

$$\sup_{\tilde{\Delta},\Delta\in\boldsymbol{\Delta^c}} \|\mathcal{G}_{\tilde{\Delta}}(s) - \mathcal{G}_{r\Delta}(s)\|_\infty \le 2(\gamma_1^t + \cdots + \gamma_q^t + \overline{\theta}), \tag{15}$$

where $\overline{\theta} = \sum_{i=1}^k \sqrt{\theta_{oi}\overline{\theta}_{ci}}$, $\theta_{oi}, \overline{\theta}_{ci}$ are the repeated entries of $\Lambda_o, \overline{\Lambda}_c$ respectively, as defined in (2).

Proof: We will utilize [16, Theorem 1] to prove the above result. For any $\tilde{\Delta}, \Delta \in \boldsymbol{\Delta^c}$, define

$$\overline{\Delta} = \text{diag}(s^{-1}I_d, s^{-1}I_{n-d}, \tilde{\Delta}), \quad \hat{\Delta} = \text{diag}(s^{-1}I_d, 0_{n-d}, \Delta),$$
$$\overline{M}_{11} = \begin{bmatrix} \overline{A} & \overline{E} \\ \overline{K} & 0 \end{bmatrix}, \ \overline{M}_{12} = \begin{bmatrix} \overline{B} \\ G \end{bmatrix}, \ \overline{M}_{21} = \begin{bmatrix} \overline{C} & D \end{bmatrix}, \ \overline{M}_{22} = 0,$$
$$\overline{M} = \begin{bmatrix} \overline{M}_{11} & \overline{M}_{12} \\ \overline{M}_{21} & \overline{M}_{22} \end{bmatrix}, \Pi_1 = \begin{bmatrix} 0 & \Sigma_1^{-1} \\ \Sigma_1^{-1} & 0 \end{bmatrix}, \ \Pi_2 = \begin{bmatrix} 0 & \Sigma_2^{-1} \\ \Sigma_2^{-1} & 0 \end{bmatrix},$$
$$\Pi_3 = \begin{bmatrix} \overline{\Lambda}_c^{-1} & 0 \\ 0 & -\overline{\Lambda}_c^{-1} \end{bmatrix}, \overline{\Sigma} = \text{diag}\left(\Sigma_1, \Sigma_2, (\Lambda_o\overline{\Lambda}_c)^{\frac{1}{2}}\right),$$

where $\overline{A}, \overline{E}, \overline{B}, \overline{K}, \overline{G}, \Sigma_1, \Sigma_2, \overline{\Lambda}_c, \Lambda_o$ are obtained from Procedure 11. It is easy to check that $\mathcal{G}_{\tilde{\Delta}} = \mathcal{F}_u(\overline{M}, \overline{\Delta})$, $\mathcal{G}_{r\Delta} = \mathcal{F}_u(\overline{M}, \hat{\Delta})$, and Π_1, Π_2, Π_3 are corresponding IQC multipliers for the uncertainty blocks in $\overline{\Delta}, \hat{\Delta}$. Then $\overline{\Delta}, \hat{\Delta}$ satisfy the IQCs defined by $\Pi = \begin{bmatrix} \Pi_{(1,1)} & \Pi_{(1,2)} \\ \Pi_{(2,1)} & \Pi_{(2,2)} \end{bmatrix}$, $\Pi_{(i,j)} = \text{diag}(\Pi_{1(i,j)}, \cdots, \Pi_{3(i,j)})$, $i, j = 1, 2$, where $\Pi_i = \begin{bmatrix} \Pi_{i(1,1)} & \Pi_{i(1,2)} \\ \Pi_{i(2,1)} & \Pi_{i(2,2)} \end{bmatrix}$. Note that (11) and (12) are equivalent to the following two matrix inequalities respectively,

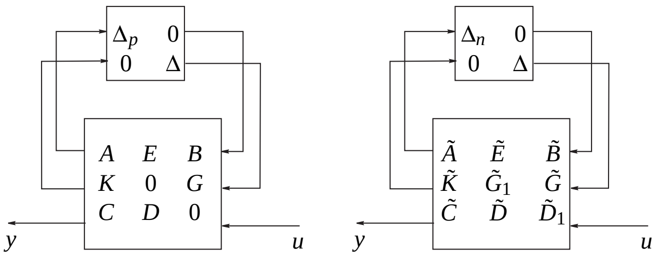
$$\begin{bmatrix} \overline{M}_{11} & \overline{M}_{12} \\ I & 0 \end{bmatrix}^* \Pi \begin{bmatrix} \overline{M}_{11} & \overline{M}_{12} \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$
$$\begin{bmatrix} \overline{M}_{11} \\ I \end{bmatrix}^* \begin{bmatrix} \overline{\Sigma}^2 & 0 \\ 0 & \overline{\Sigma}^2 \end{bmatrix} \Pi \begin{bmatrix} \overline{M}_{11} \\ I \end{bmatrix} + \overline{M}_{21}^*\overline{M}_{21} < 0.$$

Therefore, the error bound (15) holds by invoking [16, Theorem 1]. ∎

Definition 14: The Hausdorff distance $d_H(\mathcal{F}, \mathcal{H})$ between the sets $\mathcal{F}$ and $\mathcal{H}$ is defined as

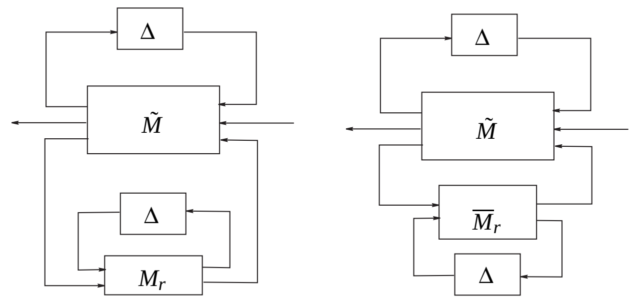
$$d_H(\mathcal{F}, \mathcal{H}) := \max(\vec{d}(\mathcal{F}, \mathcal{H}), \vec{d}(\mathcal{H}, \mathcal{F})),$$
$$\vec{d}(\mathcal{F}, \mathcal{H}) := \sup_{f(s)\in\mathcal{F}} \inf_{h(s)\in\mathcal{H}} \|f(s) - h(s)\|_\infty.$$

If we denote $\mathcal{G}_{\boldsymbol{\Delta}} := \{\mathcal{G}_\Delta : \Delta \in \boldsymbol{\Delta^c}\}$ and $\mathcal{G}_{r\boldsymbol{\Delta}} := \{\mathcal{G}_{r\Delta} : \Delta \in \boldsymbol{\Delta^c}\}$, the above result provides an upper bound on the Hausdorff distance between these two sets: $d_H(\mathcal{G}_{\boldsymbol{\Delta}}, \mathcal{G}_{r\boldsymbol{\Delta}}) \le 2(\gamma_1^t + \cdots + \gamma_q^t + \overline{\theta})$.



(a) Continuous-time system. (b) Discrete-time system.

Fig. 1. Bilinear transformation.



(a) Original configuration. (b) Equivalent configuration.

Fig. 2. LFT configuration.

A. Connection to Discrete-time Cases

Following [11], [16], we include a passive integral operator $\Delta_p = s^{-1}I$ and a norm-bounded shift operator $\Delta_n = \lambda I$, as seen in Fig. 1, in the upper uncertainty blocks; the lower blocks are constant matrices. A bilinear transformation $\Delta_n = (\Delta_p - I)(\Delta_p + I)^{-1}$ can be applied to convert the continuous-time system in Fig. 1(a) to the discrete-time system in Fig. 1(b) as follows,

$$\begin{aligned}\tilde{A} &= (I-A)^{-1}(I+A),\\ [\tilde{E} \quad \tilde{B}] &= \sqrt{2}(I-A)^{-1}[E \quad B], \quad \begin{bmatrix}\tilde{K}\\ \tilde{C}\end{bmatrix} = \sqrt{2}\begin{bmatrix}K\\ C\end{bmatrix}(I-A)^{-1}, \\ \begin{bmatrix}\tilde{G}_1 & \tilde{G}\\ \tilde{D} & \tilde{D}_1\end{bmatrix} &= \begin{bmatrix}K\\ C\end{bmatrix}(I-A)^{-1}[E \quad B] + \begin{bmatrix}0 & G\\ D & 0\end{bmatrix}.\end{aligned} \tag{16}$$

Using the discrete-time results in [10], [11], the Lyapunov inequality associated with generalized controllability Gramian for discrete-time systems in Fig. 1(b) is

$$\begin{aligned}\begin{bmatrix}\tilde{A} & \tilde{E}\\ \tilde{K} & \tilde{G}_1\end{bmatrix}\begin{bmatrix}S & 0\\ 0 & \Lambda_c^{-1}\end{bmatrix}\begin{bmatrix}\tilde{A} & \tilde{E}\\ \tilde{K} & \tilde{G}_1\end{bmatrix}^*\\ - \begin{bmatrix}S & 0\\ 0 & \Lambda_c^{-1}\end{bmatrix} + \begin{bmatrix}\tilde{B}\\ \tilde{G}\end{bmatrix}\begin{bmatrix}\tilde{B}\\ \tilde{G}\end{bmatrix}^* < 0,\end{aligned} \tag{17}$$

where $S > 0$, and $\Lambda_c \in \boldsymbol{P_\Theta}$. By using the Schur complement, (17) is equivalent to

$$\begin{bmatrix}\tilde{A}S\tilde{A}^* - S & \tilde{A}S\tilde{K}^* & \tilde{E} & \tilde{B}\\ \star & \tilde{K}S\tilde{K}^* - \Lambda_c^{-1} & \tilde{G}_1 & \tilde{G}\\ \star & \star & -\Lambda_c & 0\\ \star & \star & \star & -I\end{bmatrix} < 0. \tag{18}$$

Left and right multiplying (18) by diag$\left(\begin{bmatrix}\frac{\sqrt{2}}{2}(I-A) & 0\\ -\frac{\sqrt{2}}{2}K & I\end{bmatrix}, I, I\right)$ and its transpose and using (16), the continuous-time ARI (6) can be derived. Note that ARI (6) is related to generalized controllability Gramians for our continuous uncertain systems. This derivation illustrates the connection between our continuous-time results and those in [11] for discrete-time systems, and provides a different perspective on our balanced truncation approach.

V. H_∞ Model Reduction

As shown in [24, Theorem 4.20], for a nominal system without uncertainties, generalized Gramians can be used to characterize H_∞ model reduction problems; see also the original paper [5]. This is also true for our uncertain system (1), as stated in the following theorem.

Theorem 15: Given a robustly stable uncertain system (1), there exists a reduced dimension uncertain system defined by $M_r = \left[\begin{array}{c|cc}A_r & E_r & B_r\\ \hline K_r & D_{r11} & D_{r12}\\ C_r & D_{r21} & D_{r22}\end{array}\right]$ of order d such that $\sup_{\Delta \in \boldsymbol{\Delta^c}} \|\mathcal{F}_u(M,\Delta) - \mathcal{F}_u(M_r,\Delta)\|_\infty < \varepsilon$, if there exist $S > 0, P > 0, \Lambda_c \in \boldsymbol{P_\Theta}, \Lambda_o \in \boldsymbol{P_\Theta}$ solving Riccati inequalities (6), (7) and satisfying

$$\Lambda_o \geq \varepsilon^2 \Lambda_c, \quad \lambda_{min}(SP) = \varepsilon^2, \quad rank(SP - \varepsilon^2 I_n) \leq d. \tag{19}$$

Proof: Let $M_r = \begin{bmatrix}M_{r11} & M_{r12}\\ M_{r21} & M_{r22}\end{bmatrix}$ and define

$$\tilde{M} = \begin{bmatrix}M_{11} & M_{12} & 0\\ M_{21} & M_{22} & -I\\ 0 & I & 0\end{bmatrix} = \left[\begin{array}{c|ccc}A & E & B & 0\\ \hline K & 0 & G & 0\\ C & D & 0 & -I\\ 0 & 0 & I & 0\end{array}\right].$$

Then the configuration of the error system $\mathcal{F}_u(M,\Delta) - \mathcal{F}_u(M_r,\Delta)$ is shown in Fig. 2(a), which is equivalent to the one shown in Fig. 2(b) for $\overline{M}_r = \begin{bmatrix}M_{r22} & M_{r21}\\ M_{r12} & M_{r11}\end{bmatrix}$. Now the result of the theorem can be proved by using [26, Theorem 5.1] for an equivalent LPV H_∞ synthesis problem. ■

Remark 16: Note that (6), (7) and (19) are equivalent to (11), (12) and the conditions below,

$$\begin{bmatrix}\Lambda_o & \varepsilon I_h\\ \star & \Lambda_c\end{bmatrix} \geq 0, \quad \begin{bmatrix}S & \varepsilon I_n\\ \star & P\end{bmatrix} \geq 0, \quad rank\begin{bmatrix}S & \varepsilon I_n\\ \star & P\end{bmatrix} \leq n + d. \tag{20}$$

Those are referred to as rank constrained LMIs and can be solved by LMIRank [27].

VI. LQG Balanced Truncation

The balanced truncation and H_∞ model reduction techniques introduced above require uncertain systems be robustly stable. An LQG balanced truncation approach, taking into account *closed-loop* control considerations, was presented in [6] to overcome the stability requirement for LTI systems. In this section, we apply this approach to the uncertain system (1).

Suppose that the uncertain system (1) is robustly stabilizable and detectable. Consider the following LQG control and filter Riccati inequalities for the uncertain system (1), for all $\Delta \in \boldsymbol{\Delta^c}$,

$$\begin{aligned}W(\mathcal{A}_\Delta - \mathcal{B}_\Delta \mathcal{R}_\Delta^{-1} \mathcal{D}_\Delta^* \mathcal{C}_\Delta) + (\mathcal{A}_\Delta - \mathcal{B}_\Delta \mathcal{R}_\Delta^{-1} \mathcal{D}_\Delta^* \mathcal{C}_\Delta)^* W\\ - W \mathcal{B}_\Delta \mathcal{R}_\Delta^{-1} \mathcal{B}_\Delta^* W + \mathcal{C}_\Delta^* \tilde{\mathcal{R}}_\Delta^{-1} \mathcal{C}_\Delta < 0,\end{aligned} \tag{21}$$

$$\begin{aligned}(\mathcal{A}_\Delta - \mathcal{B}_\Delta \mathcal{R}_\Delta^{-1} \mathcal{D}_\Delta^* \mathcal{C}_\Delta) V + V(\mathcal{A}_\Delta - \mathcal{B}_\Delta \mathcal{R}_\Delta^{-1} \mathcal{D}_\Delta^* \mathcal{C}_\Delta)^*\\ - V \mathcal{C}_\Delta^* \tilde{\mathcal{R}}_\Delta^{-1} \mathcal{C}_\Delta V + \mathcal{B}_\Delta \mathcal{R}_\Delta^{-1} \mathcal{B}_\Delta^* < 0,\end{aligned} \tag{22}$$

where $\mathcal{R}_\Delta = I + \mathcal{D}_\Delta^* \mathcal{D}_\Delta$, $\tilde{\mathcal{R}}_\Delta = I + \mathcal{D}_\Delta \mathcal{D}_\Delta^*$.

It is shown that LQG control and filter algebraic Riccati equations or inequalities are closely related to coprime factorization problems [8], [25] and some special $\mathcal{H}_2$ control problems [28], [29]. In what follows, we will establish these connections and provide a numerical approach to obtain solutions to Riccati inequalities (21) and (22).

Motivated by [29], [28], the filter Riccati inequality (22) is related to an output injection $\mathcal{H}_2$ problem. This problem involves finding an observer gain L, such that $\|\mathcal{F}_l(\mathcal{G}_{OI\Delta}, L)\|_{\mathcal{H}_2} < \gamma$ with a given $\gamma > 0$. Here[2] $\mathcal{G}_{OI\Delta} = \left[\begin{array}{c|cc} \mathcal{A}_\Delta & [0 \;\; \mathcal{B}_\Delta] & I \\ \hline I & [0 \;\; 0\;] & 0 \\ \mathcal{C}_\Delta & [I \;\; \mathcal{D}_\Delta] & 0 \end{array}\right]$, and the state space description is

$$\mathcal{G}_{OI\Delta} : \begin{cases} \dot{x} = Ax + E\xi + Bu_2 + w, \\ z = Kx + Gu_2, \\ y = x, \\ p = Cx + D\xi + u_1, \\ \xi = \Delta z, \quad \Delta \in \boldsymbol{\Delta^c}. \end{cases} \tag{23}$$

Now apply Proposition 8 to $\mathcal{G}_{OI\Delta}$ with $w = Lp$. That is, make the following substitution in (11),

$$A + LC \to A, \; E + LD \to E, \; [L \;\; B] \to B, \; [0 \;\; G] \to G. \tag{24}$$

Defining variables $\overline{S} = S^{-1}$, $\Lambda_c = \overline{\Lambda}_c^{-1}$, $Y = \overline{S}L$, we have the following result.

Theorem 17: If matrices $\overline{S} > 0$, $\Lambda_c \in \boldsymbol{P_\Theta}$ and $Y \in \mathbf{R}^{n\times l}$ solve the following LMI:

$$\begin{bmatrix} (1,1) & \overline{S}E + YD & Y & \overline{S}B + K^*\Lambda_c G \\ \star & -\Lambda_c & \mathbf{0}_{h\times l} & \mathbf{0}_{h\times m} \\ \star & \star & -I_l & \mathbf{0}_{l\times m} \\ \star & \star & \star & -I_m + G^*\Lambda_c G \end{bmatrix} < 0, \tag{25}$$

where $(1,1) = A^*\overline{S} + \overline{S}A + YC + C^*Y^* + K^*\Lambda_c K$, then $\overline{S}^{-1}$ satisfies (22).

Proof: By Proposition 8, the solution S to (11) satisfies Lyapunov inequality (4). Since (25) is derived by substituting (24) into (11), the solution $\overline{S}$ to (25) satisfies

$$\begin{aligned} &(\mathcal{A}_\Delta + L\mathcal{C}_\Delta)\overline{S}^{-1} + \overline{S}^{-1}(\mathcal{A}_\Delta + L\mathcal{C}_\Delta)^* \\ &\quad + [L \;\; \mathcal{B}_\Delta + L\mathcal{D}_\Delta][L \;\; \mathcal{B}_\Delta + L\mathcal{D}_\Delta]^* < 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(\mathcal{A}_\Delta - \mathcal{B}_\Delta\mathcal{R}_\Delta^{-1}\mathcal{D}_\Delta^*\mathcal{C}_\Delta)\overline{S}^{-1} + \overline{S}^{-1}(\mathcal{A}_\Delta - \mathcal{B}_\Delta\mathcal{R}_\Delta^{-1}\mathcal{D}_\Delta^*\mathcal{C}_\Delta)^* \\ &- \overline{S}^{-1}\mathcal{C}_\Delta^*\tilde{\mathcal{R}}_\Delta^{-1}\mathcal{C}_\Delta\overline{S}^{-1} + \mathcal{B}_\Delta\mathcal{R}_\Delta^{-1}\mathcal{B}_\Delta^* \\ &+ (\tilde{\mathcal{R}}_\Delta Y\overline{S}^{-1} + \mathcal{C}_\Delta\overline{S}^{-1} + \mathcal{D}_\Delta\mathcal{B}_\Delta^*)^*\tilde{\mathcal{R}}_\Delta^{-1}(\tilde{\mathcal{R}}_\Delta Y\overline{S}^{-1} + \mathcal{C}_\Delta\overline{S}^{-1} + \mathcal{D}_\Delta\mathcal{B}_\Delta^*) < 0. \end{aligned}$$

This implies that $\overline{S}^{-1}$ satisfies (22). ■

The following result on the control Riccati inequality (21) can be obtained similarly.

Theorem 18: If matrices $\overline{P} > 0, \overline{\Lambda}_o \in \boldsymbol{P_\Theta}$ and $X \in \mathbf{R}^{m\times n}$ solve the LMI:

$$\begin{bmatrix} (1,1) & \overline{P}K^* + X^*G^* & X^* & \overline{P}C^* + E\overline{\Lambda}_o D^* \\ \star & -\overline{\Lambda}_o & \mathbf{0}_{h\times m} & \mathbf{0}_{h\times l} \\ \star & \star & -I_m & \mathbf{0}_{m\times l} \\ \star & \star & \star & -I_l + D\overline{\Lambda}_o D^* \end{bmatrix} < 0, \tag{26}$$

where $(1,1) = A\overline{P} + \overline{P}A^* + BX + X^*B^* + E\overline{\Lambda}_o E^*$, then $\overline{P}^{-1}$ verifies (21).

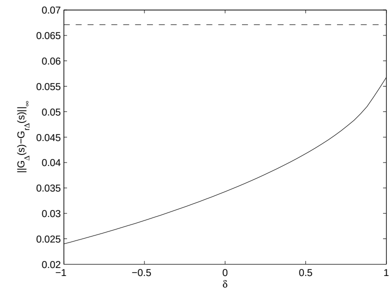
Note that solutions to LMIs (25) and (26) are not unique. A possible heuristic is, taking (25) for example, to solve the following SDP problem: minimize $trace(Z)$, subject to (25) and $\begin{bmatrix} Z & I_n \\ I_n & \overline{S} \end{bmatrix} > 0$; see e.g. [25]. We now summarize the proposed LQG balanced truncation algorithm as follows.

Procedure 19 (LQG Balanced Truncation):

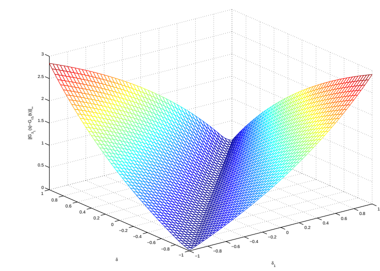
1) Obtain $\overline{S}$ and $\overline{P}$ by solving LMIs (25) and (26) or the associated SDP problems, and let $S = \overline{S}^{-1}, P = \overline{P}^{-1}$;

2) Follow Steps 2-4 in Procedure 11.

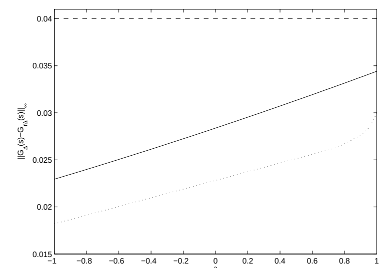
[2]This expression is a slight abuse of notation for state space realizations since here $\mathcal{A}_\Delta$, $\mathcal{B}_\Delta$, $\mathcal{C}_\Delta$, and $\mathcal{D}_\Delta$ are operators.



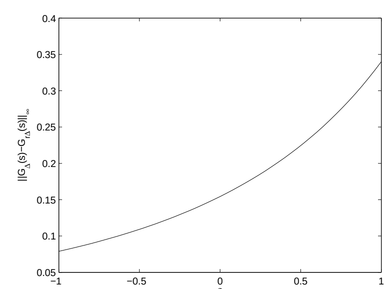
(a) Balanced truncation



(b) Balanced truncation



(c) H_∞ model reduction



(d) LQG Balanced truncation

Fig. 3. H_∞-norm of the error system.

VII. EXAMPLE

Consider the following uncertain system of the form (1) with $\Delta = \delta \in [-1,1]$, and

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$
$$K = C = [1 \ \ 1 \ \ 1], \quad G = 1, \quad D = 0.1. \tag{27}$$

Following the balanced truncation procedure in Section IV, the balanced Gramian is $\Sigma = \mathrm{diag}(2.1728, 0.0319, 0.0017)$. A natural choice in model reduction would be to truncate the last 2 states. Figure 3(a) shows the actual H_∞-norm of the error system, and the dashed line indicates the error bound given by (14) as $\sup_{\delta\in[-1,1]} \|\mathcal{G}_\Delta(s) - \mathcal{G}_{r\Delta}(s)\|_\infty \leq 0.0672$. If different uncertainties are allowed, letting $\Delta_1 = \delta_1, \delta_1 \in [-1,1]$, Figure 3(b) shows the actual H_∞-norm of the error system as a function of δ, δ_1. The error bound is given by (15) as $\sup_{\delta,\delta_1\in[-1,1]} \|\mathcal{G}_{\Delta_1}(s) - \mathcal{G}_{r\Delta}(s)\|_\infty \leq 22.8896$.

Now, we apply the H_∞ model reduction algorithm in Section V to the uncertain system (27), with comparison to the technique in [15]. The LMIRank solver [27] is used to solve the associated rank constrained LMI problems. For $\varepsilon = 0.04$, the solid line and the dotted line in Figure 3(c) show the results by our method and [15] respectively, and the dashed line is the upper bound $\varepsilon = 0.04$. The result using [15] is slightly better than ours. However, this is at the expense of solving more matrix inequalities at all vertices of the underlying polytope.

Finally, the LQG balanced truncation algorithm in Section VI is applied to the uncertain system (27); see the result in Figure 3(d). We remark here that, as introduced in Sections I and VI, LQG balanced truncation is a model reduction method in the *closed-loop* sense. Therefore, the *open-loop* results (i.e. no controllers involved) in Fig. 3 should not be interpreted as that LQG balanced truncation is outperformed by the other two methods; see [6] for more details.

VIII. CONCLUSIONS

In this paper Gramian-based approaches to model reduction for a class of uncertain systems with norm bounded structured uncertainty are presented. We introduce notions of controllability and observability Gramians in terms of certain parameterized algebraic Riccati inequalities. This enables us to develop a series of model reduction methods for uncertain systems, namely, balanced truncation and H_∞ model reduction for robustly stable uncertain systems, or LQG balanced truncation for uncertain systems which are not robustly stable.

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