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A Graph Theoretic Approach for Certain Properties of Spectral Null Codes

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1. Introduction

In this chapter, we look at the spectral null codes from another angle, using graph theory, where we present a few properties that have been published. The graph theory will help us to understand the structure of spectral null codes and analyze their properties differently.

Graph theory [1]–[2] is becoming increasingly important as it plays a growing role in electrical engineering for example in communication networks and coding theory, and also in the design, analysis and testing of computer programs.

Spectral null codes [3] are codes with nulls in the power spectral density function and they have great importance in certain applications such as transmission systems employing pilot tones for synchronization and track-following servos in digital recording [4]–[5].

Yeh and Parhami [6] introduced the concept of the index-permutation graph model, which is an extension of the Cayley graph model and applied it to the systematic development of communication-efficient interconnection networks. Inspiring the concept of building a relationship between an index and a permutation symbol, we make use in this chapter of the spectral null equations variables in each grouping by representing only their corresponding indices in a permutation sequence form. In another way, these indices will be presented by a permutation sequence, where the symbols refer to the position of the corresponding variables in the spectral null equation.

Presenting a symmetric-permutation codebook graphically, Swart *et al.* [7] allocated states to all symbols of a permutation sequence and presented all possible transpositions between these symbols by links as depicted for a few examples in Fig. 1 [7].

The Chapter is organized as follows: Section II introduces definitions and notations to be used for spectral null codes. Section III presents few graph theory definitions. Section IV presents the index-graphic presentation of spectral null codes. Section V makes an approach between graph theory and spectral null codes where we focus on the relationship between the cardinalities of the spectral null codebooks and the concepts of distances in graph theory and also we elaborate the concept of subgraph and its corresponding to the structure of the spectral null codebooks. We conclude with some final remarks in Section VI.

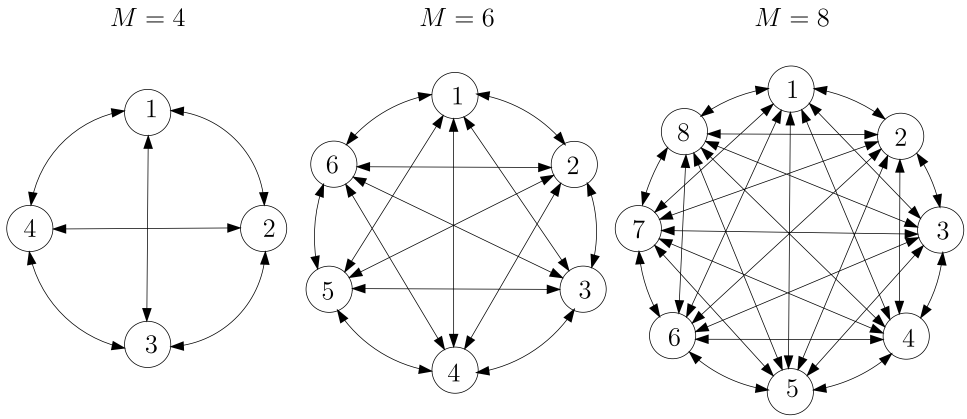


Fig. 1. Graph representation for permutation sequences

2. Spectral null codes

The technique of designing codes to have a spectrum with nulls occurring at certain frequencies, i.e. having the power spectral density (PSD) function equal to zero at these frequencies, started with Gorog [8], when he considered the vector $\mathbf{X} = (x_1, x_2, ..., x_M)$, $x_i \in \{-1, +1\}$ with $1 \leq i \leq M$, to be an element of a set S, which is called a codebook of codewords with elements in $\{-1, +1\}$. We investigate codewords of length, M, as an integer multiple of N, thus let

$$M = Nz,$$

where N represents the number of groupings in the spectral null equation and z represents the number of elements in each grouping. The values of $f = r/N$ are frequencies at spectral nulls (SN) at the rational submultiples r/N [9]. To ensure the presence of these nulls in the continuous component at the spectrum, it is sufficient to satisfy the following spectral null equation [10],

$$A_1 = A_2 = \cdots = A_N, \tag{1}$$

where

$$A_i = \sum_{\lambda=0}^{z-1} x_{i+\lambda N}, \quad i = 1, 2, \ldots, N, \tag{2}$$

which can also be presented differently as,

$$\begin{array}{lcl} A_1 &=& \overbrace{x_1 + x_{1+N} + x_{1+2N} + x_{1+3N} + \cdots + x_{1+(z-1)N}}^{z} \\ A_2 &=& x_2 + x_{2+N} + x_{2+2N} + x_{2+3N} + \cdots + x_{2+(z-1)N} \\ A_3 &=& x_3 + x_{3+N} + x_{3+2N} + x_{3+3N} + \cdots + x_{3+(z-1)N} \\ \vdots & & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ A_N &=& x_N + x_{2N} + x_{3N} + x_{4N} + \cdots + x_{zN}. \end{array} \tag{3}$$

If all the codewords in a codebook satisfy these equations, the codebook will exhibit nulls at the required frequencies. henceforth we present the channel symbol -1 with binary symbol 0.

Definition 2.1. *A spectral null binary block code of length M is a subset $\mathcal{C}_b(M,N) \subseteq \{0,1\}^M$ of all binary M-tuples of length M which have spectral nulls at the rational submultiples of the symbol frequency $1/N$.*

Definition 2.2. *The spectral null binary codebook $\mathcal{C}_b(M,N)$ is a subset of the* M *dimensional vector space $(\mathbb{F}_2)^M$ of all binary M-tuples, where $\mathbb{F}_2$ is the finite field with two elements, whose arithmetic rules are those of mod-2 arithmetic.*

For codewords of length M consisting of N interleaved subwords of length z, the cardinality of the codebook $\mathcal{C}_b(M,N)$ for the case where N is a prime number is presented by the following formula [10],

$$|\mathcal{C}_b(M,N)| = \sum_{i=0}^{M/N} \binom{M/N}{i}^N, \tag{4}$$

where $\binom{M/N}{i}$ denotes the combinatorial coefficient $\frac{(M/N)!}{i!(M/N-i)!}$.

Example 2.3. *If we consider the case of $M = 6$, we can predict two types of spectral with different nulls since N can take the value of $N = 2$ or $N = 3$. Their corresponding spectral null equations are presented respectively as follows:*

$$x_1 + x_3 + x_5 = x_2 + x_4 + x_6 \tag{5}$$

$$x_1 + x_4 = x_2 + x_5 = x_3 + x_6 \tag{6}$$

The corresponding codebooks for (5) *and* (6) *are respectively as follows:*

$$\mathcal{C}_b(6,2) = \left\{ \begin{array}{c} 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,1\,1\,0 \\ 0\,0\,1\,0\,0\,1 \\ 0\,0\,1\,1\,0\,0 \\ 0\,0\,1\,1\,1\,1 \\ 0\,1\,0\,0\,1\,0 \\ 0\,1\,1\,0\,0\,0 \\ 0\,1\,1\,0\,1\,1 \\ 0\,1\,1\,1\,1\,0 \\ 1\,0\,0\,0\,0\,1 \\ 1\,0\,0\,1\,0\,0 \\ 1\,0\,0\,1\,1\,1 \\ 1\,0\,1\,1\,0\,1 \\ 1\,1\,0\,0\,0\,0 \\ 1\,1\,0\,0\,1\,1 \\ 1\,1\,0\,1\,1\,0 \\ 1\,1\,1\,0\,0\,1 \\ 1\,1\,1\,1\,0\,0 \\ 1\,1\,1\,1\,1\,1 \end{array} \right\},$$

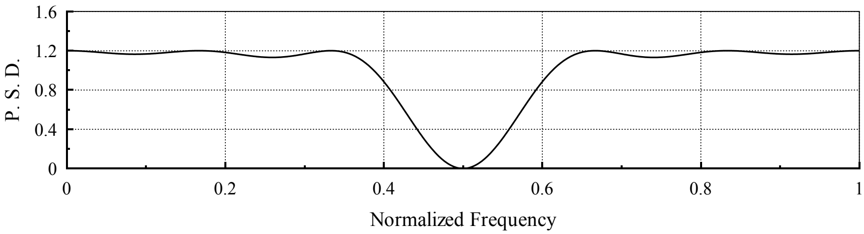


Fig. 2. Power spectral density of codebook $N = 2, M = 6$.

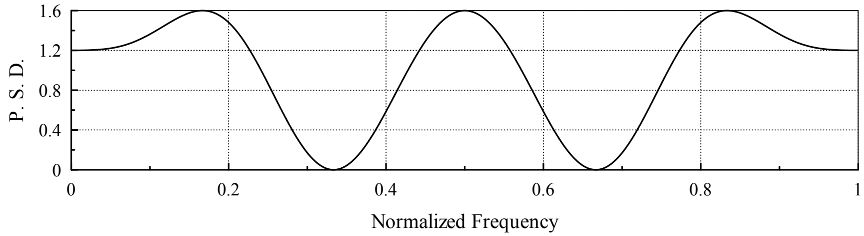


Fig. 3. Power spectral density of codebook $N = 3$, $M = 6$.

and

$$\mathcal{C}_b(6,3) = \begin{Bmatrix} 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,1\,1 \\ 0\,0\,1\,1\,1\,0 \\ 0\,1\,0\,1\,0\,1 \\ 0\,1\,1\,1\,0\,0 \\ 1\,0\,0\,0\,1\,1 \\ 1\,0\,1\,0\,1\,0 \\ 1\,1\,0\,0\,0\,1 \\ 1\,1\,1\,0\,0\,0 \\ 1\,1\,1\,1\,1\,1 \end{Bmatrix}.$$

The cardinalities of $\mathcal{C}_b(6,2)$ and $\mathcal{C}_b(6,3)$ are respectively equal to 20 *and* 10*. This also can be easily verified from* (4)*.*

We can see clearly the power spectral density $\mathcal{C}_b(6,2)$ and $\mathcal{C}_b(6,3)$ respectively presented in Figures 2 and 3 where the nulls appear to be multiple of $1/N$ as presented in Definition 2.1.

3. Graph theory: Preliminary

We present a brief overview of related definitions for certain graph theory fundamentals which will be used in the following sections.

Definition 3.1. *[1]–[2]*

(a) *A graph* G = (V, E) *is a mathematical structure consisting of two finite sets* V *and* E*. The elements of* V *are called vertices, and the elements of* E *are called edges. Each edge has a set of one or two vertices associated with it.*

(b) *A graph $G' = (V', E')$ is a subgraph of another graph $G = (V, E)$ iff $V' \subseteq V$ and $E' \subseteq E$.*

Definition 3.2. *[1]–[2] The graph distance denoted by $\mathcal{G}_d(u, v)$ between two vertices u and v of a finite graph is the minimum length of the paths connecting them.*

Definition 3.3. *[1]–[2] The adjacency matrix of a graph is an $M \times M$ matrix $\mathcal{A}_d = [a_{i,j}]$ in which the entry $a_{i,j} = 1$ if there is an edge from vertex i to vertex j and is 0 if there is no edge from vertex i to vertex j.*

4. Index-graphic presentation of spectral null codes

The idea of the index-graphic presentation of the spectral null codes is actually based on the presentation of the indices of the variables in each grouping of the spectral null equation (1).

Definition 4.1. *We denote by $I_p(i, \lambda)$ the permutation symbol of the corresponding index of the variable $x_{i+\lambda N}$ in (2).*

$$I_p(i, \lambda) = i + \lambda N \quad \textit{where} \quad \begin{cases} i = 1, 2, \ldots, N, \\ \lambda = 0, 1, \ldots, z-1. \end{cases} \tag{7}$$

Definition 4.2. *We denote by $\mathcal{P}_{I_p}(M, N)$ the index-permutation sequence from a spectral null equation for variables of length $M = Nz$ as presented.*

$$\mathcal{P}_{I_p}(M, N) = \prod_{i=1}^{N} \prod_{\lambda=0}^{z-1} I_p(i, \lambda). \tag{8}$$

The product sign in (8) is not used in its traditional way, but just to give an idea about the sequence and the order of the permutation symbols.

Example 4.3. *To explain the relationship between the spectral nulls equation, the index-permutation sequences and their graph presentation, we take the case of $M = 4$ where we have only two groupings since $N = 2$.*

$$A_1 = A_2 \rightarrow x_1 + x_3 = x_2 + x_4 \tag{9}$$

We can see from (9), that the indices of the variables x_i, using (8), are represented by the symbols $I_p(1, 0) = 1$, $I_p(1, 1) = 3$, $I_p(2, 0) = 2$ and $I_p(2, 1) = 4$. The index-permutation sequence is then $\mathcal{P}_{I_p}(4, 2) = (13)(24)$.

An index-permutation symbol is presented graphically by just being lying on a circle, which it is called a state. The state design follow the order of appearance of the indices in (9). The symbols are connected in respect of the addition property of their corresponding variables in (9) as depicted in Fig. 4.

Spectral null codebooks have the all-zeros and all-ones codewords [10], where all the variables y_i are equal. We call the corresponding spectral null equation, which is $x_1 = x_2 = x_3 = x_4$ as the all-zeros spectral null equation, which still satisfying (9) since it is a special case of it. If we substitute the variables in (9) by using the all-zeros spectral null equation, we obtain the following relationships:

$$\begin{cases} x_1 + x_3 = x_2 + x_4, \\ x_1 = x_2 = x_3 = x_4, \end{cases} \Rightarrow \begin{cases} x_2 + x_3 = x_1 + x_4, \\ x_1 + x_2 = x_3 + x_4. \end{cases} \tag{10}$$

Equation (10) shows the resultant equations derived from (9) and the all-zeros spectral null equation. Fig. 5 shows that the same graph G_1 in Fig. 4 is actually a special case of the graph G_2 when we take into consideration the all-zeros spectral null equation.

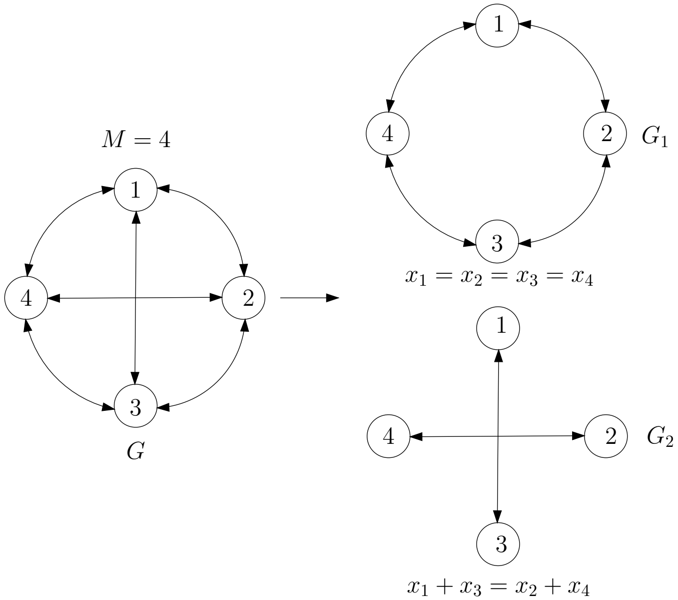


Fig. 4. Equation representation for Graph $M = 4$

Since the obtained relationship between the variables $x_1 = x_2 = x_3 = x_4$ is a special case of the equation representing the graph G_2 in Fig. 4, we limit our studies to (1) *and to its corresponding graph to study the cardinality and other properties of the code.*

Fig. 4 shows that the graph G, which is the general form of all possible permutations is the combinations or the union, $G = G_1 \cup G_2$, of other subgraphs related to the spectral null equation.

5. Graph theory and spectral null codes

In this section we will present certain concepts and properties for spectral null codes and try to confirm and very them from a graph theoretical approach.

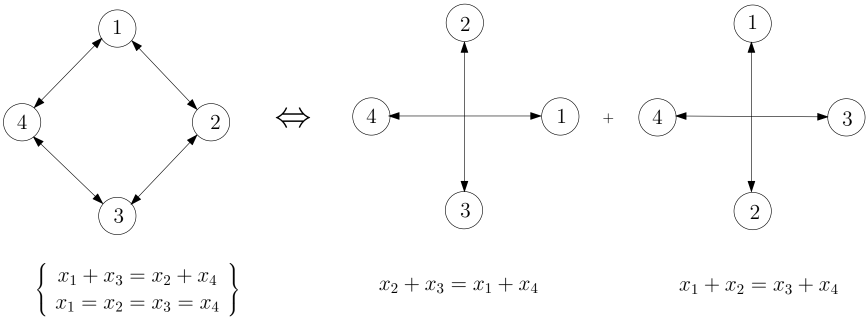


Fig. 5. All-zero equation representation for Graph $M = 4$

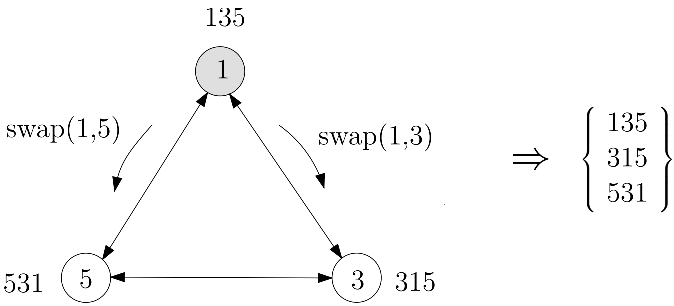


Fig. 6. Index-permutation sequences

5.1 Cardinalities approach

5.1.1 Hamming distance approach

The use of the Hamming distance [11] in this section is just to refer to the number of places that two permutation sequences representing the index-permutation symbols of each grouping A_i of the spectral null equation differ, and not in the study of the error correction properties of the spectral null codes.

To generate the permutation sequences, we start with any state representing an index-permutation symbol in each grouping as appearing in (1). A permutation sequence used as a starting point, contains the symbol from the start state followed by the rest of symbols from the other states taking into consideration the order of the symbols as appearing in (1). Fig. 6 shows the starting permutation sequence as 135. We swap the state-symbol with the following state-symbol in the permutation sequence based on the *k*-cube construction [12]. We end the swapping process at the last state in the graph. We do not swap symbols between the last state and the starting state for the reason to not disturb the obtained sequences at each state. As an example, for $M = 6$, Fig. 6 depicts the swaps and shows the resultant index-permutation codebooks for one grouping.

Definition 5.1. *The Hamming distance $d_H(\mathbf{Y}^i, \mathbf{Y}^j)$ is defined as the number of positions in which the two sequences $\mathbf{Y}^i$ and $\mathbf{Y}^j$ differ. We denote by $\mathcal{H}_d(M, N)$ the distance matrix, whose entries are the distances between index-permutation sequences from a spectral null code of length $M = Nz$ defined as follows:*

$$\mathcal{H}_d(M, N) = [h_{i,j}] \quad \text{with} \quad h_{i,j} = d_H(\mathbf{Y}^i, \mathbf{Y}^j). \tag{11}$$

Definition 5.2. *The Hamming distance between the same sequences or between sequences with non connected symbols is always equal to zero.*

Definition 5.3. *The sum on the Hamming distances in the $\mathcal{H}_d(M, N)$ distance matrix is*

$$|\mathcal{H}_d(M, N)| = \sum_{i=1}^{M} \sum_{j=1}^{M} h_{i,j}. \tag{12}$$

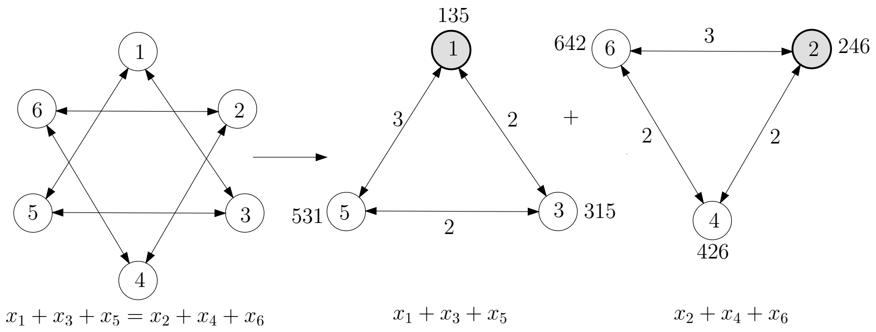


Fig. 7. Distances for Graph $M = 6$ with $N = 2$

In the following examples we consider different cases of number of groupings and number of elements in each grouping and we discuss their impact on the resultant Hamming distance and its relationship with the cardinalities of the spectral null codebooks.

Example 5.4. *We consider the case of $M = 6$ where the number of groupings is $N = 2$ and the number of variables in each grouping is $z = 3$. The corresponding spectral null equation is*

$$\overbrace{x_1 + x_3 + x_5}^{A_1} = \overbrace{x_2 + x_4 + x_6}^{A_2} \tag{13}$$

The equation (13) *is presented by the graph in Fig. 7, where the index-permutation symbols are presented with their corresponding Hamming distances.*

$$\mathcal{H}_d(6,2) = \begin{array}{c} \\ 135 \\ 315 \\ 513 \\ 246 \\ 426 \\ 624 \end{array} \begin{array}{c} \begin{array}{cccccc} 135 & 315 & 513 & 246 & 426 & 624 \end{array} \\ \begin{bmatrix} 0 & 2 & 3 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 2 & 0 \end{bmatrix} \end{array} \tag{14}$$

Each grouping in (13) *is represented by a subgraph as depicted in Fig. 7. The Hamming distance matrix for all possible index-permutation sequences is presented in* (14)*, where "0" represents the Hamming distance between same sequences or sequences with non connected symbols as defined in Definition 5.2. From Definition 5.3, we have,*

$$|\mathcal{H}_d(6,2)| = 28.$$

Example 5.5. *For the case of $M = 6$ where $N = 3$ and $z = 2$, the corresponding spectral null equation is*

$$\overbrace{x_1 + x_4}^{A_1} = \overbrace{x_2 + x_5}^{A_2} = \overbrace{x_3 + x_6}^{A_3}. \tag{15}$$

The equation (15) *is presented by the graph in Fig. 8. Using the concept of graph distance and the permutation sequences, we can have the distance values as depicted in Fig. 8.*

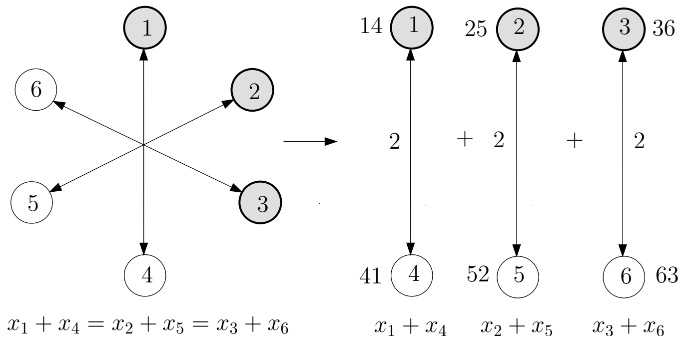


Fig. 8. Distances for Graph $M = 6$ for $N = 3$

The corresponding subgraphs for each grouping A_1, A_2 and A_3 are presented in Fig. 8.

$$\mathcal{H}_d(6,3) = \begin{array}{c} \\ 14 \\ 41 \\ 25 \\ 52 \\ 36 \\ 63 \end{array} \begin{array}{c} \begin{array}{cccccc} 14 & 41 & 25 & 52 & 36 & 63 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix} \end{array} \tag{16}$$

The Hamming distance matrix for all possible index-permutation sequences is presented in (16). *From Definition 5.3, we have,*

$$|\mathcal{H}_d(6,3)| = 12.$$

Comparing the two results we have,

$$|\mathcal{H}_d(6,2)| > |\mathcal{H}_d(6,3)|.$$

Example 5.6. *In this example we take the case of N not a prime number, where we have to suppose that $N = cd$, where c and d are integer factors of N. The equation, which leads to nulls, is*

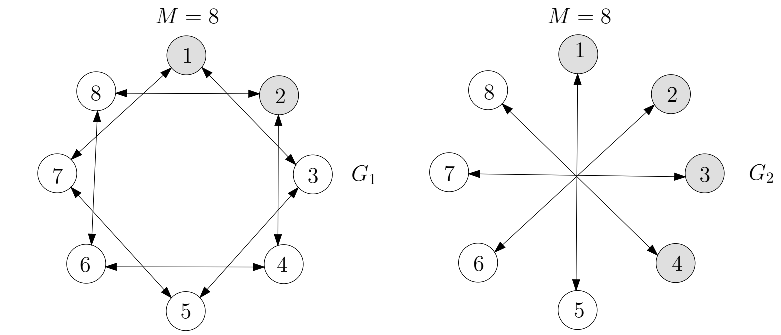
$$\begin{aligned} A_u &= A_{u+vc}, \\ u &= 0,1,2,\ldots,c-1, \\ v &= 1,2,\ldots,d-1, \\ N &= cd, \end{aligned} \tag{17}$$

We consider the case of $M = 8$, where N can be whether $N = 2$ or $N = 4$. The corresponding graph of each case is respectively depicted depicted in Fig. 9 as G_1 and G_2. From Definition 5.3, we have,

$$|\mathcal{H}_d(8,2)| = 40.$$

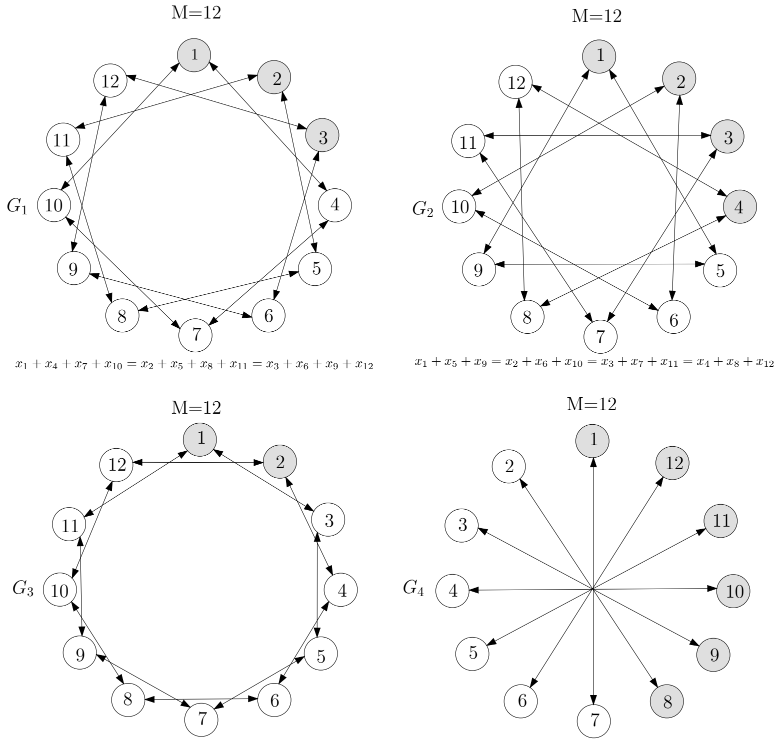
and

$$|\mathcal{H}_d(8,4)| = 16.$$



$$x_1 + x_3 + x_5 + x_7 = x_2 + x_4 + x_6 + x_8 \qquad x_1 + x_5 = x_2 + x_6 = x_3 + x_7 = x_4 + x_8$$

Fig. 9. Equation representation for Graph $M = 8$



$$x_1 + x_4 + x_7 + x_{10} = x_2 + x_5 + x_8 + x_{11} = x_3 + x_6 + x_9 + x_{12}$$

$$x_1 + x_5 + x_9 = x_2 + x_6 + x_{10} = x_3 + x_7 + x_{11} = x_4 + x_8 + x_{12}$$

$$x_1 + x_3 + x_5 + x_7 + x_9 + x_{11} = x_2 + x_4 + x_6 + x_8 + x_{10} + x_{12}$$

$$x_1 + x_7 = x_2 + 8 = x_3 + x_9 = x_4 + x_{10} = x_5 + x_{11} = x_6 + x_{12}$$

Fig. 10. Equation representation for Graph $M = 12$

Comparing the two results we have,

$$|\mathcal{H}_d(8,2)| > |\mathcal{H}_d(8,4)|.$$

Example 5.7. *In the case of $M = 12$, we have four combinations where the value of N could be $N = 4$, $N = 3$, $N = 2$ or $N = 6$ as depicted in* (17). *In each case we have a graph representing the spectral null equation as depicted in Fig. 10.*

From Definition 5.3, we have,

$$|\mathcal{H}_d(12,2)| = 64,$$

$$|\mathcal{H}_d(12,6)| = 24,$$

$$|\mathcal{H}_d(12,3)| = 60,$$

and

$$|\mathcal{H}_d(12,4)| = 56.$$

Comparing all the results we have,

$$|\mathcal{H}_d(12,2)| > |\mathcal{H}_d(12,6)|,$$

$$\text{and } |\mathcal{H}_d(12,3)| > |\mathcal{H}_d(12,4)|.$$

Theorem 5.8. *The sum on the Hamming distances for all index-permutation sequences is*

$$|\mathcal{H}_d(M,N)| = \begin{cases} 4N, & \text{for } z = 2, \\ 2N(3z-2), & \text{for } z \geq 3. \end{cases}$$

Proof. Since the matrix $\mathcal{H}_d(M,N)$ is clearly symmetric, we can just prove half of the results of the theorem and then the final will be the double. For the case of $z = 2$ the proof is trivial since we swap only two symbols in each index-permutation sequence. Thus the sum on the distances is $4 \times N$. For the case of $z \geq 3$ we have a cycle graph [1]-[2], where the number of edges is equal to the number of vertices. Since we swap two symbols each time we move from one state to another, the distance at each edge is equal to two, except for the last edge connecting the first state to the last state where all symbols are swapped and the distance is equal to the length of the index-permutation sequences, which is z. The sum on the Hamming distances for a cycle graph for each grouping is $2 \times (z-1) + z = 3 \times z - 2$. Thus the result on the sum of the Hamming distances in the matrix is $2 \times N \times (3 \times z - 2)$. □

5.1.2 Graph-swap distance approach

The length of each grouping A_i, which is equal to the value of z plays an important role in cardinalities of the corresponding codebooks. We make use of the graph distance theory to see how z also plays an important role in the value of the graph distance.

Definition 5.9. *The graph-swap distance denoted by $\mathcal{G}_d$ between two index-permutation symbols represented by the vertices u and v of a finite graph is the minimum number of times of swaps that symbol u can take the position of symbol v in the graph.*

Definition 5.10. *The graph-swap distance between the same index-permutation symbol or between non connected symbols is always equal to zero.*

Definition 5.11. *We denote by $\mathcal{M}_{\mathcal{G}_d}(M,N)$ the graph-swap distance matrix, whose entries $m_{i,j}$ are the graph distances between two index-permutation symbols from a spectral null code of length $M = Nz$.*

Definition 5.12. *The sum on the graph-swap distances in the* $\mathcal{M}_{\mathcal{G}_d}(M,N)$ *distance matrix is*

$$|\mathcal{M}_{\mathcal{G}_d}(M,N)| = \sum_{i=1}^{M}\sum_{j=1}^{M} m_{i,j}. \tag{18}$$

Example 5.13. *We consider the case of* $M = 8$ *with* $N = 2$ *or* $N = 4$*, the corresponding graph-swap distance matrices are respectively as*

$$\mathcal{M}_{\mathcal{G}_d}(8,2) = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc}1&2&3&4&5&6&7&8\end{array}\\ \begin{bmatrix} 0&0&1&0&2&0&1&0\\ 0&0&0&1&0&2&0&1\\ 1&0&0&0&1&0&2&0\\ 0&1&0&0&0&1&0&2\\ 2&0&1&0&0&0&1&0\\ 0&2&0&1&0&0&0&1\\ 1&0&2&0&1&0&0&0\\ 0&1&0&2&0&1&0&0 \end{bmatrix}\end{array}, \quad \text{and} \quad \mathcal{M}_{\mathcal{G}_d}(8,4) = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc}1&2&3&4&5&6&7&8\end{array}\\ \begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{bmatrix}\end{array}.$$

From Definition 5.12, we have $|\mathcal{M}_{\mathcal{G}_d}(8,2)| = 32$ *and* $|\mathcal{M}_{\mathcal{G}_d}(8,4)| = 8$*. where we can see clearly that*

$$|\mathcal{M}_{\mathcal{G}_d}(8,2)| > |\mathcal{M}_{\mathcal{G}_d}(8,4)|.$$

Theorem 5.14. *The sum on the graph distances for all index-permutation symbols is*

$$|\mathcal{M}_{\mathcal{G}_d}(M,N)| = \begin{cases} \left(\frac{z}{2}\right)^2 M, & \text{for } z \text{ even,} \\ \frac{z^2-1}{4} M, & \text{for } z \text{ odd.} \end{cases}$$

Proof. The graphs that we are using are cycle graphs. As long as we go through the edges of a graph the graph distance is incremented by one. When z is even, the first state has the farthest state to it located at $\frac{z}{2}$. So the graph distances from the first state to the $\frac{z}{2}$ state are in a numerical series of ratio one from one to $\frac{z}{2}$. From the state at the position $\frac{z}{2} - 1$ till the first state, the graph distances are in a numerical series of ratio one from one to $\frac{z}{2} - 1$. Adding the two series we get the final sum equal to $\left(\frac{z}{2}\right)^2 M$. Same analogy for the case of z as odd with a numerical series from one till $\frac{z-1}{2}$. □

5.1.3 Adjacency-swap matrix approach

We introduce the adjacency-swap matrix inspired by graph theory as follows.

Definition 5.15. *The adjacency-swap matrix of index-permutation symbols is an* $M \times M$ *matrix* $\mathcal{N}_{A_d}(M,N) = (n_{i,j})$ *in which the entry* $n_{i,j} = 1$ *if there is a swap between an index symbol i and an index symbol j and is* 0 *if there is no swap between index symbol i and index symbol j as presented in each grouping of a spectral null equation.*

Example 5.16. *For the case of $M = 6$ with $N = 2$ or $N = 3$, the corresponding adjacency-swap matrices are*

$$\mathcal{N}_{A_d}(6,2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\!\!\begin{array}{c} 1\;2\;3\;4\;5\;6 \\ \left[\begin{array}{cccccc} 0&0&1&0&1&0 \\ 0&0&0&1&0&1 \\ 1&0&0&0&1&0 \\ 0&1&0&0&0&1 \\ 1&0&1&0&0&0 \\ 0&1&0&1&0&0 \end{array}\right] \end{array}, \quad \text{and} \quad \mathcal{N}_{A_d}(6,3) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\!\!\begin{array}{c} 1\;2\;3\;4\;5\;6 \\ \left[\begin{array}{cccccc} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{array}\right] \end{array}.$$

We can see that $|\mathcal{N}_{A_d}(6,2)| = 12 > |\mathcal{N}_{A_d}(6,3)| = 6$.

M	N	z	$\|\mathcal{C}_b(M,N)\|$	$\|\mathcal{H}_d(M,N)\|$	$\|\mathcal{M}_{\mathcal{G}_d}(M,N)\|$	$\|\mathcal{N}_{A_d}(M,N)\|$
6	3	2	10	12	4	6
6	2	3	20	28	12	12
8	4	2	36	16	8	8
8	2	4	70	40	32	24
10	5	2	34	20	10	10
10	2	5	252	52	60	40
12	6	2	250	24	12	12
12	4	3	300	56	24	24
12	3	4	346	60	48	36
12	2	6	924	64	108	60
15	5	3	488	70	30	30
15	3	5	2252	78	90	60

Table 1. Graph Distances and Cardinalities of Different Codebooks

Theorem 5.17. *The total number of swaps in an adjacency-swap matrix is*

$$|\mathcal{N}_{A_d}(M,N)| = (z-1)M$$

Proof. The proof is trivial as per grouping we have z index-permutation symbols. Thus we have $z-1$ ones in each row of the matrix $\mathcal{N}_{A_d}(M,N)$ which refer to the possible swaps of each symbol with others in the same grouping. The total number of swaps is $(z-1) \times M$. □

Table 1 presents few examples of the relationship between the cardinalities of spectral null codes denoted by $\mathcal{C}_b(M,N)$ and their correspondences of graph distances. It is clear from Table 1 that the cardinalities of different codebooks with the same length of codewords, increase when the number of swaps increases. This results is also verified in Table 1 based on the concept of distances from graph theory perspective.

5.2 Subsets approach

5.2.1 Subgraph theory

In this section we make use of one of the properties in graph theory related to the design of subgraphs as presented in Definition 3.1.

The elimination of states from any graph corresponding to the index-permutation symbols is in fact the same as eliminating the corresponding variables from the spectral null equation (1). The elimination of the variables is performed in such a way that the spectral null equation is always satisfied. This leads to the basic idea of eliminating an equivalent number of variable equal to N as a total number from different groupings in the spectral null equation. This is true when we eliminate only one variable from each grouping. In the case when we eliminate t variables with $1 < t < z$ from each grouping, we have a total number of eliminated variables of tN.

$\frac{1}{2}\mathcal{C}_b(8,2) =$ (19)

N bits		N bits			
0	0	0	0	0 0 0 0	$\mathcal{C}_b(4,2)$ (rows 1–6); $\mathcal{C}_b(6,2)$ (rows 1–20)
0	0	0	0	0 0 1 1	
0	0	0	0	0 1 1 0	
0	0	0	0	1 0 0 1	
0	0	0	0	1 1 0 0	
0	0	0	0	1 1 1 1	
0	0	0	1	0 0 1 0	
0	0	0	1	1 0 0 0	
0	0	0	1	1 0 1 1	
0	0	0	1	1 1 1 0	
0	0	1	0	0 0 0 1	
0	0	1	0	0 1 0 0	
0	0	1	0	0 1 1 1	
0	0	1	0	1 1 0 1	
0	0	1	1	0 0 0 0	$\mathcal{C}_b(4,2)$ (rows 15–20)
0	0	1	1	0 0 1 1	
0	0	1	1	0 1 1 0	
0	0	1	1	1 0 0 1	
0	0	1	1	1 1 0 0	
0	0	1	1	1 1 1 1	
0	1	0	0	0 0 1 0	
0	1	0	0	1 0 0 0	
0	1	0	0	1 0 1 1	
0	1	0	0	1 1 1 0	
0	1	0	1	1 0 1 0	
0	1	1	0	0 0 0 0	
0	1	1	0	0 0 1 1	
0	1	1	0	0 1 1 0	
0	1	1	0	1 0 0 1	
0	1	1	0	1 1 0 0	
0	1	1	0	1 1 1 1	
0	1	1	1	0 0 1 0	
0	1	1	1	1 0 0 0	
0	1	1	1	1 0 1 1	
0	1	1	1	1 1 1 0	

Example 5.18. *We construct the code for the case of* $M = 8$*, with* $N = 2$ *and* $z = 4$*, which is represented by the codebook* $\mathcal{C}_b(8,2)$ *in* (19) *(we present only the half of the codebook because of space*

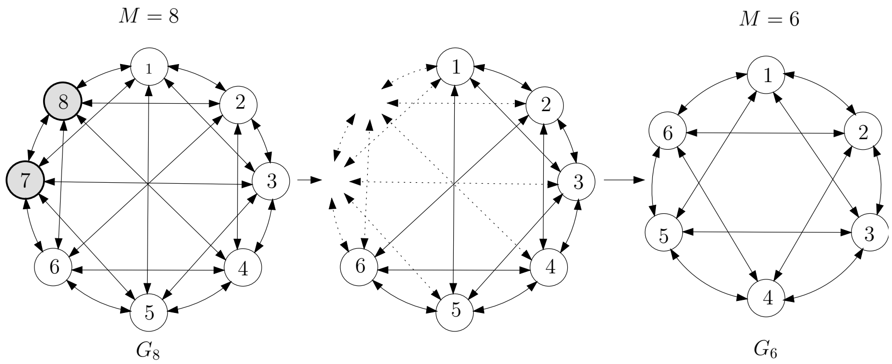


Fig. 11. Subgraph design from $M = 8$ to $M = 6$ with $N = 2$

limitation in the page) and which is designed from the spectral null equation presented as follows:

$$\overbrace{\overbrace{y_1 + y_3 + y_5 + y_7}^{z=4} = \overbrace{y_2 + y_4 + y_6 + y_8}^{z=4}}^{N=2}. \tag{20}$$

The corresponding graph for $\mathcal{C}_b(8,2)$ is G_8 as presented in Fig. 11.

From the spectral null equation (20) *we eliminate the variables y_7 and y_8 using the addition property. Thus we get,*

$$\overbrace{\overbrace{y_1 + y_3 + y_5}^{z=3} = \overbrace{y_2 + y_4 + y_6}^{z=3}}^{N=2}. \tag{21}$$

This resultant equation is the spectral null equation for the case of $M = 6$ with $N = 2$ and the corresponding codebook is denoted by $\mathcal{C}_b(6,2)$. Fig. 11 depicts the elimination of the states from a graph theory perspective.

Based on the same approach, we eliminate the variables y_5 and y_6 from the equation (21)*. The resultant spectral equation for the case of $M = 4$, with $N = 2$ and $z = 2$ is presented as follows:*

$$\overbrace{\overbrace{y_1 + y_3}^{z=2} = \overbrace{y_2 + y_4}^{z=2}}^{N=2}. \tag{22}$$

The code generated from the spectral null equation (22) *is denoted by the codebook $\mathcal{C}_b(4,2)$ as depicted in* (19)*. The corresponding graph for $\mathcal{C}_b(4,2)$ is G_4 as presented in Fig. 12.*

It is clear that from the codebook presented in (19)*, we have $\mathcal{C}_b(4,2) \subset \mathcal{C}_b(6,2) \subset \mathcal{C}_b(8,2)$ in terms of the existence of elements from the codebooks $\mathcal{C}_b(4,2)$ and $\mathcal{C}_b(6,2)$ in the codebook $\mathcal{C}_b(8,2)$, which is the same as for the subgraps where we have $G_4 \subset G_6 \subset G_8$.*

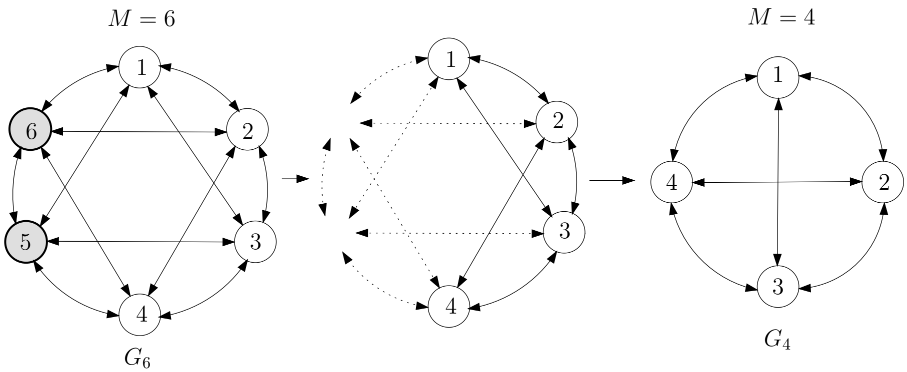


Fig. 12. Subgraph design from $M = 6$ to $M = 4$ with $N = 2$

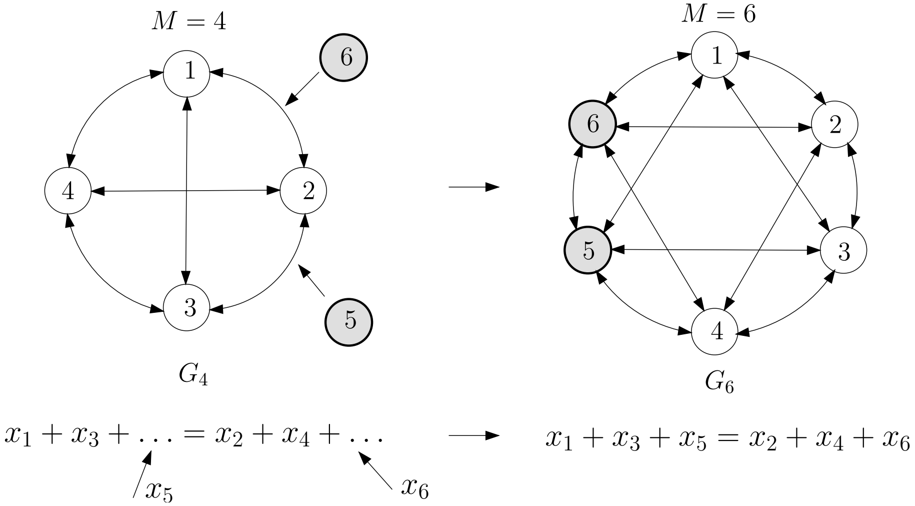


Fig. 13. Supergraph design: From $M = 4$ to $M = 6$ with $N = 2$

5.2.2 Supergraph theory

The concept of supergraphs is totally opposite to what was introduced with the subgraphs. Although this concept is not treated in graph theory because of its complexity and the conditions that we should have to add vertices to any graph. This problem is already solved in the design of spectral null codes since we are dealing with spectral null equations where it is easy to add variables in all groupings in such a way the spectral null equations are satisfied. Thus it results in the addition of the corresponding states of the symbols in the corresponding permutation equation.

Definition 5.19. *A spectral null preserving supergraph is an extension of a graph with a multiple of N states, which always keeps the spectral null equation satisfied.*

Fig. 13 presents the mechanism of the addition of states to an existing graph. The example of a graph of six states, which is related to the case of $M = 6$, is actually an extension of the graph of four states which corresponds to the case of $M = 4$. An addition of a state corresponds to the addition of its corresponding variable in a way to keep the equation (1) satisfied.

6. Conclusion

Spectral shaping technique that design codes with certain power spectral density properties is used to construct codes called spectral null codes that can generate nulls at rational submultiples of the symbol frequency. These codes have great importance in certain applications like in the case of transmission systems employing pilot tones for synchronization and that of track-following servos in digital recording. these codes are not confined to magnetic recorders but they ware taken further to their utilization in write-once recording systems.

In this investigation we have shown how the use of graphs can give a new insight into the analysis and understanding the structure of the spectral null codes, where with incisive observations to spectral null codebooks, we could derive important properties that can be useful in the field of digital communications.

The relationship between the spectral null equations for our designed codes and the permutation sequences corresponding to the indices of the variables in those equations have lead to a very important derivation of certain properties based on graph theory approach.

The properties that we have presented could potentially lead to the discovery of other interesting properties for specific applications like those that we have investigated in [13].

The use of certain graph theory properties helped in understanding certain properties of spectral null codes. The introduction of the index-permutation sequences and the use of the concept of distances gave us an idea about the structure and the design conditions of spectral null codes.

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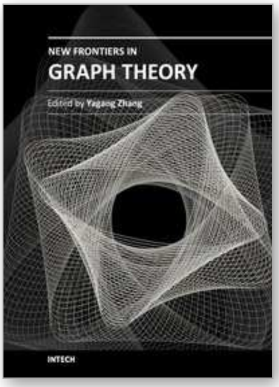

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Nowadays, graph theory is an important analysis tool in mathematics and computer science. Because of the inherent simplicity of graph theory, it can be used to model many different physical and abstract systems such as transportation and communication networks, models for business administration, political science, and psychology and so on. The purpose of this book is not only to present the latest state and development tendencies of graph theory, but to bring the reader far enough along the way to enable him to embark on the research problems of his own. Taking into account the large amount of knowledge about graph theory and practice presented in the book, it has two major parts: theoretical researches and applications. The book is also intended for both graduate and postgraduate students in fields such as mathematics, computer science, system sciences, biology, engineering, cybernetics, and social sciences, and as a reference for software professionals and practitioners.