A Graph and its Complement with Specified Properties. IV. Counting Self-Complementary Blocks

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Dedicated to Robert W. Robinson

ABSTRACT

In this series, we investigate the conditions under which both a graph G and its complement $\bar{G}$ possess certain specified properties. We now characterize all the graphs G such that both G and $\bar{G}$ have the same number of endpoints, and find that this number can only be 0 or 1 or 2. As a consequence, we are able to enumerate the self-complementary blocks.

1. NOTATIONS AND BACKGROUND

In the first paper [1] in this series, we found all graphs G such that both G and its complement $\bar{G}$ have connectivity 1, and other properties. In the second paper [2], we determined the graphs G for which G and $\bar{G}$ are obtained from some graph by the same unary operation. More recently [3] we characterized the graphs such that both G and $\bar{G}$ have the same girth and the same circumference 3 or 4.

An *endpoint* of graph has degree 1. We denote the number of endpoints in G by $e = e(G)$ and in $\bar{G}$ by $\bar{e}$. We characterize all the graphs G with $e = \bar{e}(\geq 2)$ in the next section, and count the number of self-complementary blocks in the last section.

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Following the notation and terminology of [5], we define the *join* G_1+G_2 of two graphs to be the union of G_1 and G_2 with the complete bigraph having point sets V_1 and V_2, and the *corona* $G \circ H$ of two graphs G with p points v_i and H is obtained from G and p copies of H by joining each point v_i of G with all the points of the ith copy of H. For our result later we need a *ternary operation* written $F+G\circ H$ which is defined in [3] as the union of the join $F+G$ with the corona $G\circ H$. Thus this resembles the composition of the path P_3 not with just one other graph but with three graphs, one for each point, for example, Figure 1 illustrates the graph $A=K_1+K_2\circ K_1$.

2. ENDPOINTS

Let g_p be the number of graphs of order p.

Lemma 1. For $n \geq 1$, the mapping $F \to F+K_n \circ K_1$ which takes graphs F of order p to graphs $G=F+K_n\circ K_1$ of order $p+2n$ is one-to-one.

Proof. Suppose G can be written in the form $F+K_n\circ K_1$. We will show that F is uniquely recoverable from G. Let S be the set of points of G which are adjacent to endpoints. Clearly S is the point set of the distinguished subgraph K_n. Let H be the subgraph induced by $V(G)-S$. Then H has at least n isolates, and removing exactly n isolates from H leaves F. ∎

Lemma 2. If G has two endpoints, then $\bar{G}$ has at most two endpoints.

Proof. Let v_0 and v_1 be two endpoints of G, adjacent to u_0 and u_1, respectively. Then obviously the only candidates for endpoints in $\bar{G}$ are u_0 and u_1. ∎

Theorem 1. A graph G of order $p \geq 4$ has $e=\bar{e}=2$ iff G is of the form $F+K_2\circ K_1$, where F is a graph of order $p-4$.

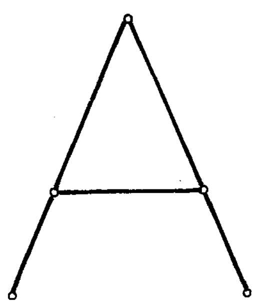

FIGURE 1. $A=K_1+K_2\circ K_1$

Proof. If $e = \bar{e} = 2$, then G has exactly two points v_0 and v_1 of degree $p-2$ and exactly two points u_0 and u_1 of degree 1, where u_0, u_1 are not adjacent to v_0, v_1, respectively. Since $\deg v_0 = \deg v_1 = p-2$, v_i is adjacent to every point other than u_i for $i = 0, 1$. On the other hand, u_i is not adjacent to any point other than v_{1-i} for $i = 0, 1$, since $\deg u_0 = \deg u_1 = 1$. Denote by F the subgraph of G induced by the point set $V(G) - \{v_0, v_1, u_0, u_1\}$. Then in G any point v of F must be adjacent to both v_0 and v_1 which are adjacent to each other by the above observations. Thus G is a graph of the form $F + K_2 \circ K_1$.

The converse follows immediately from the proof of Lemma 1. ∎

Corollary 1. The number of graphs of order p with $e = \bar{e} = 2$ is g_{p-4}.

Proof. By Theorem 1, G is of the form $F + K_2 \circ K_1$ where F has $p-4$ points. Hence by the $1-1$ correspondence of Lemma 1, the number of graphs G with $e = \bar{e} = 2$ is g_{p-4}. ∎

Corollary 2. All graphs with $e = \bar{e} = 2$ have diameter 3.

Proof. The maximum distance between two points of $F + K_2 \circ K_1$ is 3, as this is the distance between the two endpoints. ∎

3. SELF-COMPLEMENTARY GRAPHS

A graph G is *self-complementary* (or briefly, s-c) if it is isomorphic to its complement $\bar{G}$. The isomorphism between G and $\bar{G}$ can be represented as a permutation, α , on $V(G)$. We will write $\alpha(G) = \bar{G}$ and call α a *complementing permutation for* G as in Gibbs [6]. We will assume that all permutations are expressed as the product of disjoint cycles. We first state the result obtained independently by Ringel [8] and Sachs [10], which gives the cycle structure of a complementing permutation.

Theorem RS. If G is s-c of order p and $\alpha(G) = \bar{G}$, then if $p \equiv 0 \pmod 4$, each cycle of α has length divisible by 4 and if $p \equiv 1 \pmod 4$, α has exactly one cycle of length 1 and all other cycles have length divisible by 4.

We begin with the result concerning the number of endpoints of a s-c graph, which was communicated to us by R. W. Robinson and proved nicely by one of the referees.

Lemma 3. A self-complementary graph does not have exactly one endpoint.

Proof. Suppose G is s-c with a unique point of degree 1. Then G must have a unique point of degree $p-2$ and these observations hold for $\bar{G}$ as well. In G let $\deg v_1 = 1$ and $\deg v_2 = p-2$. Hence in $\bar{G}$, $\deg v_1 = p-2$ and $\deg v_2 = 1$. But v_1 and v_2 are adjacent in exactly one of G and $\bar{G}$, a contradiction. ▮

We now characterize all s-c graphs with two endpoints.

Lemma 4. All s-c graphs of order $p+4$ having two endpoints can be constructed using the ternary operation $G = F + K_2 \circ K_1$, where F is a s-c graph of order p.

Proof. Let G be any s-c graph of order $p+4$ having 2 endpoints. Since $G \cong \bar{G}$ and G has exactly 2 endpoints, we know that G is of the form $F + K_2 \circ K_1$ for some graph F of order p by Theorem 1. On the other hand, it is easy to see that $G = F + K_2 \circ K_1$ is s-c iff F is s-c. Thus, G can be constructed using the ternary operation $G = F + K_2 \circ K_1$ for some s-c graph F of order p. ▮

We denote by s_p the number of all s-c graphs of order p and by s_p'' the number of s-c graphs of order p which have 2 endpoints. Since the ternary operation $G = F + K_2 \circ K_1$ is $1-1$ as proved in Lemma 1, we have the following equality from Lemma 4.

Lemma 5. For any positive integer p,

$$s_{p+4}'' = s_p. \quad ▮$$

Recall [5, p. 24] that G is a *block* if G is connected and has no cutpoint. The number of blocks was determined by Robinson [9]. Our object is to derive the number of self-complementary blocks.

Lemma 6. If G is a s-c graph with no endpoints, then G is a block.

Proof. Assume that G is s-c with no endpoints but has a cutpoint v. The removal of v from G results in a subgraph with at least 2 components. Let G_1 be a component of $G-v$ and let $\overline{G-v} = G_1 \cup G_2$. Thus $\overline{G-v}$ contains a complete spanning bigraph B whose point sets are $V(G_1)$ and $V(G_2)$. The cardinalities of both $V(G_1)$ and $V(G_2)$ are at least 2 by the hypothesis that G has no endpoints. Therefore $\bar{G}$ is 2-connected and hence $G = \bar{G}$ cannot have a cutpoint, a contradiction. ▮

Read [7] found a formula for the number of self-complementary graphs s_p. Frucht and Harary [4] derived an alternative equation. We now see how to count s-c blocks in terms of the numbers s_p.

Theorem 2. For any positive integer $p \geq 5$, the number of s-c blocks of order p is $s_p - s_{p-4}$.

Proof. Let G be a self-complementary block of order p, so that $p \geq 5$. By Lemmas 3 and 6, the number of s-c blocks equals s_p less the number of s-c graphs with $e = 2$. But this is s_{p-4} by Lemma 5. ∎

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