A Growth Framework Using the Constant Elasticity of Substitution Model

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Abstract

Some results in growth theory based on the Cobb-Douglas production function model are generalized when the production function is chosen to be the Constant Elasticity of Substitution (CES) function. Such a generalization is of considerable interest because it is known that the Cobb-Douglas function cannot be used as a suitable model for some production technologies (like the US economy and climate changes). It is shown that in the steady state the growth rate of the output is equal to the Solow residual and that the capital deepening term becomes zero. The CES function is a homogeneous function of degree two and a result is obtained on the wage of a worker using the Euler's theorem.

1. Introduction

In growth theory, a "production function" is taken to be a mathematical expression that is used to model a production technology with distinct inputs and outputs. Several types of production functions have been proposed in the past based on either empirical or theoretical considerations (for general surveys, see e.g., [1] [2]). For example, the Cobb-Douglas production function ([3]) is widely used as a simple model to study economic growth in spite of some of its limitations (that we shall discuss later).

The CES ("Constant Elasticity of Substitution") function was introduced by Solow ([4]), and later expounded by Arrow *et al.* ([5]) to synthesize several types of production functions. The CES function has been applied extensively to study

economic growth (e.g., [6]-[17]). It has also applications in high energy physics [18]. Many generalizations of the format of the CES function (including the multi-input case) have been proposed, for details see e.g. [6] [7]. The shape of the frontier of the CES function is also of significant interest in economic analysis and its connection with the differential geometry of hyperspaces has been studied. (e.g., [19] [20] [21]).

The *elasticity of substitution* between any two input variables in a production function measures how easily one variable can be substituted for the other variable and it measures the curvature of the isoquant (the concept was first introduced by Hicks [22]). So, an elasticity of substitution equal to 0 indicates no substitution between the input variables can be possible and an elasticity of substitution equal to infinity indicates the perfect substitution. More formally, the elasticity of substitution between two factors of production is an index that measures the percentage of response of the relative marginal products of the two factors to a percentage of change in the ratio of the two quantities. In order to make the paper self-contained, we shall briefly review in Appendix I the formal definition of the elasticity of substitution for the case of a production function with *n* input variables.

For the Cobb-Douglas function the elasticity of substitution between the input variables is always equal to 1 (for a proof see, e.g., [7]) and this fact restricts its use as a suitable production model in several applications, as claimed by many authors. For example, Antrás [23] has shown that the US economy is not amenable to the elasticity of substitution being taken as 1. Also Werf [24] has shown that it is not suitable to take the Cobb-Douglas function as a production function for modeling climate change policies. Furthermore, Young [25] has shown that the elasticity of substitution for U.S. aggregate and of most industries cannot be equal to 1 and it is estimated to be less than 0.620; thus it follows that the Cobb-Douglas production model (whose elasticity of substitution is fixed to be 1) is not suitable for such applications.

The CES function has a constant elasticity of substitution (as the name suggests) and it can have any pre-determined value as its elasticity of sub-stitution (as we shall show later). Thus, it offers a wider flexibility than the Cobb-Douglas function and is still computationally tractable, as remarked in ([6], p. 54). These reasons have partially motivated us to extend some results of the neoclassical growth theory based on the Cobb-Douglas function by using the more general setting of the CES production function.

We now briefly review some definitions and results.

Definition 1 ([5]) The *Constant Elasticity of Substitution* (CES) production function for the three factors-capital K, labor L and the total factor of productivity F, is given by

$$Y = F\left(\alpha K^{\gamma} + \beta L^{\gamma}\right)^{\frac{1}{\gamma}} \quad \text{with } \alpha + \beta = 1 \qquad (1)$$

where Y is the output and K, L, F are smooth functions of time t; α is a

certain constant, called the *share parameter* between the capital and labor; and γ is another constant, called the *substitution parameter*.

The following result indicates that the CES production function is a generalization of the Cobb-Douglas function:

Proposition 1 ([5]) *When* $\gamma \to 0$, *the CES production function* (1) *approaches the Cobb-Douglas production function*

$$Y = FK^{\alpha}L^{\beta} \tag{2}$$

where α, β are constants such that $\alpha + \beta = 1$.

It is known that the elasticity of substitution of the CES production function as defined by (1), is equal to $1/(1-\gamma)$ ([5], p. 230) when $\gamma \neq 1$, and using this result we can easily construct (as indicated in Example 1) an infinite family of CES production functions each of whose members has the same elasticity of substitution equal to any given nonzero number.

Example 1 Suppose we want to construct a CES production function whose elasticity of substitution, ϵ , is equal to, say, 2. Solving the equation

$$1/(1-\gamma) = 2 \tag{3}$$

gives $\gamma = 0.5$. Substituting $\gamma = 0.5$ into (1) and taking F to be any smooth function of *t*, and by varying α , we get an infinite family of CES production functions given by

$$Y = F\left\{\alpha\sqrt{K} + (1-\alpha)\sqrt{L}\right\}^2 \tag{4}$$

where $0 < \alpha < 1$ and each member of the family has the same elasticity of substitution equal to 2. Similarly, if $\epsilon = 1$, then by solving the equation $1/(1-\gamma) = 1$ we get $\gamma = 0$ and this corresponds to the Cobb-Douglas production function (compare with Proposition 1).

In defining the CES production function, in the form given by (1), many authors take F to be a parameter (e.g., [5], p. 230; [7], p. 397; [26], p. 397). However, we shall consider here a more general model where F is assumed to be a function of time. Such a model would be able to handle some situations that cannot be accommodated by a CES function where F is a parameter. For example, the output of a factory may increase at a time when the production manager is replaced by a more efficient one (*i.e.* when F increases) even when there are no increases in investments in capital and labor. We note that ([6], p. 54) takes F to be a function of time, like us. Furthermore we shall exclude the cases when the output is either identically zero, or a negative number as these cases are not of interest. Thus we make the following assumption:

Assumption 1 *We assume in* (1) *that F is a function of time and that* $F > 0$.

As remarked in ([27], p. 107), the wages and salaries in USA and many other countries form about 70 percent of the national income. Consequently, the value of $\alpha = 0.3$ has been used in ([27], p. 107-109) as the share of the capital for the Cobb-Douglas production function model (2) to estimate the growth rate for a number of countries. However, such an estimate for the growth rate of countries

based solely on a Cobb-Douglas production function model may not be realistic because the Cobb-Douglas production function (2) has the unitary elasticity of substitution and as [23] has shown, it is not suitable to model the US economy with the elasticity of substitution equal to 1 (also, it is not known whether we can realistically assume that the elasticity of substitution is 1 for all the other countries involved in that study). So, it would be of significant interest to estimate the economic growth for various countries using the same data but for a more general setting involving the CES production function model for a range of values of the parameter γ in (1) with $\alpha = 0.3$, and then to estimate an optimal value of γ to fit the data set.

The structure of the rest of the paper is as follows. In Section 2 we obtain a growth equation for the CES production function and define the (generalized) Solow residual and the corresponding capital deepening term. In Section 3 we obtain some bounds for the (generalized) Solow residual and the capital deepening term. In Section 4 we investigate the growth rates corresponding to the CES production function. In Section 5 we investigate the homogeneous property of the CES production function. Section 6 gives our conclusions. Appendix I reviews the definition of the elasticity of substitution, The proofs of all the results are given in the Appendix II.

2. Growth Equation for CES Production Model

In this section, we generalize some results obtained earlier in the setting of the Cobb-Douglas production model (e.g., as in [27], Chap. 5) to the case of the CES production function. First, we shall derive a growth equation corresponding to the CES production function. Continuing with the notation introduced in Definition 1, we now define the variables y and k given by

$$y = Y/L, \ k = K/L \tag{5}$$

where we assume that L is nonzero (for, if $L = 0$, the CES production function takes the simple form $Y = FK$ and that is a special case of the Cobb-Douglas function with $\alpha = 1$ and $\beta = 0$; and so we omit this case). Thus, y and k are well-defined and y represents the output per worker (*i.e.*, per capita output) and k is the capital stock per worker. Also, L cannot be negative because we have not given any interpretation to negative labor. So, we shall make the following assumption:

Assumption 2
We assume that $L > 0$.

We note that ([6], p. 36) also makes a similar assumption.

From (1), dividing both sides of the equation by L, and using (5), we get

$$y = F \left(\alpha k^{\gamma} + \beta \right)^{\frac{1}{\gamma}} \tag{6}$$

Log differentiating both sides of (6) with respect to t and denoting $\dot{y}/y$ by G, we get the following *growth equation* when the CES production function is taken as the production model:

$$G = \frac{\dot{F}}{F} + \frac{\alpha k^{\gamma-1}\dot{k}}{\alpha k^{\gamma} + \beta} \tag{7}$$

where $\cdot$ denotes differentiation with respect t. Both the terms on the right hand side of (7) are well-defined because their denominators are nonzero (F is nonzero by the Assumption 1, and $\left(\alpha k^{\gamma} + \beta\right)$ is nonzero since otherwise it will follow from (6) that $y = 0$ and we shall exclude this trivial case when the output is identically equal to zero); also G is well-defined (since otherwise from the expression $G = \dot{y}/y$, it will follow that $y = 0$).

Taking the limit as $\gamma \to 0$ in (7) and using Proposition 1, we can easily obtain the growth equation corresponding to the Cobb-Douglas production function (2) and the resulting equation matches with the corresponding Equation (5.3) derived in ([27], p. 106).

Definition 2 For the CES production function, the expression

$$G = \dot{y}/y \tag{8}$$

is called the *growth rate of the output per worker*, and the growth rate of the total factor of productivity, $\dot{F}/F$, will be called the *Solow residual* corresponding to the CES production function.

Using (7), the Solow residual can be expressed as:

$$\frac{\dot{F}}{F} = G - D \tag{9}$$

where

$$D = \frac{\alpha k^{\gamma-1}\dot{k}}{\alpha k^{\gamma} + \beta} \tag{10}$$

is called the *capital deepening term* corresponding to the CES production function. From (9), we observe that G, the growth rate of the output per worker, is the sum of two components: (i) the Solow residual, and (ii) the capital deepening component D (we note that a similar observation is made for the Cobb-Douglas production function (2), in ([27], p. 106)).

Next, we consider the form of the growth equation (7) in the steady state.

Proposition 2 (i) *When a steady state of production is reached, the growth rate of the output per worker is equal to the Solow residual, and the capital deepening term is zero. As a partial converse, if the production is not entirely labor intensive, a steady state of production is reached when the growth rate of the output per worker is equal to the Solow residual (or equivalently when the capital deepening term is zero).*

(ii) The total factor of productivity is in a steady state if and only if the growth rate of the output per worker is equal to the capital deepening term.

3. Estimates for Solow Residual

We now obtain some bounds for the capital deepening term D and the Solow residual.

Proposition 3 *For the CES production function given by* (6), *if the ratio of the share parameters is less than* k^γ, *i.e. if*

$$k > \left(\beta/\alpha\right)^{1/\gamma} \tag{11}$$

then (i) the capital deepening term is less than the growth rate of capital per worker, and

(ii) the Solow residual is greater than the difference between the growth rates per worker, of the output and the capital.

Corollary 1 (i) *When* $\gamma \to \infty$, *the estimates given in Proposition 3 hold for any* $k > 1$.

(ii) *When* $\gamma \to 0$, (*i.e.*, when the production function is approaching the Cobb-Douglas function (2), see Proposition 1), the estimates given in Proposition 3 hold for any $\alpha > 1/2$.

We now give some examples to illustrate what happens to the inequality (11) as we progressively increase the value of γ.

Example 2 (i) Suppose, as an illustration, we choose $\alpha = 0.3$ and so $\beta = 0.7$. In **Figure 1**, we plot the values of $u := \left(\beta/\alpha\right)^{1/\gamma}$ where the horizontal axis corresponds to γ and the vertical axis corresponds to *u*. We know from Corollary 1 (i) that as γ becomes larger and larger, the value of *u* would tend to 1 and this property is being exhibited in **Figure 1**. We mention that similar illustrations can also be given by taking other values of α.

(ii) If $\alpha = 0.5$ (that is, when the capital and labor are shared equally in the CES production model), the condition (11) reduces to $k > 1$ and so the estimates described in Proposition 3 hold for any $k > 1$ and for *any* value of γ. In other words, when the distributions of resources between the capital and labor in the production are equal and the capital stock per worker is greater than one, the results of Proposition 3 hold for *any* value of γ.

Now we obtain further interpretations of the estimates that were given in Proposition 3.

Proposition 4 *For any set of values of* α, β, γ *and k satisfying the condition*
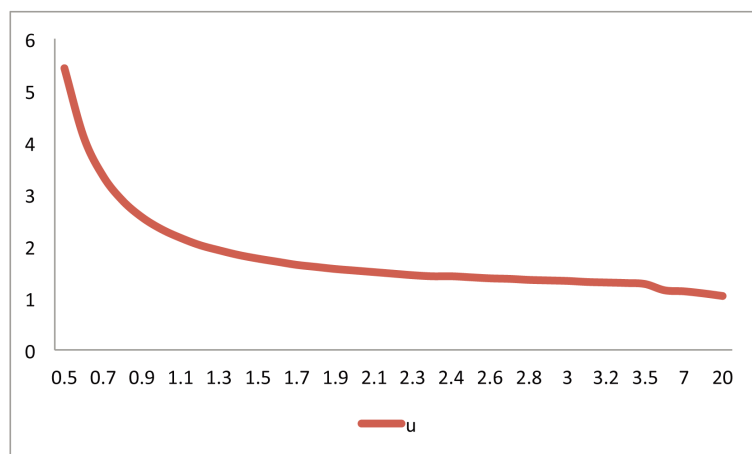

Figure 1. Plot of $u = \left(\beta/\alpha\right)^{1/\gamma}$ against γ.

(11), *the estimates given in Proposition 3 would continue to hold when we keep increasing the value of either, α or, k.*

4. Marginal Rate of Substitution

For any arbitrary production function with output Y and the factors of production, K, L and F, the marginal products with respect to the capital, labor and the factor of productively are defined as $\partial Y/\partial K$, $\partial Y/\partial L$ and $\partial Y/\partial F$ respectively. Also, the marginal rate of substitution (MRS) is defined as the ratio of the marginal product with respect to the capital by the marginal product with respect to labor, *i.e.*,

$$\text{MRS} = \left(\partial Y/\partial K\right)/\left(\partial Y/\partial L\right) \tag{12}$$

We now consider the growth of the marginal products with respect to the three factors appearing in the CES production function.

Proposition 5

For the CES production function, the marginal products with respect to all the three factors of production are increasing functions of time. Further, the marginal rate of substitution (MRS) is given by

$$\text{MRS} = \left(\alpha/\beta\right)k^{\gamma-1} \tag{13}$$

We note that (13) is well defined since by Assumption 2 we have $L \neq 0$ and so $\beta \neq 0$. It follows from (13) that the MRS is independent of both the total factor of productivity and the output per worker. Also, for $\gamma > 1$, (13) implies that $\text{MRS} \to +\infty$ as $k \to +\infty$. For $\gamma < 1$, (13) implies that $\text{MRS} \to 0$ as $k \to +\infty$.

For the Cobb-Douglas function (2), the marginal rate of substitution can be easily obtained by taking the limit as $\gamma \to 0$ in (13) and using Proposition 1; and the result thus obtained matches with the corresponding expression in ([27], p. 107).

5. Homogeneity of CES Function and Wages

We recall the standard definition that a function $Y = f\left(x_1, x_2, \cdots, x_n\right)$ of n independent variables $x_1, \cdots, x_n$ is a homogenous function of degree k if

$$f\left(tx_1, tx_2, \cdots, tx_n\right) = t^k f\left(x_1, x_2, \cdots, x_n\right)$$

for any positive scalar t. A classical theorem due to L. Euler (1703-1783) on homogeneous functions (see, e.g., [28] for a proof) states that if $Y = f\left(x_1, x_2, \cdots, x_n\right)$ is a homogeneous function of degree k with continuous partial derivatives then

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf\left(x_1, x_2, \cdots, x_n\right) \tag{14}$$

Now, it is easy to see that the CES production function, given by (1), is a homogeneous function of degree 2 in the variables K, L and F. So, from (14) with $k = 2, n = 3$ and $x_1 = K, x_2 = L, x_3 = F$ and writing Y for f, we have

$$\frac{\partial Y}{\partial K}K + \frac{\partial Y}{\partial L}L + \frac{\partial Y}{\partial F}F = 2Y \qquad (15)$$

Using (15) we now obtain a result for the wage of a worker in the context of the CES production function model (the result is closely along the lines of ([29], p. 7), Equation (15)):

Proposition 6 *Assume that in the short run the relative price of the factors adjust so that capital and labor are fully employed. Then, for the CES production function, the wage of a worker is equal to the balance remaining from the output per worker when we spend the rental price of capital times the capital per worker (assuming that there is no wage differentiation, i.e. all the workers receive the same wages).*

We remark that if *F* is a constant (in temporary contravention of Assumption 1), then the degree of homogeneity of the CES function is unity, and the Euler's theorem on homogeneous function (14) now gives (compare with (15))

$$\frac{\partial Y}{\partial K}K + \frac{\partial Y}{\partial L}L = Y \qquad (16)$$

and it is easy to verify that the statement of Proposition 6 still holds for this case by rearranging slightly the proof of Proposition 6.

6. Conclusion

We have extended some results of the neoclassical growth theory when the production function is taken to be the CES function instead of the Cobb-Douglas function. Such generalizations are of considerable interest because the Cobb-Douglas function is not suitable for some application areas because its elasticity of substation has always the fixed value 1 whereas a CES function can be designed to have any pre-determined value as its elasticity of substitution. We assume that the total factor of productivity is a variable instead of being a parameter and under this assumption the CES production function becomes a homogenous function of degree two, and so it gives increasing returns to scale. When the total factor of productivity is steady, we show that the growth rate is equal to the Solow residual. We obtain some estimates of the Solow residual and the capital deepening term. We show that when the production is not entirely capital-intensive, an increase in capital implies an increase in the ratio of the production rate. We have considered here a CES production function where the input variables are the capital, labor and the total factor of productivity; and it would be of interest to extend our results to the more general cases of several input variables and also several output variables.

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APPENDIX I: Elasticity of Substitution-Review

We briefly review here the definition of the elasticity of substitution (for further details see, e.g., [6] [7]). Consider a general production function with output Y given by

$$Y = f\left(x_1, x_2, \cdots, x_n\right) \tag{17}$$

where x_i ($1 \le i \le n$) are some independent variables and f is an arbitrary function that is differentiable partially with respect to each of the variables x_i. The *elasticity of substitution* σ_{ij} between any two distinct variables x_i and x_j measures the percentage of response of the relative marginal products of the two factors to a percentage of change in the ratio of the two quantities. It is defined as (e.g., [29], p. 509):

$$\sigma_{ij} = \frac{\partial \log_e\left(x_i/x_j\right)}{\partial \log_e\left(\left(\partial f/\partial x_i\right)/\left(\partial f/\partial x_j\right)\right)} \tag{18}$$

along the curve $f\left(x_1, x_2, \cdots, x_n\right) = \lambda$ where λ is a constant; the logarithm being taken to the base e (*i.e.*, the natural logarithm). For the CES production function, (17) takes the form given by (1) with $n = 3$ and x_1, x_2, x_3 to be the variables K, L, F respectively. For the CES function, it can be shown ([5]) that

$$\sigma_{ij} = 1/(1 - \gamma) \tag{19}$$

when we take i, j to be any two of the variables K, L, F.

APPENDIX II: Proofs

1) Proof of Proposition 2:

(i) In the steady state, $\dot{k} = 0$, and we get from (7) that $G = \dot{F}/F$. Also, when $\dot{k} = 0$, we get from (10) that $D = 0$.

For the partial converse, we note that if the growth rate is equal to the rate of the total factor of productivity, *i.e.* if $G = \dot{F}/F$, then we get from (7) and (10) that

$$D = \frac{\alpha k^{\gamma-1}\dot{k}}{\alpha k^{\gamma} + \beta} = 0 \tag{20}$$

and (20) implies that $\dot{k} = 0$ because both α and k are nonzero by our assumptions; also $\left(\alpha k^{\gamma} + \beta\right)$ is nonzero since otherwise from (6) we would get $y = 0$ and this would imply from (5) that $Y = 0$, a trivial case.

(ii) It follows from (9) that the total factor of productivity is in a steady state (*i.e.*, $\dot{F} = 0$) if and only if $G = D$.

2) Proof of Proposition 3:

From (10) on expanding by Taylor's theorem and stopping after one term, we have as a linear approximation

$$D = \frac{\dot{k}}{k}\left(1 + \frac{\beta}{\alpha k^{\gamma}}\right)^{-1} \approx \frac{\dot{k}}{k}\left(1 - \frac{\beta}{\alpha k^{\gamma}}\right) \tag{21}$$

provided $\beta/(\alpha k^{\gamma}) < 1$; or, equivalently, provided (11) holds (here $\approx$ denotes

approximately). Now, the condition $\beta/(\alpha k^\gamma) < 1$ can be expressed as:

$$0 < \left(1 - \frac{\beta}{\alpha k^\gamma}\right) < 1 \tag{22}$$

Thus, from (21) and (22) we get

$$D \leq \dot{k}/k \tag{23}$$

provided (11) holds. This proves (i).

(We remark that if instead of a linear approximation, we had taken a second degree approximation from (10), *i.e.* had stopped after the second term in the Taylor's expansion (21), then it is easy to verify that we would obtain the same result (23) provided (11) holds; we omit the details.)

Using (9) and (23) we get

$$\frac{\dot{F}}{F} \geq \left(G - \frac{\dot{k}}{k}\right) \tag{24}$$

provided (11) holds. This proves (ii).

3) Proof of Corollary 1:

(i) When $\gamma \to \infty$, we have $(\beta/\alpha)^{1/\gamma} \to 1$ for a fixed value of α, and the inequality (11) reduces to $k > 1$. (ii) When $\gamma \to 0$, we have that $k^\gamma \to 1$ and (11) gives $\alpha > \beta$, *i.e.*, $\alpha > 1/2$ since $\alpha + \beta = 1$.

4) Proof of Proposition 4:

If α_1, α_2 are any two non-zero values, then it is easy to see that for a given non-zero value of γ, we have $\alpha_2 > \alpha_1$ implies that

$$\left(\frac{1-\alpha_1}{\alpha_1}\right)^{1/\gamma} > \left(\frac{1-\alpha_2}{\alpha_2}\right)^{1/\gamma} \tag{25}$$

It follows from (25) that for any $\alpha_2 > \alpha_1$,

$$k > \left(\frac{1-\alpha_1}{\alpha_1}\right)^{1/\gamma} \Rightarrow k > \left(\frac{1-\alpha_2}{\alpha_2}\right)^{1/\gamma} \tag{26}$$

So, once the inequality (11) holds for a certain set of values of α, k, and γ, the estimates (23)-(24) in the statement of Proposition 3 would continue to hold if we keep on increasing the value of α while keeping γ fixed. Further, if we increase the value of k, the estimates (23)-(24) would still hold.

5) Proof of Proposition 5:

From (1) we get after some simplifications

$$R_K := \partial Y/\partial K = \alpha F^\gamma (K/Y)^{(\gamma-1)} = \alpha F^\gamma (k/y)^{\gamma-1} \tag{27}$$

If the production is not entirely labor intensive, then α and K are not identically zero (also $F > 0$ by the Assumption 1); so it follows from (27) that $R_K > 0$; thus the production rate with respect to capital is increasing. Similarly,

$$R_L := \partial Y/\partial L = \beta F^\gamma (L/Y)^{(\gamma-1)} = \beta F^\gamma y^{1-\gamma} \tag{28}$$

By the Assumption 2, the production is not entirely capital-intensive, and so

$\beta \neq 0$ and $L \neq 0$. We thus have from (28) that $R_L := \partial Y/\partial L > 0$; thus the production rate with respect to labor is increasing (note that by the Assumption 2, $L \neq 0$).

Again, we get that

$$R_F := \partial Y/\partial F = Y/F > 0 \qquad (29)$$

since otherwise $Y = 0$, a trivial case that we exclude (recall that $F > 0$ by the Assumption 1). Thus the production rate with respect to the total factor of productivity is also increasing. From (27) and (28) we now obtain

$$R_K/R_L = (\alpha/\beta) k^{\gamma-1} \qquad (30)$$

and this proves (13).

6) Proof of Proposition 6:

Assuming that the capital and labor are fully employed, in the short run the wage is given by $R_L = \partial Y/\partial L$. Also, we have

$$R_K = \partial Y/\partial K \quad \text{and} \quad \partial Y/\partial F = Y/F \qquad (31)$$

So, (15) can be written as

$$LR_L + KR_K = Y \qquad (32)$$

and rewriting (32) by using (5), we get

$$\text{wage} = (Y - KR_K)/L = (y - kR_K) \qquad (33)$$

and this implies that in the short run, the wage of a worker is equal to the balance remaining from the output per worker when we spend the rental price of capital times the capital per worker since y is the output per worker and R_K is the rental price and k is the capital stock per worker. This proves the result.