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A GRAPH AND ITS COMPLEMENT WITH SPECIFIED PROPERTIES I: CONNECTIVITY

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Dedicated to Karl Menger

ABSTRACT. We investigate the conditions under which both a graph G and its complement $\overline{G}$ possess a specified property. In particular, we characterize all graphs G for which G and $\overline{G}$ both (a) have connectivity one, (b) have line-connectivity one, (c) are 2-connected, (d) are forests, (e) are bipartite, (f) are outerplanar and (g) are eulerian. The proofs are elementary but amusing.

KEY WORDS AND PHRASES. Graphs, Complement.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 05C99.

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1. CONNECTIVITY.

The connectivity (or line-connectivity) $\kappa = \kappa(G)$ (or $\lambda = \lambda(G)$) of a graph G is the minimum number of points (or lines) whose removal results in a disconnected or a trivial graph. We write $\bar{\kappa}$ (or $\bar{\lambda}$) for $\kappa(\bar{G})$ (or $\lambda(\bar{G})$) where $\bar{G}$ is the complement of G. We follow the graph theoretic terminology and notation of the book [1]. Recall that Δ denotes the maximum degree among all points of G.

LEMMA 1. The complement $\bar{G}$ of a connected graph G is connected if and only if G has no spanning complete bipartite subgraph.

PROOF. If G has a spanning complete bipartite subgraph, then $\bar{G}$ clearly contains no line joining the two parts, hence must be disconnected. Conversely, if $\bar{G}$ is disconnected, then any bipartition of V(G) in which one part consists of the points of precisely one component of $\bar{G}$ gives a spanning complete bipartite subgraph of G.

The next statement is an easy consequence of the lemma.

THEOREM 1. A graph G with p points satisfies the condition $\kappa = \bar{\kappa} = 1$ if and only if G is a graph with either

(1) $\kappa = 1$ and $\Delta = p - 2$, or

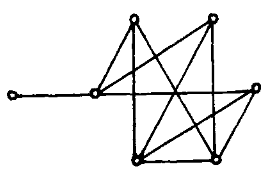
(2) $\kappa = 1$, $\Delta \leq p - 3$ and G has a cutpoint v with endline e and endpoint u such that G − u contains a spanning complete bipartite subgraph.

PROOF. We note that if $\kappa = \bar{\kappa} = 1$, then the degree of each point of G is at most p − 2, since otherwise $\bar{G}$ would contain an isolated point which would make $\bar{\kappa} = 0$.

(1) Let G be a graph with $\Delta = p - 2$ and $\kappa = 1$, as in Figure 1a.

(a) Figure 1. (b)

The removal of any cutpoint v from G results in a disconnected graph, so that $\overline{G - v}$ is connected. Since $\Delta = p - 2$ by hypothesis, v is adjacent in $\bar{G}$ to a point of $\overline{G - v}$. Thus $\bar{G}$ is connected. Furthermore $\bar{G}$ has an endline since $\Delta = p - 2$, and hence $\bar{G}$ has a cutpoint (as illustrated in Figure 1b), so that $\bar{\kappa} = 1$.
(2) Let G be a graph with $\kappa = \bar{\kappa} = 1$ and $\Delta \leq p - 3$. By the definition of κ, G is connected and has a cutpoint v. We see that H = G - v has just two components, since otherwise every two points of $\bar{G}$ would lie on a common cycle of $\bar{G}$ and thus $\bar{G}$ would have no cutpoint, contradicting $\bar{\kappa} = 1$. Denote by H_1 and H_2 the two components of H, with p_1 and p_2 points respectively. Assume that both $p_1, p_2 \geq 2$. Then $\bar{G}$ would have no cutpoint since every two points of $\bar{G}$ would lie on a common cycle of $\bar{G}$. Thus it is sufficient to consider only a connected graph G which has a cutpoint with endline e and endpoint u. We now show that G - u contains a spanning complete bipartite subgraph. If G - u does not contain such a subgraph, then $\overline{G - u}$ is connected by Lemma 1. Moreover, the endpoint u of e is adjacent in $\bar{G}$ to every point of $\bar{G}$ lie on a common cycle and so $\bar{G}$ has no cutpoint, which again contradicts $\bar{\kappa} = 1$. Thus G - u contains a spanning complete bipartite subgraph.


(a)

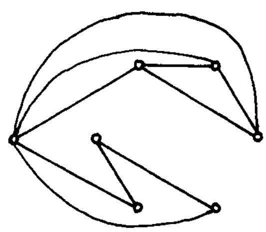

(b)

Figure 2.

Conversely, let G satisfy the condition (2) as shown in Figure 2a. Then $\bar{G}$ is connected and the removal of the endpoint u from $\bar{G}$ results in at least two components by Lemma 1. Hence we see that $\kappa = \bar{\kappa} = 1$.

A graph G is a block if G is connected and has no cutpoint. From Theorem 1 and Lemma 1, we obtain two consequences whose proofs are ommitted or outlined.

COROLLARY 1a. If G is a block, then $\bar{G}$ is also a block if and only if

(1) $2 \leq \deg v \leq p - 3$ for every point v of G, and

(2) G has no spanning complete bipartite subgraph.

CORLLARY 1b. A graph G with p points satisfies the condition $\lambda = \bar{\lambda} = 1$ if and only if G is a connected graph with a bridge and $\Delta = p - 2$.

PROOF. Let G be a graph with $\lambda = \bar{\lambda} = 1$. Then G satisfies the condition $\kappa = \bar{\kappa} = 1$ by the relation $\kappa \leq \lambda$. Hence the graph G satisfies either (1) or (2) of Theorem 1. It is clear that (2) cannot hold, since $\bar{G}$ can possess an endline only if the spanning bipartite subgraph of G - u is a star, in which case $\Delta = p - 2$, and so (1) must obtain.

Conversely, if G is a graph with $\lambda = 1$ and $\Delta = p - 2$, then $\bar{G}$ is connected and has an endline, that is, $\bar{\lambda} = 1$.

2. BIPARTITE GRAPHS AND OUTERPLANAR GRAPHS.

A graph G is a forest if G has no cycles. An outerplanar graph is planar and can be embedded in the plane so that all its points lie on the same face.

THOEREM 2. All the graphs G such that both G and $\bar{G}$ are bipartite are:

are shown in Figure 3.

K_1 K_2 $\bar{K}_2$ $K_1 \cup K_2$ P_3 P_4 $2K_2$ C_4

Figure 3.

PROOF. The number k of components of G is at most two, since otherwise $\bar{G}$ would contain a triangle.

CASE 1: $k = 2$. Let G have components G_1 and G_2. Both G_1 and G_2 are complete, since otherwise $\bar{G}$ would contain a triangle. Furthermore, the order of each of the complete graphs G_1 and G_2 is at most two, since otherwise G would contain a triangle. Hence we obtain $G = \bar{K}_2$, $K_1 \cup K_2$ and $2K_2$.

CASE 2: $k = 1$. Since G is bipartite, the point set of G can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. The cardinalities of V_1 and V_2 are at most two, since otherwise $\bar{G}$ would contain a triangle. Furthermore, each subgraph induced by any three points of G contains one or two lines. Hence we get $G = K_1, K_2, P_3, P_4$, and C_4.

COROLLARY 2a. All the graphs G such that both G and $\bar{G}$ are forests are:

$$G = K_1, K_2, \bar{K}_2, K_1 \cup K_2, P_3 \text{ and } P_4$$

We have determined in Theorem 2 all eight graphs such that both G and $\bar{G}$ are bipartite, and note that for none of these graphs G is both G and $\bar{G}$ have even cycles. We now show that for just two graphs G, both G and $\bar{G}$ have an odd cycle.

THEOREM 3. The two self-complementary graphs of order 5, A and C_5, are the only G such that both G and $\bar{G}$ have only odd cycles (Figure 4).

PROOF. If the number of points of G is at least 6, either G or $\bar{G}$ contains C_4 since the ramsey number $r(C_4) = 6$. It is easily verified that the two self-complementary graphs of order 5, A and C_5 shown in Figure 4, are the only G such that both G and $\bar{G}$ have odd cycles.

THEOREM 4. All the graphs G such that neither G nor $\bar{G}$ are forests but both are outerplanar are the following 32 graphs:

(1) the two self-complementary graphs A and C_5 of order 5 (Figure 4), and

(2) the 15 graphs shown in Figure 5 and their complements.

THEOREM 5. Both G and $\bar{G}$ are eulerian if and only if both are connected, p is odd, and G is eulerian.

Of course p must be odd so that the degree of each point in both G and $\bar{G}$ is even. Lemma 1 already gives a simple condition for both G and $\bar{G}$ to be connected. The result follows at once.

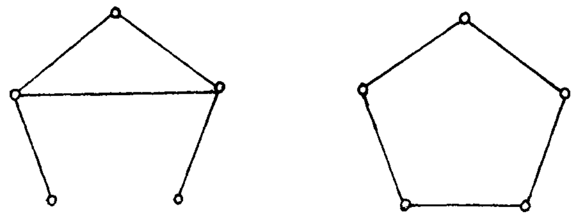


Figure 4.

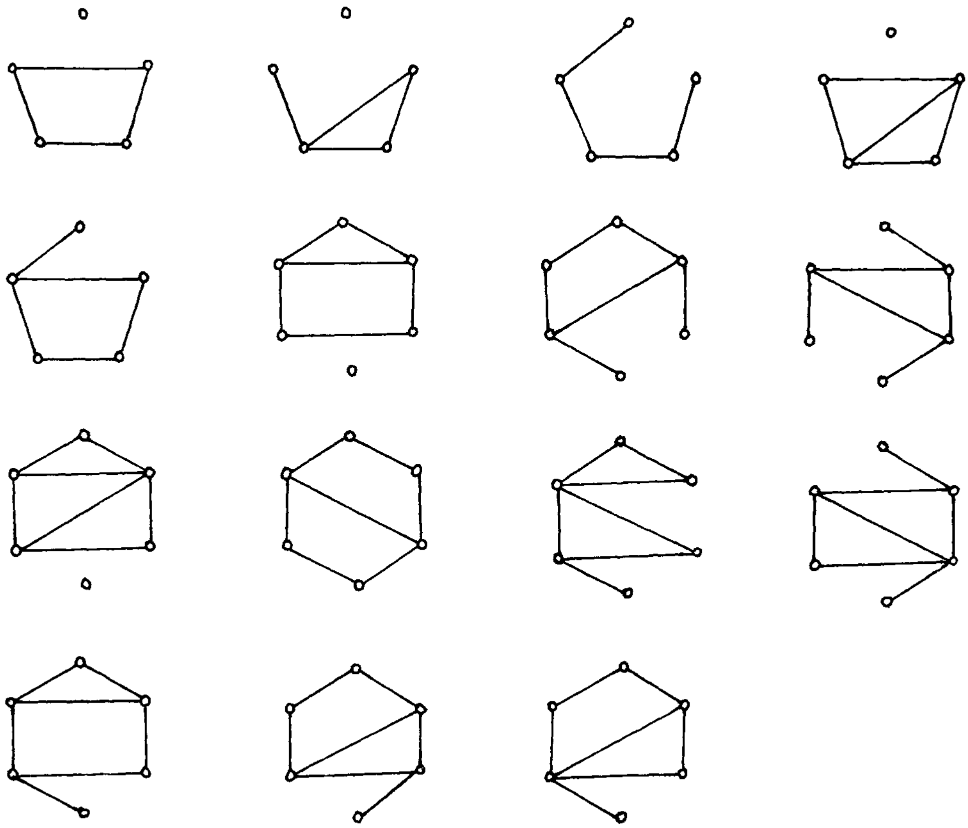


Figure 5.

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