# A group action on derangements* 

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#### Abstract

In this paper we define a cyclic analogue of the MFS-action on derangements, and give a combinatorial interpretation of the expansion of the $n$-th derangement polynomial on the basis $\left\{q^{k}(1+q)^{n-1-2 k}\right\}, k=0,1, \ldots,\lfloor(n-1) / 2\rfloor$.


Keywords: derangement polynomials; group action

## 1 Introduction

Let $[n]$ denote the set $\{1,2, \ldots, n\}$ and let $\mathfrak{S}_{n}$ denote the set of all permutations of $[n]$. For $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in \mathfrak{S}_{n}$ and $x \in[n]$, we write $\pi$ as the concatenation $\pi=w_{1} w_{2} x w_{3} w_{4}$, where $w_{2}$ is the maximal contiguous subword immediately to the left of $x$ whose letters are all smaller than $x$, and $w_{3}$ is the maximal contiguous subword immediately to the right of $x$ whose letters are all smaller than $x$. Following Foata and Strehl [4, 5], this concatenation is called the $x$-factorization of $\pi$. For example, let $\pi=714358296$ and $x=5$. Then $w_{1}=7, w_{2}=143, w_{3}=\emptyset$ and $w_{4}=8296$.

Foata and Strehl [4,5] defined an involution acting on $\mathfrak{S}_{n}$ by $\varphi_{x}(\pi)=w_{1} w_{3} x w_{2} w_{4}$ for $x \in[n]$ and $\varphi_{S}(\pi)=\prod_{x \in S} \varphi_{x}(\pi)$ for $S \subseteq[n]$. The group $\mathbb{Z}_{2}^{n}$ acts on $\mathfrak{S}_{n}$ via the functions $\varphi_{S}$ for $S \subseteq[n]$.

Definition 1. Let $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in \mathfrak{S}_{n}$ and denote $\pi_{0}=\pi_{n+1}=n+1$. The entry $\pi_{k}$ is called a valley if $\pi_{k-1}>\pi_{k}<\pi_{k+1}$; a peak if $\pi_{k-1}<\pi_{k}>\pi_{k+1}$; a double ascent if $\pi_{k-1}<\pi_{k}<\pi_{k+1}$; a double descent if $\pi_{k-1}>\pi_{k}>\pi_{k+1}$.

[^0]Let $\operatorname{Val}(\pi), \operatorname{Peak}(\pi), \operatorname{Dasc}(\pi), \operatorname{Des}(\pi)$ denote the set of all valley, peaks, double ascents and double descents of $\pi$, respectively. The corresponding cardinalities are $\operatorname{val}(\pi)$, $\operatorname{peak}(\pi), \operatorname{dasc}(\pi)$ and $\operatorname{ddes}(\pi)$, respectively. Shapiro et al. [6] modified the Foata-Strehl action in the following way. For $x \in[n]$, let

$$
\varphi_{x}^{\prime}(\pi)= \begin{cases}\varphi_{x}(\pi) & \text { if } x \text { is a double ascent or a double descent }  \tag{1}\\ \pi & \text { if } x \text { is a valley or a peak. }\end{cases}
$$

For any subset $S \subseteq[n]$, define $\varphi_{S}^{\prime}(\pi)=\prod_{x \in S} \varphi_{x}^{\prime}(\pi)$. From the definition, if $x$ is a double ascent (double descent, resp.) of $\pi$, then $x$ is a double descent (double ascent, resp.) of $\varphi_{x}^{\prime}(\pi)$. The group $\mathbb{Z}_{2}^{n}$ acts on $\mathfrak{S}_{n}$ via the functions $\varphi_{S}^{\prime}, S \in[n]$ and call this action the MFS-action.

By the theory of symmetric functions, Brenti [2] showed that derangement polynomials are symmetric and unimodal polynomials. Using the method of continued fractions, Shin and Zeng [7] gave a combinatorial interpretation for coefficients in the expansion of the $n$-th derangement polynomial on the basis $\left\{q^{k}(1+q)^{n-1-2 k}\right\}, k=0,1, \ldots,\lfloor(n-1) / 2\rfloor$. In this note, we define a cyclic analogous of the MFS-action on derangements and give a new proof for the result of Shin and Zeng.

## 2 Main results

Let $\pi \in \mathfrak{S}_{n}$. We say that $\pi$ is a derangement of $[n]$ if $\pi_{i} \neq i$ for all $i \in[n]$. Denote by $D_{n}$ the set of all derangements of $[n]$. An element $i \in[n]$ is an excedance of $\pi$ if $\pi_{i}>i$. Denote by $\operatorname{Exc}(\pi)$ the set of all excedances in $\pi$ and let $\operatorname{exc}(\pi)=|\operatorname{Exc}(\pi)|$. The $n$-derangement polynomial $D_{n}(q)$ is the generating function of statistic excedance over the set $D_{n}$, i.e.,

$$
\begin{equation*}
D_{n}(q)=\sum_{\pi \in D_{n}} q^{e x c(\pi)}=\sum_{j=1}^{n-1} d(n, j) q^{j} \tag{2}
\end{equation*}
$$

where $d(n, j)=\left|\left\{\pi \in D_{n}: \operatorname{exc}(\pi)=j\right\}\right|$.
Recall that a permutation $\pi \in \mathfrak{S}_{n}$ may be regarded as a disjoint union of its distinct cycles $C_{1}, C_{2}, \ldots, C_{k}$, written $\pi=C_{1} C_{2} \cdots C_{k}$. Let $c(\pi)$ denote the number of cycles of $\pi$. For a derangement $\pi$, each cycle contains at least two elements. The standard cycle representation of $\pi$ is defined by requiring that (i) each cycle is written with its largest element first, and (ii) the cycles are written in increasing order of their largest elements [8]. For example, the standard cycle representation of $\pi=456321 \in D_{6}$ is (52)(6143). Throughout the paper all permutations are written in standard cycle representation.

Definition 2 ([7]). Let $\pi \in \mathfrak{S}_{n}$. The entry $x=\pi_{i}(i \in[n])$ is called a cyclic valley if $i=\pi^{-1}(x)>x<\pi(x)$; a cyclic peak if $i=\pi^{-1}(x)<x>\pi(x)$; a cyclic double ascent if $i=\pi^{-1}(x)<x<\pi(x)$; a cyclic double descent if $i=\pi^{-1}(x)>x>\pi(x)$; a fixed point if $\pi(x)=x$.

Let $\operatorname{Cval}(\pi), \operatorname{Cpeak}(\pi), \operatorname{Cdasc}(\pi), \operatorname{Cddes}(\pi)$ and $\operatorname{Fix}(\pi)$ denote the set of all cyclic valley, cyclic peaks, cyclic double ascents, cyclic double descents and fixed points of $\pi$, respectively. The corresponding cardinalities are $\operatorname{cval}(\pi), \operatorname{cpeak}(\pi), \operatorname{cdasc}(\pi), \operatorname{cddes}(\pi)$ and $\operatorname{fix}(\pi)$, respectively. It is easy to see that the union of sets $\operatorname{Cval}(\pi), \operatorname{Cpeak}(\pi)$, $\operatorname{Cdasc}(\pi), \operatorname{Cddes}(\pi)$ and $\operatorname{Fix}(\pi)$ is $[n]$ for any $\pi \in \mathfrak{S}_{n}$. For a derangement $\pi$, the set $\operatorname{Fix}(\pi)$ is empty. The following proposition is immediate by Definition 2.

Proposition 3. Let $\pi=C_{1} C_{2} \cdots C_{k}$ be a permutation of $[n]$. Then

$$
\operatorname{Exc}(\pi)=C \operatorname{val}(\pi) \cup C d a s c(\pi)
$$

and

$$
\operatorname{exc}(\pi)=\operatorname{cval}(\pi)+\operatorname{cdasc}(\pi)
$$

Let $\pi=C_{1} C_{2} \cdots C_{k}$. Following Stanley [8], let $o(\pi)$ be the permutation obtained from $\pi$ by erasing the parentheses of cycles. For example, if $\pi=(71435)(826)$, then $o(\pi)=71435862$. The map $o: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ defined above is a bijection. The following result is direct.

Proposition 4. Let $\pi=C_{1} C_{2} \cdots C_{k} \in D_{n}$. Suppose that $o(\pi)(0)=0$ and $o(\pi)(n+1)=$ $n+1$. Then

$$
\begin{gathered}
\operatorname{Cpeak}(\pi)=\operatorname{Peak}(o(\pi)), \quad \operatorname{Cval}(\pi)=\operatorname{Val}(o(\pi)), \\
\operatorname{Cdasc}(\pi)=\operatorname{Dasc}(o(\pi)) \quad \text { and } \quad \operatorname{Cddes}(\pi)=\operatorname{Ddes}(o(\pi)),
\end{gathered}
$$

where the sets Peak $(o(\pi)), \operatorname{Val}(o(\pi)), \operatorname{Dasc}(o(\pi))$ and $\operatorname{Ddes}(o(\pi))$ are defined similar to Definition 1 with the only difference $o(\pi)(0)=0$.

We define the cyclic analogous of the MFS-action on derangements in the following way. Let $\pi=C_{1} C_{2} \cdots C_{k}$. Suppose that $o(\pi)(0)=0$ and $o(\pi)(n+1)=n+1$. For $x \in[n]$, define the map $\theta_{x}: D_{n} \rightarrow D_{n}$ by

$$
\theta_{x}(\pi)=o^{-1}\left(\varphi_{x}^{\prime}(o(\pi))\right)
$$

The map is well-defined. To see this, let $\pi=C_{1} C_{2} \cdots C_{k} \in D_{n}$. If $x$ is a cyclic valley of $\pi$, then $x$ is a valley of $o(\pi), \varphi_{x}^{\prime}(o(\pi))=o(\pi)$ and $\theta_{x}(\pi)=\pi$. If $x$ is a cyclic peak of $\pi$, then $x$ is a peak of $o(\pi), \varphi_{x}^{\prime}(o(\pi))=o(\pi)$ and $\theta_{x}(\pi)=\pi$. If $x$ is a cyclic double ascent of $C_{i}$ in $\pi$, where $C_{i}=\left(w_{0} w_{1} x w_{2}\right)$ and $w_{1}$ denotes the maximal contiguous subword immediately to the left of $x$ whose letters are all smaller than $x$. Then $x$ is a double ascent of $o(\pi), \varphi_{x}^{\prime}(o(\pi))=o\left(C_{1} C_{2} \cdots C_{i-1} \bar{C}_{i} C_{i+1} \cdots C_{k}\right)$ and $\theta_{x}(\pi)=C_{1} C_{2} \cdots C_{i-1} \bar{C}_{i} C_{i+1} \cdots C_{k} \in D_{n}$, where $\bar{C}_{i}=\left(w_{0} x w_{1} w_{2}\right)$. If $x$ is a cyclic double descent of $C_{i}$ in $\pi$, where $C_{i}=\left(w_{0} x w_{1} w_{2}\right)$ and $w_{1}$ denotes the maximal contiguous subword immediately to the right of $x$ whose letters are all smaller than $x$. Then $x$ is a double descent of $o(\pi), \varphi_{x}^{\prime}(o(\pi))=o\left(C_{1} C_{2} \cdots C_{i-1} C_{i} C_{i+1} \cdots C_{k}\right)$ and $\theta_{x}(\pi)=$ $C_{1} C_{2} \cdots C_{i-1} \bar{C}_{i} C_{i+1} \cdots C_{k} \in D_{n}$, where $\bar{C}_{i}=\left(w_{0} w_{1} x w_{2}\right)$. Hence the map $\theta_{x}$ is welldefined for all $x \in[n]$.

Table 1 gives an example of the maps $\theta_{x}$ on $\pi=(623)(87514)$ for all $x \in$ [8], where $o(\pi)=62387514$.

| $x$ | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: |
| $\varphi_{x}^{\prime}(o(\pi))$ | 62387514 | 62387514 | 63287514 | 62387514 |
| $\theta_{x}(\pi)$ | $(623)(87514)$ | $(623)(87514)$ | $(632)(87514)$ | $(623)(87514)$ |
| $x$ | 5 | 6 | 7 | 8 |
| $\varphi_{x}^{\prime}(o(\pi))$ | 62387145 | 62387514 | 62385147 | 62387514 |
| $\theta_{x}(\pi)$ | $(623)(87145)$ | $(623)(87514)$ | $(623)(85147)$ | $(623)(87514)$ |

Table 1.
The function $\theta_{x}$ is an involution and $\theta_{x} \theta_{y}=\theta_{y} \theta_{x}$ for all $x, y \in[n]$. For any subset $S \subseteq[n]$, define the function $\theta_{S}(\pi): D_{n} \rightarrow D_{n}$ by

$$
\theta_{S}(\pi)=\prod_{x \in S} \theta_{x}(\pi)
$$

The group $\mathbb{Z}_{2}^{n}$ acts on $D_{n}$ via the functions $\theta_{S}, S \in[n]$ and call this action the CMFSaction.

For $\pi \in D_{n}$, let $\operatorname{Orb}^{c}(\pi)$ denote the orbit including $\pi$ under the CMFS-action. There is a unique derangement in $\operatorname{Or}^{c}(\pi)$, denoted by $\tilde{\pi}$, such that $\tilde{\pi}$ has no cyclic double ascents. The next is the main results of this note.

Theorem 5. Let $\pi \in D_{n}$. Then

$$
\sum_{\sigma \in O r b^{c}(\pi)} q^{\operatorname{exc}(\sigma)}=q^{\operatorname{exc}(\tilde{\pi})}(1+q)^{n-2 \operatorname{exc}(\tilde{\pi})}=q^{\operatorname{cpeak}(\pi)}(1+q)^{n-2 \operatorname{cpeak}(\pi)} .
$$

Proof. If $x$ is a cyclic double descent of some cycle $C_{i}$ in $\pi$, then $x$ is a cyclic double ascent of cycle $C_{i}^{\prime}$ in $\theta_{x}(\pi)$, where $\pi=C_{1} C_{2} \cdots C_{k}$ and $\theta_{x}(\pi)=C_{1}^{\prime} C_{2}^{\prime} \cdots C_{k}^{\prime}$. We have $\operatorname{Cdasc}\left(\theta_{x}(\pi)\right)=\operatorname{Cdasc}(\pi) \cup\{x\}$ and $\operatorname{Cval}\left(\theta_{x}(\pi)\right)=\operatorname{Cval}(\pi)$. It follows that $\operatorname{Exc}\left(\theta_{x}(\pi)\right)=$ $\operatorname{Exc}(\pi) \cup\{x\}$ and $\operatorname{exc}\left(\theta_{x}(\pi)\right)=\operatorname{exc}(\pi)+1$ from Proposition 3. Then

$$
\sum_{\sigma \in O r b^{c}(\pi)} q^{\operatorname{exc}(\sigma)}=q^{\operatorname{exc}(\tilde{\pi})}(1+q)^{\operatorname{cddes}(\tilde{\pi})}
$$

For any $\pi=C_{1} C_{2} \cdots C_{k} \in D_{n}$, delete all double descents and double ascents of $o(\pi)$, then we get an alternating permutation

$$
0<x_{1}>x_{2}<x_{3}>\cdots>x_{n-\operatorname{cddes}(\pi)-\operatorname{cdasc}(\pi)}<n+1,
$$

where $o(\pi)(0)=0$ and $o(\pi)(n+1)=n+1$. Thus

$$
\operatorname{cpeak}(\pi)=\operatorname{peak}(o(\pi))=\operatorname{val}(o(\pi))=\operatorname{cval}(\pi) .
$$

Note that the union of sets $\operatorname{Cval}(\tilde{\pi}), \operatorname{Cpeak}(\tilde{\pi})$ and $\operatorname{Cddes}(\tilde{\pi})$ is the set $[n]$. Hence $\operatorname{exc}(\tilde{\pi})=\operatorname{cpeak}(\tilde{\pi})=\operatorname{cpeak}(\pi)$ and $\operatorname{cddes}(\tilde{\pi})=n-2 \operatorname{exc}(\tilde{\pi})=n-2 \operatorname{cpeak}(\pi)$.

The following corollary is an immediate consequence of Theorem 5 .
Corollary 6 ([7]). The derangement polynomials can be expanded as

$$
D_{n}(q)=\sum_{i=0}^{\lfloor n / 2\rfloor} b_{i} q^{i}(1+q)^{n-2 i}
$$

where $b_{i}=2^{-n+2 i}\left|\left\{\pi \in D_{n}: \operatorname{cpeak}(\pi)=i\right\}\right|$ and $b_{0}=0$.

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