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A GROUP-EMBEDDABLE NON-AUTOMATIC SEMIGROUP WHOSE UNIVERSAL GROUP IS AUTOMATIC

ALAN J. CAIN
School of Mathematics and Statistics, University of St Andrews, North Haugh, St Andrews, Fife KY16 9SS, United Kingdom
e-mail: alanc@mcs.st-andrews.ac.uk

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Abstract. Answering a question of Hoffmann and of Kambites, an example is exhibited of a finitely generated semigroup S such that S embeds in a group and S is not automatic, but the universal group of S is automatic.

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1. Introduction. The concept of an automatic structure has been generalized from groups [**5**] to semigroups [**2**]. Several authors [**1**, **6**, **7**] have asked the following question. Let S be a finitely generated semigroup embeddable in a group and let G be its universal group [**3**, Chapter 12]. If G is automatic, must S be automatic?

Examples in favour of this implication include: free groups and semigroups; braid groups and semigroups [**5**, Chapter 9]; abelian groups and their subsemigroups [**1**, Proposition 3.15].

A similar question asked whether the automatism of a group implied the automatism of its positive subsemigroups. (A *positive subsemigroup* is a subsemigroup generated by a group generating set.) This question was recently answered negatively [**1**, Section 7]. However, the techniques used cannot be adapted to answer the original question.

The purpose of this paper is to answer the original question in the negative. Section 3 contains an example of a finitely generated semigroup S such that S embeds in a group and S is not automatic, but the universal group of S is automatic.

2. Definitions and preliminaries. This section contains the definitions required for automatic semigroups and states the various results on automatism required for the remainder of the paper. This paper assumes familiarity with regular languages and finite automata; Sections 1.1 and 1.2 of [**5**] contain all the basic theory for automata and languages needed hereafter.

Definition 2.1. Let A be an alphabet representing a set of generators for a semigroup S. For any word $w \in A^+$, denote by $\overline{w}$ the element of S represented by w. For any set of words W, $\overline{W}$ is the set of all elements represented by at least one word in W.

Let \$ be a new symbol not in A. Let

$$A(2, \$) = \{(a, b) : a, b \in A \cup \{\$\}\} - \{(\$, \$)\}$$

be a new alphabet. Define the mapping $\delta_A : A^+ \times A^+ \to A(2, \$)^+$ by

$$(u_1 \cdots u_m, v_1 \cdots v_n) \mapsto \begin{cases} (u_1, v_1) \cdots (u_m, v_n) & \text{if } m = n, \\ (u_1, v_1) \cdots (u_n, v_n)(u_{n+1}, \$) \cdots (u_m, \$) & \text{if } m > n, \\ (u_1, v_1) \cdots (u_m, v_m)(\$, v_{m+1}) \cdots (\$, v_n) & \text{if } m < n, \end{cases}$$

where $u_i, v_i \in A$. The symbol \$ is usually called the *padding symbol*.

DEFINITION 2.2. An *automatic structure* for S is a pair (A, L), where A is a finite alphabet representing a set of generators for S and $L \subseteq A^+$ is a regular language with $\overline{L} = S$ and such that, for each $a \in A \cup \{\varepsilon\}$,

$$L_a = \{(u, v) : u, v \in L, \overline{ua} = \overline{v}\}\delta_A$$

is a regular language over $A(2, \$)$. An *automatic semigroup* is a semigroup that admits an automatic structure.

PROPOSITION 2.3 ([**2**, Proposition 3.5]). *If a semigroup S is automatic, then so is S^1, the semigroup formed by adjoining an identity to S.*

PROPOSITION 2.4 ([**4**, Theorem 1.1]). *Let M be a monoid with automatic structure (A, L) and let B represent a finite* [*semigroup*] *generating set for M. Then there exists an automatic structure (B, K) for M.*

PROPOSITION 2.5 ([**5**, Corollary 4.1.6]). *All finitely generated virtually abelian groups are automatic.*

PROPOSITION 2.6 ([**2**, Proposition 2.3]). *Let U and V be subsets of $A^+ \times A^+$ such that $U\delta_A$ and $V\delta_A$ are regular. Then*

$$U\delta_A \circ V^{-1}\delta_A = \{(u, v)\delta_A : (\exists w \in A^+)((u, w) \in U \wedge (v, w) \in V)\}$$

is also a regular language over $A(2, \$)$.

In addition to the results above on automatism, Section 3 requires some information about universal groups. If S is a semigroup that embeds in a group, then the universal group U of S is the largest group into which S embeds and which it generates, in the sense that all other such groups are homomorphic images of U. Alternatively, the universal group of S is the group defined by treating any semigroup presentation for S as a group presentation. (Actually, universal groups are defined for all semigroups, not just those embeddable into groups. For the formal definition of universal groups of semigroups, and for further information on the subject, see [**3**, Chapter 12].)

Given a subsemigroup S of a group G, the subgroup of G generated by S does not, in general, coincide with the universal group of S. However, in certain special cases, they are isomorphic.

PROPOSITION 2.7 ([**1**, Corollary 4.4]). *Let G be a group that satisfies a non-trivial semigroup law. Let S be a subsemigroup of G and let H be the subgroup of G generated by S. Then H coincides with the universal group of S.*

3. The example. Let $\mathcal{S}_8$ be the symmetric group on eight elements. Let $\mathbb{Z}^8$ be the direct product of eight copies of the integers under addition. View elements of $\mathbb{Z}^8$ as octuples of integers. Let $G = \mathcal{S}_8 \ltimes \mathbb{Z}^8$, where $\mathcal{S}_8$ acts (on the right) by permuting the components of elements of $\mathbb{Z}^8$. (The $\mathbb{Z}$-components are indexed from 1 at the left to 8 at the right.) The abelian subgroup $\mathbb{Z}^8$ of G has index 8!, so that G is a virtually abelian group.

Let $A = \{a, b, c, d, e, f, g, h\}$ be an alphabet representing elements of G in the following way:

$$\begin{aligned}
\overline{a} &= [(1\ 3),\ (0, 1, 1, 0, 0, 0, 1, 0)], & & \\
\overline{b} &= [\mathrm{id},\ (0, 0, 1, 0, 0, 0, 0, 0)], & \overline{f} &= [(1\ 5)(2\ 6),\ (0, 0, 0, 0, 1, 1, 2, 0)], \\
\overline{c} &= [(1\ 3)(2\ 4),\ (1, 0, 0, 0, 0, 0, 1, 1)], & \overline{g} &= [\mathrm{id},\ (0, 0, 0, 0, 1, 1, 0, 0)], \\
\overline{d} &= [\mathrm{id},\ (0, 0, 0, 1, 0, 0, 0, 0)], & \overline{h} &= [(1\ 5)(2\ 6),\ (1, 1, 0, 0, 0, 0, 0, 2)]. \\
\overline{e} &= [(2\ 4),\ (0, 1, 0, 0, 0, 0, 0, 1)], & &
\end{aligned}$$

Let S be the subsemigroup of G generated by $\overline{A}$.

PROPOSITION 3.1. *The semigroup S is not automatic.*

Proof. Let $A' = \{a, c, e, f, h\}$. Observe that only letters from A' have non-identity $\mathcal{S}_8$-components or non-zero seventh and eighth $\mathbb{Z}$-components. Note further that the seventh and eighth $\mathbb{Z}$-components are not affected by any of the $\mathcal{S}_8$-components in $\overline{A}$, and that there are no negative integers amongst the $\mathbb{Z}$-components.

Now, for any $\alpha \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
\overline{ab^{\alpha}cd^{\alpha}e} &= [(1\ 3),\ (0, 1, 1, 0, 0, 0, 1, 0)][\mathrm{id},\ (0, 0, \alpha, 0, 0, 0, 0, 0)]\overline{cd^{\alpha}e} \\
&= [(1\ 3),\ (0, 1, \alpha+1, 0, 0, 0, 1, 0)][(1\ 3)(2\ 4),\ (1, 0, 0, 0, 0, 0, 1, 1)]\overline{d^{\alpha}e} \\
&= [(2\ 4),\ (\alpha+2, 0, 0, 1, 0, 0, 2, 1)][\mathrm{id},\ (0, 0, 0, \alpha, 0, 0, 0, 0)]\overline{e} \\
&= [(2\ 4),\ (\alpha+2, 0, 0, \alpha+1, 0, 0, 2, 1)][(2\ 4),\ (0, 1, 0, 0, 0, 0, 0, 1)] \\
&= [\mathrm{id},\ (\alpha+2, \alpha+2, 0, 0, 0, 0, 2, 2)]
\end{aligned}$$

and

$$\begin{aligned}
\overline{fg^{\alpha}h} &= [(1\ 5)(2\ 6),\ (0, 0, 0, 0, 1, 1, 2, 0)][\mathrm{id},\ (0, 0, 0, 0, \alpha, \alpha, 0, 0)]\overline{h} \\
&= [(1\ 5)(2\ 6),\ (0, 0, 0, 0, \alpha+1, \alpha+1, 2, 0)][(1\ 5)(2\ 6),\ (1, 1, 0, 0, 0, 0, 0, 2)] \\
&= [\mathrm{id},\ (\alpha+2, \alpha+2, 0, 0, 0, 0, 2, 2)].
\end{aligned}$$

Hence $\overline{ab^{\alpha}cd^{\alpha}e} = \overline{fg^{\alpha}h}$ for all $\alpha \in \mathbb{N} \cup \{0\}$.

LEMMA 3.2. *For each $\alpha \in \mathbb{N} \cup \{0\}$, the elements of S represented by $ab^{\alpha}cd^{\alpha}$ and fg^{α} are represented by those words alone.*

Proof. Suppose that w represents

$$s = \overline{fg^{\alpha}} = [(1\ 5)(2\ 6),\ (0, 0, 0, 0, \alpha+1, \alpha+1, 2, 0)].$$

Since the seventh and eighth $\mathbb{Z}$-components are 2 and 0, the only letters from A' in w are either two letters a or one letter f. The first option is impossible, since the $\mathcal{S}_8$-component of $\overline{w}$ would then be the identity permutation. Since the third and fourth

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$\mathbb{Z}$-components of s are zero, and these components are unaffected by the $\mathcal{S}_8$-component of $\overline{f}$, the rest of w must consist of letters g. Hence w is a rearrangement of fg^β for some β. Since the first two $\mathbb{Z}$-components of s are 0, no letters g can precede the letter f. Thus $w = fg^\beta$. Considering the fifth and sixth $\mathbb{Z}$-components of s shows that $\beta = \alpha$. Therefore fg^α is the unique word over A representating s.

Now suppose that v represents

$$t = \overline{ab^\alpha cd^\alpha} = [(2\ 4),\ (\alpha + 2, 0, 0, \alpha + 1, 0, 0, 2, 1)].$$

The first task is to determine what letters from A' appear in v. The letter h is ruled out by the last $\mathbb{Z}$-component of t being 1. If an f is present, the only other letter from A' must be e, since the last two $\mathbb{Z}$-components of t are 2 and 1. However, this gives the wrong $\mathcal{S}_8$-component. The other possibilities are a, a, and e; or a and c. Suppose the former. If the letter e is the last of these three letters, then $\overline{v}$ has a non-zero second $\mathbb{Z}$-component. If one of the letters a is the last of the three, then $\overline{v}$ has non-zero second and third $\mathbb{Z}$-component. Hence the letters from A' must be a and c.

Since the fifth and sixth $\mathbb{Z}$-components of t are 0, and these components are unaffected by the $\mathcal{S}_8$ components of $\overline{a}$ or $\overline{c}$, no letters g can be present in v. Hence v is a rearrangement of $acb^\beta d^\gamma$, for some $\beta, \gamma \in \mathbb{N} \cup \{0\}$. The letter a must precede the letter c, for otherwise $\overline{v}$ would have non-zero second and third $\mathbb{Z}$-components. Similarly, the third $\mathbb{Z}$-component of s being zero forces the letters b to lie between the letter a and the letter c (since the $\mathcal{S}_8$-components of $\overline{a}$ and $\overline{c}$ together send the third $\mathbb{Z}$-component to itself). The letters d must lie to the right of the letter c, since otherwise $\overline{v}$ would have non-zero second $\mathbb{Z}$-component. Hence $w = ab^\beta cd^\gamma$. The values of the first and fourth $\mathbb{Z}$-components of t together force $\beta = \gamma = \alpha$, so that $ab^\alpha cd^\alpha$ is the unique word over A representing t. □

Suppose that S is automatic. Then, by Proposition 2.3, so is S^1. Proposition 2.4 implies that S^1 has an automatic structure (C, L), where $C = A \cup \{1\}$. (The new symbol 1 represents the adjoined identity of S^1.) Let $\phi : C^* \to A^*$ map $w \in C^*$ to the word over A formed by deleting any symbols 1 from w. Obviously $\overline{w\phi} = \overline{w}$.

Proposition 2.6 shows that the language

$$\begin{aligned} L_e \circ L_h^{-1} &= \{(u, w)\delta_C : u, w \in L, \overline{ue} = \overline{w}\} \circ \{(w, v)\delta_C : w, v \in L, \overline{vh} = \overline{w}\} \\ &= \{(u, v)\delta_C : u, v \in L, \overline{ue} = \overline{vh}\} \end{aligned}$$

is regular. Let N be the number of states in a finite state automaton $\mathcal{A}$ recognizing $L_e \circ L_h^{-1}$.

For each $\alpha \in \mathbb{N} \cup \{0\}$, let u_α and v_α be representatives in L of the elements $\overline{ab^\alpha cd^\alpha}$ and $\overline{fg^\alpha}$, respectively. Since $\overline{ab^\alpha cd^\alpha}$ has a unique representative over A by Lemma 3.2, it is clear that $u_\alpha\phi = ab^\alpha cd^\alpha$. Similarly, $v_\alpha\phi = fg^\alpha$. (Hence u_α and v_α are the words $ab^\alpha cd^\alpha$ and fg^α with some symbols 1 possibly inserted.) By its definition, the language $L_e \circ L_h^{-1}$ contains $(u_\alpha, v_\alpha)\delta_C$ for all $\alpha \in \mathbb{N} \cup \{0\}$.

Fix $\alpha > N$. Consider the automaton $\mathcal{A}$ reading $(u_\alpha, v_\alpha)\delta_C$, and the states it enters immediately after reading each of the letters b from the word u_α. Since the number of letters b exceeds N, the automaton enters the same state after reading two different letters b. Let u' and $u'u''$ be the prefixes of u_α up to and including these two different letters b. That is, $u'\phi = ab^\beta$, $(u'u'')\phi = ab^\beta b^\gamma$, for some $\beta, \gamma \in \mathbb{N}$. Let v' and $v'v''$ be prefixes of v_α of the same lengths as u' and $u'u''$, respectively. The subword v'' is such that

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$v''\phi = fg^\eta$ or $v''\phi = g^\eta$ for some $\eta \in \mathbb{N} \cup \{0\}$. (The former possibility arises because v' may be a string of symbols 1.)

Suppose that $v''\phi = fg^\eta$. Then pumping $(u'', v'')\delta_C$ shows that

$$\overline{ab^\beta b^{2\gamma} b^{\alpha-\beta-\gamma} cd^\alpha e} = \overline{fg^\eta fg^\alpha h}.$$

This is a contradiction, since the S_8-component of the left-hand side is the identity permutation and that of the right-hand side is (1 5)(2 6).

Therefore suppose that $v''\phi = g^\eta$. Again, pumping $(u'', v'')\delta_C$ implies that

$$\overline{ab^\beta b^{2\gamma} b^{\alpha-\beta-\gamma} cd^\alpha e} = \overline{fg^\eta g^\alpha h}.$$

This too is a contradiction, since γ is at least 1, but

$$\overline{ab^\beta b^{2\gamma} b^{\alpha-\beta-\gamma} cd^\alpha e} = [\mathrm{id},\ (\alpha+\gamma+2, \alpha+2, 0, 0, 0, 0, 2, 2)],$$

while

$$\overline{fg^\eta g^\alpha h} = [\mathrm{id},\ (\alpha+\eta+2, \alpha+\eta+2, 0, 0, 0, 0, 2, 2)].$$

Therefore S is not automatic. This completes the proof of Proposition 3.1. □

Let H be the subgroup of G generated by $\overline{A}$. The group H, which contains S, is a subgroup of the virtually abelian group G and is therefore itself virtually abelian. It is finitely generated, and so is automatic by Proposition 2.5.

Furthermore, the group G satisfies the non-trivial semigroup law $x^{8!}y^{8!} = y^{8!}x^{8!}$. (Recall that $\mathbb{Z}^8$ is an index 8! normal subgroup of G.) By Proposition 2.7, the subgroup H coincides with the universal group of S.

Therefore S is a finitely generated non-automatic semigroup that is embeddable in a group but whose universal group is automatic.

4. Further observations.

PROPOSITION 4.1. *The semigroup S from Section 3 is not finitely presented.*

Proof. For all $\alpha \in \mathbb{N} \cup \{0\}$, the relation $(ab^\alpha cd^\alpha e, fg^\alpha h)$ holds in S. Lemma 3.2 showed that $\overline{fg^\alpha}$ was represented by the word fg^α alone. Similar reasoning shows that, for each $\alpha \in \mathbb{N} \cup \{0\}$, the element of S represented by $g^\alpha h$ is represented by that word alone.

Therefore no non-trivial relation in S can be applied to a proper subword of $fg^\alpha h$. Therefore in any presentation for S on the generating set $\overline{A}$, each word $fg^\alpha h$ must appear as one side of a defining relation. Thus S is not finitely presented. □

QUESTION 4.2. Is there a *finitely presented* group-embeddable non-automatic semigroup whose universal group is automatic?

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REFERENCES

1. A. J. Cain, E. F. Robertson and N. Ruškuc, Subsemigroups of groups: presentations, Malcev presentations, and automatic structures, *J. Group Theory* **9** (3) (2006), 397–426.

2. C. M. Campbell, E. F. Robertson, N. Ruškuc and R. M. Thomas, Automatic semigroups, *Theoret. Comput. Sci.* **250** (1–2) (2001), 365–391.

3. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups* (Vol. II), Mathematical Surveys, No. 7. (American Mathematical Society, Providence, R.I., 1967).

4. A. J. Duncan, E. F. Robertson and N. Ruškuc, Automatic monoids and change of generators, *Math. Proc. Cambridge Philos. Soc.* **127** (3) (1999), 403–409.

5. D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston, *Word processing in groups* (Jones and Bartlett Publishers, Boston, 1992).

6. M. Hoffmann, *Automatic semigroups*, Ph.D. Thesis (University of Leicester, 2000).

7. M. E. Kambites, *Combinatorial aspects of partial algebras*, Ph.D. Thesis (University of York, 2003).