

## A GRAPH-THEORETIC APPROACH TO THE METHOD OF GLOBAL LYAPUNOV FUNCTIONS

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ABSTRACT. A class of global Lyapunov functions is revisited and used to resolve a long-standing open problem on the uniqueness and global stability of the endemic equilibrium of a class of multi-group models in mathematical epidemiology. We show how the group structure of the models, as manifested in the derivatives of the Lyapunov function, can be completely described using graph theory.

### 1. INTRODUCTION

Let a function  $x \mapsto f(x) \in \mathbb{R}^N$  be defined in an open region  $D \subset \mathbb{R}^N$  such that the differential equation

$$(1.1) \quad x' = f(x), \quad x \in D,$$

has a unique solution  $x(t, x_0)$  for each initial point  $x_0 \in D$ . An equilibrium  $\bar{x} \in D$  is *globally stable* in  $D$  if it is locally stable and  $x(t, x_0) \rightarrow \bar{x}$  as  $t \rightarrow \infty$  for all  $x_0 \in D$ . A function  $x \mapsto V(x) \in \mathbb{R}^1$  is said to be a global Lyapunov function of (1.1) for  $D$  if

$$\dot{V}(x) = \text{grad}V(x) \cdot f(x) \leq 0, \quad x \in D.$$

A classical theorem of Lyapunov states that if (1)  $V(x) \geq 0$  for  $x \in D$  and  $V(x) = 0$  iff  $x = \bar{x}$ , and (2)  $\dot{V}(x) \leq 0$  for  $x \in D$  and  $\dot{V}(x) = 0$  iff  $x = \bar{x}$ , then  $\bar{x}$  is globally stable in  $D$ . Lyapunov's theorem was further extended as the LaSalle Invariance Principle [12]: if  $V(x)$  is a global Lyapunov function in  $D$ , then all omega limit sets of (1.1) are contained in the maximal compact invariant subset  $K$  of  $G = \{x \in D : \dot{V}(x) = 0\}$ . In particular, if  $D$  is positively invariant and  $K = \{\bar{x}\}$ , then  $x(t, x_0) \rightarrow \bar{x}$  as  $t \rightarrow \infty$ . We note that this also implies the local stability of  $\bar{x}$ , since otherwise  $K$  would contain a non-constant full orbit.

In the literature of ecological models, the region  $D$  is typically in the positive cone of  $\mathbb{R}^N$ , and a class of Lyapunov functions

$$(1.2) \quad V(x) = \sum_{k=1}^N a_k (x_k - \bar{x}_k \ln x_k)$$

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has proven useful; see e.g. [3, 9] and the references therein. Recently, this form of Lyapunov functions was applied to several single-group epidemic models and used to prove the global stability of a unique endemic equilibrium [4, 10]. In the present paper, to further demonstrate the applicability of this form of Lyapunov functions, we apply them to a class of  $n$ -group ( $n \geq 2$ ) epidemic models of SEIR type with bilinear incidence, described by the following system of equations:

$$(1.3) \quad \begin{cases} S'_k = \Lambda_k - d_k^S S_k - \sum_{j=1}^n \beta_{kj} S_k I_j, \\ E'_k = \sum_{j=1}^n \beta_{kj} S_k I_j - (d_k^E + \epsilon_k) E_k, \\ I'_k = \epsilon_k E_k - (d_k^I + \gamma_k) I_k. \end{cases} \quad k = 1, 2, \dots, n,$$

Here  $S_k, E_k,$  and  $I_k$  denote the population in the  $k$ -th group that are susceptible to the disease, infected but non-infectious, and infectious, respectively. The parameters in the model are non-negative constants and summarized in the following list:

- $\beta_{kj}$  : transmission coefficient between compartments  $S_k$  and  $I_j,$
- $d_k^S, d_k^E, d_k^I$  : natural death rates of  $S, E, I$  compartments in the  $k$ -th group, respectively,
- $\Lambda_k$  : influx of individuals into the  $k$ -th group,
- $\epsilon_k$  : rate of becoming infectious after latent period in the  $k$ -th group,
- $\gamma_k$  : recovery rate of infectious individuals in the  $k$ -th group.

In particular,  $\beta_{kj} \geq 0,$  and  $\beta_{kj} = 0$  if there is no transmission of the disease between compartments  $S_k$  and  $I_j.$  The matrix  $B = (\beta_{kj})$  encodes the patterns of contact and transmission among groups that are built into the model. Associated to  $B,$  one can construct a directed graph  $L = G(B)$  whose vertex  $k$  represents the  $k$ -th group,  $k = 1, \dots, n.$  A directed edge exists from vertex  $k$  to vertex  $j$  if and only if  $\beta_{kj} > 0.$  Throughout the paper, we assume that  $B$  is irreducible. This is equivalent to  $G(B)$  being strongly connected (see Section 2). Biologically, this is the same as assuming that any two groups  $k$  and  $j$  have a direct or indirect route of transmission. More specifically, individuals in  $I_j$  can infect ones in  $S_k$  directly or indirectly. We also assume that  $\epsilon_k > 0$  and  $d_k^* > 0,$  where  $d_k^* = \min\{d_k^S, d_k^E, d_k^I + \gamma_k\}.$  For more detailed discussions of the model and interpretations of parameters, we refer the reader to [16].

For each  $k,$  adding the three equations in (1.3) gives  $(S_k + E_k + I_k)' \leq \Lambda_k - d_k^*(S_k + E_k + I_k).$  Hence  $\limsup_{t \rightarrow \infty} (S_k + E_k + I_k) \leq \Lambda_k / d_k^*.$  Similarly, from the  $S_k$  equation we obtain  $\limsup_{t \rightarrow \infty} S_k \leq \Lambda_k / d_k^S.$  Therefore, omega limit sets of system (1.3) are contained in the following bounded region in the non-negative cone of  $\mathbb{R}^{3n}:$

$$(1.4) \quad \Gamma = \left\{ (S_1, E_1, I_1, \dots, S_n, E_n, I_n) \in \mathbb{R}_+^{3n} \mid S_k \leq \frac{\Lambda_k}{d_k^S}, S_k + E_k + I_k \leq \frac{\Lambda_k}{d_k^*}, 1 \leq k \leq n \right\}.$$

It can be verified that region  $\Gamma$  is positively invariant. System (1.3) always has the disease-free equilibrium  $P_0 = (S_1^0, 0, 0, \dots, S_n^0, 0, 0)$  on the boundary of  $\Gamma,$  where  $S_k^0 = \Lambda_k / d_k^S.$  An equilibrium  $P^* = (S_1^*, E_1^*, I_1^*, \dots, S_n^*, E_n^*, I_n^*)$  in the interior  $\overset{\circ}{\Gamma}$  of  $\Gamma$  is called an *endemic equilibrium,* where  $S_k^*, E_k^*, I_k^* > 0$  satisfy the equilibrium

equations

$$(1.5) \quad \Lambda_k = d_k^S S_k^* + \sum_{j=1}^n \beta_{kj} S_k^* I_j^*,$$

$$(1.6) \quad (d_k^E + \epsilon_k) E_k^* = \sum_{j=1}^n \beta_{kj} S_k^* I_j^*, \quad \epsilon_k E_k^* = (d_k^I + \gamma_k) I_k^*,$$

and

$$(1.7) \quad \frac{(d_k^E + \epsilon_k)(d_k^I + \gamma_k)}{\epsilon_k} I_k^* = \sum_{j=1}^n \beta_{kj} S_k^* I_j^*,$$

which follows from (1.6).

Let

$$(1.8) \quad R_0 = \rho(M_0)$$

denote the spectral radius of the matrix

$$M_0 = \left( \frac{\beta_{kj} \epsilon_k S_k^0}{(d_k^E + \epsilon_k)(d_k^I + \gamma_k)} \right)_{1 \leq k, j \leq n}.$$

The parameter  $R_0$  is referred to as the basic reproduction number. Its biological significance is that if  $R_0 < 1$  the disease dies out while if  $R_0 > 1$  the disease becomes endemic ([16, 17]). A long-standing open question in mathematical epidemiology is whether a multi-group model such as system (1.3) has a unique endemic equilibrium  $P^*$  when  $R_0 > 1$ , and whether  $P^*$  is globally stable when it is unique [16]. We prove the following theorem, which settles this open problem for system (1.3), as well as other multi-group models that can be converted to the same form.

**Theorem 1.1.** *Assume that  $B = (\beta_{kj})$  is irreducible. If  $R_0 > 1$ , then system (1.3) has a unique endemic equilibrium  $P^*$ , and  $P^*$  is globally stable in  $\overset{\circ}{\Gamma}$ .*

One of the earliest results on multi-group models is by Lajmanovich and Yorke [11] on a class of  $n$ -group SIS models for gonorrhoea. The global stability of the unique endemic equilibrium is proved using a quadratic global Lyapunov function. Global stability results also exist for other types of multi-group models; see e.g., [1, 5, 6, 13, 15]. The most recent result is Lin and So [13] for a class of SIRS models, in which the endemic equilibrium is shown to be globally stable if all  $\beta_{kj}$ ,  $k \neq j$ , are small or if all  $\gamma_k$ ,  $1 \leq k \leq n$ , are small. Results in the opposite direction also exist in the literature. For a class of  $n$ -group SIR models with proportionate incidence, uniqueness of endemic equilibria may not hold when  $R_0 > 1$  [7, 16].

Our proof of Theorem 1.1 uses the form of global Lyapunov functions given in (1.2). Compared to results in [4, 10], the group structure in system (1.3) greatly increases the complexity exhibited in the derivatives of the Lyapunov function  $V$ . The key to our analysis is a complete description of the patterns exhibited in the derivative  $\dot{V}$  using graph theory. As structured models are used to describe more complicated biological problems, we expect that this class of Lyapunov functions, together with our graph theoretical analysis, will have much wider applicability.

2. PRELIMINARIES

A non-negative matrix  $E$  is *reducible* if, for some permutation matrix  $Q$ ,

$$QEQT = \begin{bmatrix} E_1 & 0 \\ E_2 & E_3 \end{bmatrix},$$

and  $E_1, E_3$  are square matrices. Otherwise,  $E$  is *irreducible*. Irreducibility of  $E$  can be checked using the associated directed graphs. The *directed graph*  $G(E)$  associated with  $E = (e_{kj})$  has vertices  $\{1, 2, \dots, n\}$  with a directed arc  $(k, j)$  from  $k$  to  $j$  iff  $e_{kj} \neq 0$ . It is *strongly connected* if any two distinct vertices are joined by an oriented path. Matrix  $E$  is irreducible if and only if  $G(E)$  is strongly connected.

A *tree* is a connected graph with no cycles. A subtree  $T$  of a graph  $G$  is said to be *spanning* if  $T$  contains all the vertices of  $G$ . A *directed tree* is a tree in which each edge has been replaced by an arc directed one way or the other. A directed tree is said to be *rooted* at a vertex, called the root, if every arc is oriented in the direction towards the root. An *oriented cycle* in a directed graph is a simple closed oriented path. A *unicyclic graph* is a directed graph consisting of a collection of disjoint rooted directed trees whose roots are on an oriented cycle. We refer the reader to [8, 14] for more details.

When a directed arc from the root to any non-root vertex is added to a directed rooted tree, we obtain a unicyclic graph. See Figure 1. When we perform this operation in all possible ways to all possible directed rooted trees on a given set of vertices, we obtain all possible unicyclic graphs on these vertices with each unicyclic graph counted separately for each cyclic arc it contains. This observation will play a crucial role in the proof of Theorem 1.1.

Now consider the linear system

$$(2.1) \quad \bar{B}v = 0,$$

where

$$(2.2) \quad \bar{B} = \begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix},$$

and  $\bar{\beta}_{kj} \geq 0, 1 \leq k, j \leq n$ . Let  $L = G(B)$  denote the directed graph associated with matrix  $B$  (and  $(\bar{\beta}_{kj})$ ), and  $C_{jk}$  denote the cofactor of the  $(j, k)$  entry of  $\bar{B}$ .

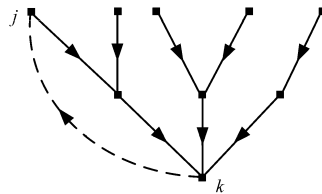


FIGURE 1. A unicyclic graph formed by adding a directed arc  $(k, j)$  to a directed tree rooted at  $k$ .

**Lemma 2.1.** *Assume  $(\bar{\beta}_{kj})_{n \times n}$  is irreducible and  $n \geq 2$ . Then the following results hold:*

- (1) *The solution space of system (2.1) has dimension 1, with a basis  $(v_1, v_2, \dots, v_n) = (C_{11}, C_{22}, \dots, C_{nn})$ .*
- (2) *For  $1 \leq k \leq n$ ,*

$$(2.3) \quad C_{kk} = \sum_{T \in \mathbb{T}_k} \prod_{(r,m) \in E(T)} \bar{\beta}_{rm} > 0,$$

where  $\mathbb{T}_k$  is the set of all directed spanning subtrees of  $L$  that are rooted at vertex  $k$ , and  $E(T)$  denotes the set of directed arcs in a directed tree  $T$ .

*Proof.* Since the sum of each column in  $\bar{B}$  equals zero, we have  $C_{jk} = C_{lk}$ ,  $1 \leq j, k, l \leq n$ . From  $\det(\bar{B}) = 0$ , we know that  $(C_{11}, C_{12}, \dots, C_{1n})$ , and thus  $(C_{11}, C_{22}, \dots, C_{nn})$ , is a solution of (2.1). By the Matrix-Tree Theorem ([14, Theorem 5.5] or [8, page 378]), the expansion of each  $(n - 1)$  principal minor of  $\bar{B}$  can be expressed as

$$C_{kk} = \sum_{T \in \mathbb{T}_k} \prod_{(r,m) \in E(T)} \bar{\beta}_{rm}.$$

Since  $(\bar{\beta}_{kj})$  is irreducible, its associated directed graph  $L$  is strongly connected. For each  $k$ , at least one term in  $\sum_{T \in \mathbb{T}_k} \prod_{(r,m) \in E(T)} \bar{\beta}_{rm}$  is positive, and thus  $C_{kk} > 0$ . This implies  $\text{rank}(\bar{B}) = n - 1$ , and the solution space of (2.1) has dimension 1.  $\square$

As an illustration of (2.3), let  $n = 3$  and  $\mathbb{T}_1 = \{T_1^1, T_1^2, T_1^3\}$  be the set of all directed trees with vertices  $\{1, 2, 3\}$  rooted at vertex 1. See Figure 2. Then,  $E(T_1^1) = \{(3, 2), (2, 1)\}$ ,  $E(T_1^2) = \{(2, 1), (3, 1)\}$ ,  $E(T_1^3) = \{(2, 3), (3, 1)\}$ . Therefore,

$$\begin{aligned} C_{11} &= \det \begin{vmatrix} \bar{\beta}_{21} + \bar{\beta}_{23} & -\bar{\beta}_{32} \\ -\bar{\beta}_{23} & \bar{\beta}_{31} + \bar{\beta}_{32} \end{vmatrix} \\ &= \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31} \\ &= \sum_{T_1^i \in \mathbb{T}_1} \prod_{(r,m) \in E(T_1^i)} \bar{\beta}_{rm}. \end{aligned}$$

Let  $R_0$  be defined in (1.8). The following result for system (1.3) is known in the literature, at least for some special classes of system (1.3), and its proof is standard (see [6, 15, 17]).

**Proposition 2.2.** *Assume  $B = (\beta_{kj})$  is irreducible. Then the following statements hold:*

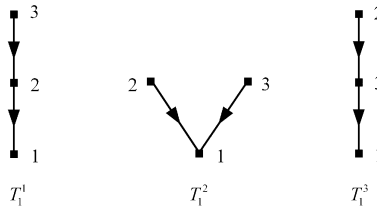


FIGURE 2. Three-vertex directed trees that are rooted at 1.

- (1) If  $R_0 \leq 1$ , then  $P_0$  is the unique equilibrium and it is globally stable in  $\Gamma$ .
- (2) If  $R_0 > 1$ , then  $P_0$  is unstable and system (1.3) is uniformly persistent in  $\overset{\circ}{\Gamma}$ .

Uniform persistence of (1.3), together with uniform boundedness of solutions in  $\overset{\circ}{\Gamma}$ , implies the existence of an equilibrium of (1.3) in  $\overset{\circ}{\Gamma}$  ([2, Theorem 2.8.6]).

**Corollary 2.3.** *Assume  $B = (\beta_{kj})$  is irreducible. If  $R_0 > 1$ , then (1.3) has at least one endemic equilibrium.*

### 3. PROOF OF THEOREM 1.1

Let  $P^* = (S_1^*, E_1^*, I_1^*, \dots, S_n^*, E_n^*, I_n^*), S_k^*, E_k^*, I_k^* > 0$  for  $1 \leq k \leq n$ , denote an endemic equilibrium whose existence is established in Corollary 2.3. We prove that  $P^*$  is globally stable when  $R_0 > 1$ . In particular, this implies that the endemic equilibrium is unique in  $\overset{\circ}{\Gamma}$  when it exists. Choose

$$(3.1) \quad \bar{\beta}_{kj} = \beta_{kj} S_k^* I_j^*, \quad 1 \leq k, j \leq n, \quad n \geq 2,$$

and matrix  $\bar{B}$  as given in (2.2). Since  $B = (\beta_{kj})$  is irreducible, we know matrix  $(\bar{\beta}_{kj})$  is irreducible. Let  $\{v_1, \dots, v_n\}, v_k > 0$ , be a basis for system (2.1),  $\bar{B}v = 0$ , as described in Lemma 2.1. Set

$$(3.2) \quad V = \sum_{k=1}^n v_k \left[ (S_k - S_k^* \ln S_k) + (E_k - E_k^* \ln E_k) + \frac{d_k^E + \epsilon_k}{\epsilon_k} (I_k - I_k^* \ln I_k) \right].$$

Differentiating  $V$  along solutions to (1.3) and using (1.5), (1.6), (1.7), we obtain

$$(3.3) \quad \begin{aligned} \dot{V} &= \sum_{k=1}^n v_k \left[ \Lambda_k - d_k^S S_k - \sum_{j=1}^n \beta_{kj} S_k I_j - \Lambda_k \frac{S_k^*}{S_k} + d_k^S S_k^* + \sum_{j=1}^n \beta_{kj} S_k^* I_j \right. \\ &\quad + \sum_{j=1}^n \beta_{kj} S_k I_j - (d_k^E + \epsilon_k) E_k - \sum_{j=1}^n \beta_{kj} \frac{E_k^* S_k I_j}{E_k} + (d_k^E + \epsilon_k) E_k^* \\ &\quad + (d_k^E + \epsilon_k) E_k - \frac{(d_k^E + \epsilon_k)(d_k^I + \gamma_k)}{\epsilon_k} I_k - (d_k^E + \epsilon_k) \frac{I_k^* E_k}{I_k} \\ &\quad \left. + \frac{(d_k^E + \epsilon_k)(d_k^I + \gamma_k)}{\epsilon_k} I_k^* \right] \\ &= \sum_{k=1}^n v_k \left[ d_k^S S_k^* \left( 2 - \frac{S_k}{S_k^*} - \frac{S_k}{S_k^*} \right) + \left( \sum_{j=1}^n \beta_{kj} S_k^* I_j - \frac{(d_k^E + \epsilon_k)(d_k^I + \gamma_k)}{\epsilon_k} I_k \right) \right. \\ &\quad \left. + \left( 3 \sum_{j=1}^n \beta_{kj} S_k^* I_j^* - \sum_{j=1}^n \beta_{kj} I_j^* \frac{(S_k^*)^2}{S_k} - \sum_{j=1}^n \beta_{kj} S_k I_j \frac{E_k^*}{E_k} - (d_k^E + \epsilon_k) E_k \frac{I_k^*}{I_k} \right) \right] \\ &\leq \sum_{k=1}^n v_k \left[ \left( \sum_{j=1}^n \beta_{kj} S_k^* I_j - \frac{(d_k^E + \epsilon_k)(d_k^I + \gamma_k)}{\epsilon_k} I_k \right) \right. \\ &\quad \left. + \left( 3 \sum_{j=1}^n \beta_{kj} S_k^* I_j^* - \sum_{j=1}^n \beta_{kj} I_j^* \frac{(S_k^*)^2}{S_k} - \sum_{j=1}^n \beta_{kj} S_k I_j \frac{E_k^*}{E_k} - (d_k^E + \epsilon_k) E_k \frac{I_k^*}{I_k} \right) \right], \end{aligned}$$

since  $S_k^*/S_k + S_k/S_k^* \geq 2$ , with equality holding if and only if  $S_k = S_k^*$ .

To simplify this we observe that since  $\bar{B}v = 0$ , it follows readily that

$$\sum_{j=1}^n \bar{\beta}_{jk} v_j = \sum_{i=1}^n \bar{\beta}_{ki} v_k$$

or, appealing to (3.1) and (1.7), that

$$\sum_{j=1}^n \beta_{jk} S_j^* I_k^* v_j = \sum_{i=1}^n \beta_{ki} S_k^* I_i^* v_k = \frac{(d_k^E + \epsilon_k)(d_k^I + \gamma_k)}{\epsilon_k} I_k^* v_k$$

for  $1 \leq k \leq n$ . This implies that

$$\sum_{k=1}^n v_k \sum_{j=1}^n \beta_{kj} S_k^* I_j = \sum_{k=1}^n I_k \sum_{j=1}^n \beta_{jk} S_j^* v_j = \sum_{k=1}^n \frac{(d_k^E + \epsilon_k)(d_k^I + \gamma_k)}{\epsilon_k} I_k v_k.$$

Consequently, the contributions from the first two double sums in the right-hand-side of (3.3) cancel, so it follows from (3.3), appealing again to relation (3.1), that

$$\begin{aligned} \dot{V} &\leq \sum_{k=1}^n v_k \left( 3 \sum_{j=1}^n \bar{\beta}_{kj} - \sum_{j=1}^n \bar{\beta}_{kj} \frac{S_k^*}{S_k} - \sum_{j=1}^n \bar{\beta}_{kj} \frac{S_k I_j E_k^*}{S_k^* I_j^* E_k} - (d_k^E + \epsilon_k) E_k \frac{I_k^*}{I_k} \right) \\ &= \sum_{k,j=1}^n v_k \bar{\beta}_{kj} \left( 3 - \frac{S_k^*}{S_k} - \frac{S_k I_j E_k^*}{S_k^* I_j^* E_k} - \frac{E_k I_k^*}{E_k^* I_k} \right) =: H_n(S_1, E_1, I_1, \dots, S_n, E_n, I_n). \end{aligned} \tag{3.4}$$

In the following, we show  $H_n \leq 0$  for all  $(S_1, E_1, I_1, \dots, S_n, E_n, I_n) \in \overset{\circ}{\Gamma}$ .

The products of factors  $\bar{\beta}_{rm}$  in the expression for  $v_k$  given in (2.3) can be interpreted as weights of directed spanning subtrees  $T$  of  $L$  that are rooted at vertex  $k$ . Consequently, each product in  $v_k \bar{\beta}_{kj}$  can be interpreted as the weight of a unicyclic graph  $Q$  obtained from such a tree  $T$  by adding an arc  $(k, j)$  directed from the root vertex  $k$  to vertex  $j$ ; see Figure 1. Such a product can be written as

$$\prod_{(r,m) \in E(Q)} \bar{\beta}_{rm},$$

where the product is over the arcs  $(r, m)$  in  $Q$ , and the same product is obtained for each arc  $(k, j)$  in the unique cycle  $CQ$  of  $Q$ . It is not difficult to see that the double sum in  $H_n$  can be interpreted as a sum over all the arcs in the cycles of all the unicyclic subgraphs  $Q$  of  $L$ . Hence  $H_n$  can be rewritten as

$$H_n = \sum_Q H_{n,Q},$$

where

$$\begin{aligned} H_{n,Q} &= \prod_{(r,m) \in E(Q)} \bar{\beta}_{rm} \cdot \sum_{(k,j) \in E(CQ)} \left( 3 - \frac{S_k^*}{S_k} - \frac{S_k I_j E_k^*}{S_k^* I_j^* E_k} - \frac{E_k I_k^*}{E_k^* I_k} \right) \\ &= \prod_{(r,m) \in E(Q)} \bar{\beta}_{rm} \cdot \left( 3l - \sum_{(k,j) \in E(CQ)} \left( \frac{S_k^*}{S_k} - \frac{S_k I_j E_k^*}{S_k^* I_j^* E_k} - \frac{E_k I_k^*}{E_k^* I_k} \right) \right), \end{aligned}$$

and where  $l = l(Q)$  denotes the number of arcs in  $CQ$ .

It is not difficult to see that

$$\prod_{(k,j) \in E(CQ)} \frac{S_k^*}{S_k} \cdot \frac{S_k I_j E_k^*}{S_k^* I_j^* E_k} \cdot \frac{E_k I_k^*}{E_k^* I_k} = \prod_{(k,j) \in E(CQ)} \frac{I_j I_k^*}{I_j^* I_k} = 1$$

for each unicyclic graph  $Q$ . Therefore,

$$\sum_{(k,j) \in E(CQ)} \left( \frac{S_k^*}{S_k} + \frac{S_k I_j E_k^*}{S_k^* I_j^* E_k} + \frac{E_k I_k^*}{E_k^* I_k} \right) \geq 3l,$$

and thus  $H_{n,Q} \leq 0$  for each  $Q$ , and

$$(3.5) \quad H_{n,Q} = 0 \iff \frac{S_k^*}{S_k} = \frac{S_k I_j E_k^*}{S_k^* I_j^* E_k} = \frac{E_k I_k^*}{E_k^* I_k}, \quad (k, j) \in E(CQ).$$

Therefore,

$$H_n = \sum_Q H_{n,Q} \leq 0.$$

We have shown that  $\dot{V} \leq 0$  for all  $(S_1, E_1, I_1, \dots, S_n, E_n, I_n) \in \overset{\circ}{\Gamma}$ , and that  $\dot{V} = 0$  iff  $S_k = S_k^*$  and  $H_n = 0$ . We claim that if  $S_k = S_k^*, 1 \leq k \leq n$ , then

$$(3.6) \quad H_n = 0 \iff E_k = aE_k^*, \quad I_k = aI_k^*, \quad k = 1, 2, \dots, n,$$

where  $a$  is an arbitrary positive number. We first note that it follows from (3.5) that

$$(3.7) \quad \frac{I_k}{I_k^*} = \frac{E_k}{E_k^*} = \frac{I_j}{I_j^*}$$

for every arc  $(k, j)$  that belongs to the cycle of some unicyclic subgraph  $Q$  of  $L$ . But  $L$  is strongly connected, so, clearly, every arc  $(k, j)$  of  $L$  belongs to the cycle of at least one such subgraph  $Q$  and, hence, (3.7) holds for every arc  $(k, j)$ . Now let  $p$  and  $q$  denote any two distinct vertices of the graph. Then, by strong connectivity, there exists an oriented path from  $p$  to  $q$ . If we apply (3.7) to the arcs of this path consecutively, we find that  $\frac{I_p}{I_p^*} = \frac{I_q}{I_q^*}$ ; then it follows from (3.7) that  $\frac{E_p}{E_p^*} = \frac{I_p}{I_p^*} = \frac{I_q}{I_q^*} = \frac{E_q}{E_q^*}$  as well. Since  $p$  and  $q$  are arbitrary, this suffices to prove (3.6).

From (3.3) and (3.6), we know that  $\dot{V} = 0$  iff  $S_k = S_k^*, E_k = aE_k^*, I_k = aI_k^*, k = 1, 2, \dots, n$ . Substituting  $S_k = S_k^*, E_k = aE_k^*$ , and  $I_k = aI_k^*$  into the first equation of system (1.3), we obtain

$$(3.8) \quad 0 = \Lambda_k - d_k^S S_k^* - a \sum_{j=1}^n \beta_{kj} S_k^* I_j^*.$$

Since the right-hand-side of (3.8) is strictly decreasing in  $a$ , we know, by (1.5), that (3.8) holds iff  $a = 1$ , namely at  $P^*$ . Therefore, the only compact invariant subset of the set where  $\dot{V} = 0$  is the singleton  $\{P^*\}$ . By the LaSalle Invariance Principle,  $P^*$  is globally stable in  $\overset{\circ}{\Gamma}$  if  $R_0 > 1$ . This completes the proof of Theorem 1.1.



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