A GUIDE TO THE BURR TYPE XII DISTRIBUTIONS

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SUMMARY

Diagrams of areas in the $(\sqrt{\beta_1}, \beta_2)$ plane corresponding to the Burr Type XII distributions are presented.

Some key words: Burr Type XII distribution; beta coefficient diagram; Weibull distribution.

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Suppose that Z is a positive random variable with probability density function

$$p_Z(z) = \frac{k}{(1+z)^{k+1}} \qquad (z > 0)$$

where k is a positive parameter. Then for $c > 0$, $X = Z^{1/c}$ has the cumulative distribution function

$$P\{X \leq x\} = 1 - \frac{1}{(1+x^c)^k} \qquad (x > 0).$$

X is said to have the Burr distribution of Type XII with parameters c and k. The purpose of this exercise is to develop a diagram in the $(\sqrt{\beta_1}, \beta_2)$ plane corresponding to distributions of Type XII variables (and their powers). The treatment follows the approach used by Johnson and Kotz [8] for power transformations of gamma variables.

Burr [1] introduced a system of distributions in 1942 by considering distribution functions $F(x)$ satisfying the differential equation

$$\frac{dF}{dx} = A(F)\ g(x).$$

This is a generalization of the Pearson equation; the object was to fit a distribution function, rather than a density, to data and then obtain the density by differentiation. In the special case where $A(F) = F(1-F)$, the solution of Burr's equation is

$$F(x) = \frac{1}{1+\exp\{-G(x)\}}$$

where $G(x) = \int_{-\infty}^{x} g(t)dt$. The Type XII distribution is a particular form of this solution and was discussed in some detail by Burr, who noticed that the family gives rise to a useful range of values of skewness (α_3) and kurtosis (α_4). He recommended fitting by a method of cumulative moments and presented a short table of α_3 and α_4 as functions of c and k.

Burr Type XII distributions were further investigated by Hatke [6] in 1949. She presented a diagram of the moment ratio "coverage" using the (α_3^2, δ) coordinate system due to Craig [5]. It appears that the Type XII family was not studied again until 1968, when Burr [2] and Burr and Cislak [4] examined the distributions of sample median and range. By then the computation of moments had been simplified by the use of electronic computers. Burr and Cislak pointed out that the coverage is actually much greater than that claimed by Hatke. They presented a revised (α_3^2, δ) diagram which also relates the coverage to that of the Pearson system. A similar chart is to be found in a short communication by Burr [3] in which c and k values giving combinations of α_3 and α_4 are tabled.

However use of (α_3^2, δ) coordinates does not provide the clearest exposition of the Type XII family. In particular the upper and lower bounds are not apparent, (nor have they been derived analytically). Ord [9] presented a (β_1, β_2) version of the chart constructed by Burr and Cislak which does not appear to clarify the region of coverage. In fact, there is some doubt as to the validity of the shape of the lower bound. Finally it should be noted that none of the diagrams published so far indicates the effect of modifying c and k.

The rth moment about the origin of $X = Z^{1/c}$ is

$$E[Z^{r/c}] = kB(\tfrac{r}{c}+1,\ k-\tfrac{r}{c}) \qquad (r < ck)$$

so that the fourth moment is finite if $ck > 4$. Burr originally used the restrictions $c \geq 1$ and $k \geq 1$ in [1], possibly since the density function corresponding to X

$$f_X(x) = \frac{kcx^{c-1}}{(1+x^c)^{k+1}} \qquad (x > 0)$$

is unimodal at $x = [(c-1)/(ck+1)]^{1/c}$ if $c > 1$ and L-shaped if $c = 1$. In what follows k and c are taken to be positive with $ck > 4$.

The parametric equations for $\sqrt{\beta_1}$ and β_2 are:

$$\sqrt{\beta_1} = \frac{\Gamma^2(k)\lambda_3-3\Gamma(k)\lambda_2\lambda_1+2\lambda_1^3}{[\Gamma(k)\lambda_2-\lambda_1^2]^{3/2}}$$

$$\beta_2 = \frac{\Gamma^3(k)\lambda_4-4\Gamma^2(k)\lambda_3\lambda_1+6\Gamma(k)\lambda_2\lambda_1^2-3\lambda_1^4}{[\Gamma(k)\lambda_2-\lambda_1^2]^{3/2}},$$

where $\lambda_j = \Gamma(\frac{j}{c}+1)\Gamma(k-\frac{j}{c})$, $j = 1,2,3,4$. Tables of the moment-ratios were obtained for fixed values of the power parameter $1/c$. These numerical results extended the table of α_3 and α_4 constructed originally by Burr in [1] and were used in the preparation of Figures 1 and 2 which are the analogues of the diagrams in Johnson and Kotz [8]. Clearly a $(\sqrt{\beta_1}, \beta_2)$ coordinate system is the best choice for displaying the results of the calculations.

Figure 1 shows the loci of $(\sqrt{\beta_1}, \beta_2)$ points for constant c with k varying. The Type XII Burr distributions occupy a region shaped like

FIG. 1. VALUES OF $(\sqrt{\beta_1}, \beta_2)$ FOR THE TYPE XII BURR DISTRIBUTIONS

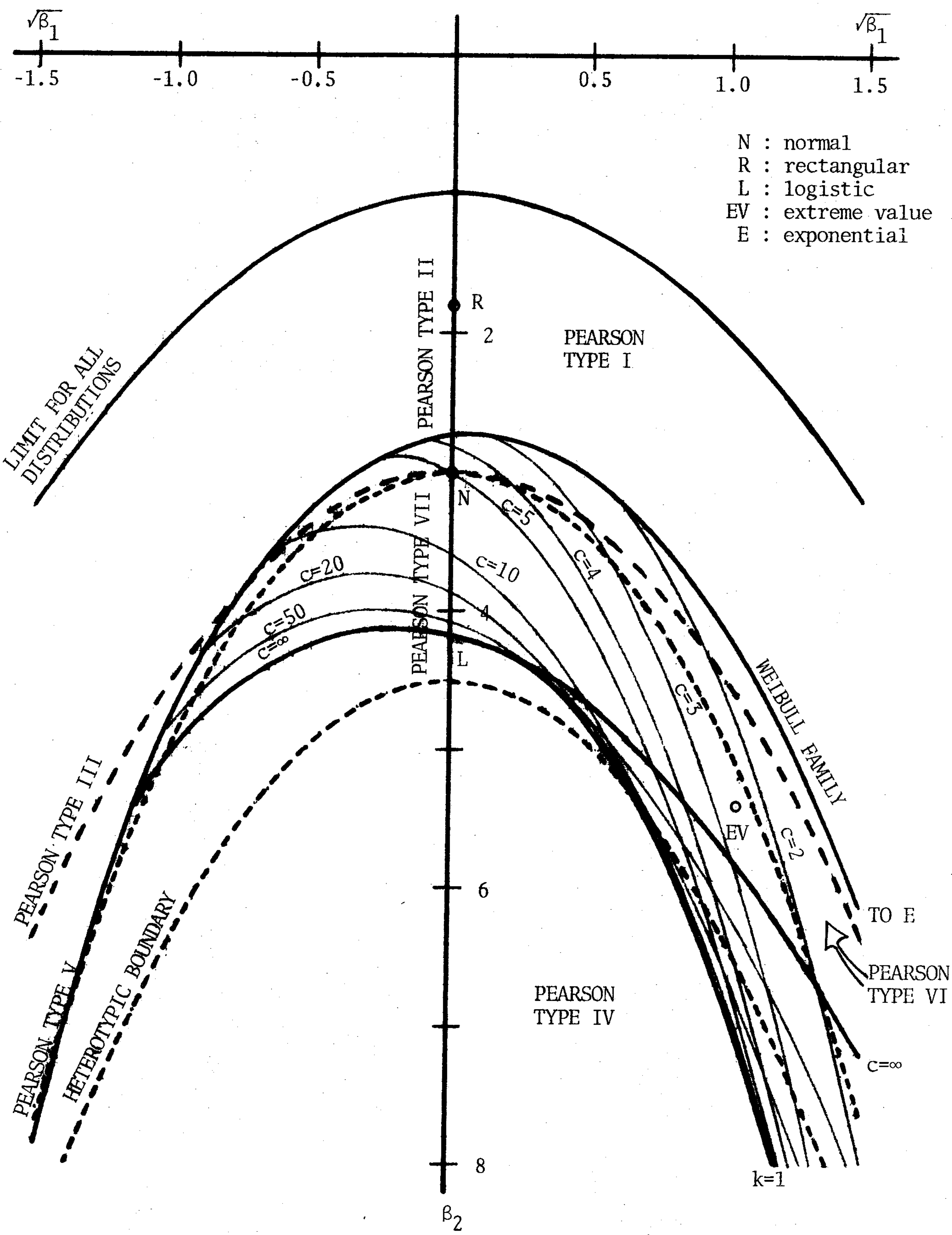
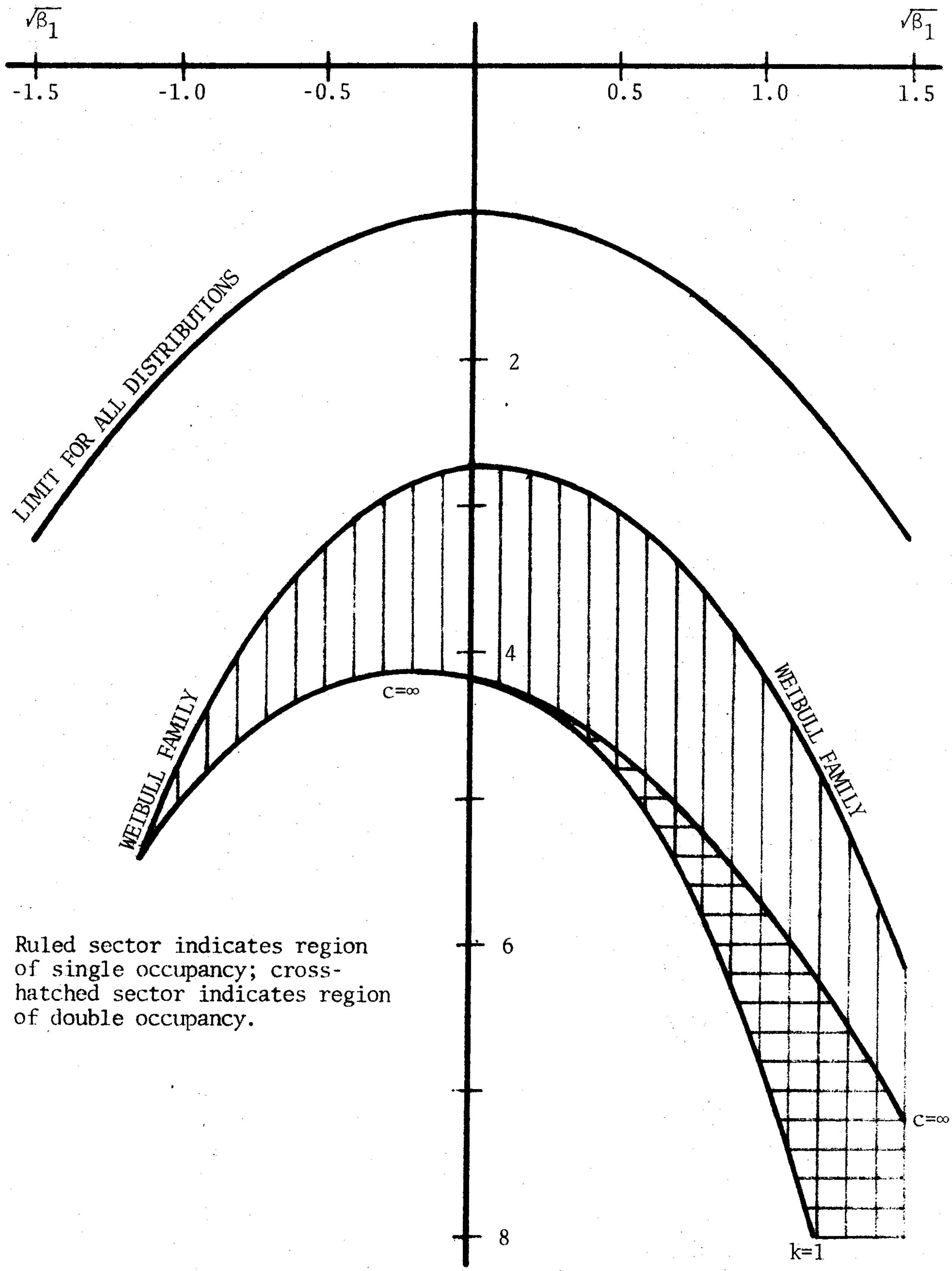


FIG. 2. EXISTENCE REGIONS FOR THE TYPE XII BURR DISTRIBUTIONS



Ruled sector indicates region of single occupancy; cross-hatched sector indicates region of double occupancy.

the prow of an ancient Roman galley. As k varies from $4/c$ to $+\infty$ the c-constant curve moves in a counter-clockwise direction toward the tip of the "prow". For $c < 3.6$ the c-constant lines terminate at endpoints with positive $\sqrt{\beta_1}$. For $c > 3.6$ the c-constant lines terminate at endpoints with negative $\sqrt{\beta_1}$.

These endpoints form the lower bound† of the Type XII region. Their parametric equations are

$$\lim_{k\to\infty} \sqrt{\beta_1} = \frac{\Gamma(\frac{3}{c}+1)-3\Gamma(\frac{2}{c}+1)\Gamma(\frac{1}{c}+1)+2\Gamma^3(\frac{1}{c}+1)}{[\Gamma(\frac{2}{c}+1)-\Gamma^2(\frac{1}{c}+1)]^{3/2}}$$

$$\lim_{k\to\infty} \beta_2 = \frac{\Gamma(\frac{4}{c}+1)-4\Gamma(\frac{3}{c}+1)\Gamma(\frac{1}{c}+1)+6\Gamma(\frac{2}{c}+1)\Gamma^2(\frac{1}{c}+1)-3\Gamma^4(\frac{1}{c}+1)}{[\Gamma(\frac{2}{c}+1)-\Gamma^2(\frac{1}{c}+1)]^2} .$$

These limits are easily obtained from the parametric equations for $\sqrt{\beta_1}$ and β_2 by applying Stirling's formula. The lower bound for the Type XII region forms part of the Weibull curve in the $(\sqrt{\beta_1}, \beta_2)$ plane. Values for the moment ratios of the Weibull distributions are tabulated in Chapter 20 of Johnson and Kotz [7]. β_2 has a minimum value of about 2.71 when $c \doteq 3.35$. The Weibull curve passes through the exponential point $(\sqrt{\beta_1} = 2, \beta_2 = 9)$, which is the endpoint for the Burr curve with $c = 1$. The Weibull density functions are unimodal; for $0 < c \leq 1$ the mode is at zero, and for $c > 1$ the mode is located to the right of the origin.

The identification of the lower bound with the Weibull family can

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† We are following the convention of presenting moment ratio diagrams upside-down. Thus the upper bounds in Figures 1 and 2 are referred to as "lower bounds" and conversely.

be explained as follows:

$$P\{X \le (\tfrac{1}{k})^{1/c}\, y\} = 1 - (1 + \frac{y^c}{k})^{-k}$$

$$= 1 - \exp\{-k \log(1 + \frac{1}{k}\, y^c\}$$

$$= 1 - \exp\{-k[\frac{y^c}{k} - \tfrac{1}{2}(\frac{y^c}{k})^2 + \ldots]\}$$

$$\to 1 - e^{-y^c} \qquad \text{as} \quad k \to \infty.$$

On the other hand, the Type XII distribution can be obtained as a smooth mixture of Weibull distributions

$$P\{W \le w\} = 1 - \exp\{-\theta w^c\} \qquad (\theta > 0;\ w > 0)$$

compounded with respect to θ, where θ has a standard gamma distribution with parameter k.

As $c \to \infty$ the endpoints of the c-constant Burr curves approach a point on the Weibull curve. In order to derive the limiting coordinates we use the fact that the gamma function is analytic in the positive half-plane, so that

$$\Gamma(1+z) = \Gamma(1) + z\Gamma'(1) + \frac{z^2}{2}\Gamma''(1) + \frac{z^3}{3!}\,\Gamma^{(3)}(1) + \frac{z^4}{4!}\,\Gamma^{(4)}(1) + \ldots ,$$

where the coefficients needed are:

$$\Gamma(1) = 1$$

$$\Gamma'(1) = \Psi(1) = -\gamma$$

$$\Gamma''(1) = \gamma^2 + \zeta(2) = \gamma^2 + \frac{\pi^2}{6}$$

$$\Gamma^{(3)}(1) = -\gamma^3 - 3\gamma\zeta(2) - 2\zeta(3)$$

and

$$\Gamma^{(4)}(1) = \gamma^4 + 6\gamma^2\zeta(2) + 3\zeta^3(2) + 8\gamma\zeta(3) + 6\zeta(4).$$

Here $\psi(z)$ is the digamma function, $\gamma = 0.57722$ is Euler's constant, and $\zeta(z)$ is the Riemann zeta function. (See Whittaker and Watson [10].) It can be shown that

$$\lim_{c\to\infty}\lim_{k\to\infty} \sqrt{\beta_1} = \frac{\Gamma^{(3)}(1)+3\gamma\Gamma''(1)-2\gamma^3}{[\Gamma''(1)-\gamma^2]^{3/2}} \doteq -1.14$$

$$\lim_{c\to\infty}\lim_{k\to\infty} \beta_2 = \frac{\Gamma^{(4)}(1)+4\gamma\Gamma^{(3)}(1)+6\gamma^2\Gamma''(1)-3\gamma^4}{[\Gamma''(1)-\gamma^2]^2}$$

$$\doteq 5.35.$$

The limiting Burr curve $c = +\infty$ forms part of the upper bound for the Burr Type XII region, passing through the logistic point $(\sqrt{\beta_1} = 0, \beta_2 = 4.2)$ and approaching the Weibull curve asymptotically as $k \to 0^+$. In the Burr region below the curve $c = \infty$ each point $(\sqrt{\beta_1}, \beta_2)$ corresponds to a unique pair (c,k) and hence a unique Type XII distribution. (See Figure 2.) Above the curve $c = \infty$ there are two Type XII distributions for each $(\sqrt{\beta_1}, \beta_2)$ point, one with $c > 3.6$ and one with $c < 3.6$.

The upper bound for the Burr region in the positive $\sqrt{\beta_1}$ half-plane presents a special problem. It is clearly not a c-limiting member of the Type XII family. Initially it was thought that this section of the upper bound might represent the moment ratios of some generalization of the logistic distribution, and several known generalizations were considered unsuccessfully. For example, a natural "candidate" is the Burr Type II family with distribution function $(e^{-y} + 1)^{-k}$. Tabulation of moment ratios shows that the Type II curve lies well within the Type XII region.

Actually the upper bound corresponds to the Burr Type XII distributions for which $k = 1$ and $c > 4$. This can be shown by applying the method of Lagrange multipliers in order to minimize $\sqrt{\beta_1}$ subject to the constraint $\beta_2 = \text{constant} \geq 4.2$. Using the parametric equations for $\sqrt{\beta_1}$ and β_2, one obtains

$$\left[\frac{\partial\beta_1}{\partial c}\right]\left[\frac{\partial\beta_2}{\partial c}\right]^{-1} = \left[\frac{\partial\beta_1}{\partial k}\right]\left[\frac{\partial\beta_2}{\partial k}\right]^{-1}.$$

This is an extremely complicated equation, and the algebraic details are omitted, but it can be shown that $k = 1$ necessarily. (Undoubtedly a simpler theoretical derivation can be given, although the result is substantiated numerically.) Consequently the upper bound for $\sqrt{\beta_1} > 0$ corresponds to the distributions

$$P\{X \leq x\} = 1 - \frac{1}{1+x^c} \qquad (x>0;\ c>4),$$

where $c = \infty$ corresponds to the logistic distribution.

As indicated by Figure 1 the Burr Type XII region covers sections of the areas corresponding to the main Pearson Types I, IV, and VI. The section of the Type III line joining the normal and exponential points is also included, as are those Type VII distributions represented by the line segment joining the normal and logistic points. Part of the Type II line segment is included in the Burr region, but not the rectangular endpoint ($\sqrt{\beta_1} = 0$, $\beta_2 = 1.8$) as claimed by Burr and Cislak [4]. On the other hand, the extreme value point is included.

The Burr region contains points in the heterotypic region. The lognormal curve (omitted for simplicity) lies between the Type III and

Type V curves, so it is clear that the Burr region covers areas corresponding to each of the three families in the Johnson system.

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