

A guide to tropicalizations

Walter Gubler

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Abstract

Tropicalizations form a bridge between algebraic and convex geometry. We generalize basic results from tropical geometry which are well-known for special ground fields to arbitrary non-archimedean valued fields. To achieve this, we develop a theory of toric schemes over valuation rings of rank 1. As a basic tool, we use techniques from non-archimedean analysis.

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1 Introduction

Let us consider a field K endowed with a non-archimedean absolute value $|\cdot|$. We fix coordinates x_1, \dots, x_n on the split multiplicative torus \mathbb{G}_m^n over K . Using logarithmic coordinates $-\log|x_1|, \dots, -\log|x_n|$, any closed subscheme X of \mathbb{G}_m^n transforms into a finite union $\text{Trop}(X)$ of polyhedra in \mathbb{R}^n . This process is called *tropicalization* and it can be used to transform a problem from algebraic geometry into a corresponding problem in convex geometry which is usually easier. If the toric coordinates are well suited to the problem, it is sometimes possible to use a solution of the convex problem to solve the original algebraic problem. Another strategy is to vary the ambient torus to compensate the loss of information due to the tropicalization process.

Tropicalization originates from a paper of Bergman [Berg] on logarithmic limit sets. The convex structure of the tropical variety $\text{Trop}(X)$ was worked out by Bieri–Groves [BG] with applications to geometric group theory in mind. Sturmfels [Stu] pointed out that $\text{Trop}(X)$ is a subcomplex of the Gröbner complex. In fact, the polyhedral complex $\text{Trop}(X)$ has some natural weights satisfying a balancing condition which appears first in Speyer’s thesis [Spe]. This relies on the description of the Chow cohomology of a toric variety given by Fulton and Sturmfels [FS]. An intrinsic approach to tropical geometry was proposed by Mikhalkin. The idea is to develop tropical geometry as some sort of algebraic geometry based on the min-plus algebra where every $\text{Trop}(X)$ occurs as a natural object. This approach was used by Mikhalkin to prove celebrated results in enumerative geometry (see [Mik]). These results popularized tropical geometry generating a huge amount of interesting results and applications.

In tropical literature, one usually considers tropicalizations under severe restrictions for the ground field suitable to its own applications. These restrictions can be subdivided into four groups. First, many papers are written in case of the trivial valuation on K . Then $\text{Trop}(X)$ is a finite union of rational cones. This is a setting which occurs very often in algebraic geometry. A second group of people is working under the assumption that K is the field of Puiseux series or a related field. This is often used by people interested in the combinatorial structure and effective computation of $\text{Trop}(X)$. In this case, K has a natural grading, contains the residue field and the valuation has a canonical section which makes many arguments easier.

A third group is assuming that the valuation is discrete. Most of the valuations occurring in applications (e.g. in number theory) are discrete. This makes it possible to use noetherian models over the valuation ring. Finally, a fourth group of people is working with algebraically closed ground fields endowed with a non-trivial complete absolute value. This is suitable for using arguments from the theory of rigid analytic spaces. As an excellent source for this case, we refer the reader to the recent paper of Baker–Payne–Rabinoff [BPR].

The goal of this paper is to survey basic results about tropicalizations and to generalize them to arbitrary non-archimedean absolute values on K . This will make these results accessible to all kind of applications. To handle the difficulties mentioned above, we will use methods from the theory of Berkovich analytic spaces which are very well-suited for this general framework. This is not surprising as even in the original Bieri–Groves paper, the analytification X^{an} of X with its Berkovich analytic topology was implicitly used before Berkovich introduced his new concept in rigid analytic geometry. Most parts of the paper can be read having just a topological understanding of X^{an} which is rather elementary. The paper is not meant as an introduction to the subject. For this purpose, we refer the reader to the forthcoming book of Maclagan–Sturmfels [MS].

The paper is organized as follows: In Section 2, we introduce the analytification of an algebraic scheme X over K and we sketch how this fits into the theory of Berkovich analytic spaces. In Section 3, we define the tropicalization map and the tropicalization of a closed subscheme X of the split torus \mathbb{G}_m^n over K . In Section 4, we study models over the valuation ring. For a potentially integral point of the generic fibre, we define its reduction to the special fibre. We compare this with the reduction map from the theory of strictly affinoid algebras. In Section 5, the initial degeneration of a closed subscheme of \mathbb{G}_m^n at $\omega \in \mathbb{R}^n$ is studied leading to an alternative characterization of the tropicalization.

In Section 6, we investigate normal affine toric schemes over a valuation ring associated to polyhedra. In Section 7, we globalize these results assigning a normal toric scheme to every admissible fan in $\mathbb{R}^n \times \mathbb{R}_+$. This is rather new and it generalizes the theory from [KKMS] worked out in the special case of a discrete valuation. In Section 8, we introduce the tropical cone of X as a subset of $\mathbb{R}^n \times \mathbb{R}_+$. This new notion can be seen as a degeneration of the tropical variety with respect to the given valuation to the tropical variety with respect to the trivial valuation. It is very convenient to work with the tropical cone in the framework of tropical schemes over a valuation ring.

In Section 9, we study projectively embedded toric varieties over the valuation ring which are not necessarily normal. This generalizes work of Eric Katz. In Section 10, we show that the tropical variety is a subcomplex of the Gröbner complex. In Section 11, we study the closure of X in a toric scheme over the valuation ring and in Section 12, we generalize Tevelev’s tropical compactifications to our setting. We introduce tropical multiplicities in Section 13 leading to the Sturmfels–Tevelev multiplicity formula for tropical cycles. In Section 14, we characterize proper compactifications of X in a toric scheme which intersect all orbits properly. In the appendix, we collect results from convex geometry which are needed in the paper.

Terminology

In $A \subset B$, A may be equal to B . The complement of A in B is denoted by $B \setminus A$ as we reserve $-$ for algebraic purposes. The zero is included in \mathbb{N} and in \mathbb{R}_+ .

All occurring rings and algebras are commutative with 1. If A is such a ring, then the group of multiplicative units is denoted by A^\times . A variety over a field is a separated reduced scheme of finite type. We denote by \bar{F} an algebraic closure of the field F .

For $\mathbf{m} \in \mathbb{Z}^n$, let $\mathbf{x}^{\mathbf{m}} := x_1^{m_1} \cdots x_n^{m_n}$ and $|\mathbf{m}| := |m_1| + \cdots + |m_n|$. The standard scalar product of $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^n$ is denoted by $\mathbf{u} \cdot \mathbf{u}' := u_1 u'_1 + \cdots + u_n u'_n$. The terminology from convex geometry is explained in the appendix.

In the whole paper, the base field is a *valued field* (K, v) which means that the field K is endowed with a non-archimedean absolute value $|\cdot|$ which might be trivial. The corresponding valuation is $v := -\log(|\cdot|)$ with value group $\Gamma := v(K^*)$. We get a valuation ring $K^\circ := \{\alpha \in K \mid v(\alpha) \geq 0\}$ with maximal ideal $K^{\circ\circ} := \{\alpha \in K \mid v(\alpha) > 0\}$ and residue field $\tilde{K} := K^\circ / K^{\circ\circ}$.

We fix a free abelian group M of rank n with dual $N := \text{Hom}(M, \mathbb{Z})$. An element of M is usually denoted by u and an element of N is usually denoted by ω . We get the duality pairing $\langle u, \omega \rangle := \omega(u)$. We have the split torus $\mathbb{T} = \text{Spec}(K^\circ[M])$ over K° with generic fibre T . Then M might be seen as the character group of this torus and the character corresponding to $u \in M$ is denoted by χ^u . If G is an abelian group, then $N_G := N \otimes_{\mathbb{Z}} G$ denotes the base change of N to G . Similarly, \mathbb{T}_A denotes the base change of \mathbb{T} to a K° -algebra A .

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2 Analytification

In this section, we recall the construction of the Berkovich analytic space X^{an} associated to an algebraic scheme X over the field K with non-archimedean complete absolute value $|\cdot|$ and corresponding valuation $v := -\log(|\cdot|)$. Note that completeness is no restriction of generality as analytic constructions are always performed over complete fields. In general, this may be achieved by base change to the completion of K . The topological part 2.1–2.6 of the construction is elementary and essential for the understanding of the whole paper. The remaining analytic part is more technical and may be skipped in a first reading. Details and proofs for this section may be found in [Berk1] and [Tem].

2.1 We start with the construction for an affine scheme $X = \text{Spec}(A)$ of finite type over K . Then the *Berkovich analytic space associated to X* is the set of multiplicative seminorms on A extending the given absolute value on K . In other words, we consider maps $p : A \rightarrow \mathbb{R}_+$ characterized by the properties

- (a) $p(fg) = p(f)p(g)$
- (b) $p(1) = 1$
- (c) $p(f + g) \leq p(f) + p(g)$
- (d) $p(\alpha) = |\alpha|$

for all $f, g \in A$ and $\alpha \in K$. It is easy to see that the triangle inequality (c) is equivalent to the ultrametric triangle inequality

$$p(f + g) \leq \max(p(f), p(g)).$$

We denote this analytification of X by X^{an} and we endow it with the coarsest topology such that the maps $X^{\text{an}} \rightarrow \mathbb{R}, p \mapsto p(f)$ are continuous for every $f \in A$. We embed the set of closed points of X into X^{an} by mapping P to the seminorm p given by $p(f) = |f(P)|$.

Remark 2.2 Let $X = \text{Spec}(A)$ be an affine scheme of finite type over K . For $p \in X^{\text{an}}$, the integral domain $A/\{a \in A \mid p(a) = 0\}$ is endowed with a canonical multiplicative norm induced by p . We conclude that its quotient field L is endowed with an absolute value $|\cdot|_w$ extending $|\cdot|$. The canonical homomorphism $A \rightarrow L$ gives an L -rational point P of $\text{Spec}(A)$ and we may retrieve the seminorm by $p(a) = |a(P)|_w$ for any $a \in A$.

Conversely, any valued field (L, w) extending (K, v) and any L -rational point P of X give rise to an element $p \in X^{\text{an}}$ by $p(a) := |a(P)|_w$. Obviously, different L -valued points might induce the same seminorm on A .

Lemma 2.3 *Let $X = \text{Spec}(A)$ be an affine scheme of finite type over K and let (F, u) be a complete valued field extending (K, v) . Then the restriction map of seminorms gives a continuous surjective map from $(X_F)^{\text{an}}$ onto X^{an} .*

Proof: Continuity is obvious from the definitions. For $p \in X^{\text{an}}$, there is a valued field (L, w) and $P \in X(L)$ as in Remark 2.2. By Lemma 5.2 below, there is a valued field (F', u') extending both (L, w) and (F, u) . We conclude that P is also an F' -rational point of X_F and hence it gives rise to a seminorm $p' \in (X_F)^{\text{an}}$. By construction, p' extends p proving surjectivity. \square

2.4 For any scheme X of finite type over K , we choose an open affine covering $\{U_i\}_{i \in I}$. Then we define the *Berkovich analytic space* X^{an} associated to X by glueing the spaces U_i^{an} obtained in 2.1. We get a topological space which is locally compact. It is Hausdorff if and only if X is separated over K .

2.5 We will use analytifications also in situations where the valuation v is not complete. Then we define X^{an} over the completion K_v of K . Let \bar{K} be the algebraic closure of K in an algebraic closure of K_v and hence every \bar{K} -rational point of X induces a point in X^{an} as in Remark 2.2. If the valuation is non-trivial, then the closed points of X are dense in X^{an} . If v is complete, then this follows from [Berk1], Proposition 2.1.15. The general case may be deduced from the complete case using that any variety over an algebraically closed field is birational to a hypersurface and continuity of roots ([BGR], 3.4.1).

2.6 If $\varphi : X \rightarrow Y$ is a morphism of schemes of finite type over K , then we have a canonical map $\varphi^{\text{an}} : X^{\text{an}} \rightarrow Y^{\text{an}}$ between the associated Berkovich analytic spaces. It is easy to see that it is enough to define the map locally, i.e. we may assume that X and Y are affine. Then we set $\varphi^{\text{an}}(p) := p \circ \varphi^{\sharp}$ for any multiplicative seminorm p on $\mathcal{O}(X)$.

2.7 We will explain below how X^{an} is endowed with an analytic structure. Of course, the analytic structure will depend on the underlying scheme structure. First we handle the case of the affine space $\mathbb{A}^n := \text{Spec}(K[x_1, \dots, x_n])$. For $f(\mathbf{x}) = \sum_{\mathbf{m}} \alpha_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in K[x_1, \dots, x_n]$, we have the *Gauss norm*

$$|f(\mathbf{x})| = \max_{\mathbf{m}} |\alpha_{\mathbf{m}}|. \quad (1)$$

The *Tate algebra* is defined as

$$K\langle x_1, \dots, x_n \rangle := \left\{ \sum_{\mathbf{m}} \alpha_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in K[[x_1, \dots, x_n]] \mid \lim_{|\mathbf{m}| \rightarrow \infty} |\alpha_{\mathbf{m}}| = 0 \right\}$$

and it is the completion of $K[x_1, \dots, x_n]$ with respect to the Gauss norm. The corresponding Banach norm $|\cdot|$ on $K\langle x_1, \dots, x_n \rangle$ is also defined by (1). The closed ball \mathbb{B}^n of radius 1 in \mathbb{A}^n is defined as the set of multiplicative seminorms on $K\langle x_1, \dots, x_n \rangle$ which are bounded by the Gauss norm, i.e. we have again properties

(a)–(d) from 2.1 for all $a, b \in K\langle x_1, \dots, x_n \rangle$ and the additional property $p(f) \leq |f|$. Note that a closed point of \mathbb{A}^n is in \mathbb{B}^n if and only if all its coordinates have absolute value at most 1. It is easy to see that the supremum norm on \mathbb{B}^n is equal to the Gauss-norm.

More generally, we may consider $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ and the Banach algebra $K\langle r_1^{-1}x_1, \dots, r_n^{-1}x_n \rangle$ which is given by completion of $K[x_1, \dots, x_n]$ with respect to the weighted Gauss norm

$$|f(\mathbf{x})|_{\mathbf{r}} := \max_{\mathbf{m}} |\alpha_{\mathbf{m}}| \mathbf{r}^{\mathbf{m}}.$$

If we repeat the above construction, we get the closed ball $\mathbb{B}_{\mathbf{r}}^n$ of radius \mathbf{r} in \mathbb{A}^n . It is easy to see that $(\mathbb{A}^n)^{\text{an}}$ may be covered by a union of such balls. They serve as compact charts for the analytic structure of $(\mathbb{A}^n)^{\text{an}}$.

2.8 An *affinoid algebra* is a Banach algebra $(\mathcal{A}, \|\cdot\|)$ which is isomorphic to $K\langle r_1^{-1}x_1, \dots, r_n^{-1}x_n \rangle/I$ for an ideal I and such that the norm $\|\cdot\|$ is equivalent to the quotient norm

$$\|f + I\|_{\text{quot}} := \inf\{\|g\| \mid g \in f + I\}$$

on \mathcal{A}/I . It is called a *strictly affinoid algebra* if we may choose $r_i = 1$ for all $i = 1, \dots, n$.

The Banach norm does not matter if the valuation is non-trivial (see [Tem], Fact 3.1.15). In this case, all such Banach norms on \mathcal{A} are equivalent and every homomorphism between affinoid algebras is bounded. In classical rigid geometry as in [BGR], one considers only strictly affinoid algebras and they are called affinoid algebras there.

2.9 For an affinoid algebra as above, the *spectral radius* is defined by $\rho(a) := \inf\{\|a\|^{1/n} \mid n \geq 1\}$ for $a \in \mathcal{A}$. We set

$$\mathcal{A}^\circ := \{a \in \mathcal{A} \mid \rho(a) \leq 1\}, \quad \mathcal{A}^{\circ\circ} := \{a \in \mathcal{A} \mid \rho(a) < 1\}.$$

and the *residue algebra* is defined by $\tilde{\mathcal{A}} := \mathcal{A}^\circ / \mathcal{A}^{\circ\circ}$.

2.10 The *Berkovich spectrum* $\mathcal{M}(\mathcal{A})$ of a \mathbb{K} -affinoid algebra \mathcal{A} is defined as the set of multiplicative bounded seminorms p on \mathcal{A} , i.e. for all $a, b \in \mathcal{A}$, we have

- (a) $p(ab) = p(a)p(b)$
- (b) $p(1) = 1$
- (c) $p(a + b) \leq p(a) + p(b)$
- (d) $p(a) \leq \rho(a)$.

It is endowed with the coarsest topology such that the maps $p \mapsto p(a)$ are continuous for all $a \in \mathcal{A}$. We get a compact space. The spectral radius $\rho(a)$ turns out to be equal to the supremum seminorm of $a \in \mathcal{A}$ on $\mathcal{M}(\mathcal{A})$.

Example 2.11 Let $\mathcal{A} = K\langle r_1^{-1}x_1, \dots, r_n^{-1}x_n \rangle/I$. A *rational subdomain* of $X := \mathcal{M}(\mathcal{A})$ is defined by

$$X \left(\mathbf{s}^{-1} \frac{\mathbf{f}}{g} \right) := \{x \in X \mid |f_j(x)| \leq s_j |g(x)|, j = 1, \dots, m\}$$

where g, f_1, \dots, f_m generate the unit ideal in \mathcal{A} and $s_1, \dots, s_m > 0$. The corresponding affinoid algebra is

$$\mathcal{A} \left\langle \mathbf{s}^{-1} \frac{\mathbf{f}}{g} \right\rangle := K\langle \mathbf{r}^{-1}\mathbf{x}, s_1^{-1}y_1, \dots, s_m^{-1}y_m \rangle / \langle I, g(\mathbf{x})y_j - f_j \mid j = 1, \dots, m \rangle$$

(see [Berk1], Remarks 2.2.2).

2.12 We will not give the precise definition of a *Berkovich analytic space* X (see [Berk2] for details). Roughly speaking it is a topological space endowed with an atlas such that each chart is homeomorphic to the Berkovich spectrum of an affinoid algebra and then there are some compatibility conditions. Analytic functions on such a chart $\mathcal{M}(\mathcal{A})$ are given by the elements of \mathcal{A} .

A morphism $\varphi : X_1 \rightarrow X_2$ between Berkovich spaces X_1 and X_2 is a continuous map such that for every chart U_1 of X_1 with $\varphi(U_1)$ contained in a chart U_2 of X_2 and every analytic function f_2 on U_2 , the function $\varphi^\sharp(f_2) := f_2 \circ \varphi$ is an analytic function on U_1 .

2.13 If X is a scheme of finite type over K as at the beginning, then X^{an} is a Berkovich analytic space. As charts, we may choose $U^{\text{an}} \cap \mathbb{B}_{\mathbb{R}}^n$, where U is an affine open subset of X realized as a closed subset of \mathbb{A}^n and $\mathbb{B}_{\mathbb{R}}^n$ is a closed ball in \mathbb{A}^n . Serre's GAGA principle holds also in the non-archimedean framework. For details, we refer to [Berk1], §3.4 and §3.5.

Remark 2.14 In Example 2.11, we have defined rational subdomains of the Berkovich spectrum $X := \mathcal{M}(\mathcal{A})$. More generally, one can define *affinoid subdomains* of X by a certain universal property. They are Berkovich spectra contained in X which are used for localization arguments on Berkovich spaces. For details, we refer to [Berk1], Section 2.2. By the Gerritzen–Grauert theorem, every affinoid subdomain is a union of rational domains if the valuation v is non-trivial.

Roughly speaking, an *analytic subdomain* of a Berkovich analytic space X is a subset which behaves locally like an affinoid subdomain. For a precise definition and for properties, we refer to [Berk1], Section 3.1. In this paper, we need only analytic functions on affinoid subdomains of X^{an} where they are just elements of the corresponding affinoid algebra. However, it should be noted that analytic functions form a sheaf on open subsets giving X^{an} the structure of a locally ringed space (see [Berk1], or [BPS], §2.2, for a neat description).

3 Tropicalization

In this section, we consider a closed subscheme X of the multiplicative torus T over the valued field (K, v) and we will define the tropical variety $\text{Trop}_v(X)$ associated to X . The tropicalization process is a transfer from algebraic geometry to convex geometry in \mathbb{R}^n . We will use the analytifications X^{an} and T^{an} from the previous section which are always performed over the completion of K .

3.1 Let M be a free abelian group of rank n and let $N = \text{Hom}(M, \mathbb{Z})$ be the dual group. Then we consider the multiplicative torus $T := \text{Spec}(K[M])$ with character group M . We have the *tropicalization map*

$$\text{trop}_v : T^{\text{an}} \rightarrow N_{\mathbb{R}}, \quad p \mapsto \text{trop}_v(p),$$

where $\text{trop}_v(p)$ is the element of $N_{\mathbb{R}} = \text{Hom}(M, \mathbb{R})$ given by

$$\langle u, \text{trop}_v(p) \rangle := -\log(p(\chi^u))$$

with χ^u the character of T corresponding to $u \in M$. Choosing coordinates x_1, \dots, x_n on $T = \mathbb{G}_m^n$, we may identify M and N with \mathbb{Z}^n and we get an explicit description

$$\text{trop}_v : T^{\text{an}} \rightarrow \mathbb{R}^n, \quad p \mapsto (-\log(p(x_1)), \dots, -\log(p(x_n))).$$

It is immediate from the definitions that the map trop_v is continuous. This is the big advantage of working with Berkovich analytic spaces in this framework as we may use their nice topological properties.

Definition 3.2 We define the *tropical variety associated to X* by $\text{Trop}_v(X) := \text{trop}_v(X^{\text{an}})$. In Section 13, we will complete the definition of a tropical variety by assigning certain weights.

In the following result, we refer the reader to the appendix for the terminology borrowed from convex geometry.

Theorem 3.3 (Bieri–Groves) *$\text{Trop}_v(X)$ is a finite union of Γ -rational polyhedra in $N_{\mathbb{R}}$. If X is of pure dimension d , then we may choose all the polyhedra d -dimensional.*

Proof: The proof is given in [BG], Theorem A. Note that even the definition of X^{an} occurs implicitly in this paper. For a translation into tropical language, we refer to [EKL], Theorem 2.2.3. In Theorem 10.14, we will give a proof of the first statement using the Gröbner fan. A proof for dimensionality is given in [Gub3], Proposition 5.4, which generalizes to closed analytic subvarieties. \square

Remark 3.4 If the absolute value on K is trivial, then a Γ -rational polyhedron is just a rational polyhedral cone. In this case, we conclude that $\text{Trop}_v(X)$ is a finite union of such cones.

We illustrate the advantage of Berkovich spaces by giving the proof of the following well-known result (see [EKL], Theorem 2.2.7):

Proposition 3.5 If X is connected, then $\text{Trop}_v(X)$ is connected.

Proof: If X is connected, then X^{an} is connected ([Berk1], Theorem 3.4.8 and Theorem 3.5.3). This is a rather nontrivial fact. By continuity of the tropicalization map, we conclude that $\text{Trop}_v(X)$ is connected. \square

Proposition 3.6 *Let (L, w) be a valued field extending (K, v) . Then we have $\text{Trop}_w(X_L) = \text{Trop}_v(X)$.*

Proof: Let $\varphi : (X_L)^{\text{an}} \rightarrow X^{\text{an}}$ be the restriction map of seminorms. We have seen in Lemma 2.3 that φ is surjective. Using $\text{trop}_w = \text{trop}_v \circ \varphi$, we get the claim. \square

The following result shows that our definition of a tropical variety agrees with the usual one.

Proposition 3.7 *Let (L, w) be an algebraically closed valued field extending (K, v) endowed with a non-trivial absolute value $|\cdot|_w$ and let x_1, \dots, x_n be torus coordinates on T . Then $\text{Trop}_v(X)$ is equal to the closure of*

$$\{(-\log |x_1|_w, \dots, -\log |x_n|_w) \mid \mathbf{x} \in X(L)\}$$

in \mathbb{R}^n .

Proof: By base change and Proposition 3.6, we may assume that K is algebraically closed and that v is non-trivial. We have seen in 2.5, that $X(K)$ is dense in X^{an} and hence continuity of the tropicalization map yields the claim. \square

4 Models over the valuation ring and reduction

In this section, (K, v) is a valued field with valuation ring K° and residue field \tilde{K} . We will study models of a scheme X of finite type over K . The models are flat schemes over K° but not necessarily of finite type. We will obtain a model of a closed subscheme of X by taking the closure. For integral points of a model, there is always a reduction modulo the maximal ideal $K^{\circ\circ}$ which is a point in the special fibre. We will compare it with the reduction from the theory of strictly affinoid algebras.

Definition 4.1 A K° -model of a scheme X over K is a flat scheme \mathcal{X} over K° with generic fibre $\mathcal{X}_\eta := \mathcal{X}_K = X$. The special fibre $\mathcal{X}_{\bar{K}}$ of \mathcal{X} is denoted by \mathcal{X}_s . The model \mathcal{X} is called *algebraic* if it is of finite type over K° .

Lemma 4.2 A module M over K° is flat if and only if M is torsion-free.

Proof: Any flat module is obviously torsion-free. If the base is a valuation ring, then the converse holds by [Bou], ch. VI, §3.6, Lemma 1. \square

4.3 Let $\mathcal{X} = \text{Spec}(A)$ be a flat scheme over K° with generic fibre $X = \mathcal{X}_\eta$. Then we have $X = \text{Spec}(A_K)$ for $A_K := A \otimes_{K^\circ} K$. Note that flatness implies $A \subset A_K$. A closed subscheme Y of X is given by an ideal I_Y in A_K . We define the *closure* \bar{Y} of Y in \mathcal{X} as the closed subscheme of \mathcal{X} given by the ideal $I_Y \cap A$.

Proposition 4.4 The closure of Y is the unique closed subscheme of \mathcal{X} with generic fibre Y which is flat over K° .

Proof: It is clear that $A/(I_Y \cap A)$ is K° -torsion free and hence flat over K° by Lemma 4.2. For every $f \in A_K$, there is a non-zero $\lambda \in K^\circ$ with $\lambda f \in A$. We conclude that $I_Y \cap A$ generates I_Y as an ideal in A_K and hence the generic fibre of \bar{Y} is Y .

Let \mathcal{Y} be any closed subscheme of \mathcal{X} with generic fibre Y which is flat over K° . Then \mathcal{Y} is given by an ideal J in A such that J generates I_Y as an ideal in A_K . We conclude that $J \subset I_Y \cap A$. Hence we have a canonical homomorphism $h : A/J \rightarrow A/(I_Y \cap A)$. By flatness over K° , we have $A/J \subset A_K/I_Y$ and h factors through this inclusion. We deduce that h is one-to-one proving $J = I_Y \cap A$. \square

Corollary 4.5 Let $\psi : \mathcal{X}' \rightarrow \mathcal{X}$ be a flat morphism of affine schemes with generic fibre $\psi_\eta : X' \rightarrow X$. Then we have $(\psi_\eta)^{-1}(Y) = \psi^{-1}(\bar{Y})$ for the closures in \mathcal{X}' and \mathcal{X} .

Proof: As $\psi^{-1}(\bar{Y})$ is a closed subscheme of \mathcal{X}' with generic fibre $(\psi_\eta)^{-1}(Y)$ which is flat over K° , the claim follows from Proposition 4.4. \square

Remark 4.6 In particular, this shows that localization is compatible with taking the closure. Therefore we may define the closure in any flat scheme \mathcal{X} over K° . Indeed, let Y be a closed subscheme of $X := \mathcal{X}_\eta$. First, we define \bar{Y} locally on affine charts as in 4.3 and then we glue the affine pieces to get a closed subscheme \bar{Y} of \mathcal{X} by compatibility of the affine construction with localization. The closure is still characterized by Proposition 4.4. Moreover, Corollary 4.5 immediately yields that the formation of the closure is compatible with flat pull-back. Note also that the underlying set of \bar{Y} is the topological closure of Y in \mathcal{X} .

Corollary 4.7 Let (L, w) be a valued field extension of (K, v) and let \mathcal{X} be a flat scheme over K° . For a closed subscheme Y of $X = \mathcal{X}_\eta$, we have $(\bar{Y})_{L^\circ} = \bar{Y}_L$ with closures taken in \mathcal{X} and \mathcal{X}_{L° .

Proof: Note that the base change morphism \mathcal{X}_{L° is flat. Taking the closure depends only on the model and not on the base and hence compatibility with flat pull-back (Corollary 4.5) yields the claim. \square

4.8 For an L° -integral point P of \mathcal{X} , the reduction $\pi(P) \in \mathcal{X}_s$ is defined as the image of the closed point of $\text{Spec}(L^\circ)$ with respect to the morphism $\text{Spec}(L^\circ) \rightarrow \mathcal{X}$ defining P . If $\mathcal{X} = \text{Spec}(A)$ is affine, then $\pi(P)$ is given by the prime ideal $\{a \in A \mid |a(P)|_w < 1\}$ in A .

4.9 Let X be a scheme of finite type over K with K° -model \mathcal{X} . Using that every p in X^{an} is induced by an L -rational point for a valued field extension (L, w) of (K, v) (see Remark 2.2), we can define the reduction on the subset X° of points of X^{an} given as the union of all $U^\circ := \{p \in U^{\text{an}} \mid p(f) \leq 1 \forall f \in A\}$, where $\mathcal{U} = \text{Spec}(A)$ ranges over the affine open subsets of \mathcal{X} and $U := \mathcal{U}_\eta$. Indeed, X° is the set of points in X^{an} which are induced by an L° -integral point of X for some valued field extension (L, w) . Such points of X are called *potentially integral*. If $p \in U^\circ$, then the reduction $\pi(p) \in \mathcal{X}_s$ is given by the prime ideal $\{a \in A \mid p(a) < 1\}$ in A .

Note that if \mathcal{X} is an algebraic K° -model, then X° is an analytic subdomain of X^{an} . If \mathcal{X} is a proper scheme over K° , then rational and integral points are the same and hence $X^\circ = X^{\text{an}}$.

4.10 For a scheme X of finite type over K with algebraic K° -model \mathcal{X} , the reduction map π can be described algebraically in the following way: We consider an integral point P of X . Integrality means that there is an affine chart \mathcal{U} of \mathcal{X} with affine coordinates x_1, \dots, x_n such that $x_1(P), \dots, x_n(P) \in K^\circ$. Then $\pi(P)$ is the point of the special fibre \mathcal{U}_s given by using the coordinates modulo the maximal ideal $K^{\circ\circ}$ of K° .

In the theory of strictly affinoid algebras introduced in 2.8, there is a similar concept of reduction which we study next. For this, we assume that the valuation v on K is non-trivial and complete.

4.11 Let \mathcal{A} be a strictly affinoid K -algebra with Berkovich spectrum $Y = \mathcal{M}(\mathcal{A})$. We define the *reduction* of Y by $\tilde{Y} := \text{Spec}(\tilde{\mathcal{A}})$ and the *special fibre* of Y by $Y_s := \text{Spec}(\mathcal{A}^\circ / (K^{\circ\circ} \mathcal{A}^\circ))$. Since the maximal ideal $K^{\circ\circ}$ of K° generates an ideal in \mathcal{A}° contained in $\mathcal{A}^{\circ\circ}$, we get a canonical surjective homomorphism $\mathcal{A}^\circ / (K^{\circ\circ} \mathcal{A}^\circ) \rightarrow \tilde{\mathcal{A}}$. This induces a canonical morphism $\tilde{Y} \rightarrow Y_s$. Since the spectral radius is power-multiplicative, it is clear that this morphism is a bijection.

We have a map $Y \rightarrow \tilde{Y}$, given by mapping the seminorm p to the prime ideal $\{a \in \mathcal{A}^\circ \mid p(a) < 1\} / \mathcal{A}^{\circ\circ}$ of $\tilde{\mathcal{A}}$. It induces a *reduction map* $\pi : Y \rightarrow Y_s$.

Lemma 4.12 *For a Zariski open and dense subset S of Y , we have $\pi(S) = Y_s$.*

Proof: We first note that the reduction map π is surjective (see [Berk1], Proposition 2.4.4). If z is a closed point of Y_s , then $\pi^{-1}(z)$ is an open non-empty subset of Y ([Berk1], Lemma 2.4.1). By density, there is $y \in S$ with $\pi(y) = z$. In general, there is a complete valued field (L, w) extending (K, v) and an L -rational point P of Y such that $\pi(P) = z$ (see Remark 2.2). Then the reduction $\pi_L(P)$ of P in the special fibre of $Y_L = \mathcal{M}(\mathcal{A} \hat{\otimes}_K L)$ is \tilde{L} -rational. The preimage S_L of S is Zariski open and dense in Y_L . Using the above, there is $y_L \in S_L$ with $\pi_L(y_L) = \pi_L(P)$. Then we have $\pi(y) = z$ for the image y of y_L in Y . \square

4.13 We compare the two concepts for a reduction map in the following situation: Let (K, v) be an arbitrary valued field and let (L, w) be a complete valued field extending (K, v) with w non-trivial. We consider a flat affine scheme $\mathcal{X} = \text{Spec}(A)$ of finite type over K° with generic fibre $X = \text{Spec}(A_K)$. For convenience, we choose coordinates x_1, \dots, x_n on \mathcal{X} , i.e. $A = K^\circ[x_1, \dots, x_n]/I$ for an ideal I in $K^\circ[x_1, \dots, x_n]$. Then we complete the base change \mathcal{X}_{L° along the special fibre (more precisely, we take the ν -adic completion for some non-zero $\nu \in K^{\circ\circ}$) to get a flat formal scheme $\mathcal{Y} = \text{Spf}(L^\circ\langle x_1, \dots, x_n \rangle / \langle I \rangle)$ over L° (see [Ull]). The generic fibre Y of \mathcal{Y} is the Berkovich spectrum of the strictly affinoid algebra \mathcal{A} defined by

$$\mathcal{A} := (L^\circ\langle x_1, \dots, x_n \rangle / \langle I \rangle) \otimes_{L^\circ} L = L\langle x_1, \dots, x_n \rangle / \langle I \rangle.$$

By construction, Y is the affinoid subdomain $\{p \in (X_L)^{\text{an}} \mid p(x_1) \leq 1, \dots, p(x_n) \leq 1\}$ in $(X_L)^{\text{an}}$. It is easy to see that we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & Y_s \\ \downarrow & & \downarrow \\ X^\circ & \xrightarrow{\pi} & \mathcal{X}_s \end{array} \quad (2)$$

where the vertical maps are induced by base change and the horizontal maps are the reduction maps. It follows from [BGR], Theorem 6.3.4/2 that the canonical morphism $\tilde{Y} \rightarrow \mathcal{Y}_s$ is a finite map. Both spaces have dimension equal to $\dim(X)$ and an easy localization argument shows that this finite map is surjective. As the base change morphism $\mathcal{Y}_s = (\mathcal{X}_L)_s \rightarrow \mathcal{X}_s$ is also surjective and since the canonical morphism $\tilde{Y} \rightarrow Y_s$ is a bijection, we deduce that the morphism $Y_s \rightarrow \mathcal{X}_s$ is surjective. By Lemma 4.12, we conclude that the reduction map $\pi : X^\circ \rightarrow \mathcal{X}_s$ is surjective.

Proposition 4.14 *Let \mathcal{X} be a flat scheme of finite type over K° with generic fibre X and let U be an open dense subset of X . Then we have $\pi(U^{\text{an}} \cap X^\circ) = \mathcal{X}_s$. If K is algebraically closed and v is non-trivial, then every \tilde{K} -rational point of \mathcal{X}_s is the reduction of a K° -integral point contained in U .*

Proof: We may assume that \mathcal{X} is affine and hence we are in the situation of 4.13. We choose a valued field (L, w) extending (K, v) with w complete and non-trivial. Since the base change U_L is open and dense in X_L , we conclude that $S := (U_L)^{\text{an}} \cap Y$ is Zariski open and dense in the analytic space Y . Using surjectivity of the map $Y_s \rightarrow \mathcal{X}_s$ and Lemma 4.12, we deduce $\pi(U^{\text{an}} \cap X^\circ) = \mathcal{X}_s$ from the commutative diagram (2).

If K is algebraically closed and v is non-trivial, then we note first that \tilde{K} is algebraically closed (see [BGR], 3.4.1). For any closed point z of the special fibre, the above shows that $\pi^{-1}(z) \cap U^{\text{an}}$ is a non-empty open subset of X^{an} and density of the K -rational points yields the claim (see 2.5).

Example 4.15 We assume that the absolute value is trivial on K . Let $X = \mathbb{P}_K^1$ with projective coordinates x_0, x_1 . For $i = 0, 1$, we consider the affine charts $U_i := \{x \in X \mid x_i \neq 0\}$ isomorphic to \mathbb{A}_K^1 . For any $r > 0$, we get an element $p_r \in U_0^{\text{an}}$ given as the seminorm $p_r(f) := \max_i |a_i| r^i$ for $f(y) = \sum_i a_i y^i \in K[y]$ with $y := \frac{x_1}{x_0}$. If $r \leq 1$, then $p_r(f) = r^j$ for j minimal with $a_j \neq 0$. Then we have $p_r \leq 1$ and hence the reduction of p_r is defined by $\pi(p_r) = \{f \in K[y] \mid p_r(f) < 1\} \in \text{Spec}(K[y]) \subset \mathbb{P}_K^1$. If $r < 1$, then $\pi(p_r) = (1 : 0)$. If $r = 1$, then $\pi(p_r)$ is the generic point of \mathbb{P}_K^1 . If $r > 1$, then we use the other chart U_1 with affine coordinate $z := \frac{x_0}{x_1}$. For $g(z) = \sum_i a_i z^i \in K[z]$, we have $p_r(g) = \max_i |a_i| r^{-i}$ and hence $\pi(p_r) = (0 : 1)$.

5 Initial degeneration

In this section, we study the initial degeneration $\text{in}_\omega(X)$ of a closed subscheme X of the multiplicative torus $T = \mathbb{G}_m^n$ over the valued field (K, v) at $\omega \in N_{\mathbb{R}}$. We follow here the original definition of the initial degeneration using a translation to the origin of the torus. Then $\text{in}_\omega(X)$ is a closed subscheme of the torus $\mathbb{T}_{\tilde{K}}$ which is only well-defined up to translations. This approach fits very well to Hilbert schemes as we will see in Section 10. For an intrinsic approach, we refer to [OP].

Definition 5.1 Let (L, w) be a valued field extending (K, v) and let $t \in X(L)$. Then the *initial degeneration of X at t* is defined as the special fibre of the closure

of $t^{-1}X_L$ in the split multiplicative torus \mathbb{T}_{L° over the valuation ring L° . It is a closed subscheme of the split torus $\mathbb{T}_{\tilde{L}}$ over the residue field \tilde{L} which we denote by $\text{in}_t(X)$.

Lemma 5.2 *Let (L, w) and (L', w') be valued fields extending (K, v) . Then there is a valued field (L'', w'') extending (L, w) and (L', w') .*

Proof: This is proved in [Duc], §0.3.2, using Berkovich's theory. \square

Proposition 5.3 *Let (L, w) and (L', w') be valued fields extending (K, v) . Suppose that there is $\omega \in N_{\mathbb{R}}$ such that $\omega = \text{trop}_w(t) = \text{trop}_{w'}(t')$ for $t \in T(L)$ and $t' \in T(L')$. For any field (L'', w'') as in Lemma 5.2, there is $g \in \mathbb{T}(\tilde{L}'')$ with*

$$\text{in}_{t'}(X)_{\tilde{L}''} = g \cdot \text{in}_t(X)_{\tilde{L}''}. \quad (3)$$

Proof: Since t, t' have the same tropicalizations, the point $t/t' \in T(L'')$ is in fact an $(L'')^\circ$ -integral point of \mathbb{T} and hence it has a well-defined reduction $g \in \mathbb{T}(\tilde{L}'')$. The relation

$$(t')^{-1}X_{L''} = (t/t') \cdot t^{-1}X_{L''}$$

gives immediately the claim. \square

5.4 The proposition shows that the initial degeneration depends essentially only on ω . For any $\omega \in N_{\mathbb{R}}$, there is a valued field (L, w) extending (K, v) and $t \in T(L)$ with $\text{trop}_w(t) = \omega$. We define the *initial degeneration* $\text{in}_\omega(X)$ of X at ω as $\text{in}_t(X)$ which is well-defined as an equivalence class for the equivalence relation (3). We call the residue field \tilde{L} or any extension of it a *field of definition* for $\text{in}_\omega(X)$.

Proposition 5.5 *Let (L, w) be a valued field extending (K, v) and let $\omega \in N_{\mathbb{R}}$. Then we have $\text{in}_\omega(X_L) = \text{in}_\omega(X)_{\tilde{L}}$.*

Proof: By Corollary 4.7, the formation of the closure is compatible with base change and this yields easily the claim. \square

The next result is called the *fundamental theorem* of tropical algebraic geometry. It is due to Kapranov in the hypersurface case (unpublished manuscript, later incorporated in [EKL]) and to Speyer–Sturmfels [SS], Draisma [Dra], Payne [Pay] in general.

Theorem 5.6 *For a closed subscheme X of T , the tropical variety $\text{Trop}_v(X)$ may be characterized in the following two equivalent ways:*

- (a) $\text{Trop}_v(X) = \text{trop}_v(X^{\text{an}})$
- (b) The set $\{\omega \in N_{\mathbb{R}} \mid \text{in}_\omega(X) \neq \emptyset\}$ in $N_{\mathbb{R}}$ is equal to $\text{Trop}_v(X)$.

Proof: We have to prove that $\omega \in N_{\mathbb{R}}$ is in $\text{trop}_v(X)$ if and only if $\text{in}_\omega(X) \neq \emptyset$. By base change, we may assume that (K, v) is a non-trivially valued complete algebraically closed field such that $\omega = \text{trop}_v(t)$ for some $t \in T(K)$ (see Propositions 3.6 and 5.5). Passing to $t^{-1}X$, we may assume that $t = e$ and $\omega = 0$. Let \mathcal{X} be the closure of X in \mathbb{T} . It is an algebraic K° -model of X . We have seen in Proposition 4.14 that the reduction map gives a surjective map from the K° -integral points of \mathcal{X} onto the \tilde{K} -rational points of the special fibre \mathcal{X}_s . Note that \mathcal{X}_s represents $\text{in}_0(X)$. We conclude that $\text{in}_0(X)$ is empty if and only if \mathcal{X} has no integral points which means that $X^{\text{an}} \cap \text{trop}_v^{-1}(0) = \emptyset$ using density of K -rational points in X^{an} (see 2.5). This proves the claim. \square

6 Affine toric schemes over a valuation ring

First, we recall some facts from the theory of normal toric varieties which will be very important in the sequel. We refer to [CLS], [Ful2], [KKMS] or [Oda] for details. They are independent of any valuations on the field K . Then we assume that K is endowed with a non-archimedean absolute value $|\cdot|$ with valuation $v := -\log|\cdot|$ and value group $\Gamma := v(K^\times)$. We consider the split torus $\mathbb{T} = \text{Spec}(K^\circ[M])$ over the valuation ring K° with generic fibre T . The main focus will be laid on the theory of affine \mathbb{T} -toric schemes over K° associated to a pointed Γ -rational polyhedron. While the generic fibre of such a scheme is a T -toric variety over K , the geometry of the special fibre is more complicated and is closely related to the combinatorics of the polyhedron. This section can be seen as a generalization of §4.3 in [KKMS], where the case of a discrete valuation is handled. Further references: [Rab], [BPR].

Definition 6.1 A T -toric variety over a field K is a variety Y over K containing T as an open dense subset such that the translation action of T on itself extends to an algebraic T -action on Y .

6.2 There are bijective correspondences between

- (a) rational polyhedral cones σ in $N_{\mathbb{R}}$ which do not contain a line;
- (b) finitely generated saturated semigroups S in M which generate M as a group;
- (c) affine normal T -toric varieties Y over K (up to equivariant isomorphisms restricting to the identity on T).

The correspondences are given by $S = \check{\sigma} \cap M$ and $Y = \text{Spec}(K[S])$. We refer the reader to the appendix for the terminology from convex geometry.

6.3 In general, there is a bijective correspondence between normal T -toric varieties Y over K (up to equivariant isomorphisms restricting to the identity on T) and rational fans in $N_{\mathbb{R}}$. We denote the toric variety associated to the fan Σ by Y_{Σ} . Every cone σ of Σ induces an open affine toric subset U_{σ} of Y_{Σ} by the affine case considered above and Y_{Σ} is covered by such affine charts.

We extend the above definition to the case of valuation rings:

Definition 6.4 A \mathbb{T} -toric scheme over the valuation ring K° is an integral separated flat scheme \mathcal{Y} over K° such that the generic fibre \mathcal{Y}_{η} contains T as an open subset and such that the translation action of T on T extends to an algebraic action of \mathbb{T} on \mathcal{Y} over K° . We call it a \mathbb{T} -toric variety if \mathcal{Y} is of finite type over K° .

Definition 6.5 For a Γ -rational polyhedron Δ in $N_{\mathbb{R}}$, we set

$$K[M]^{\Delta} := \left\{ \sum_{u \in M} \alpha_u \chi^u \in K[M] \mid v(\alpha_u) + \langle u, \omega \rangle \geq 0 \ \forall \omega \in \Delta \right\}.$$

The algebra $K[M]^{\Delta}$ was studied by [KKMS] in case of a discrete valuation and by [BPR] in case of an algebraically closed ground field endowed with a non-trivial complete absolute value. We will see in the following that most of their results hold in our more general setting.

6.6 It is easy to show that $K[M]^{\Delta}$ is an integral domain with quotient field $K(\check{\sigma} \cap M) = K(\rho^{\perp} \cap M)$, where $\rho = \sigma \cap (-\sigma)$ is the largest linear subspace contained in the recession cone σ of Δ . Since $K[M]^{\Delta}$ has no K° -torsion, it is a flat K° -algebra by Lemma 4.2.

Proposition 6.7 *If the value group Γ is either discrete or divisible in \mathbb{R} , then the algebra $K[M]^\Delta$ is of finite presentation over K° .*

Proof: It is enough to prove that $K[M]^\Delta$ is a finitely generated K° -algebra. This follows from the fact that every finitely generated flat algebra over an integral domain is of finite presentation ([RG], Corollaire 3.4.7).

If the valuation is discrete, then we may assume $\Gamma = \mathbb{Z}$. We consider the cone σ in $N_{\mathbb{R}} \times \mathbb{R}_+$ generated by $\Delta \times \{1\}$. It is a rational polyhedral cone. If π is a uniform parameter for K° , then $K[M]^\Delta$ is generated by $\pi^k \chi^u$ with (u, k) ranging over the semigroup $S_\sigma := \sigma \cap (M \times \mathbb{Z})$. This semigroup is finitely generated (see 6.2) and hence we get the claim in the case of a discrete valuation.

If the value group is divisible in \mathbb{R} , we argue as follows: We reduce to the case of a pointed Γ -rational polyhedron by the procedure described in 6.11 below. Then the same proof as for Proposition 4.11 in [BPR] works. Indeed, the crucial point in this proof is that the vertices $\omega_1, \dots, \omega_r$ of Δ are in N_Γ which is always the case for Γ divisible in \mathbb{R} . Then it is shown that $K[M]^\Delta$ is generated by the functions $\alpha_{ij} \chi^{u_{ij}}$, where $(u_{ij})_j$ is a finite set of generators for $\sigma_i \cap M$ with σ_i equal to the local cone $\text{LC}_{\omega_i}(\Delta)$ and where $\alpha_{ij} \in K$ with $v(\alpha_{ij}) + \langle u_{ij}, \omega_i \rangle = 0$. \square

6.8 For $\omega \in N_{\mathbb{R}}$, we will use the ω -weight

$$v_\omega\left(\sum_u \alpha_u \chi^u\right) := \min_u v(\alpha_u) + \langle u, \omega \rangle$$

on $K[M]$ which extends obviously to a valuation on the field $K(T)$. We may view it as a weighted Gauss-valuation similarly as in 2.7. For a polyhedron Δ in $N_{\mathbb{R}}$, it is clear that $v_\Delta := \min_{\omega \in \Delta} v_\omega$ is not necessarily a valuation.

Example 6.9 We show that the assumptions on the value group Γ are really necessary in Proposition 6.7. So we assume that Γ is dense and not divisible in \mathbb{R} . The latter means that there is $\omega \in \mathbb{R} \setminus \Gamma$ and a non-zero $n \in \mathbb{N}$ with $n\omega \in \Gamma$. Then

$$A := \left\{ f = \sum_u \alpha_u x^u \in K[x] \mid v(\alpha_u) + u \cdot \omega \geq 0 \forall u \in \mathbb{Z} \right\}$$

is equal to $K[x]^\Delta$ for the Γ -rational polytope $\Delta := \{\omega\}$. We claim that A is not finitely generated over K° . In fact, this holds for any $\omega \notin \Gamma$.

To prove this, we argue by contradiction. We assume that A is generated by g_1, \dots, g_r as a K° -algebra. Obviously, we may assume that $g_i = \alpha_i x^{u_i}$ with $u_i \in \mathbb{Z}$ for $i = 1, \dots, r$. For any $\beta \in K$ with $v_\omega(\beta x) \geq 0$, there is $p \in K^\circ[x_1, \dots, x_r]$ with $\beta x = p(g_1, \dots, g_r)$. We may assume that $p(\mathbf{x}) = \sum_{\mathbf{m}} p_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ with $m_1 u_1 + \dots + m_r u_r = 1$ for every \mathbf{m} . We get

$$v(\beta) + \omega \geq \min_{\mathbf{m}} v_\omega(p_{\mathbf{m}}(g_1)^{m_1} \cdots (g_r)^{m_r}).$$

For every \mathbf{m} , there is $j \in \{1, \dots, r\}$ with $m_j v_\omega(g_j) \notin \Gamma$, otherwise we would have

$$\sum_j m_j v(\alpha_j) + \omega = \sum_j m_j (v(\alpha_j) + u_j \cdot \omega) = \sum_j m_j v_\omega(g_j) \in \Gamma$$

which contradicts $\omega \notin \Gamma$. In particular, we get

$$v_\omega(\beta x) = v(\beta) + \omega \geq \min_j v_\omega(g_j) > 0,$$

where j ranges over all elements of $\{1, \dots, r\}$ with $v_\omega(g_j) \neq 0$. This contradicts the density of Γ in \mathbb{R} .

Proposition 6.10 *For a Γ -rational polyhedron Δ in $N_{\mathbb{R}}$, the algebra $K[M]^{\Delta}$ is integrally closed.*

Proof: Using the procedure described in 6.11 below, we may assume that Δ is pointed. Since every affine form on Δ takes its minimum in a vertex, we deduce that $K[M]^{\Delta} = \bigcap_{\omega} K[M]^{\omega}$ with ω ranging over the vertices of Δ . Hence it is enough to show that $K[M]^{\omega}$ is integrally closed in $K[M]$. Since the ω -weight v_{ω} from 6.8 is a valuation and $K[M]^{\omega} = \{f \in K[M] \mid v_{\omega}(f) \geq 0\}$, the same argument as in the case of valuation rings proves the claim. Indeed, let $f^m + a_{m-1}f^{m-1} + \dots + a_0 = 0$ with $f \in K[M]$ and all $a_i \in K[M]^{\omega}$. Then the ultrametric triangle inequality applied to $f^m = -a_{m-1}f^{m-1} - \dots - a_0$ and $v_{\omega}(a_i) \geq 0$ yield $v_{\omega}(f) \geq 0$. \square

6.11 Let Δ be a Γ -rational polyhedron in $N_{\mathbb{R}}$ with recession cone σ . We call $\mathcal{U}_{\Delta} := \text{Spec}(K[M]^{\Delta})$ the *polyhedral scheme over K° associated to Δ* . By 6.6 and Proposition 6.10, \mathcal{U}_{Δ} is a normal scheme which is flat over K° . If the value group is discrete or divisible then Proposition 6.7 shows that \mathcal{U}_{Δ} is of finite type over K° .

The K° -algebra $K[M]^{\Delta}$ is M -graded and hence \mathbb{T} acts on \mathcal{U}_{Δ} . It follows from 6.6 that \mathcal{U}_{Δ} is a \mathbb{T} -toric scheme over K° if and only if Δ is a pointed polyhedron in the sense of A.8. In this case, the generic fibre is the affine T -toric variety U_{σ} associated to σ (as in [BPR], Lemma 4.9).

In general, we consider the smallest linear subspace $\rho = \sigma \cap (-\sigma)$ contained in σ . Then \mathcal{U}_{Δ} is a toric scheme over K° with respect to the split torus $\text{Spec}(K^{\circ}[M(\rho)])$. Here, we have used the lattice $M(\rho) := M \cap \rho^{\perp}$ with dual lattice $N(\rho) = N/N_{\rho}$ where N_{ρ} is the group generated by $N \cap \rho$. The image of Δ in $N(\rho)_{\mathbb{R}}$ is a pointed polyhedron whose associated polyhedral scheme is \mathcal{U}_{Δ} . This procedure is often used to reduce to the case of pointed polyhedra. In particular, we deduce from the above that U_{σ} is always the generic fibre of \mathcal{U}_{Δ} .

Proposition 6.12 *Let Δ' be a closed face of the Γ -rational polyhedron Δ in $N_{\mathbb{R}}$. Then the canonical \mathbb{T} -equivariant morphism $\mathcal{U}_{\Delta'} \rightarrow \mathcal{U}_{\Delta}$ is an open immersion.*

Proof: There is a halfspace $\{\omega \in N_{\mathbb{R}} \mid \langle u, \omega \rangle + v(\alpha) \geq 0\}$ containing Δ such that the face Δ' is cut out from Δ by the hyperplane $\{\omega \in N_{\mathbb{R}} \mid \langle u, \omega \rangle + v(\alpha) = 0\}$ for suitable $u \in M$ and $\alpha \in K$. We conclude that $\mathcal{U}_{\Delta'}$ is the complement of the closed subscheme of \mathcal{U}_{Δ} given by the equation $\alpha\chi^u = 0$. \square

Let Δ be any Γ -rational polyhedron in $N_{\mathbb{R}}$. It follows from 6.11 that the split torus $\mathbb{T}_{\tilde{K}}$ acts on the special fibre of \mathcal{U}_{Δ} . Our goal is the description of the orbits of this action and hence only the induced reduced structure $((\mathcal{U}_{\Delta})_s)_{\text{red}}$ is relevant.

Lemma 6.13 *The reduced induced structure on the special fibre is given by*

$$((\mathcal{U}_{\Delta})_s)_{\text{red}} = \text{Spec}(K[M]^{\Delta}/\{f \in K[M]^{\Delta} \mid v_{\Delta}(f) > 0\}).$$

Proof: If v is the trivial valuation, then the special fibre is equal to the generic fibre and the claim is obvious. So we may assume that v is non-trivial. The special fibre of \mathcal{U}_{Δ} is a closed subscheme of \mathcal{U}_{Δ} given by the ideal $I = K^{\circ\circ}K[M]^{\Delta}$ in $K[M]^{\Delta}$. Since v_{Δ} is power-multiplicative, it is clear that the radical ideal \sqrt{I} of I is contained in the ideal $J = \{f \in K[M]^{\Delta} \mid v_{\Delta}(f) > 0\}$. On the other hand, J is a M -homogeneous ideal in $K[M]^{\Delta}$ and so it is enough to show that every $f = \alpha\chi^u \in J$ is contained in \sqrt{I} . Since the valuation v on K is non-trivial, there is β in the maximal ideal $K^{\circ\circ}$ of K and $v(\beta) \leq v(f^m)$ for $m \in \mathbb{N}$ sufficiently large. We conclude that $f^m \in I$ proving $I \subset \sqrt{J}$ and the claim. \square

Proposition 6.14 *Let Δ be a pointed Γ -rational polyhedron in $N_{\mathbb{R}}$. Then there is a bijection between the vertices of Δ and the irreducible components of $(\mathcal{U}_{\Delta})_s$. The irreducible component corresponding to the vertex ω is the closed subscheme Y_{ω} of \mathcal{U}_{Δ} given by the prime ideal $\{f \in K[M]^{\Delta} \mid v_{\omega}(f) > 0\}$ of $K[M]^{\Delta}$.*

Proof: Since v_ω is a valuation on $K[M]^\Delta$ for any $\omega \in \Delta$, it is clear that $I_\omega := \{f \in K[M]^\Delta \mid v_\omega(f) > 0\}$ is a prime ideal in $K[M]^\Delta$. Since Δ is a pointed polyhedron, the restriction of any affine form $v_\omega(\alpha\chi^u)$ to Δ takes its minimum in a vertex ω and for every vertex, there is such an affine form which has its minimum precisely in this vertex. This means that the set of prime ideals I_ω , with ω ranging over the vertices of Δ , is a minimal primary decomposition of the ideal $\{f \in K[M]^\Delta \mid v_\Delta(f) > 0\}$. We have seen in Lemma 6.13 that the latter is the ideal of the special fibre $(\mathcal{U}_\Delta)_s$ in $K[M]^\Delta$ and hence the I_ω are the ideals of the irreducible components of $(\mathcal{U}_\Delta)_s$. \square

Corollary 6.15 *The irreducible component Y_ω of $(\mathcal{U}_\Delta)_s$ is $\mathbb{T}_{\tilde{K}}$ -equivariantly isomorphic to $((\mathcal{U}_{\Delta(\omega)})_s)_{\text{red}}$ where $\Delta(\omega) = \omega + \text{LC}_\omega(\Delta)$. Moreover, $M_\omega := \{u \in M \mid \langle u, \omega \rangle \in \Gamma\}$ is a sublattice of finite index in M and Y_ω is equivariantly isomorphic to the $\text{Spec}(\tilde{K}[M_\omega])$ -toric variety over \tilde{K} associated to the local cone $\text{LC}_\omega(\Delta)$.*

Proof: Since $\Delta \subset \Delta(\omega)$, we have a canonical injective homomorphism

$$\varphi : K[M]^{\Delta(\omega)} / \{f \in K[M]^{\Delta(\omega)} \mid v_\omega(f) > 0\} \rightarrow K[M]^\Delta / \{f \in K[M]^\Delta \mid v_\omega(f) > 0\}.$$

To show surjectivity, it is enough to show that the residue class of $f = \alpha\chi^u \in K[M]^\Delta$ is in the image of φ . We may assume that $v_\omega(f) = 0$ otherwise this is trivial. Then the affine form $\Delta \rightarrow \mathbb{R}, \nu \mapsto v_\nu(f)$ takes its minimum in the vertex ω . This even holds if we extend the affine form to $\Delta(\omega)$ by definition of the local cone $\text{LC}_\omega(\Delta)$. We conclude that $f \in K[M]^{\Delta(\omega)}$ proving that φ is an isomorphism. By Lemma 6.13 and Proposition 6.14, we deduce $Y_\omega \cong ((\mathcal{U}_{\Delta(\omega)})_s)_{\text{red}}$. Equivariance of this isomorphism follows from the fact that φ is an M -graded isomorphism.

Since Δ is Γ -rational, there is a non-zero $m \in \mathbb{N}$ with $m\omega \in N_\Gamma$ and hence M_ω is a sublattice of finite index in M . It is trivial to show that the canonical homomorphism from $K[M_\omega]^{\Delta(\omega)} / \{f \in K[M_\omega]^{\Delta(\omega)} \mid v_\omega(f) > 0\}$ to $K[M]^{\Delta(\omega)} / \{f \in K[M]^{\Delta(\omega)} \mid v_\omega(f) > 0\}$ is an isomorphism. We conclude that we may replace M by M_ω and so we may assume $M = M_\omega$. Then there is $t \in T(K)$ with $\text{trop}_v(t) = \omega$. We may replace Δ by $\Delta - \omega$ which means geometrically that we use translation by t^{-1} on T . Then $\omega = 0$ is the given vertex of Δ . By the first claim, the irreducible component Y_ω is equivariantly isomorphic to

$$\text{Spec}(K[M]^{\Delta(\omega)} / \{f \in K[M]^{\Delta(\omega)} \mid v_\omega(f) > 0\}) = \text{Spec}(\tilde{K}[M]^{\text{LC}_\omega(\Delta)})$$

which is the $\mathbb{T}_{\tilde{K}}$ -toric variety associated to $\text{LC}_\omega(\Delta)$. \square

6.16 Next, we describe the reduction map with respect to the \mathbb{T} -toric scheme \mathcal{U}_Δ over K° associated to the pointed Γ -rational polyhedron Δ in $N_\mathbb{R}$. Recall from 6.11 that the T -toric variety U_σ is the generic fibre of \mathcal{U}_Δ where σ is the recession cone of Δ . We have seen in 4.9 that the reduction is a map to the special fibre $(\mathcal{U}_\Delta)_s$ which is defined on the set $U_\sigma^\circ := \{p \in U_\sigma^{\text{an}} \mid p(f) \leq 1 \forall f \in K[M]^\Delta\}$. The points of U_σ° are induced by the potentially integral points of \mathcal{U}_Δ .

We will describe the analytic structure of U_σ° using the following result of Joe Rabinoff.

Proposition 6.17 *We assume that the valuation v on K is complete. Let Δ be a pointed Γ -rational polyhedron in $N_\mathbb{R}$ and let $\|\cdot\|$ be any norm on $M_\mathbb{R}$. Then the set of Laurent series*

$$\mathcal{A}_\Delta := \left\{ \sum_{u \in \sigma \cap M} a_u \chi^u \mid \lim_{\|u\| \rightarrow \infty} v(a_u) + \langle u, \omega \rangle = \infty \forall \omega \in \Delta \right\}$$

is a strictly affinoid algebra with spectral radius

$$\rho\left(\sum_{u \in \check{\sigma} \cap M} a_u \chi^u\right) = \sup_{\omega \in \Delta, u \in \check{\sigma} \cap M} |a_u| e^{-\langle u, \omega \rangle} = \max_{\omega \text{ vertex}, u \in \check{\sigma} \cap M} |a_u| e^{-\langle u, \omega \rangle}. \quad (4)$$

Proof: In the case of a non-trivial valuation, we use [Rab], Proposition 6.9. If v is trivial, then the sums in the definition of \mathcal{A}_Δ are finite and the claim is obvious. \square

Remark 6.18 The special case of polytopal domains was studied in [Gub3]. Using Hochster's theorem for toric varieties, Rabinoff has shown that \mathcal{A}_Δ is Cohen-Macaulay for any Γ -rational polyhedron Δ (see [Rab], Proposition 6.9). If the valuation is discrete or K algebraically closed, then Wilke [Wil] has shown that \mathcal{A}_Δ is a factorial ring for Γ -rational polytopes Δ .

Proposition 6.19 *Using the notation from Proposition 6.17, the Berkovich spectrum $\mathcal{M}(\mathcal{A}_\Delta)$ is an affinoid subdomain of U_σ^{an} which is equal to U_σ° . Moreover, the special fibres of $\mathcal{M}(\mathcal{A}_\Delta)$ and \mathcal{U}_Δ agree which means $\text{Spec}(\mathcal{A}_\Delta^\circ / (K^{\circ\circ} \mathcal{A}_\Delta^\circ)) = (\mathcal{U}_\Delta)_s$.*

Proof: By Proposition 6.17, $\mathcal{O}(U_\sigma) = K[\check{\sigma} \cap M]$ is dense in \mathcal{A}_Δ and hence $\mathcal{M}(\mathcal{A}_\Delta)$ may be seen as a subset of U_σ^{an} . In fact, it is shown in [Rab], Proposition 6.17, that $\mathcal{M}(\mathcal{A}_\Delta)$ is an affinoid subdomain of U_σ^{an} . Moreover, we deduce from Rabinoff's result that $K[M]^\Delta$ is a subset of \mathcal{A}_Δ° and hence $\mathcal{M}(\mathcal{A}_\Delta) \subset U_\sigma^\circ$. We will prove the reverse inclusion and so we choose $p \in U_\sigma^\circ$.

We claim first that $p(\chi^u) \leq \rho(\chi^u)$ for any $u \in \check{\sigma} \cap M$ where $\rho(\chi^u)$ is the spectral radius in \mathcal{A}_Δ . There is a vertex ω_0 of Δ such that the halfspace $H := \{\omega \in N_{\mathbb{R}} \mid \langle u, \omega \rangle \geq 0\} + \omega_0$ contains Δ . By Γ -rationality of Δ , there is a non-zero $m \in \mathbb{N}$ such that $m\omega_0 \in N_\Gamma$. We conclude that there is a non-zero $\alpha \in K$ such that $v(\alpha) + \langle mu, \omega_0 \rangle = 0$. Using $H = \{\omega \in N_{\mathbb{R}} \mid v(\alpha) + \langle mu, \omega \rangle \geq 0\}$, we get $\alpha \chi^{mu} \in K[M]^\Delta$ and $\rho(\alpha \chi^{mu}) = |\alpha| e^{-\langle mu, \omega_0 \rangle} = 1$ follows from Proposition 6.17. Using power multiplicativity of both $p \in U_\sigma^\circ$ and ρ , we get

$$|\alpha| p(\chi^u)^m = p(\alpha \chi^{mu}) \leq 1 = \rho(\alpha \chi^{mu}) = |\alpha| \rho(\chi^u)^m.$$

This proves $p(\chi^u) \leq \rho(\chi^u)$ for any $u \in \check{\sigma} \cap M$.

Next, we will prove $p(f) \leq \rho(f)$ for any $f \in \mathcal{O}(U_\sigma)$. Note that $f = \sum_u \alpha_u \chi^u$ where u ranges over a finite subset of $\check{\sigma} \cap M$. Using the above and Proposition 6.17, we get

$$p(f) \leq \max_u |\alpha_u| p(\chi^u) \leq \max_u |\alpha_u| \rho(\chi^u) = \max_u \rho(\alpha_u \chi^u) = \rho(f)$$

as desired. Now $p \leq \rho$ yields that p extends uniquely to a multiplicative seminorm of $\mathcal{M}(\mathcal{A}_\Delta)$. This proves $\mathcal{M}(\mathcal{A}_\Delta) = U_\sigma^\circ$. The claim about the special fibres follows immediately from Proposition 6.17. \square

In the following, we do not necessarily assume that the valuation v on K is complete as the analytifications are defined on the completion of K anyway.

Corollary 6.20 *Let Δ be any Γ -rational polyhedron in $N_{\mathbb{R}}$ with recession cone σ . Then reduction modulo the maximal ideal $K^{\circ\circ}$ maps $U_\sigma^\circ \cap T^{\text{an}}$ onto $(\mathcal{U}_\Delta)_s$.*

Proof: Using the procedure described in 6.11, we may assume that Δ is a pointed polyhedron. Passing to the completion does not change the special fibre and so we may assume K complete. By Proposition 6.19, the special fibre of \mathcal{U}_Δ agrees with the special fibre of $U_\sigma^\circ = \mathcal{M}(\mathcal{A}_\Delta)$. Since T is the dense orbit in the generic fibre U_σ , it is clear that $T^{\text{an}} \cap U_\sigma^\circ$ is Zariski open and dense in the affinoid subdomain U_σ° . Now the claim follows from Lemma 4.12. \square

Lemma 6.21 *For a pointed Γ -rational polyhedron Δ , we get $U_\sigma^\circ \cap T^{\text{an}} = \text{trop}_v^{-1}(\Delta)$.*

Proof: By definition, $U_\sigma^\circ \cap T^{\text{an}}$ is the set of multiplicative seminorms p on $K[M]$ extending $|\cdot|$ with the additional condition that $p(f) \leq 1$ for every $f \in K[M]^\Delta$. The latter is equivalent to the condition that if there are $u \in M$, $\alpha \in K$ with $v(\alpha) + \langle u, \omega \rangle \geq 0$ for every $\omega \in \Delta$, then $p(\alpha\chi^u) \leq 1$. Since Δ is defined by such inequalities and since $-\log p(\chi^u) = \langle u, \text{trop}_v(p) \rangle$, this is also equivalent to $\text{trop}_v(p) \in \Delta$. \square

As we have defined the tropicalization map only on T^{an} , we restrict the reduction map in the following proposition to $U_\sigma^\circ \cap T^{\text{an}}$. By abuse of notation, we denote this restriction $U_\sigma^\circ \cap T^{\text{an}} \rightarrow (\mathcal{U}_\Delta)_s$ also by π . In the following result, we use the partial order on the set of orbits (resp. open faces) given by inclusion of closures.

Proposition 6.22 *Let Δ be a pointed Γ -rational polyhedron in $N_{\mathbb{R}}$ and let \mathcal{U}_Δ be the associated \mathbb{T} -toric scheme over K° . Then there is a bijective order reversing correspondence between \mathbb{T} -orbits Z of $(\mathcal{U}_\Delta)_s$ and open faces τ of Δ given by*

$$Z = \pi(\text{trop}_v^{-1}(\tau)), \quad \tau = \text{trop}_v(\pi^{-1}(Z)).$$

Moreover, we have $\dim(Z) + \dim(\tau) = n$.

Proof: Let τ be an open face of Δ . The affinoid torus $\{p \in T^{\text{an}} \mid p(\chi^u) = 1 \ \forall u \in M\}$ operates on $\text{trop}_v^{-1}(\tau)$. By passing to the reductions, we see that $Z := \pi(\text{trop}_v^{-1}(\tau))$ is $\mathbb{T}_{\tilde{K}}$ -invariant. Note that Z is well-defined by Lemma 6.21. It is clear that distinguished open faces give rise to distinguished $\mathbb{T}_{\tilde{K}}$ -invariant subsets of $(\mathcal{U}_\Delta)_s$. It remains to show that Z is an orbit. We are allowed to pass to a \mathbb{T} -invariant open subset and hence we may assume that $\tau = \text{relint}(\Delta)$ by using Proposition 6.12. Since Δ is pointed, it has a vertex ω . We have seen in Corollary 6.15 that the irreducible component Y_ω is the $\text{Spec}(\tilde{K}[M_\omega])$ -toric variety over \tilde{K} associated to the local cone $\text{LC}_\omega(\Delta)$. We claim that Z is the unique closed orbit Z' of Y_ω . Since Z is invariant and Z' is an orbit, it is enough to show that $\pi(p) \in Z'$ for every $p \in \text{trop}_v^{-1}(\tau)$.

We will first prove that $\pi(p) \in Y_\omega$. By Proposition 6.14, the latter is given by the M -graded prime ideal $\{f \in K[M]^\Delta \mid v_\omega(f) > 0\}$. So let us choose $f = \alpha\chi^u \in K[M]^\Delta$ with $v_\omega(f) > 0$. Then $v_\nu(f) > 0$ for all $\nu \in \tau = \text{relint}(\Delta)$. In particular, this holds for $\nu = \text{trop}_v(p)$ and hence $-\log p(f) = v_\nu(f) > 0$. We conclude $p(f) < 1$ which means that f is contained in the prime ideal of $\pi(p)$ in $K[M]^\Delta$. This proves $\pi(p) \in Y_\omega$.

The well-known orbit-cone correspondence for toric varieties over a field shows that the closed orbit Z' of Y_ω is given by the ideal in $\tilde{K}[Y_\omega]$ generated by $\{\chi^u \mid u \in M_\omega, \langle u, \omega' \rangle > 0 \ \forall \omega' \in \tau'\}$ where $\tau' := \text{LC}_\omega(\tau)$. Taking into account how Y_ω is defined as a $\text{Spec}(\tilde{K}[M_\omega])$ -toric variety, we conclude that Z' is given as a closed subscheme of \mathcal{U}_Δ by the ideal generated by $\{f = \beta\chi^u \mid \beta \in K, u \in M_\omega, v_{\omega'}(f) > 0 \ \forall \omega' \in \omega + \tau'\}$. For such an f , we conclude $-\log p(f) = v_\nu(f) > 0$ as above and hence $p(f) < 1$. Again this means $\pi(p) \in Z'$ proving that $Z = Z'$. We conclude that $Z = \pi(\text{trop}_v^{-1}(\tau))$ is a \mathbb{T} -orbit.

Since π maps $U_\sigma^\circ \cap T^{\text{an}} = \text{trop}_v^{-1}(\Delta)$ onto $(\mathcal{U}_\Delta)_s$ by Corollary 6.20, we get a bijective correspondence between open faces of Δ and torus orbits of $(\mathcal{U}_\Delta)_s$. Since Z is the torus orbit of Y_ω corresponding to the open cone τ' , we get $\dim(\tau) + \dim(Z) = \dim(\tau') + \dim(Z) = n$ from the theory of toric varieties over a field. Moreover, we conclude from the special case Y_ω that the correspondence is order reversing.

Conversely, let Z be any torus orbit of $(\mathcal{U}_\Delta)_s$. By the above, we have $Z = \pi(\text{trop}_v^{-1}(\tau))$ for a unique open face τ of Δ . This yields $\tau \subset \text{trop}_v(\pi^{-1}(Z))$. Equality follows from the fact that the torus orbits (resp. open faces) form a partition of $(\mathcal{U}_\Delta)_s$ (resp. Δ). \square

Remark 6.23 The bijective correspondence between open faces and orbits holds more generally for the polyhedral scheme \mathcal{U}_Δ associated to any Γ -rational polyhedron Δ in $N_{\mathbb{R}}$. This follows from the reduction to the case of pointed polyhedra described in 6.11.

If Δ' is any Γ -rational polyhedron contained in Δ , then the canonical equivariant morphism $\mathcal{U}_{\Delta'} \rightarrow \mathcal{U}_\Delta$ is an open immersion if and only if Δ' is a closed face of Δ . We have seen one direction in Proposition 6.12 and the converse follows easily from the orbit-face correspondence.

7 Toric schemes over a valuation ring

In this section, K is a field endowed with a non-archimedean valuation v and Γ is the valuation group of v . We extend the theory of toric schemes over a discrete valuation ring from [KKMS] to this more general situation. More precisely, we will use the affine toric schemes associated to pointed polyhedra from the previous section to define toric schemes. For the glueing process, it is necessary to work with fans in $N_{\mathbb{R}} \times \mathbb{R}_+$ rather than polyhedral complexes in $N_{\mathbb{R}}$. Recall that $\mathbb{T} = \text{Spec}(\mathbb{G}_m^n)$ is the split multiplicative torus over K° with generic fibre T . The character group of T is M and N is the dual lattice. Further references for the special case of a discrete valuation are [BPS] (with a lot of arithmetic applications) and [Smi] (from the projective point of view).

7.1 As a building block, we will use the affine \mathbb{T} -toric scheme \mathcal{U}_Δ over K° from 6.11. For glueing, it is better to replace Δ by the closed cone $\sigma = c(\Delta)$ in $N_{\mathbb{R}} \times \mathbb{R}_+$ generated by $\Delta \times \{1\}$. For $s \in \mathbb{R}_+$, let $\sigma_s := \{\omega \in N_{\mathbb{R}} \mid (\omega, s) \in \sigma\}$. For $s > 0$, we have $\sigma_s = s\Delta$ and σ_0 is the recession cone of Δ .

7.2 A cone σ in $N_{\mathbb{R}} \times \mathbb{R}_+$ is called Γ -admissible if it may be written as

$$\sigma = \bigcap_{i=1}^N \{(\omega, s) \mid \langle u_i, \omega \rangle + sc_i \geq 0\}$$

for $u_1, \dots, u_N \in M$ and $c_1, \dots, c_N \in \Gamma$ and if σ does not contain a line. For $s \in \mathbb{R}_+$, we define σ_s as above.

Definition 7.3 For a Γ -admissible cone σ in $N_{\mathbb{R}} \times \mathbb{R}_+$, we define

$$K[M]^\sigma := \left\{ \sum_{u \in M} \alpha_u \chi^u \in K[M] \mid cv(\alpha_u) + \langle u, \omega \rangle \geq 0 \ \forall (\omega, c) \in \sigma \right\}$$

and $\mathcal{V}_\sigma := \text{Spec}(K[M]^\sigma)$.

Proposition 7.4 \mathcal{V}_σ is an affine normal \mathbb{T} -toric scheme over K° with generic fibre equal to the affine toric variety U_{σ_0} associated to σ_0 . If the value group Γ of K is discrete or divisible in \mathbb{R} , then \mathcal{V}_σ is an affine normal \mathbb{T} -toric variety over K° .

Proof: If σ is contained in the hyperplane $N_{\mathbb{R}} \times \{0\}$, then \mathcal{V}_σ is the normal toric variety U_{σ_0} over K associated to σ_0 . Since K is of finite type over the valuation ring K° , it is also a normal toric variety over K° .

If σ is not contained in $N_{\mathbb{R}} \times \{0\}$, then σ_1 is a non-empty Γ -rational polyhedron Δ in $N_{\mathbb{R}}$ with $\mathcal{V}_\sigma = \mathcal{U}_\Delta$ and the claim follows from 6.11. \square

Definition 7.5 A Γ -admissible fan Σ in $N_{\mathbb{R}} \times \mathbb{R}_+$ is a fan of Γ -admissible cones. For $s \geq 0$, let Σ_s be the fan $\{\sigma_s \mid \sigma \in \Sigma\}$ in $N_{\mathbb{R}}$.

Remark 7.6 It was noticed by Burgos and Sombra [BS] that if \mathcal{C} is a Γ -rational polyhedral complex in $N_{\mathbb{R}}$, then $c(\mathcal{C}) := \{c(\Delta) \mid \Delta \in \mathcal{C}\}$ is not necessarily a fan in $N_{\mathbb{R}} \times \mathbb{R}_+$. However if the support of \mathcal{C} is convex, then they prove that $c(\mathcal{C})$ is a Γ -admissible fan. This gives a bijective correspondence between complete Γ -rational pointed polyhedral complexes of $N_{\mathbb{R}}$ and complete Γ -admissible fans of $N_{\mathbb{R}} \times \mathbb{R}_+$.

7.7 Let Σ be a Γ -admissible fan in $N_{\mathbb{R}} \times \mathbb{R}_+$. Then the affine \mathbb{T} -toric schemes \mathcal{V}_{σ} , $\sigma \in \Sigma$, may be glued along the open subschemes \mathcal{V}_{ρ} from common subfaces ρ to get a normal \mathbb{T} -toric scheme \mathcal{B}_{Σ} over K° . The generic fibre of \mathcal{B}_{Σ} is the normal T -toric variety Y_{Σ_0} over K associated to the fan Σ_0 in $N_{\mathbb{R}}$. This follows all from the affine case except separatedness which we shall prove next:

Lemma 7.8 *The scheme \mathcal{B}_{Σ} is separated over K° .*

Proof: Let $\sigma := \sigma' \cap \sigma''$ for $\sigma', \sigma'' \in \Sigma$. We have to show that the canonical morphism $\mathcal{V}_{\sigma} \rightarrow \mathcal{V}_{\sigma'} \times_{K^{\circ}} \mathcal{V}_{\sigma''}$ is a closed embedding. To prove that we may assume that σ', σ'' are not contained in $N_{\mathbb{R}} \times \{0\}$ (as the claim is well-known for toric varieties over a field). Then we have pointed Γ -rational polyhedra $\Delta' := \sigma'_1$, $\Delta'' := \sigma''_1$ and $\Delta := \Delta' \cap \Delta'' = \sigma_1$ in $N_{\mathbb{R}}$ with $\mathcal{U}_{\Delta'} = \mathcal{V}_{\sigma'}$, $\mathcal{U}_{\Delta''} = \mathcal{V}_{\sigma''}$ and $\mathcal{U}_{\Delta} = \mathcal{V}_{\sigma}$. We have to show that $K[M]^{\Delta}$ is generated by $K[M]^{\Delta'}$ and $K[M]^{\Delta''}$ as a K° -algebra. It is enough to consider $f \in K[M]^{\Delta}$ of the form $f = \alpha \chi^u$ with $\alpha \in K^{\times}$ and $u \in M \cap (\sigma_0)^{\vee}$.

Let $G := \{\gamma \in \mathbb{R} \mid \exists k \in \mathbb{N} \setminus \{0\} \ k\gamma \in \Gamma\}$, then the affine subspace of $N_{\mathbb{R}}$ generated by the face of a Γ -rational polyhedron is a N_G -translate of a rational linear subspace. We conclude that there is $u_0 \in M$ and $\omega_0 \in N_G$ such that $\Delta = \Delta' \cap (\omega_0 + \{u_0\}^{\perp})$, $\Delta' \subset \omega_0 + \{\omega \in N_{\mathbb{R}} \mid \langle u_0, \omega \rangle \geq 0\}$ and $\Delta'' \subset \omega_0 + \{\omega \in N_{\mathbb{R}} \mid \langle u_0, \omega \rangle \leq 0\}$. There is $\alpha_0 \in K$ and a non-zero $k \in \mathbb{N}$ with $v(\alpha_0) = k\langle u_0, \omega_0 \rangle$. We consider a vertex ω' of Δ' . By construction,

$$v_{\omega'}((\chi^{ku_0}/\alpha_0)^m f) = km\langle u_0, \omega' - \omega_0 \rangle + v_{\omega'}(f)$$

is non-negative for $m \gg 0$. We conclude that $v_{\Delta'}(g) \geq 0$ for $g := (\chi^{ku_0}/\alpha_0)^m f$.

Since

$$v_{\omega''}(\alpha_0 \chi^{-ku_0}) = k\langle u_0, \omega_0 - \omega'' \rangle \geq 0$$

for every $\omega'' \in \Delta''$, we conclude that $\alpha_0 \chi^{-ku_0} \in K[M]^{\Delta''}$. Using $f = (\alpha_0 \chi^{-ku_0})^m g$, we get the claim proving that \mathcal{B}_{Σ} is separated. \square

7.9 We have a bijective correspondence between torus orbits of \mathcal{B}_{Σ} and open faces of Σ . The torus orbits in the generic fibre correspond to the open faces contained in $N_{\mathbb{R}} \times \{0\}$ using the theory of toric varieties over a field. The torus orbits in the special fibre correspond to the remaining open faces of Σ using the fact that the latter are the open faces of the polyhedral complex Σ_1 in $N_{\mathbb{R}}$. Indeed, the orbits are contained in an affine \mathbb{T} -toric scheme \mathcal{V}_{σ} for some $\sigma \in \Sigma$ and so we may use Proposition 6.22. We will later describe the orbit correspondence for \mathcal{B}_{Σ} in a neat way (see Proposition 8.8).

If Z_{τ} is the torus orbit corresponding to the open face τ of Σ , then we have

$$\dim(\tau) = \text{codim}(Z_{\tau}, \mathcal{B}_{\Sigma}).$$

In particular, the \mathbb{T} -invariant prime divisors on \mathcal{B}_{Σ} correspond to the halflines in Σ . The irreducible components of the special fibre of \mathcal{B}_{Σ} correspond to the halflines of Σ not contained in $N_{\mathbb{R}} \times \{0\}$ or in other words to the vertices of Σ_1 .

Lemma 7.10 *Suppose that the valuation on K has value group $\Gamma = \mathbb{Z}$ and suppose $v(\pi) = 1$ for $\pi \in K$. Let Y_{σ} be an irreducible component of the special fibre of \mathcal{B}_{Σ} corresponding to the halfline σ of Σ . Then the multiplicity of the divisor Y_{σ} in \mathcal{B}_{Σ} is equal to k , where (ω, k) is the primitive generator of the monoid $\sigma \cap (N \times \mathbb{Z})$.*

Proof: See [KKMS], §4.3. \square

Proposition 7.11 *If the valuation on K is discrete, then the following conditions are equivalent for a Γ -admissible fan Σ in $N_{\mathbb{R}} \times \mathbb{R}_+$:*

- (a) *The vertices of Σ_1 are contained in N_{Γ} .*
- (b) *The special fibre $(\mathcal{Y}_{\Sigma})_s$ is reduced.*
- (c) *$(\mathcal{Y}_{\Sigma})_s$ is geometrically reduced.*
- (d) *For all valued fields (L, w) extending (K, v) , the formation of \mathcal{Y}_{Σ} is compatible with base change to L° .*
- (e) *For all $\Delta \in \Sigma_1$, the canonical map $K[M]^{\Delta} \otimes_{K^{\circ}} \tilde{K} \rightarrow \mathcal{A}_{\Delta}^{\circ} / \mathcal{A}_{\Delta}^{\circ\circ}$ is an isomorphism, where we refer to Proposition 6.17 for the definition of \mathcal{A}_{Δ} .*

Proof: The equivalence of (a) and (b) follows from Lemma 7.10. Clearly, (c) implies (b).

Now let $\sigma \in \Sigma$ and $\Delta := \sigma_1$. Suppose that the vertices ω_j of Δ are contained in N_{Γ} . In this case, we may use also the last part of the proof of Proposition 6.7 to get a set of generators of $K[M]^{\Delta}$ which depends only on the combinatorics of Δ and hence it generates also $L[M]^{\Delta}$. This proves

$$L[M]^{\Delta} = K[M]^{\Delta} \otimes_{K^{\circ}} L^{\circ}$$

and hence (a) implies (d).

Now suppose that (d) holds. There is a finite extension L/K such that (a) holds for the value group of L . By the equivalence of (a) and (b), we conclude that the special fibre of $(\mathcal{Y}_{\Sigma})_L$ is reduced and hence the special fibre of \mathcal{Y}_{Σ} is also reduced. We may repeat this for any finite extension of K and hence (d) yields (c). Since the residue algebra $\mathcal{A}_{\Delta}^{\circ} / \mathcal{A}_{\Delta}^{\circ\circ}$ of a strictly affinoid algebra is always reduced, we see that (e) implies (b).

Finally we show that (a) implies (e). Since the vertices are in N_{Γ} , it is easy to see that the kernel of the quotient homomorphism $K[M]^{\Delta} \rightarrow K[M]^{\Delta} / \langle K^{\circ\circ} \rangle = K[M]^{\Delta} \otimes_{K^{\circ}} \tilde{K}$ is equal to $\{\sum_{u \in M} a_u \chi^u \in K[M] \mid v(a_u) + \langle u, \omega \rangle > 0 \ \forall \omega \in \Delta\}$. By density of $K[M]^{\Delta}$ in $\mathcal{A}_{\Delta}^{\circ}$, we deduce (e). \square

Proposition 7.12 *If v is not a discrete valuation, then (a) still yields (b)–(e) in Proposition 7.11. In particular, if Γ is a divisible group in \mathbb{R} , then (a)–(e) hold.*

Proof: The above proof shows that (a) implies (d) and (e). Moreover, we have seen that (e) yields (b) without using that v is a discrete valuation. Using this for any valued field (L, w) extending (K, v) , we deduce that (a) yields (c). If Γ is a divisible group in \mathbb{R} , then the vertices of Σ are always in N_{Γ} proving also the last claim. \square

7.13 For a Γ -rational polyhedron Δ in $N_{\mathbb{R}}$, we introduce the following notation: The affine space generated by Δ is a translate of $\mathbb{L}_{\mathbb{R}}$ for a rational linear subspace \mathbb{L} of $N_{\mathbb{Q}}$. Then $N_{\Delta} := N \cap \mathbb{L}_{\Delta}$ and $N(\Delta) := N/N_{\Delta}$ are free abelian groups of finite rank with quotient homomorphism $\pi_{\Delta} : N \rightarrow N(\Delta)$. Dually, we have $M(\Delta) := \mathbb{L}_{\Delta}^{\perp} \cap M = \text{Hom}(N(\Delta), \mathbb{Z})$.

We return to an arbitrary valued field (K, v) . Let Σ be a Γ -admissible cone in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let Z be an orbit of \mathcal{Y}_{Σ} contained in the generic fibre. By 7.9, Z corresponds to the relative interior of a rational cone $\sigma \in \Sigma_0$

Proposition 7.14 *Under the hypothesis above, the closure \overline{Z} of Z in \mathcal{Y}_Σ is isomorphic to the $\text{Spec}(K^\circ[M(\sigma)])$ -toric scheme over K° associated to the Γ -admissible fan $\Sigma_\sigma := \{(\pi_\sigma \times \text{id}_{\mathbb{R}_+})(\nu) \mid \nu \in \Sigma, \nu \supset \sigma\}$ in $N(\sigma)_\mathbb{R} \times \mathbb{R}_+$.*

Proof: Let $\nu \in \Sigma$ with $\nu \supset \sigma$ and let $\nu_\sigma := (\pi_\sigma \times \text{id}_{\mathbb{R}_+})(\nu)$. Then there is a canonical surjective K° -algebra homomorphism

$$K[M]^\nu \rightarrow K[M(\sigma)]^{\nu_\sigma}, \quad \alpha\chi^u \mapsto \begin{cases} \alpha\chi^u, & \text{if } u \in M(\sigma), \\ 0, & \text{if } u \in M \setminus M(\sigma). \end{cases}$$

We conclude that the $\text{Spec}(K^\circ[M(\sigma)])$ -toric scheme over K° associated to the Γ -admissible fan Σ_σ in $N(\sigma)_\mathbb{R} \times \mathbb{R}_+$ is a closed subscheme of \mathcal{Y}_Σ . By [Ful2], §3.1, its generic fibre is the closure of Z in the generic fibre of \mathcal{Y}_Σ . By Proposition 4.4, we get the claim. \square

Now we assume that the orbit Z of \mathcal{Y}_Σ is contained in the special fibre. By 7.9, Z corresponds to the relative interior τ of $\Delta \in \Sigma_1$. Similarly as in Proposition 6.15, Γ -rationality of Δ yields that $M(\Delta)_\tau := \{u \in M(\sigma) \mid \langle u, \omega \rangle \in \Gamma \forall \omega \in \tau\}$ is a lattice of finite index in $M(\Delta)$. For $\nu \in \Sigma_1$ with face Δ , we define $\text{LC}_\tau(\nu) := \text{LC}_\omega(\nu)$ which is independent of the choice of $\omega \in \tau$ and where we use the local cones from A.6.

Proposition 7.15 *Under the hypothesis above, the closure \overline{Z} of Z in \mathcal{Y}_Σ is equivariantly isomorphic to the $\text{Spec}(K^\circ[M(\Delta)_\tau])$ -toric scheme over \tilde{K} associated to the rational fan $\{\pi_\Delta(\text{LC}_\tau(\nu)) \mid \nu \in \Sigma_1, \nu \supset \Delta\}$ in $N(\Delta)_\mathbb{R}$.*

Proof: If ω is a vertex of Σ_1 , this follows immediately from Corollary 6.15. The general case follows from the corresponding generalization of Corollary 6.15 which can be proved completely analogous. We leave the details to the reader. \square

8 Tropical cone of a variety

In this section, K denotes a field with a non-trivial non-archimedean absolute value $|\cdot|_v$, corresponding valuation $v := -\log |\cdot|_v$ and valuation ring K° . We consider $W := \{\varepsilon v \mid \varepsilon \geq 0\}$ which is induced by all valuations equivalent to v together with the trivial absolute which we denote by 0. We may identify W with \mathbb{R}_+ using $\varepsilon v \leftrightarrow \varepsilon$. The value group of $w \in W$ is denoted by Γ_w and the residue field by $k(w)$. Obviously, we have $k(w) = k(v)$ for $w \neq 0$ and $k(0) = K$. At the end of this section, we show how to adjust the notation so that everything works also for the trivial valuation.

We have seen the advantage of using fans Σ in $N_\mathbb{R} \times \mathbb{R}_+$ rather than polyhedral complexes in $N_\mathbb{R}$ to define an associated toric scheme \mathcal{Y}_Σ over K° . It is not surprising that the consideration of the closed cone in $N_\mathbb{R} \times \mathbb{R}_+$ generated by $\text{Trop}_v(X) \times \{1\}$ is useful to describe information about the closure of the closed subscheme X of T in \mathcal{Y}_Σ in a uniform way. Moreover, we will see that the tropical variety of X with respect to the trivial valuation is just the intersection of this tropical cone with $N_\mathbb{R} \times \{0\}$.

8.1 For an algebraic scheme X over K , we have defined in Section 2 the analytification with respect to the valuation w which we denote here by X_w^{an} . In fact, we can define the *analytification* X_W^{an} of X with respect to W by the same process allowing all multiplicative seminorms p with restriction $w_p := p|_K \in W$. This gives again a locally compact Hausdorff space which is as a set equal to the disjoint union of all X_w^{an} with w ranging over W .

8.2 For $w \in W$, let $\text{trop}_w : T_w^{\text{an}} \rightarrow N_{\mathbb{R}}$ be the tropicalization map. Proceeding fibrewise, we get the W -tropicalization map

$$\text{trop}_W : T_W^{\text{an}} \rightarrow N_{\mathbb{R}} \times W, \quad t \mapsto (\text{trop}_{w_t}(t), w_t).$$

It is clear that trop_W is continuous.

Definition 8.3 Let X be a closed subscheme of T . Then we define the *tropical cone* associated to X as $\text{trop}_W(X_W^{\text{an}})$ and we denote it by $\text{Trop}_W(X)$.

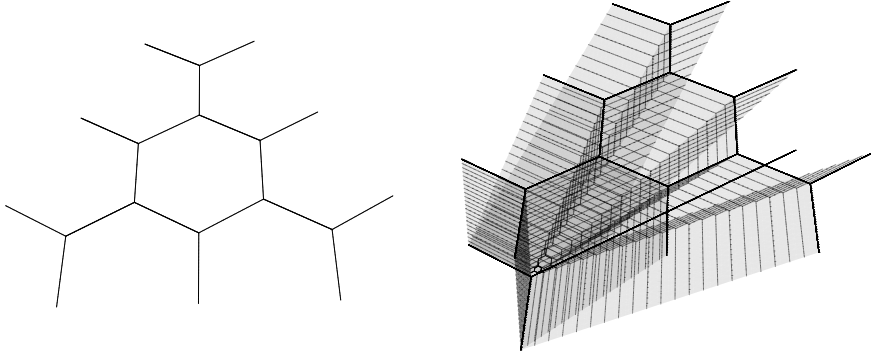


Figure 1: Tropical curve and its tropical cone

Proposition 8.4 We have $\text{Trop}_W(X) \cap (N_{\mathbb{R}} \times \{w\}) = \text{Trop}_w(X) \times \{w\}$ for all $w \in W$. Moreover, the tropical cone $\text{Trop}_W(X)$ is indeed a cone in $N_{\mathbb{R}} \times W$ and it is the closure of the cone generated by $\text{Trop}_v(X) \times \{v\}$.

Proof: The first claim is clear from the definition of the tropical cone. For any $w \in W$, $t \in T_w^{\text{an}}$ and $\varepsilon \geq 0$, the seminorm t^ε is contained in $T_{\varepsilon w}^{\text{an}}$. Then

$$\text{trop}_W(t^\varepsilon) = (\text{trop}_{\varepsilon w}(t^\varepsilon), \varepsilon w) = \varepsilon(\text{trop}_w(t), w) = \varepsilon \text{trop}_W(t)$$

shows that $\text{Trop}_W(X)$ is a cone and that $\text{Trop}_W(X) \cap (N_{\mathbb{R}} \times (W \setminus \{0\}))$ is generated by $\text{Trop}_v(X) \times \{v\}$. Using that T_W^{an} is locally compact and that trop_W is continuous, we easily deduce that $\text{Trop}_W(X)$ is closed.

It remains to show that every $\omega_0 \in \text{Trop}_0(X) \times \{0\}$ is contained in the closure of $\text{Trop}_W(X) \cap (N_{\mathbb{R}} \times (W \setminus \{0\}))$. By the fundamental theorem of tropicalization (see Theorem 5.6), the initial degeneration $\text{in}_{\omega_0}^0(X)$ with respect to the trivial valuation 0 is non-empty, i.e. there is $f = \sum_{u \in S} a_u \chi^u$ in the ideal I_X of X with non-zero coefficients $a_u \in K$ and at least two terms with minimal ω_0 -weight $\langle u, \omega_0 \rangle$. Let S_0 be the subset of S with minimal ω_0 -weight, i.e.

$$\langle u, \omega_0 \rangle = \langle u', \omega_0 \rangle, \quad \langle u, \omega_0 \rangle < \langle u'', \omega_0 \rangle \quad (5)$$

for all $u, u' \in S_0$ and $u'' \in S \setminus S_0$. Let us choose $u \in S_0$ with $v(a_u)$ minimal. Since $v(a_u) + \langle u, \omega_0 \rangle$ is also minimal on S_0 , there is $u' \in S_0 \setminus \{u\}$ and an $\omega \in N_{\mathbb{R}}$ such that

$$v(a_u) + \langle u, \omega \rangle = v(a_{u'}) + \langle u', \omega \rangle \leq v(a_{u''}) + \langle u'', \omega \rangle \quad (6)$$

for all $u'' \in S_0$. Indeed, this follows by choosing $(u', \omega) \in (S_0 \setminus \{u\}) \times N_{\mathbb{R}}$ with $v(a_u) + \langle u, \omega \rangle = v(a_{u'}) + \langle u', \omega \rangle$ such that the distance from ω to ω_0 is minimal. Replacing ω by $\omega + \lambda\omega_0$ with $\lambda \gg 0$, we keep (6) and we get additionally

$$v(a_u) + \langle u, \omega \rangle \leq v(a_{u''}) + \langle u'', \omega \rangle \quad (7)$$

for all $u'' \in S$ by using (5). Now we choose a field extension L/K with a complete absolute value extending $|\cdot|_v$ and with value group \mathbb{R} . For $\varepsilon > 0$, let $\omega_\varepsilon := \omega_0 + \varepsilon\omega$. Then the relations (6) and (7) hold also for $(\varepsilon v, \omega_\varepsilon)$ instead of (v, ω) . Normalizing f , we may assume that the minimal v_ω -weight $v(a_u) + \langle u, \omega \rangle$ is equal to 1. Then we deduce that the initial degeneration $\text{in}_{\omega_\varepsilon}^{\varepsilon v}(X)$ with respect to the valuation εv is non-empty. The fundamental theorem again shows that $\omega_\varepsilon \in \text{Trop}_{\varepsilon v}(X)$. Since $\lim_{\varepsilon \rightarrow 0} \omega_\varepsilon = \omega_0$, we get the claim. \square

As a consequence, we get the Bieri–Groves theorem for the tropical cone:

Corollary 8.5 *The tropical cone $\text{Trop}_W(X)$ of X in $N_{\mathbb{R}} \times \mathbb{R}_+$ is a finite union of Γ -admissible cones in $N_{\mathbb{R}} \times \mathbb{R}_+$. If X is of pure dimension d , then we may choose these cones $d + 1$ -dimensional.*

Proof: Using Proposition 8.4, this follows immediately from Theorem 3.3. \square

Let Σ be a Γ -admissible fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let \mathcal{Y}_Σ^v be the associated toric scheme over K° . We recall that we may identify T with the dense open orbit in \mathcal{Y}_Σ^v and this orbit is contained in the generic fibre Y_{Σ_0} .

Proposition 8.6 *Let (L, u) be a valued field extending (K, v) . Then $P \in T(L)$ is an L° -integral point of \mathcal{Y}_Σ if and only if $\text{trop}_u(P)$ is contained in the support of Σ_1 .*

Proof: Suppose that P is an L° -integral point of \mathcal{Y}_Σ . Then the closed point of $\text{Spec}(L^\circ)$ maps to \mathcal{U}_Δ for some $\Delta \in \Sigma_1$ and we have $P \in \mathcal{U}_\Delta(L^\circ)$. By Lemma 6.21, we get $\text{trop}_u(P) \in \Delta$.

Conversely, we assume that $\text{trop}_u(P) \in \Delta$ for $\Delta \in \Sigma_1$. Then Lemma 6.21 shows that P is an L° -integral point of \mathcal{U}_Δ proving the claim. \square

8.7 We conclude from Proposition 8.6 that we have a well-defined reduction map $\pi_W : \text{trop}_W^{-1}(|\Sigma|) \rightarrow \mathcal{Y}_\Sigma^v$. Indeed, we have $\mathcal{Y}_{\Sigma}^{\varepsilon v} = \mathcal{Y}_\Sigma^v$ for all $\varepsilon > 0$ and so we may use the reduction map $\pi_w : \text{trop}_w^{-1}(|\Sigma_\varepsilon|) \rightarrow (\mathcal{Y}_{\Sigma}^w)_s = (\mathcal{Y}_\Sigma^v)_s$ in the fibre over $w = \varepsilon v$. For $w = 0$, the special fibre agrees with the generic fibre Y_{Σ_0} and we use the reduction $\pi_0 : \text{trop}_0^{-1}(|\Sigma_0|) \rightarrow Y_{\Sigma_0}$. Note that we may use Proposition 8.6 also for the trivial valuation $v = 0$.

Then we can describe the orbit-face correspondence in the following uniform way.

Proposition 8.8 *There is a bijective order reversing correspondence between \mathbb{T} -orbits Z of \mathcal{Y}_Σ^v and open faces τ of Σ given by*

$$Z = \pi_W(\text{trop}_W^{-1}(\tau)), \quad \tau = \text{trop}_W(\pi_W^{-1}(Z)).$$

Proof: We easily reduce to the case of an invariant open subset \mathcal{V}_σ of \mathcal{Y}_Σ^v for $\sigma \in \Sigma$. Then the claim follows from Proposition 6.22 applied to every $w \in W$. \square

Remark 8.9 If v is the trivial valuation, then we have to adjust the notation of this section by using the set \mathbb{R}_+ rather than $W = \{0\}$. We define $X_W^{\text{an}} := X_0^{\text{an}} \times \mathbb{R}_+$ which is a locally compact Hausdorff space. Then everything works as above.

9 Projectively embedded toric varieties

In this section, K denotes a field with a non-archimedean absolute value $|\cdot|$, corresponding valuation $v := -\log|\cdot|$ and value group $\Gamma := v(K^\times)$. We have defined toric varieties in Definition 6.1. Here, we consider projective toric varieties over K° with an equivariant embedding into projective space. These toric varieties are not necessarily normal. This section is inspired by the introductory article of E. Katz ([Kat], section 4) and we will generalize his results. Further references: [CLS], §2.1, §3.A; [GKZ], Chapter 5.

Recall that $\mathbb{T} = \text{Spec}(\mathbb{G}_m^n)$ is a split multiplicative torus over K° with generic fibre T . The character group of T is M and the character corresponding to $u \in M$ is denoted by χ^u . For convenience, we always choose coordinates on the projective space $\mathbb{P}_{K^\circ}^N$ defined over the valuation ring K° .

9.1 We first recall the following well-known way to construct a not necessarily normal toric subvariety Y from $A = (u_0, \dots, u_N) \in M^{N+1}$ and $\mathbf{y} = (y_0 : \dots : y_N) \in \mathbb{P}^N(K)$ (see [GKZ], Chapter 5). The torus T acts on $\mathbb{P}_{K^\circ}^N$ by

$$t \cdot \mathbf{x} := (\chi^{u_0}(t)x_0 : \dots : \chi^{u_N}(t)x_N)$$

and we define Y as the closure of the orbit $T\mathbf{y}$. Then there is a bijective correspondence between T -orbits of Y and faces of the weight polytope $\text{Wt}(\mathbf{y})$ which is defined as the convex hull of $A(\mathbf{y}) := \{u_j \mid y_j \neq 0\}$. If Q is a face of $\text{Wt}(\mathbf{y})$, then the corresponding orbit is given by

$$Z := \{\mathbf{z} \in Y \mid z_j \neq 0 \iff u_j \in Q\}.$$

Duality gives also a bijective correspondence to the normal fan Σ of $\text{Wt}(\mathbf{y})$. The cone σ corresponding to the face Q is the set of $\omega \in N_{\mathbb{R}}$ such that the linear functional $\langle \cdot, \omega \rangle$ achieves its minimum on $\text{Wt}(\mathbf{y})$ precisely in the face Q . The torus corresponding to the orbit Z has character group $\mathbb{Z}(\sigma^\perp \cap M)$ and hence $\dim(Z) = \dim(Q) = n - \dim(\sigma)$ (see [CLS], Section 3.A).

9.2 The goal of this section is to perform a similar construction over the valuation ring K° . Let $A = (u_0, \dots, u_N) \in M^{N+1}$ and let $\mathbf{y} = (y_0 : \dots : y_N) \in \mathbb{P}^N(K)$. We define the *height function* of \mathbf{y} by

$$a : \{0, \dots, N\} \rightarrow \Gamma \cup \{\infty\}, \quad j \mapsto v(y_j).$$

The torus \mathbb{T} operates on $\mathbb{P}_{K^\circ}^N$ by

$$\mathbb{T} \times_{K^\circ} \mathbb{P}_{K^\circ}^N \rightarrow \mathbb{P}_{K^\circ}^N, \quad (t, \mathbf{x}) \mapsto (\chi^{u_0}(t)x_0 : \dots : \chi^{u_N}(t)x_N).$$

The closure of the orbit $T\mathbf{y}$ in $\mathbb{P}_{K^\circ}^N$ is a projective toric variety with respect to the split torus over K° with generic fibre $T/\text{Stab}(\mathbf{y})$. We denote this projective toric variety by $\mathcal{Y}_{A,a}$ and its generic fibre by $Y_{A,a}$. Using the base point \mathbf{y} , the torus $T/\text{Stab}(\mathbf{y})$ may be seen as an open dense subset of $\mathcal{Y}_{A,a}$. Next, we will see that $\mathcal{Y}_{A,a}$ depends only on the affine geometry of (A, a) .

Proposition 9.3 *Suppose that \mathbb{T}' is another split multiplicative torus over K° with character lattice M' and that there is an injective affine transformation $F : M \rightarrow M'$ of lattices. Let $A = (u_0, \dots, u_N) \in M^{N+1}$, $A' = (u'_0, \dots, u'_N) \in (M')^{N+1}$ and let $\mathbf{y}, \mathbf{y}' \in \mathbb{P}^N(K)$ with height functions a (resp. a'). Let $\mathcal{Y}_{A,a}$ (resp. $\mathcal{Y}_{A',a'}$) be the projective toric variety with respect to A, \mathbf{y} (resp. A', \mathbf{y}'). We assume that $F(u_j) = u'_j$ for every j with $y_j \neq 0$. If there is $\lambda \in \Gamma$ such that $a' = a + \lambda$, then $\mathcal{Y}_{A,a}$ is canonically isomorphic to $\mathcal{Y}_{A',a'}$.*

Proof: The injective linear map corresponding to F induces a surjective homomorphism $\mathbb{T}' \rightarrow \mathbb{T}$ of multiplicative tori. If $\mathbf{y} = \mathbf{y}'$, then we deduce that $\mathcal{Y}_{A,a} = \mathcal{Y}_{B,a}$. In general, we have $\mathbf{y}' = g\mathbf{y}$ for some $g = (g_0, \dots, g_N) \in K^{N+1}$ with $|g_0| = \dots = |g_N| \neq 0$. Then g induces a linear automorphism of $\mathbb{P}_{K^\circ}^N$ mapping $\mathcal{Y}_{A,a}$ onto $\mathcal{Y}_{B,b}$. If $y_j \neq 0$, then g_j is uniquely determined and hence we have constructed a canonical isomorphism. \square

Corollary 9.4 *The open dense orbit $T/\text{Stab}(\mathbf{y})$ of $\mathcal{Y}_{A,a}$ is a torus with character lattice $\{\sum m_j u_j \mid \sum m_j = 0\}$, where j ranges over $0, \dots, N$ with $y_j \neq 0$. This orbit has dimension equal to $\dim(\mathbb{A})$, where \mathbb{A} is the smallest affine subspace of $M_{\mathbb{R}}$ containing $A(\mathbf{y}) := \{u_j \mid y_j \neq 0\}$.*

The following result is well-known for fields or discrete valuation rings. We need it for arbitrary valuation rings of rank 1 which are not noetherian in general and hence we may not use algebraic intersection theory. However, there is an intersection theory with Cartier divisors in this situation (see [Gub1]) which together with the result for the generic fibre will easily imply the claim.

Proposition 9.5 *The restriction map gives an isomorphism $\text{Pic}(\mathbb{P}_{K^\circ}^N) \rightarrow \text{Pic}(\mathbb{P}_K^N)$ and pull-back with respect to the second projection gives an isomorphism $\text{Pic}(\mathbb{P}_{K^\circ}^N) \rightarrow \text{Pic}(\mathbb{T} \times_{K^\circ} \mathbb{P}_{K^\circ}^N)$*

Proof: To prove the first claim, we have to show that every line bundle \mathcal{L} on $\mathbb{P}_{K^\circ}^N$ is isomorphic to $\mathcal{O}_{\mathbb{P}^N}(m)$ over K° for some $m \in \mathbb{Z}$. We consider a Cartier divisor $\mathcal{D} = \{U_i, \gamma_i\}_{i \in I}$ which is trivial on the generic fibre \mathbb{P}_K^N and we have to prove that \mathcal{D} is trivial. We may assume that $\mathcal{U}_i = \text{Spec}(K^\circ[x_1, \dots, x_N]_{h_i})$ for a polynomial $h_i \in K^\circ[x_1, \dots, x_N]$. Obviously, we may skip all charts with empty special fibre. This means that the prime factors p_1, \dots, p_r of h_i in $K^\circ[x_1, \dots, x_N]$ are non-constant. Using unique factorization, we get $\mathcal{O}(U_i)^\times = K^\times p_1^{\mathbb{Z}} \cdots p_r^{\mathbb{Z}}$ for the generic fibre U_i of \mathcal{U}_i . By triviality of \mathcal{D} on the generic fibre \mathbb{P}_K^N , we get $\gamma_i = \lambda_i h'_i$ for some $\lambda_i \in K^\times$ and $h'_i \in p_1^{\mathbb{Z}} \cdots p_r^{\mathbb{Z}}$. We want to show that these factorizations fit on an overlapping $\mathcal{U}_i \cap \mathcal{U}_j$. As $\mathcal{U}_i \cap \mathcal{U}_j$ intersects the special fibre, there is a valued field (L, w) extending (K, v) and an L° -integral point P of $\mathcal{U}_i \cap \mathcal{U}_j$. Using $h_i \in \mathcal{O}(\mathcal{U}_i)^\times$, we get $|h'_i(P)|_w = 1$. The multiplicity $m(\mathcal{D}, \mathbb{P}_K^N)$ of \mathcal{D} along the special fibre \mathbb{P}_K^N was defined in [Gub1], Section 3. Since the special fibre is irreducible and smooth, it is shown in Proposition 7.6 of [Gub2] that

$$m(\mathcal{D}, \mathbb{P}_K^N) = -\log |\gamma_i(P)| = v(\lambda_i).$$

We conclude that $v(\lambda_i) = v(\lambda_j)$. Dividing the equations of \mathcal{D} by a fixed λ_i , we deduce that \mathcal{D} is trivial on $\mathbb{P}_{K^\circ}^N$ proving the first claim.

Similarly, we prove the second claim. The claim holds on the generic fibre and hence it is enough to show that a Cartier divisor \mathcal{D} on $\mathbb{T} \times_{K^\circ} \mathbb{P}_{K^\circ}^N$ which is trivial on the generic fibre $T \times_K \mathbb{P}_K^N$ is trivial on $\mathbb{T} \times_{K^\circ} \mathbb{P}_{K^\circ}^N$. This is done as above replacing K by the unique factorization domain $\mathcal{O}(T) = K[M]$ and using $K[M]^\times = \{\lambda \chi^u \mid u \in M, \lambda \in K^\times\}$. \square

Remark 9.6 It was pointed out to the author by Qing Liu and C. Pépin that the second claim holds more generally for any integral normal scheme \mathcal{X} instead of $\mathbb{P}_{K^\circ}^N$. Their argument is as follows: The injectivity of the pull-back homomorphism follows from the existence of a section for the second projection. To prove surjectivity, we may assume that \mathcal{X} is affine by using the same argument as for injectivity. Moreover, we may assume that \mathcal{X} is noetherian by the descent argument in [EGA IV], Proposition 8.9.1. Then the claim follows from Corollaire 21.4.13 in the list of Errata and Addendum in [EGA IV].

Lemma 9.7 *Suppose that the torus \mathbb{T} acts linearly on $\mathbb{P}_{K^\circ}^N$. Then this action lifts to a linear representation of \mathbb{T} on $\mathbb{A}_{K^\circ}^{N+1}$.*

Proof: The action $\sigma : \mathbb{T} \times_{K^\circ} \mathbb{P}_{K^\circ}^N \rightarrow \mathbb{P}_{K^\circ}^N$ is linear which means that it is given by a homomorphism $\mathbb{T} \rightarrow PGL(N)$ defined over K° . We are looking for a lift to a homomorphism $\mathbb{T} \rightarrow GL(N+1)$. This is equivalent to the existence of a \mathbb{T} -linearization on the line bundle $L := O(1)$ of $\mathbb{P}_{K^\circ}^N$, i.e. an action of \mathbb{T} on L which is compatible with the given group action σ . Here, we use the language of [Mum], §1.3, which is written for schemes over a base field. However, the argument for the existence of a \mathbb{T} -linearization in [Mum], Proposition 1.5, extends to the case of a valuation ring. Indeed, the essential point is the existence of an isomorphism $\sigma^*(L) \cong p_2^*(L)$ which follows from Proposition 9.5 and then we may conclude as at the end of the proof of [Mum], Proposition 1.5, to prove that $L = O(1)$ has a \mathbb{T} -linearization. \square

Proposition 9.8 *Let \mathcal{Y} be a closed irreducible subvariety of $\mathbb{P}_{K^\circ}^N$. Suppose that the torus \mathbb{T} operates linearly on $\mathbb{P}_{K^\circ}^N$ and leaves \mathcal{Y} invariant. If \mathbf{y} is any K -rational point in the open dense orbit of \mathcal{Y} , then after a suitable change of coordinates on $\mathbb{P}_{K^\circ}^N$, there is $A \in \tilde{T}^{N+1}$ such that $\mathcal{Y} = \mathcal{Y}_{A,a}$ for the height function $a : \{0, \dots, N\} \rightarrow \Gamma$ of \mathbf{y} .*

Proof: By Lemma 9.7, the projective representation of \mathbb{T} on $\mathbb{P}_{K^\circ}^N$ lifts to a representation S of \mathbb{T} on $\mathbb{A}_{K^\circ}^{N+1}$. Since the multiplicative torus T is split over K , it follows that the vector space $V := K^{N+1}$ has a simultaneous eigenbasis v_0, \dots, v_N for the T -action ([Bor], Proposition III.8.2). For $j = 0, \dots, N$, we have $S_t(v_j) = \chi_{u_j}(t)(v_j)$ for all $t \in T(K)$ and some $u_j \in M$. We may assume that the maximum of the absolute values of the coordinates of v_j is 1.

We consider the subgroup $U := \mathbb{T}(K^\circ) = \{t \in T(K) \mid v(t_1) = \dots = v(t_n) = 0\}$ of $T(K)$. For $t \in U$, we have $S_t \in GL(N+1, K^\circ)$ and hence the eigenvalues $\chi_{u_j}(t)$ have absolute value 1. If we use reduction modulo K° , then the U -action becomes a $(\mathbb{G}_m^n)_{\tilde{K}}$ -operation on \tilde{K}^{N+1} . The reduction of v_0, \dots, v_N will be a simultaneous eigenbasis for this action. By Nakayama's Lemma, it follows that v_0, \dots, v_N is a K° -basis for $(K^\circ)^{N+1}$. We choose the coordinates of $\mathbb{P}_{K^\circ}^N$ according to this basis and let a be the corresponding height function of \mathbf{y} . For $A = (u_0, \dots, u_N)$, we get $\mathcal{Y} = \mathcal{Y}_{A,a}$. \square

Remark 9.9 Every projective normal toric variety over a field can be equivariantly embedded into some projective space endowed with a linear torus action (see [Mum], §1.3). There are projective non-normal toric varieties over a field for which this is not true (see [GKZ], Remark 5.1.6).

9.10 In the following, we consider the toric variety $\mathcal{Y}_{A,a}$ for some $A \in M^{N+1}$ and $\mathbf{y} \in \mathbb{P}_{K^\circ}^{N+1}$ with height function a .

The *weight polytope* $\text{Wt}(\mathbf{y})$ is the convex hull of $A(\mathbf{y}) := \{u_j \mid a(j) < \infty\}$ in $M_{\mathbb{R}}$. The *induced subdivision* of $\text{Wt}(\mathbf{y})$ is given by projection of the faces of the convex hull of $\{(u_j, \lambda) \in M_{\mathbb{R}} \times \mathbb{R} \mid j = 0, \dots, N, \lambda \geq a(j)\}$. The weight subdivision is a polytopal complex denoted by $\text{Wt}(\mathbf{y}, a)$. The vertices of $\text{Wt}(\mathbf{y}, a)$ are contained in $A(\mathbf{y})$.

9.11 In the following, we need some additional notions from convex geometry which we have introduced in the appendix. By construction, there is a unique proper polyhedral function f on $M_{\mathbb{R}}$ such that the epigraph of f is equal to the convex hull of $\{(u_j, \lambda) \in M_{\mathbb{R}} \times \mathbb{R} \mid j = 0, \dots, N, \lambda \geq a(j)\}$. The domain of f is equal to $\text{Wt}(\mathbf{y})$ and $f(u_j) = a(j)$ for all vertices u_j of the weight subdivision $\text{Wt}(\mathbf{y}, a)$.

We define the *dual complex* $\mathcal{C}(A, a)$ of $\text{Wt}(\mathbf{y}, a)$ as the complete polyhedral complex in $N_{\mathbb{R}}$ characterized by the fact that the n -dimensional polyhedra in \mathcal{C} are the domains of linearity of the affine function

$$g(\omega) := \min_{j=0, \dots, N} a(j) + \langle u_j, \omega \rangle.$$

Obviously, all polyhedra in $\mathcal{C}(A, a)$ are Γ -rational. There is a bijective order reversing correspondence between the faces of $\text{Wt}(\mathbf{y}, a)$ and polyhedra in $\mathcal{C}(A, a)$. The polyhedron $\widehat{Q} \in \mathcal{C}(A, a)$ corresponding to the face Q of $\text{Wt}(\mathbf{y}, a)$ is given by

$$\begin{aligned} \widehat{Q} &= \{\omega \in N_{\mathbb{R}} \mid g(\omega) = \langle u, \omega \rangle + f(u) \ \forall u \in Q\} \\ &= \{\omega \in N_{\mathbb{R}} \mid g(\omega) = \langle u_j, \omega \rangle + a(j) \ \forall u_j \in A(\mathbf{y}) \cap Q\}. \end{aligned}$$

Conversely, the face $\widehat{\sigma}$ of $\text{Wt}(\mathbf{y}, a)$ corresponding to $\sigma \in \mathcal{C}(A, a)$ is given by

$$\widehat{\sigma} = \{u \in M_{\mathbb{R}} \mid g(\omega) = \langle u, \omega \rangle + f(u) \ \forall \omega \in \sigma\}$$

and it is also the convex hull of $\{u_j \in A \mid g(\omega) = \langle u_j, \omega \rangle + a(j) \ \forall \omega \in \sigma\}$. All this can be seen using the dual complex $\text{Wt}(\mathbf{y}, a)^f$ from A.11 and the conjugate polyhedral function f^* of f from A.10. Indeed, we have $f^*(\omega) = -g(-\omega)$ and hence $\text{Wt}(\mathbf{y}, a)^f = -\mathcal{C}(A, a)$.

In the next results, we will also use the tropicalization map $\text{trop}_v : T^{\text{an}} \rightarrow N_{\mathbb{R}}$ and the reduction map $\pi : Y_{A,a}^{\text{an}} \rightarrow (\mathcal{Y}_{A,a})_s$.

Proposition 9.12 *There are bijective order reversing correspondences between*

- (a) *faces Q of the weight subdivision $\text{Wt}(\mathbf{y}, a)$;*
- (b) *polyhedra σ of the dual complex $\mathcal{C}(A, a)$;*
- (c) *\mathbb{T} -orbits Z of the special fibre of $\mathcal{Y}_{A,a}$.*

The correspondences are given as follows: The face $Q = \widehat{\sigma}$ is the face of $\text{Wt}(\mathbf{y}, a)$ spanned by those u_j with $x_j \neq 0$ for $x \in Z$. The polyhedron σ is given by $\sigma = \widehat{Q}$ and $\text{relint}(\sigma) = \text{trop}_v^{-1}(\{t \in T^{\text{an}} \mid \pi(t\mathbf{y}) \in Z\})$. The orbit Z is equal to

$$\{\mathbf{x} \in (\mathcal{Y}_{A,a})_s \mid x_j \neq 0 \iff u_j \in A(\mathbf{y}) \cap Q\} = \{\pi(t\mathbf{y}) \mid t \in T^{\text{an}} \cap \text{trop}_v^{-1}(\text{relint}(\sigma))\}.$$

The correspondence $Q \leftrightarrow Z$ is preserving the natural orders and the other correspondences are order reversing. Moreover, we have $\dim(Q) = \dim(Z) = n - \dim(\sigma)$.

Proof: We have discussed the correspondence $Q \leftrightarrow \sigma$ in 9.11. Next, we note that every point z of $(\mathcal{Y}_{A,a})_s$ is the reduction of a point in $T^{\text{an}}\mathbf{y}$. Since $T\mathbf{y}$ is an open dense subset of the generic fibre of $\mathcal{Y}_{A,a}$, this follows from Lemma 4.12.

Now let σ be a polyhedron from $\mathcal{C}(A, a)$. We will show next that $Z := \{\pi(t\mathbf{y}) \mid t \in T^{\text{an}} \cap \text{trop}_v^{-1}(\text{relint}(\sigma))\}$ is a \mathbb{T} -invariant subset of $(\mathcal{Y}_{A,a})_s$. Let us consider the formal affinoid torus T° which is the affinoid subdomain of T^{an} given by the equations $|x_1| = \dots = |x_n| = 1$. The reduction map induces a surjective group homomorphism $T^\circ \rightarrow \mathbb{T}_s$ and $\pi : Y_{A,a}^{\text{an}} \rightarrow (\mathcal{Y}_{A,a})_s$ is equivariant with respect to this homomorphism. Since T° leaves the affinoid subdomain $\text{trop}_v^{-1}(\sigma)$ of T^{an} invariant, we conclude that Z is invariant under the \mathbb{T}_s -action.

For $z \in (\mathcal{Y}_{A,a})_s$, we have seen above that there is $t \in T^{\text{an}}$ with $z = \pi(t\mathbf{y})$. It follows from 9.11 that $\omega \in \text{relint}(\sigma)$ if and only if

$$A(\mathbf{y}) \cap Q = \{u_j \in A \mid g(\omega) = a(j) + \langle u_j, \omega \rangle\},$$

i.e. precisely the functions $a(j) + \langle u_j, \omega \rangle$ with $u_j \in A(\mathbf{y}) \cap Q$ are minimal in ω . If we apply this with $\omega := \text{trop}_v(t)$, then we deduce

$$Z = \{\mathbf{x} \in (\mathcal{Y}_{A,a})_s \mid x_j \neq 0 \iff u_j \in A(\mathbf{y}) \cap Q\}. \quad (8)$$

Next, we prove that Z is a \mathbb{T}_s -orbit. We have already seen that Z is \mathbb{T}_s -invariant. It remains to show that the action is transitive and so we consider $z_1, z_2 \in Z$. There is a complete valued field (F, u) extending (K, v) such that z_1, z_2 are \tilde{F} -rational. Let $L = F((\mathbb{R}))$ be the Mal'cev-Neumann ring. Note that L is a complete field consisting of certain power series in the variable x and with real exponents (see [Poo] for details). The advantage is that we have a canonical homomorphism $\rho : \mathbb{R} \rightarrow L^*$ with $v \circ \rho = \text{id}$. Using suitable coordinates, we get a homomorphism $N_{\mathbb{R}} \rightarrow T(L)$ which is a section of trop_v and which we also denote by ρ .

For $i = 1, 2$, there is $t_i \in T^{\text{an}}$ with $z_i = \pi(t_i \mathbf{y})$ and $\text{trop}_v(t_i) \in \text{reint}(\sigma)$. Choosing F sufficiently large, we may assume that t_i is induced by an F -rational point in T which we also denote by t_i . For $t \in T(L)$, we set $t^\circ := t \cdot \rho(-\text{trop}_v(t))$. This is an element of the formal affinoid torus $T^\circ(L)$ and hence reduces to an element $\tilde{t}^\circ \in \mathbb{T}(\tilde{L})$. The map $t \mapsto t^\circ$ is a homomorphism as well as the reduction. We will use this construction for t_1, t_2 and $t := t_2/t_1$. We claim that $\tilde{t}^\circ z_1 = z_2$. To see this, we note for $j \in A(\mathbf{y}) \cap Q$ that

$$(t^\circ t_1 \mathbf{y})_j = \chi^{u_j}(t^\circ t_1) y_j = \chi^{u_j}(\rho \circ \text{trop}_v(t_1/t_2) t_2) y_j = \lambda_j(t_2 \mathbf{y})_j$$

with factor

$$\lambda_j := \chi^{u_j}(\rho \circ \text{trop}_v(t_1/t_2)) = \rho(\langle u_j, \text{trop}_v(t_1) - \text{trop}_v(t_2) \rangle).$$

From the above considerations, we conclude that $\langle u_j, \text{trop}_v(t_i) \rangle + a(j) \in \text{reint}(\sigma)$ does not depend on the choice of $u_j \in A(\mathbf{y}) \cap Q$ and hence the factor λ_j does not depend on $j \in A(\mathbf{y}) \cap Q$ as well. For $i = 1, 2$, let $x_i \in \mathbb{P}^N$ be the point with coordinates $(t_i \mathbf{y})_j$ for $j \in A(\mathbf{y}) \cap Q$ and with all other coordinates 0. Then the above shows $t^\circ x_1 = x_2$ and hence $\tilde{t}^\circ z_1 = z_2$ by the equivariance of the reduction maps. This proves transitivity.

Conversely, if the orbit Z is given, then we may recover $A(\mathbf{y}) \cap Q$ by (8) and this set generates the face Q of $\text{Wt}(\mathbf{y}, a)$. Let $\sigma = \tilde{Q}$ be the corresponding polyhedron in the dual complex $\mathcal{C}(A, a)$, then

$$\text{trop}_v(\{t \in T^{\text{an}} \mid \pi(t \mathbf{y}) \in Z\}) \subset \text{reint}(\sigma)$$

is also clear from what we have proven in (8). Then we get immediately equality as the left hand side forms a partition of $N_{\mathbb{R}}$ for varying Z .

The torus corresponding to the orbit Z has character group $\mathbb{Z}(\sigma^\perp \cap M)$. This is clear as we may choose a base point \mathbf{y}' in Z and then apply 9.1 with A, \mathbf{y}' and with \tilde{K} replacing K . This and A.11 prove immediately the identities relating the dimensions. Finally, the claims about the orders are evident. \square

Remark 9.13 If v is the trivial valuation, then the dual complex $\mathcal{C}(A, a)$ is just the normal fan of the weight polytope $\text{Wt}(\mathbf{y})$.

Corollary 9.14 *There are bijective order correspondences between*

- (a) faces Q of the weight polytope $\text{Wt}(\mathbf{y})$;
- (b) polyhedra σ of the normal fan of $\text{Wt}(\mathbf{y})$;
- (c) T -orbits Z of the generic fibre of $\mathcal{Y}_{A,a}$.

Proof: If we replace the valuation v by the trivial valuation, then the generic fibre $Y_{A,a}$ does not change. Then we get the corollary and all the correspondences immediately from Proposition 9.12. \square

9.15 Let Z be an orbit of $\mathcal{Y}_{A,a}$ corresponding to a face Q of the weight subdivision $\text{Wt}(\mathbf{y}, a)$ (resp. the weight polytope $\text{Wt}(\mathbf{y})$). We choose a base point $\mathbf{z} \in Z(K)$. Then the closure of Z in \mathbb{P}^N is the projective toric variety $\mathcal{Y}_{A,a}(\mathbf{z})$ in \mathbb{P}_K^N (resp. in \mathbb{P}_K^N) constructed from \mathbf{z} and $A(\mathbf{y}) \cap Q$ as in 9.1. We conclude that Q is its weight polytope.

Remark 9.16 The polyhedra of $\mathcal{C}(A, a)$ are pointed if and only if $\text{Wt}(\mathbf{y})$ has dimension n . In other words, this means that the smallest affine space containing $A(\mathbf{y})$ is $N_{\mathbb{R}}$ and this is equivalent to $\dim(\text{Stab}(\mathbf{y})) = 0$ (see Corollary 9.4).

By passing to a sublattice of M , we may always achieve this situation and we may even assume that $M = \mathbb{Z}A(\mathbf{y})$ (see Proposition 9.3). Since $\mathcal{C}(A, a)$ is a complete complex, it follows from 7.6 that $\mathcal{C}(A, a) = \Sigma_1$ for a complete Γ -admissible fan Σ in $N_{\mathbb{R}} \times \mathbb{R}_+$.

10 The Gröbner complex

In this section, K denotes a field with a non-archimedean absolute value $|\cdot|$, corresponding valuation $v := -\log|\cdot|$, valuation ring K° , residue field \tilde{K} and value group $\Gamma = v(K^\times)$. Then we consider a closed subscheme X of the split multiplicative torus T over K . We will introduce its Gröbner complex on $N_{\mathbb{R}}$ which is related to the natural orbit of X in the Hilbert scheme of a projective compactification. This is a certain complete Γ -rational complex which has a subcomplex with support equal to $\text{Trop}_v(X)$. At the end, we relate the Gröbner complex to the initial degenerations of X . This section is inspired by [Kat], Section 5, which in turn was influenced by Tevelev. We work here with more general base fields, but the ideas are the same. For an elementary approach using Gröbner bases and for examples, we refer to [MS], Section 2.4.

10.1 First, we recall the property of the *Hilbert scheme* $\text{Hilb}_p(\mathbb{P}_S^m)$ for the projective space \mathbb{P}_S^m over a locally noetherian scheme S and for a Hilbert polynomial $p(x) \in \mathbb{Q}[x]$ which characterizes the Hilbert scheme up to unique isomorphism:

There is a projective scheme $\text{Hilb}_p(\mathbb{P}_S^m)$ over the base scheme S and a closed subscheme $\text{Univ}_p(\mathbb{P}_S^m)$ of $\mathbb{P}_S^m \times_S \text{Hilb}_p(\mathbb{P}_S^m)$ which is flat over $\text{Hilb}_p(\mathbb{P}_S^m)$ and which has Hilbert polynomial p such that for every scheme Z over S , the map from the set of morphisms $Z \rightarrow \text{Hilb}_p(\mathbb{P}_S^m)$ to the set of closed subschemes of \mathbb{P}_Z^m with Hilbert polynomial p and flat over Z , given by mapping f to the inverse image scheme

$$(\text{id} \times f)^{-1}(\text{Univ}_p(\mathbb{P}_S^m)) = \text{Univ}_p(\mathbb{P}_S^m) \times_{\text{Hilb}_p(\mathbb{P}_S^m)} Z,$$

is a bijection. In other words, there is a bijective correspondence $Y \mapsto [Y]$ between the set of closed subschemes of \mathbb{P}_Z^m which are flat over Z and which have Hilbert polynomial p and the set of Z -valued points of $\text{Hilb}_p(\mathbb{P}_S^m)$. For a proof, we refer to [Kol], Section 1.1.

Note that the Hilbert polynomial of a closed subscheme Y of \mathbb{P}_Z^m is defined for every fibre over a point z of Z . If Y is flat over Z and if Z is connected, then the Hilbert polynomial does not depend on the choice of z . We note also that it is enough to construct $\text{Hilb}_p(\mathbb{P}^m)$ over \mathbb{Z} as we may obtain $\text{Hilb}_p(\mathbb{P}_S^m)$ by base change to S . We may use this to obtain $\text{Hilb}_p(\mathbb{P}_S^m)$ over $S = \text{Spec}(K^\circ)$ with the same characteristic property even if the valuation ring K° is not noetherian. Indeed, every closed subscheme of \mathbb{P}_S^m which is flat over S is of finite presentation (use

[RG], Corollaire 3.4.7), hence it is defined over a noetherian subring of K° and so we may apply the above result.

10.2 We briefly sketch the construction of the Hilbert scheme as far as we need it later. For simplicity, we restrict to the case $S = \text{Spec}(F)$ for a field F . The general case follows similarly using graded ideal sheaves instead of graded ideals. For details, we refer to [Kol], Section 1.1.

Let $I_Y(k)$ be the k -th graded piece of the graded ideal I_Y in $F[x_0, \dots, x_m]$ of a closed subscheme Y of \mathbb{P}_F^m with Hilbert polynomial p . For sufficiently large k depending only on p , we have $\dim(I_Y(k)) = q(k) - p(k)$ and the map $Y \mapsto I_Y(k)$ is an injective map from the set of closed subschemes of \mathbb{P}_F^m to the Grassmannian $G(q(k) - p(k), q(k))$, where q is the Hilbert polynomial of \mathbb{P}^m . The image is $\text{Hilb}(\mathbb{P}_F^m)$ which we may endow with a suitable structure as a closed subscheme of the Grassmannian and with a family $\text{Univ}_p(\mathbb{P}_F^m)$ which satisfies the required universal property. Using the Grassmann coordinates $L \mapsto \bigwedge^{q(k)-p(k)}(L)$, we get $G(q(k) - p(k), q(k))$ as a closed subscheme of \mathbb{P}_F^N for $N := \binom{q(k)}{p(k)} - 1$ and hence $\text{Hilb}(\mathbb{P}_F^m)$ may be seen as a closed subscheme of \mathbb{P}_F^N as well.

10.3 We consider a linear action of the torus \mathbb{T} on $\mathbb{P}_{K^\circ}^m$. It follows easily from the universal property of the Hilbert scheme that \mathbb{T} operates also on $\text{Hilb}_p(\mathbb{P}_{K^\circ}^m)$ such that for any scheme Z over K° and any closed subscheme Y of \mathbb{P}_Z^m with Hilbert polynomial p and flat over Z , we have $g \cdot [Y] = [g^{-1}Y]$. It makes the following formulas more natural if the action is by pull-back with respect to multiplication by g rather than push-forward. If we use the closed embedding of $\text{Hilb}_p(\mathbb{P}_{K^\circ}^m)$ into $\mathbb{P}_{K^\circ}^N$ similarly as in 10.2, then the \mathbb{T} -action on $\text{Hilb}_p(\mathbb{P}_{K^\circ}^m)$ extends to a linear action of \mathbb{T} on $\mathbb{P}_{K^\circ}^N$. Indeed, if A_t is the $(m+1) \times (m+1)$ -matrix representing the action of t on $\mathbb{P}_{K^\circ}^m$ similarly as in the proof of Lemma 9.7, then $(A_t \mathbf{x})^{\mathbf{m}}$ is a linear combination of monomials of degree $|\mathbf{m}|$ and this shows easily the claim using the Grassmann coordinates.

Proposition 10.4 *Let Y be a closed subscheme of \mathbb{P}_K^m with Hilbert polynomial p . Then the closure of the T -orbit of $[Y]$ in $\text{Hilb}_p(\mathbb{P}_{K^\circ}^m)$ is equal to $\mathcal{Y}_{A,a}$ for suitable $A \in M^{n+1}$ and height function $a : \{0, \dots, N\} \rightarrow \Gamma \cup \{\infty\}$.*

Proof: This follows from Proposition 9.8. □

Definition 10.5 The dual complex $\mathcal{C}(A, a)$ from 9.11 is called the *Gröbner complex* of Y .

Definition 10.6 Let (L, w) be a valued field extension of (K, v) . For $t \in T(L)$, the special fibre of the closure of $t^{-1}Y_L$ in $\mathbb{P}_{L^\circ}^m$ is called the *initial degeneration* of Y in t . This is a closed subscheme of $\mathbb{P}_{\tilde{L}}^m$ defined over the residue field \tilde{L} which we denote by $\text{in}_t(Y)$.

For $\omega = \text{trop}_w(t)$, we set $\text{in}_\omega(Y) = \text{in}_t(Y)$. Similarly as in Proposition 5.3, this is independent of the choice of t up to multiplication by an element from \mathbb{T} defined over a suitable field extension of \tilde{K} . Since trop_v is surjective, $\text{in}_\omega(Y)$ is defined for every $\omega \in N_{\mathbb{R}}$.

10.7 In the situation above, $[t^{-1}Y_L] = t \cdot [Y_L]$ is an L -rational point of $\text{Hilb}(\mathbb{P}^m)$. By projectivity of the Hilbert scheme, we conclude that $[t^{-1}Y_L]$ extends uniquely to an L° -valued point h_t of $\text{Hilb}(\mathbb{P}^m)$ and hence corresponds to a closed subscheme of $\mathbb{P}_{L^\circ}^m$ which is flat over L° and has generic fibre $t^{-1}Y_L$. By Proposition 4.4 and Remark 4.6, this has to be the closure of $t^{-1}Y_L$ and hence the special fibre is $\text{in}_t(Y)$. In other words, $[\text{in}_t(Y)]$ is equal to the reduction of h_t in $\text{Hilb}(\mathbb{P}^m)(\tilde{L})$.

Proposition 10.8 *Suppose that \mathbb{T} acts linearly on $\mathbb{P}_{K^\circ}^m$. Let Y be a closed subscheme of \mathbb{P}_K^m and let (L, w) be a valued field extending (K, v) . For $t_1, t_2 \in T(L)$, the following conditions are equivalent:*

- (a) *There is a polyhedron σ of the Gröbner complex $\mathcal{C}(A, a)$ of Y with $\text{trop}_w(t_i) \in \text{relint}(\sigma)$ for $i = 1, 2$.*
- (b) *There is $g \in \mathbb{T}(\tilde{L})$ with $\text{in}_{t_2}(Y) = g \cdot \text{in}_{t_1}(Y)$.*

Proof: This follows from Proposition 9.12, Proposition 10.4 and 10.7. \square

Proposition 10.9 *Let $\omega_1 = \omega_0 + \Delta\omega$ in $N_{\mathbb{R}}$ and suppose that there is a polyhedron σ from the Gröbner complex $\mathcal{C}(A, a)$ with $\omega_0 \in \sigma$ and $\omega_1 \in \text{relint}(\sigma)$. Then we have*

$$\text{in}_{\omega_1}(Y) = \text{in}_{\Delta\omega}(\text{in}_{\omega_0}(Y)), \quad (9)$$

where we consider $\text{in}_{\omega_0}(Y)$ as a closed subscheme of \mathbb{P}^m over a trivially valued extension of the residue field \tilde{K} and then we take its initial degeneration with respect to $\Delta\omega$. In particular, we have (9) for all $\omega_1 \in N_{\mathbb{R}}$ in a sufficiently small neighbourhood of ω_0 .

Proof: It follows from Proposition 10.8 that $z_1 := [\text{in}_{\omega_1}(Y)]$ is in the orbit Z_σ of the special fibre of $\mathcal{Y}_{A,a}$ corresponding to σ . If ρ is the closed face of σ with $\omega_0 \in \text{relint}(\rho)$, then $z_0 := [\text{in}_{\omega_0}(Y)]$ is in the orbit $Z := Z_\rho$ corresponding to ρ .

Now we repeat the procedure taking the closure of the orbit Z with respect to the base point z_0 in $\text{Hilb}(\mathbb{P}_{\tilde{K}}^m)$. We have seen in 9.15 that the dual polytope $\hat{\rho}$ is the weight polytope of the projective toric variety \bar{Z} . Since we use the trivial valuation on \tilde{K} , the dual complex of $\hat{\rho}$ is the complete fan formed by the local cones $\text{LC}_{\omega_0}(\nu)$ with ν ranging over all polyhedra from $\mathcal{C}(A, a)$ containing ρ . Then $z := [\text{in}_{\Delta\omega}(\text{in}_{\omega_0}(Y))]$ is in the orbit of \bar{Z} corresponding to the fan $\text{LC}_{\omega_0}(\nu)$ containing $\Delta\omega$ in its relative interior. Obviously, this holds for $\nu = \sigma$.

Recall that $\text{Hilb}(\mathbb{P}_{\tilde{K}}^m)$ is the special fibre of $\text{Hilb}(\mathbb{P}_{K^\circ}^m)$ and we have $Z = Z_\rho$. Moreover, \bar{Z} is contained in the special fibre of $\mathcal{Y}_{A,a}$. We note that every orbit of \bar{Z} is an orbit of $(\mathcal{Y}_{A,a})_s$ and the corresponding fan $\text{LC}_{\omega_0}(\nu)$ transforms to ν taking into account that the base point has changed from $[Y]$ to z_0 . We conclude that z and z_1 are in the same orbit. This proves (9). Finally, the last claim is obvious from the fact that the above local fan in ω_1 is complete. \square

10.10 In the remaining part of this section, we consider the following important special case, where we can compare the definitions in 10.6 and in 5.1: We consider a projective toric variety $\mathcal{Y}_{B,0}$ over K° given by $B \in M^{m+1}$ and height function identically zero, i.e. the base point $\mathbf{z} \in \mathbb{P}^m(K)$ in the open dense orbit satisfies $v(z_j) = 0$ for $j = 0, \dots, m$. Recall that $\mathcal{Y}_{B,0}$ is a closed subvariety of $\mathbb{P}_{K^\circ}^m$ and the torus action extends to a linear action on $\mathbb{P}_{K^\circ}^m$ (see 9.2). We assume further that the stabilizer of \mathbf{z} is trivial and so we may identify T with the open dense orbit $T\mathbf{z}$. By Corollary 9.4, the affine span of B is $M_{\mathbb{R}}$. For example, the standard embedding of \mathbb{T} in $\mathbb{P}_{K^\circ}^n$ fulfills all these requirements.

The triviality of the height function implies that the weight polytope is equal to the weight subdivision and the dual complex is just the normal fan of $\text{Wt}(\mathbf{z})$. Moreover, we may identify \mathbb{T} with the \mathbb{T} -invariant open subset of $\mathcal{Y}_{B,0}$ whose generic fibre is the open dense orbit and whose special fibre is the orbit corresponding to the vertex 0 of the cones.

10.11 We consider a closed subscheme X of T and we denote by Y its closure in \mathbb{P}_K^m . For a valued field (L, w) extending (K, v) and $t \in T(L)$, it follows immediately from comparing Definitions 5.1 and 10.6 that

$$\text{in}_t(X_L) = \text{in}_t(Y_L) \cap \mathbb{T}_{\tilde{L}}.$$

Corollary 10.12 For $\omega_0 \in N_{\mathbb{R}}$, there is a neighbourhood Ω of ω_0 in $N_{\mathbb{R}}$ such that

$$\text{in}_{\omega_1}(X) = \text{in}_{\Delta\omega}(\text{in}_{\omega_0}(X)),$$

for every $\omega_1 \in \Omega$ and $\Delta\omega := \omega_1 - \omega_0$. On the right hand side, the initial degeneration of $\text{in}_{\omega_0}(X)$ at $\Delta\omega$ is with respect to a trivially valued field of definition for $\text{in}_{\omega_0}(X)$.

Proof: This follows from Proposition 10.9 and 10.11. \square

10.13 We apply the above for $Y = \overline{X}$ leading to a polyhedral complex $\mathcal{C}(A, a)$ in $N_{\mathbb{R}}$ which we call the *Gröbner complex* for X . It depends on the choices from 10.10.

Theorem 10.14 The Gröbner complex $\mathcal{C}(A, a)$ of X is a complete Γ -rational complex in $N_{\mathbb{R}}$ and $\{\sigma \in \mathcal{C}(A, a) \mid \sigma \subset \text{Trop}_v(X)\}$ is a subcomplex \mathcal{C}_X of $\mathcal{C}(A, a)$ with support equal to $\text{Trop}_v(X)$.

Proof: All statements are evident by construction except the claim about the support. Let $\omega \in \text{Trop}_v(X)$. By completeness of the Gröbner complex, there is $\sigma \in \mathcal{C}(A, a)$ with $\omega \in \text{relint}(\sigma)$. We have to prove that $\sigma \subset \text{Trop}_v(X)$. Since $\text{Trop}_v(X)$ is closed in $N_{\mathbb{R}}$, it is enough to show that every $\omega' \in \text{relint}(\sigma)$ is contained in $\text{Trop}_v(X)$. There is a valued field (L, w) extending (K, v) and $t, t' \in T(\tilde{L})$ with $\text{trop}_w(t) = \omega$ and $\text{trop}_w(t') = \omega'$. By Proposition 10.8, there is $g \in \mathbb{T}(\tilde{L})$ with $\text{in}_{t'}(Y) = g \cdot \text{in}_t(Y)$. By 10.11, we conclude that $\text{in}_{\omega'}(X) = \text{in}_{\omega}(X)$. Using $\omega \in \text{Trop}_v(X)$, Theorem 5.6 implies that $\text{in}_{\omega}(X)$ is non-empty and hence the same is true for $\text{in}_{\omega'}(X)$. Using this equivalence the other way round, we deduce that $\omega' \in \text{Trop}_v(X)$ proving the claim. \square

The following result is very useful for reducing local statements about the tropical variety to the case of trivial valuations. We will see in Proposition 13.7 that this is also compatible with tropical multiplicities.

Proposition 10.15 Let X be a closed subscheme of T and let $\omega \in N_{\mathbb{R}}$. Using the local cone at ω from Appendix A.6, we have

$$\text{Trop}_0(\text{in}_{\omega}(X)) = \text{LC}_{\omega}(\text{Trop}_v(X)).$$

Proof: The fundamental theorem of tropical algebraic geometry (Theorem 5.6) says that $\Delta\omega \in N_{\mathbb{R}}$ is in $\text{Trop}_0(\text{in}_{\omega}(X))$ if and only if $\text{in}_{\Delta\omega}(\text{in}_{\omega}(X))$ is non-empty. If we choose $\Delta\omega$ sufficiently small, then we deduce from Corollary 10.12 that these conditions are also equivalent to $\text{in}_{\omega + \Delta\omega}(X) \neq \emptyset$. Theorem 5.6 again shows that this is equivalent to $\omega + \Delta\omega \in \text{Trop}_v(X)$. As we are working in a sufficiently small neighbourhood of ω , this is equivalent to $\Delta\omega \in \text{LC}_{\omega}(\text{Trop}_v(X))$ proving the claim. \square

10.16 For a polyhedron Δ in $N_{\mathbb{R}}$, let us recall that $c(\Delta)$ denotes the cone in $N_{\mathbb{R}} \times \mathbb{R}_+$ generated by $\Delta \times \{1\}$. We call $\Sigma(A, a) := c(\mathcal{C}(A, a))$ the *Gröbner fan* of X in $N_{\mathbb{R}} \times \mathbb{R}_+$.

Corollary 10.17 The Gröbner fan $\Sigma(A, a)$ of X in $N_{\mathbb{R}} \times \mathbb{R}_+$ is a complete Γ -rational fan and $\Sigma_X := \{\sigma \in \Sigma(A, a) \mid \sigma \subset \text{Trop}_W(X)\}$ is a subcomplex of $\Sigma(A, a)$ with support equal to the tropical cone $\text{Trop}_W(X)$ from 8.3.

Proof: Since $\mathcal{C}(A, a)$ is a complete Γ -rational polyhedral complex, it follows from Remark 7.6 that $\Sigma(A, a)$ is a complete Γ -rational fan in $N_{\mathbb{R}} \times \mathbb{R}_+$. Then the claim follows from Theorem 10.14 and Proposition 8.4. \square

10.18 By 9.16, $\mathcal{C}(A, a)$ is a pointed polyhedral complex if and only if $\text{Stab}(\mathbf{y})$ is zero-dimensional. By definition of the torus action on the Hilbert scheme, we have $\text{Stab}(\mathbf{y}) = \text{Stab}(Y) = \text{Stab}(X)$, where Y is the closure of X in \mathbb{P}_K^m . In general, it is clear that $\mathcal{C}(A, a)$ is isomorphic to the product of an affine space and the Gröbner complex of $X/\text{Stab}(X)$. By the above, the latter is pointed and so it is obvious that $\mathcal{C}(A, a)$ has always a Γ -rational subdivision \mathcal{C} consisting of pointed polyhedra. By Corollary 10.17, $c(\mathcal{C})$ is a Γ -admissible fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ with support $\text{Trop}_W(X)$.

11 Compactifications in toric schemes

Let K be a field with a non-archimedean absolute value $|\cdot|$, corresponding valuation $v := -\log|\cdot|$, valuation ring K° , residue field \tilde{K} and value group $\Gamma = v(K^\times)$. Let \mathbb{T} be the split torus over \mathbb{K}° with generic fibre T associated to the character lattice M of rank n and dual lattice N . We keep the usual notation. In this section, we consider a closed subscheme X of T and we study its closure \mathcal{X} in the toric scheme \mathcal{Y}_Σ associated to a Γ -admissible fan Σ in $N_{\mathbb{R}} \times \mathbb{R}_+$ (see 7.5). First, we prove surjectivity of the reduction map which is called the tropical lifting lemma. Then, we show Tevelev's lemma which is a tropical characterization of the orbits intersecting \mathcal{X} . Finally, we give several equivalences for properness of the occurring schemes.

We start with a lemma due to Draisma.

Lemma 11.1 *Let (L, w) be a valued field extending (K, v) and let $r, s \in \mathbb{N}$. For $a_{ij}, b \in K$ and $\lambda_i \in \mathbb{R}$, we consider the following system of equalities*

$$a_{i1}x_1 + \cdots + a_{it}x_t = b_i \quad (1 \leq i \leq r)$$

and inequalities

$$w(a_{i1}x_1 + \cdots + a_{it}x_t) \geq \lambda_i \quad (r+1 \leq i \leq r+s).$$

If this system has a solution $y \in L^t$, then it has also a solution $z \in K^t$.

Proof: This follows from the same arguments as Lemma 4.3 in [Dra]. □

Lemma 11.2 *Let (L, w) be a valued field extending (K, v) and let \mathcal{X}' be the closure of X_L in the toric scheme over L° associated to Σ . Then the canonical morphism $\phi : (\mathcal{X}')_s \rightarrow \mathcal{X}_s$ is surjective.*

Proof: We will first prove the claim if the value group Γ is a divisible subgroup of \mathbb{R} and then we will reduce the claim to this special case in several steps.

Step 1: If the value group Γ is a divisible subgroup of \mathbb{R} , then ϕ is surjective.

In this case, we have seen in Proposition 7.12 that the toric scheme over L° associated to Σ is the base change of \mathcal{Y}_Σ to L° . By Corollary 4.7, we have $\mathcal{X}' = \mathcal{X}_{L^\circ}$ and hence $(\mathcal{X}')_s$ is the base change of \mathcal{X}_s to the residue field \tilde{L} . This yields surjectivity of ϕ .

In particular, this proves the claim for v trivial. We may assume that v is non-trivial and that $\mathcal{Y}_\Sigma = \mathcal{U}_\Delta$ for a pointed Γ -rational polyhedron Δ in $N_{\mathbb{R}}$. Let σ be the recession cone of Δ . Then X is given by an ideal I_X in $K[X]^\sigma$ and its closure \mathcal{X} is given by the ideal $I_X \cap K[X]^\Delta$ in $K[M]^\Delta$. Similarly, \mathcal{X}' is the closed subscheme given by the ideal $(I_X L[M]^\sigma) \cap L[M]^\Delta$ in $L[M]^\Delta$.

Step 2: The morphism ϕ is dominant.

Let $f \in K[M]^\Delta$ such that the residue class of f in $L[M]^\Delta / ((I_X L[M]^\sigma) \cap L[M]^\Delta) \otimes_{\tilde{K}} \tilde{L}$ is zero. We have to prove that there is $m \in \mathbb{N}$ such that $f^m \in (I_X \cap K[M]^\Delta) + K^{\circ\circ} K[M]^\Delta$. By assumption, we have

$$f = g_1 h_1 + \cdots + g_r h_r + \lambda f_1 + \cdots + \lambda f_s \quad (10)$$

with $g_i \in I_X$, $h_i \in L[M]^\sigma$, $\lambda \in L^{\circ\circ}$ and $f_j \in L[M]^\Delta$. We may assume that $h_i = \beta_i \chi^{u_i}$ for some $\beta_i \in L$ and $u_i \in \check{\sigma} \cap M$. Similarly, we may assume that $f_j = \gamma_j \chi^{v_j}$ for some $\gamma_j \in L$ and $v_j \in M$. Since the valuation v is non-trivial, there is $m \in \mathbb{N}$ such that λ^m is divisible by an element of K° . Replacing f by f^m , we may assume that $\lambda \in K^{\circ\circ}$. If we compare the coefficients on both sides of equation (10), then we get a finite system of linear equations with coefficients in K and unknowns β_1, \dots, β_r and $\gamma_1, \dots, \gamma_s$. The conditions $f_j \in L[M]^\Delta$ are equivalent to the finitely many inequalities $v(\gamma_j) + \langle v_j, \omega \rangle \geq 0$, where ω ranges over the vertices of Δ . By assumption, this system of equalities and inequalities has a solution in L^{r+s} . By Lemma 11.1, there is a solution with $\beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_s \in K$ which means that we find a representation in (10) with all $h_i \in K[M]^\sigma$ and all $f_j \in K[M]^\Delta$. We conclude that $f \in (I_X \cap K[M]^\Delta) + K^{\circ\circ} K[M]^\Delta$ proving Step 2.

Step 3: If L is an algebraic closure of K , then ϕ induces a finite surjective map $(\mathcal{X}')_s \rightarrow \mathcal{X}_s \otimes_{\tilde{K}} \tilde{L}$.

We use first that the value group Γ_L of w is equal to $\{\lambda \in \mathbb{R} \mid \exists m \in \mathbb{N} \setminus \{0\}, m\lambda \in \Gamma\}$. It follows that the vertices of Δ are in N_{Γ_L} and there is a non-zero $m \in \mathbb{N}$ such that $m\omega \in N_\Gamma$ for every vertex ω of Δ . For every $u \in \check{\sigma} \cap M$, there is $\beta_u \in L$ with $v_\Delta(\beta_u \chi^u) = 0$. For each vertex ω of Δ , we choose a finite generating set of the semigroup $\check{\sigma}_\omega \cap M$, where σ_ω is the local cone of Δ at ω . We have seen in the proof of Proposition 6.7 that $L[M]^\Delta$ is generated as an L° -algebra by $\beta_u \chi^u$, where u ranges over the union S of all these generating sets.

We claim that the finite set $H := \{\prod_{u \in S} (\beta_u \chi^u)^{k_u} \mid 0 \leq k_u < m\}$ generates $L[M]^\Delta$ as a $K[M]^\Delta \otimes_{K^\circ} L^\circ$ -module. Indeed, every $f \in L[M]^\Delta$ has the form $f = \sum_{h,k} \lambda_{hk} h \prod_{u \in S} (\beta_u \chi^u)^{mk_u}$ where h ranges over H , k over \mathbb{N}^S and only finitely many coefficients $\lambda_{hk} \in L^\circ$ are non-zero. The construction of m yields that $\langle mu, \omega \rangle \in \Gamma$ for every vertex ω of Δ and hence there is $\alpha_u \in K$ with $v_\Delta(\alpha_u \chi^{mu}) = 0$. We conclude that $\beta_u^m = \alpha_u \gamma_u$ for some $\gamma_u \in L^\circ$. Since $\alpha_u \chi^{mu} \in K[M]^\Delta$, this implies that $\prod_{u \in S} (\beta_u \chi^u)^{mk_u} \in L^\circ K[M]^\Delta$ proving that H generates the module $L[M]^\Delta$.

Since $(\mathcal{X}')_s$ is a closed subscheme of $\text{Spec}(L[M]^\Delta)$ and since $\mathcal{X}_s \otimes_{\tilde{K}} \tilde{L}$ is a closed subscheme of $\text{Spec}(K[M]^\Delta \otimes_{K^\circ} L^\circ)$, we conclude that $(\mathcal{X}')_s \rightarrow \mathcal{X}_s \otimes_{\tilde{K}} \tilde{L}$ is a finite map. It follows from Step 2 that this map is dominant and hence it is surjective proving Step 3.

We will now deduce the claim from Step 3. We endow an algebraic closure E of L with a valuation u extending w . Let F be the algebraic closure of K in E endowed with the restriction of u . Let \mathcal{X}'' (resp. \mathcal{X}''') be the closure of X_F (resp. X_E) in the toric scheme over F° (resp. E°) associated to Δ . Then we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{X}''')_s & \longrightarrow & \mathcal{X}_s \otimes_{\tilde{K}} \tilde{F} \\ \downarrow & & \downarrow \\ (\mathcal{X}')_s & \xrightarrow{\phi} & \mathcal{X}_s \end{array}$$

of canonical morphisms. The first row has the factorization $(\mathcal{X}''')_s \rightarrow (\mathcal{X}'')_s \rightarrow \mathcal{X}_s \otimes_{\tilde{K}} \tilde{F}$ and hence it is surjective by Steps 1 and 3. Since the second column is surjective as well, we deduce that ϕ is surjective. \square

Proposition 11.3 *The special fibre \mathcal{X}_s of \mathcal{X} is either empty or it has the same dimension as X . If X is of pure dimension, then \mathcal{X}_s is also of pure dimension.*

Proof: The claim is clear in case of a divisible value group Γ as in this case \mathcal{X} is a flat scheme of finite type over K° (see Proposition 6.7). In general, we will reduce to the divisible case: We may assume that v is non-trivial and that $\mathcal{Y}_\Sigma = \mathcal{U}_\Delta$ for a Γ -rational polyhedron Δ in $N_\mathbb{R}$. Then the claim follows from Step 3 in the proof of Proposition 11.2. \square

11.4 We recall from §4 that the reduction map is defined on an analytic subdomain $(Y_{\Sigma_0})^\circ$ of the generic fibre Y_{Σ_0} and maps to the special fibre of the K° -model \mathcal{Y}_Σ . The points of $(Y_{\Sigma_0})^\circ$ are induced by potentially integral points and Proposition 8.6 shows that $(Y_{\Sigma_0})^\circ \cap T^{\text{an}} = \text{trop}_v^{-1}(|\Sigma|)$. We conclude that the potentially integral points of X with respect to \mathcal{X} induce an analytic subdomain $X^\circ = \text{trop}_v^{-1}(|\Sigma|) \cap X^{\text{an}}$ of X^{an} where we have a well-defined reduction map $\pi : X^\circ \rightarrow \mathcal{X}_s$.

We have here the following generalization of Jan Draisma's tropical lifting lemma (see [Dra], Lemma 4.4).

Proposition 11.5 *Using the above notation, we have $\pi(U^{\text{an}} \cap X^\circ) = \mathcal{X}_s$ for every open dense subset U of X . Moreover, if K is algebraically closed and v is non-trivial, then every closed point of \mathcal{X}_s is the reduction of a closed point of U .*

Proof: The additional difficulty here in contrast to Draisma's paper is that \mathcal{X} and the ambient toric scheme \mathcal{Y}_Σ might be not of finite type (see Example 6.9). Let L be an algebraic closure of K and let us choose a valuation u on L extending v . Let \mathcal{X}' be the closure of X_L in the toric scheme over L° associated to the fan Σ . Then \mathcal{X}' is a flat scheme of finite type over L° by Proposition 6.7. By Proposition 4.14, the reduction map $\pi_L : (X_L)^\circ \rightarrow (\mathcal{X}')_s$ is surjective. We have a canonical commutative diagram

$$\begin{array}{ccc} (U_L)^{\text{an}} \cap (X_L)^\circ & \xrightarrow{\pi_L} & \mathcal{X}'_s \\ \downarrow & & \downarrow \\ U^{\text{an}} \cap X^\circ & \xrightarrow{\pi} & \mathcal{X}_s \end{array}$$

where the first row and the second column are surjective by Lemma 11.2. This proves surjectivity of π . The last claim follows directly from Proposition 4.14. \square

The following result is called *Tevelev's Lemma*. We will use the tropical cone $\text{Trop}_W(X)$ and the notation of the previous section. The bijective correspondence between open faces and orbits from Proposition 8.8 will be important for the understanding of the following.

Lemma 11.6 *Let $\sigma \in \Sigma$. Then the orbit Z_σ corresponding to $\text{relint}(\sigma)$ intersects \mathcal{X} if and only if $\text{Trop}_W(X) \cap \text{relint}(\sigma)$ is non-empty.*

Proof: If $\omega \in \text{Trop}_W(X) \cap \text{relint}(\sigma)$, then there is $x \in X_W^{\text{an}}$ with $\text{trop}_W(x) = \omega$. Let $\pi_W : \text{trop}_W^{-1}(|\Sigma|) \rightarrow \mathcal{Y}_\Sigma$ be the reduction map. We deduce from Proposition 8.8 that $\pi_W(x) \in Z_\sigma$. Since $\pi_W(x)$ is also contained in \mathcal{X} , we see that $\mathcal{X} \cap Z_\sigma$ is non-empty.

Suppose that $z \in \mathcal{X} \cap Z_\sigma$. By Proposition 11.5, π_W induces a surjective map $X_W^{\text{an}} \cap \text{trop}_W^{-1}(|\Sigma|) \rightarrow \mathcal{X}$ and hence there is $x \in X_W^{\text{an}}$ with $z = \pi_W(x)$. Again by Proposition 8.8, we see that $\text{trop}_W(x) \in \text{relint}(\sigma)$. \square

Remark 11.7 We will give a procedure which can be often used to reduce questions about \mathcal{X} to the case of the trivial valuation:

Let ω be a vertex of Σ_1 . By 7.9 or Proposition 6.14, we have a corresponding irreducible component Y_ω of the toric scheme \mathcal{Y}_Σ . It is the closure of the orbit Z_ω corresponding to the vertex ω . By Proposition 6.15, Y_ω may be viewed as a normal

toric variety over \tilde{K} associated to the fan $\text{LC}_\omega(\Sigma_1) := \{\text{LC}_\omega(\Delta) \mid \Delta \in \Sigma_1\}$. Note that the acting torus is $T_\omega := \text{Spec}(\tilde{K}[M_\omega])$, where $M_\omega := \{u \in M \mid \langle u, \omega \rangle \in \Gamma\}$ is a sublattice of M of finite index. To identify it with the dense open orbit Z_ω of Y_ω involves the choice of a basepoint in $Z_\omega(\tilde{K})$ which does not influence tropical varieties of closed subschemes of $Z_\omega(\tilde{K})$ as we deal with the trivial valuation on \tilde{K} .

We assume now that the vertex ω is also contained in $\text{Trop}_v(X)$. In local problems involving ω , the relevant closed subscheme of T_ω is $X_\omega := \mathcal{X} \cap Z_\omega$. By Tevelev's Lemma 11.6, X_ω is non-empty and its closure \mathcal{X}_ω is contained in $\mathcal{X} \cap Y_\omega$. We claim that $\text{Trop}_0(X_\omega)$ is the local cone of $\text{Trop}_v(X)$ at ω .

To prove the claim, we note first that the induced reduced structure of the special fibre is compatible with base change by Lemma 6.13. As the tropical variety is also invariant under base change (Proposition 3.6), we may assume that $\omega \in N_\Gamma$. Then there is $t \in T(K)$ with $\omega = \text{trop}_v(t)$ and we may choose the basepoint of Z_ω equal to $\pi(t)$. Using translation by t^{-1} , we conclude easily that X_ω is isomorphic to $\text{in}_\omega(X)$ and hence the claim follows from Proposition 10.15.

Proposition 11.8 *For a Γ -admissible fan Σ in $N_\mathbb{R} \times \mathbb{R}_+$, the following conditions are equivalent:*

- (a) $|\Sigma| = N_\mathbb{R} \times \mathbb{R}_+$;
- (b) $|\Sigma_1| = N_\mathbb{R}$;
- (c) *The special fibre of \mathcal{Y}_Σ is non-empty and proper over \tilde{K} .*

If (c) holds, then the generic fibre of \mathcal{Y}_Σ is also proper over K . If the value group Γ is either divisible or discrete in \mathbb{R}_+ , then (a)–(c) are equivalent to \mathcal{Y}_Σ proper over K° .

Proof: Clearly, (a) and (b) are equivalent. Suppose that (a) holds. Then Σ_0 is a complete fan and hence the generic fibre Y_{Σ_0} of \mathcal{Y}_Σ is complete. The special fibre of \mathcal{Y}_Σ is the union of its finitely many irreducible components corresponding to the vertices ω_j of Σ_1 (see 7.9). Such an irreducible component is a toric variety with fan $\text{LC}_\omega(\Sigma_1)$ generated by the local cones in ω_j (see Proposition 7.15). Since Σ_1 satisfies (b), all these fans are also complete and hence the special fibre is proper over \tilde{K} (see [Ful2], §2.4). This proves (c).

Next, we show that (c) implies (b). If the special fibre $(\mathcal{Y}_\Sigma)_s$ is proper over \tilde{K} , then every irreducible component of $(\mathcal{Y}_\Sigma)_s$ is complete. As we have seen above, such an irreducible component Y is associated to a vertex ω of Σ_1 and Y is a toric variety over \tilde{K} associated to the fan $\text{LC}_\omega(\Sigma_1)$. By completeness of Y , the fan $\text{LC}_\omega(\Sigma_1)$ is also complete (see [Ful2], §2.4). As this holds for every vertex of Σ_1 , we conclude that Σ_1 is complete. This proves (b).

We have seen now that (a)–(c) are equivalent and that the generic fibre of \mathcal{Y}_Σ is proper over K in this case. Finally, we prove the last statement. The following argument is adapted from a preliminary draft of [BPR]. By Lemma 7.8, \mathcal{Y}_Σ is separated over K° . As we assume now that Γ is either divisible or discrete in \mathbb{R} , Proposition 6.7 yields that \mathcal{Y}_Σ is finitely presented over K° . If the equivalent conditions (a)–(c) hold, then both the generic and the special fibre are geometrically connected and proper over the corresponding base field. By Proposition 8.6, every point in $T(K)$ is an integral point of \mathcal{Y}_Σ . Hence we may apply [EGA IV], Corollaire 15.7.11, to conclude that \mathcal{Y}_Σ is proper over K° . Conversely, if \mathcal{Y}_Σ is proper over K° , then the special fibre is proper over \tilde{K} . By the part of the valuative criterion of properness ([EGA II], Théorème 7.3.8) which holds also in the non-noetherian situation, we conclude that the special fibre is also non-empty. \square

11.9 We consider now another free abelian group M' of finite rank with dual N' and split multiplicative torus $T' = \text{Spec}(K[M'])$ over K . Then a Γ -admissible fan Σ' in $N'_{\mathbb{R}} \times \mathbb{R}_+$ induces a toric scheme $\mathcal{B}_{\Sigma'}$ over K° with dense orbit T' . We assume that $f : N' \rightarrow N$ is a homomorphism such that $f_{\mathbb{R}} \times \text{id}_{\mathbb{R}_+}$ maps each cone σ' of Σ' into a suitable cone σ of Σ . Then the dual homomorphism of f induces a canonical equivariant homomorphism $\mathcal{V}_{\sigma'} \rightarrow \mathcal{V}_{\sigma}$. We can patch these homomorphisms together to get an equivariant homomorphism $\varphi : \mathcal{B}_{\Sigma'} \rightarrow \mathcal{B}_{\Sigma}$ of toric schemes over K° which is canonically determined by f through the fact that φ restricts to the homomorphism $T' \rightarrow T$ of tori induced by f .

Proposition 11.10 *Under the hypothesis above, the following properties are equivalent:*

- (a) *The morphism φ is closed with generic fibre φ_{η} and special fibre φ_s both proper;*
- (b) $(f_{\mathbb{R}} \times \text{id}_{\mathbb{R}_+})^{-1}(|\Sigma|) = |\Sigma'|$.

Proof: We assume that (a) holds. By the criterion of properness for homomorphisms of toric varieties over a field ([Ful2], §2.4), we have $f^{-1}(|\Sigma_0|) = |\Sigma'_0|$. To prove (b), it remains to see that $f^{-1}(|\Sigma_1|) = |\Sigma'_1|$. Let $\omega' \in N'_{\mathbb{R}}$ with $f(\omega') \in |\Sigma_1|$. There is $t' \in (T')^{\text{an}}$ with $\text{trop}_v(t') = \omega'$ and hence $\text{trop}_v(t) = f(\omega') \in |\Sigma_1|$ for $t := \varphi^{\text{an}}(t')$. By Proposition 8.6, we have $t \in Y_{\Sigma_0}^{\circ}$ and hence we have a well-defined reduction $\pi(t)$ in the special fibre of the closure of $\varphi(T')$ in \mathcal{B}_{Σ} . Since φ is closed, we have $\pi(t) = \varphi_s(z')$ for some $z' \in (\mathcal{B}_{\Sigma'})_s$. By Proposition 11.5, there is $t'_0 \in (T')^{\text{an}} \cap Y_{\Sigma'_0}^{\circ}$ with reduction $\pi(t'_0) = z'$. Again Proposition 8.6 shows that $\omega'_0 := \text{trop}_v(t'_0) \in \Sigma'_1$. We have $\pi(\varphi^{\text{an}}(t'_0)) = \varphi_s(\pi(t'_0)) = \pi(t)$. The orbit correspondence in Proposition 8.8 yields that $f(\omega'_0) = \text{trop}_v(\varphi^{\text{an}}(t'_0))$ is in the same open face τ of Σ_1 as $f(\omega') = \text{trop}_v(t)$.

Arguing by contradiction, we assume that $\omega' \notin |\Sigma'_1|$. We consider now the closed segment $[\omega'_0, \omega']$ in $N'_{\mathbb{R}}$. Let ω'_1 be the point of $[\omega'_0, \omega'] \cap |\Sigma'_1|$ which is closest to ω' . Then ω'_1 is contained in an open face τ' of Σ'_1 . Let ω'_2 be a vertex of τ' . Using $[\omega'_1, \omega'] \cap |\Sigma'_1| = \{\omega'_1\}$ and moving ω' sufficiently close to ω'_1 , we may assume also that

$$[\omega'_2, \omega'] \cap |\Sigma'_1| = \{\omega'_2\} \quad (11)$$

Now we use the notation from Proposition 7.15 and we apply this result two times. The irreducible component $Y_{\omega'_2}$ corresponding to the vertex ω'_2 is the toric variety over \tilde{K} associated to the fan $\text{LC}_{\omega'_2}(\Sigma'_1)$. The closure \bar{Z} of the orbit Z of $(\mathcal{B}_{\Sigma})_s$ corresponding to τ is the toric variety over \tilde{K} associated to the fan in $N(\bar{\tau})_{\mathbb{R}}$ which is given by the projections of $\text{LC}_{\tau}(\Sigma_1) = \{\text{LC}_{\tau}(\nu) \mid \nu \in \Sigma_1, \nu \supset \tau\}$ to $N(\bar{\tau})_{\mathbb{R}}$. We have an equivariant morphism $\varphi_{\omega'_2} : Y_{\omega'_2} \rightarrow \bar{Z}$ induced by φ_s . We deduce from (11) that $|\text{LC}_{\omega'_2}(\Sigma'_1)|$ is a proper subset of $f_{\mathbb{R}}^{-1}(|\text{LC}_{\tau}(\Sigma_1)|)$. By the criterion of properness for homomorphisms of toric varieties over a field (see [Ful2], §2.4), $\varphi_{\omega'_2}$ is not proper. This contradicts properness of φ_s . We conclude that (a) implies (b).

To prove the converse, we assume that (b) holds. We get $f^{-1}(|\Sigma_0|) = |\Sigma'_0|$ and hence φ_{η} is proper again by the criterion in [Ful2], §2.4. By 7.9, the irreducible components of $(\mathcal{B}_{\Sigma'})_s$ correspond to the vertices ω' of Σ_1 . Moreover, let Z be the orbit of $(\mathcal{B}_{\Sigma})_s$ corresponding to the open face τ of Σ_1 containing $f(\omega')$. As above, we get an equivariant morphism $\varphi_{\omega'} : Y_{\omega'} \rightarrow \bar{Z}$ of toric varieties over \bar{K} . It follows from (b) that $f_{\mathbb{R}}^{-1}(|\text{LC}_{\tau}(\Sigma_1)|) = |\text{LC}_{\omega'}(\Sigma'_1)|$. The criterion in [Ful2], §2.4, shows that $\varphi_{\omega'}$ is proper. As this holds for any irreducible component $Y_{\omega'}$, we get properness of φ_s using [EGA II], Corollaire 5.4.5.

It remains to see that $\varphi(\mathcal{X}')$ is closed for any closed subset \mathcal{X}' of $\mathcal{B}_{\Sigma'}$. We may assume that \mathcal{X}' is irreducible. Since φ_s is proper, we may also assume that \mathcal{X}' is the closure of a closed subvariety X' of $(\mathcal{B}_{\Sigma'})_{\eta}$. Using Proposition 7.14, we may reduce

the claim to the case $X' \cap T' \neq \emptyset$. Since φ_η is proper, the generic fibre of $\varphi(\mathcal{X}')$ is a closed subvariety X of \mathcal{B}_Σ . It remains to show that any point z in the special fibre of the closure \mathcal{X} of $\varphi(\mathcal{X}')$ is contained in $\varphi(\mathcal{X}')$. By Proposition 11.5, the reduction map $\pi : X^\circ \cap T^{\text{an}} \rightarrow \mathcal{X}_s$ is surjective. We conclude that $z = \pi(x)$ for some $x \in X^\circ \cap T^{\text{an}}$ and hence $\text{trop}_v(x) \in |\Sigma_1|$ by the orbit correspondence in Proposition 8.8. There is $x' \in (X')^{\text{an}}$ with $\varphi^{\text{an}}(x') = x$. By Chevalley's theorem, $\varphi_\eta(X' \cap T')$ is a constructible dense subset of X and hence contains an open dense subset of X . We conclude that we may choose x and x' in the above argument such that $x' \in (T')^{\text{an}}$ additionally. Using $f(\text{trop}_v(x')) = \text{trop}_v(\varphi^{\text{an}}(x')) = \text{trop}_v(x) \in |\Sigma_1|$, our assumption (b) on the fans leads to $\text{trop}_v(x') \in |\Sigma'_1|$. By Proposition 8.6, we have $x' \in (X')^\circ$ and hence its reduction $z' := \pi(x')$ is well-defined in $(\mathcal{X}')_s$. We get $\varphi(z') = \varphi_s \circ \pi(x') = \pi \circ \varphi^{\text{an}}(x') = z$. This proves $z \in \varphi(\mathcal{X}')$ and therefore the morphism φ is closed. We conclude that (b) implies (a). \square

Proposition 11.11 *Let X be a closed subscheme of T . For the closure \mathcal{X} of X in \mathcal{B}_Σ , the following are equivalent:*

- (a) $\text{Trop}_W(X) \subset |\Sigma|$;
- (b) $\text{Trop}_v(X) \subset |\Sigma_1|$;
- (c) *The special fibre of \mathcal{X} is non-empty and proper over \tilde{K} .*

If (c) holds, then the generic fibre of \mathcal{X} is proper over K .

Proof: By Proposition 8.4, $\text{Trop}_W(X)$ is the closed cone in $N_\mathbb{R} \times \mathbb{R}_+$ generated by $\text{Trop}_v(X) \times \{1\}$. This shows the equivalence of (a) and (b).

We suppose that (a) holds. There is always a Γ -admissible fan Σ' with support $N_\mathbb{R} \times \mathbb{R}_+$ such that a subcomplex Σ'' is a refinement of Σ (as in [BS], Proposition 3.15). Since Σ'' is a subcomplex of Σ' , it is obvious that $\mathcal{B}_{\Sigma''}$ is an open subset of $\mathcal{B}_{\Sigma'}$. By Proposition 11.8, the generic and the special fibre of $\mathcal{B}_{\Sigma'}$ are proper and hence the same is true for the generic and the special fibre of the closure \mathcal{X}' of X in $\mathcal{B}_{\Sigma'}$. By Tevelev's Lemma 11.6, \mathcal{X}' is contained in $\mathcal{B}_{\Sigma''}$ and has non-empty special fibre. Since Σ'' is a refinement of Σ , we deduce from Proposition 11.10 that the canonical \mathbb{T} -equivariant morphism $\varphi : \mathcal{B}_{\Sigma''} \rightarrow \mathcal{B}_\Sigma$ is closed and hence $\varphi(\mathcal{X}') = \mathcal{X}$. This proves (c) and that the generic fibre of \mathcal{X} is proper over K .

It remains to see that (c) implies (b):

Step 1: Suppose that the valuation v is trivial. Then the closure \mathcal{X} of X in the toric variety Y_{Σ_0} associated to the rational fan Σ_0 in $N_\mathbb{R}$ is proper if and only if $\text{Trop}_0(X) \subset |\Sigma_0|$.

In the case of the trivial valuation, the special fibre agrees with the generic fibre and $\Sigma_0 = \Sigma_1$, hence it follows from (b) \Rightarrow (c) that $\text{Trop}_0(X) \subset |\Sigma_0|$ yields properness of \mathcal{X} . To prove the converse, we assume that there is a point $\omega \in \text{Trop}_0(X) \setminus |\Sigma_0|$. By Remark 2.2, there is a valued field (L, w) extending (K, v) and $t \in X(L)$ with $\text{trop}_w(t) = \omega$. We conclude that from the valuative criterion of properness that \mathcal{X} cannot be proper over K . Otherwise, t would be an L° integral point of \mathcal{X} and hence Proposition 8.6 would imply that $\omega \in \Sigma_0$. This proves the first step.

Now we show that (c) implies (b). By the first step, this follows for the trivial valuation and so we may assume that v is non-trivial. We may assume that X is irreducible. Arguing by contradiction, we assume that (c) holds and that $\text{Trop}_v(X)$ is not a subset of $|\Sigma_1|$. Since the special fibre of \mathcal{X} is non-empty, Tevelev's Lemma 11.6 yields that $\text{Trop}_v(X)$ intersects $|\Sigma_1|$. Since $\text{Trop}_v(X)$ is a connected finite union of Γ -rational polyhedra (see Theorem 3.3 and Proposition 3.5), there is $\omega \in \text{Trop}_v(X) \cap |\Sigma_1|$ such that $\Omega \cap \text{Trop}_v(X)$ is not contained in $|\Sigma_1|$ for every neighbourhood Ω of ω . Moreover, we may assume $m \cdot \omega \in N_\Gamma$ for some non-zero

$m \in \mathbb{N}$. Then there is a Γ -admissible subdivision Σ' of Σ such that ω is a vertex of Σ'_1 . Let \mathcal{X}' be the closure of X in $\mathcal{Y}_{\Sigma'}$. By Proposition 11.10, the canonical \mathbb{T} -equivariant morphism $\varphi : \mathcal{Y}_{\Sigma'} \rightarrow \mathcal{Y}_{\Sigma}$ is closed and the special fibre φ_s is proper. This shows $\varphi(\mathcal{X}') = \mathcal{X}$ and that the special fibre of \mathcal{X}' is also non-empty and proper over \tilde{K} . To simplify the notation, we may assume that $\mathcal{X} = \mathcal{X}'$.

By Remark 11.7, $X_{\omega} := \mathcal{X} \cap Z_{\omega}$ is a closed subscheme of the dense orbit Z_{ω} of the toric variety Y_{ω} over \tilde{K} with $\text{Trop}_0(X_{\omega})$ equal to the local cone of $\text{Trop}_v(X)$ at ω . This means that $\text{Trop}_0(X_{\omega})$ is not contained in the fan $\text{LC}_{\omega}(\Sigma_1)$ of the toric variety Y_{ω} . By Step 1, we conclude that the closure \mathcal{X}_{ω} of X_{ω} in Y_{ω} is not proper over \tilde{K} . On the other hand, \mathcal{X}_{ω} is a closed subscheme of $\mathcal{X} \cap Y_{\omega}$. Since the special fibre of \mathcal{X} is assumed to be proper over \tilde{K} , this has to be true also for its closed subscheme \mathcal{X}_{ω} . This is a contradiction and hence (c) implies (b). \square

Proposition 11.12 *Suppose that the value group Γ is divisible or discrete in \mathbb{R} and let X be any closed subscheme of T . Then the closure \mathcal{X} of X in \mathcal{Y}_{Σ} is proper over K° if and only if $\text{Trop}_v(X) \subset |\Sigma_1|$.*

Proof: If \mathcal{X} is proper over \tilde{K} , then its special fibre is proper over \tilde{K} and Proposition 11.11 yields $\text{Trop}_v(X) \subset |\Sigma_1|$. Conversely, we assume that $\text{Trop}_v(X) \subset |\Sigma_1|$. Then the same arguments as in the proof of (a) \Rightarrow (c) of Proposition 11.11 show that \mathcal{X} is proper over K° . \square

12 Tropical compactifications

We keep the notation from Section 11, where we have studied the closure \mathcal{X} of a closed subscheme X of T in the toric scheme \mathcal{Y}_{Σ} over K° . In this section, we study tropical compactifications \mathcal{X} related to certain fans Σ supported on the tropical cone $\text{Trop}_W(X)$ introduced in Section 8. This generalizes results of Tevelev who handled the case of an integral X over an algebraically closed field with trivial valuation (see [Tev]) and of Qu who obtained some results in the case of a discrete valuation (see [Qu]). Their definition of a tropical fan seems simpler, but our definition is better suited to handle the case of a non-reduced X and the definitions agree in the case of reduced closed subschemes.

Let (K, v) be an arbitrary valued field and let X be any closed subscheme of T .

Definition 12.1 A Γ -admissible tropical fan for X is a Γ -admissible fan Σ in $N_{\mathbb{R}} \times \mathbb{R}_+$ such that $\text{Trop}_v(X) \subset |\Sigma_1|$ and such that there is a closed subscheme \mathcal{F} of $\mathbb{T} \times_{K^{\circ}} \mathcal{Y}_{\Sigma}$ with the following properties:

- (a) The second projection induces a faithfully flat map $f : \mathcal{F} \rightarrow \mathcal{Y}_{\Sigma}$.
- (b) The map $\Phi : \mathbb{T} \times_{K^{\circ}} \mathcal{Y}_{\Sigma} \rightarrow \mathbb{T} \times_{K^{\circ}} \mathcal{Y}_{\Sigma}, (t, y) \mapsto (t^{-1}, t \cdot y)$ maps $T \times_K X$ isomorphically onto $f^{-1}(T)$.

In this case, we call the closure \mathcal{X} of X in \mathcal{Y}_{Σ} a *tropical compactification* of X .

Remark 12.2 Let us consider just multiplication $m : T \times_K X \rightarrow T$. Then this is isomorphic to the trivial fibre bundle $X \times_K T$ over T . The isomorphism is given by $(t, x) \mapsto (x, t \cdot x)$. In particular, we see that m is faithfully flat. A tropical fan asks for extension of faithful flatness for p_2 from $\Phi(T \times_K X)$ to a closed subscheme \mathcal{F} of $\mathbb{T} \times_{K^{\circ}} \mathcal{Y}_{\Sigma}$.

If \mathcal{F} is a Γ -admissible tropical fan, then it follows from flatness that the open subset $f^{-1}(T)$ of \mathcal{F} is dense. Using the isomorphism Φ , we get $(\Phi^{-1}(\mathcal{F}))_{\text{red}} = \mathbb{T} \times_{K^{\circ}} \mathcal{X}_{\text{red}}$ since the right hand side is reduced by [EGA IV], Proposition 17.5.7.

We conclude that the multiplication map $m : \mathbb{T} \times_{K^\circ} \mathcal{X}_{\text{red}} \rightarrow \mathcal{B}_\Sigma$ is surjective and has the same topological properties as f .

If X is reduced, then the closure \mathcal{X} is reduced. If we assume additionally that Γ is divisible or discrete in \mathbb{R} , then Proposition 11.12 implies that Σ is a Γ -admissible tropical fan if and only if \mathcal{X} is proper and the multiplication map induces a faithfully flat map $\mathbb{T} \times_{K^\circ} \mathcal{X} \rightarrow \mathcal{B}_\Sigma$. Hence our definition is the same as Tevelev's definition of a tropical fan for varieties over a trivially valued algebraically closed field.

We can now generalize Tevelev's result to our framework:

Theorem 12.3 *Let $\Sigma(A, a)$ be the Gröbner fan for X in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let Σ_X be the subcomplex with support $\text{Trop}_W(X)$ as in Corollary 10.17. Then every Γ -admissible fan Σ which subdivides Σ_X is a Γ -admissible tropical fan for X . In particular, Γ -admissible tropical fans exist for every closed subscheme X of T .*

Proof: We keep the notation introduced in Section 10 about Gröbner complexes. Let $\mathcal{Y}_{A,a}$ be the orbit closure of $\mathbf{y} := [\bar{X}]$ in $\text{Hilb}(\mathbb{P}_{K^\circ}^m)$. Since the Γ -admissible fan Σ subdivides the subcomplex Σ_X of $\Sigma(A, a)$, the canonical morphism $T \rightarrow T\mathbf{y}$ between the dense orbits extends to a \mathbb{T} -equivariant morphism $\varphi : \mathcal{B}_\Sigma \rightarrow \mathcal{Y}_{A,a}$ (see 11.9).

We consider the closed subscheme $\mathcal{G} := (\text{id} \times \varphi)^{-1}(\text{Univ}(\mathbb{P}_{K^\circ}^m))$ of $\mathbb{P}_{K^\circ}^m \times_{K^\circ} \mathcal{B}_\Sigma$ which is flat over \mathcal{B}_Σ . The fibre \mathcal{G}_y over $y \in T$ is equal to $y^{-1}\bar{X} \subset \mathbb{P}_K^m$. This makes it easy to check that

$$h : \mathcal{G}|_T \rightarrow \bar{X} \times_K T, \quad (z, y) \mapsto (y \cdot z, y) \quad (12)$$

is an isomorphism over T . Let \mathcal{F} be the restriction of \mathcal{G} to $\mathbb{T} \times_{K^\circ} \mathcal{B}_\Sigma$, then the second projection restricts to a flat morphism $f : \mathcal{F} \rightarrow \mathcal{B}_\Sigma$. Moreover, axiom (b) from the definition of a tropical fan follows from (12). By Corollary 10.17, $\text{Trop}_W(X)$ is the support of Σ_X and hence also from its subdivision Σ . By Tevelev's Lemma 11.6, we conclude that every orbit intersects \mathcal{X} and hence the multiplication map $m : \mathbb{T} \times_{K^\circ} \mathcal{X}_{\text{red}} \rightarrow \mathcal{B}_\Sigma$ is surjective. By the same argument as in Remark 12.2, we conclude that f is surjective and hence faithfully flat. This means that Σ is a tropical fan for X . Finally, we have seen in 10.18 that a Γ -admissible fan exists which subdivides Σ_X . \square

Proposition 12.4 *Let Σ be a Γ -admissible tropical fan for X and let Σ' be a Γ -rational fan which subdivides Σ . Then Σ' is a tropical fan for X .*

Proof: Since Σ' is a subdivision of Σ , we have $\text{Trop}_v(X) \subset |\Sigma_1| = |\Sigma'_1|$. By 11.9, we get a canonical \mathbb{T} -equivariant morphism $\varphi : \mathcal{B}_{\Sigma'} \rightarrow \mathcal{B}_\Sigma$ which is the identity on the dense open orbits T . Let us define the closed subscheme \mathcal{G} of $\mathbb{T} \times_{K^\circ} \mathcal{B}_{\Sigma'}$ by the following Cartesian diagram:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{f'} & \mathcal{B}_{\Sigma'} \\ \downarrow \varphi' & & \downarrow \varphi \\ \mathcal{F} & \xrightarrow{f} & \mathcal{B}_\Sigma \end{array}$$

Since f' is obtained from f by base change, we conclude that f' is faithfully flat. Since we have $(\varphi')^{-1}(f^{-1}(T)) = (f')^{-1}(T)$ and φ is the identity on T , we deduce easily axiom (b) from the definition of a tropical fan. This proves the claim. \square

Proposition 12.5 *Every Γ -admissible tropical fan for X in $N_{\mathbb{R}} \times \mathbb{R}_+$ has support equal to $\text{Trop}_W(X)$.*

Proof: It follows from Proposition 11.11 that the support of a Γ -admissible tropical fan Σ contains $\text{Trop}_W(X)$. We have to show that every $\sigma \in \Sigma$ is contained in $\text{Trop}_W(X)$. We argue by contradiction and so we assume that σ is not contained in Σ . Passing to a subdivision and using Proposition 12.4, we may assume that σ is disjoint from $\text{Trop}_W(X)$. It follows from Lemma 11.6 that the tropical compactification \mathcal{X} is disjoint from the orbit corresponding to $\text{relint}(\sigma)$. We conclude that the multiplication map $m : \mathbb{T} \times_{K^\circ} \mathcal{X}_{\text{red}} \rightarrow \mathcal{Y}_\Sigma$ is not surjective. This contradicts Remark 12.2. \square

Proposition 12.6 *Let X be a pure dimensional closed subscheme of T and let Σ be a Γ -admissible tropical fan for X with tropical compactification \mathcal{X} of X . If Z is any torus orbit in the generic (resp. special) fibre of \mathcal{Y}_Σ , then $Z \cap \mathcal{X}$ is a non-empty pure dimensional scheme over K (resp. \tilde{K}) with*

$$\dim(Z \cap \mathcal{X}) = \dim(X) + \dim(Z) - n.$$

In particular, Z intersects the generic (resp. special) fibre of \mathcal{X} properly.

Proof: By Proposition 11.3, the special fibre of \mathcal{X} is also pure dimensional of the same dimension as X . By flatness of f and Remark 12.2, the multiplication map m has pure dimensional fibres of constant fibre dimension. By Remark 12.2, this fibre dimension is equal to $\dim(X)$. For a closed point $z \in Z$, the fibre $m^{-1}(z) = \{(t, x) \in \mathbb{T} \times_{K^\circ} \mathcal{X} \mid t \cdot x = z\}$ projects onto $Z \cap \mathcal{X}$. The fibres of this projection are isomorphic to $\text{Stab}(z)$ and hence they have dimension $n - \dim(Z)$. By the fibre dimension theorem, we get

$$\dim(X) = \dim(m^{-1}(z)) = \dim(Z \cap \mathcal{X}) + n - \dim(Z)$$

proving the claim. The fibre dimension theorem yields also that $Z \cap \mathcal{X}$ is pure dimensional. \square

For a tropical fan and $z \in (\mathcal{Y}_\Sigma)_s$, Remark 12.2 shows $(m^{-1}(z))_{\text{red}} \cong f^{-1}(z)_{\text{red}}$ and this is closely related to a certain initial degeneration, as we will see in the next remark.

Remark 12.7 Let Σ be a Γ -admissible tropical fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let $z \in (\mathcal{Y}_\Sigma)_s$. By the tropical lifting lemma (Proposition 11.5), there is $y \in T^{\text{an}} \cap Y_{\Sigma_0}^\circ$ with reduction $\pi(y) = z$. There is a valued field (L, w) extending (K, v) such that y is an L -rational point of T in the sense of Remark 2.2. Let $f : \mathcal{F} \rightarrow \mathcal{Y}_\Sigma$ be the faithfully flat family from Definition 12.1. We claim that $f^{-1}(z)$ is isomorphic to the special fibre of the closure of $y^{-1}X$ in \mathbb{T}_{L° . This means that $\text{in}_{\text{trop}_w(y)}(X)$ is represented by $f^{-1}(z)$.

To prove the claim, we may assume $K = L$. By flatness of f , the closure of $f^{-1}(y)$ is equal to $f^{-1}(\bar{y})$ (see Corollary 4.5 and Remark 4.6). We restrict the flat family f to the closure \bar{y} of y in \mathcal{Y}_Σ . The generic fibre of this restriction is $f^{-1}(y)$ which is isomorphic to $y^{-1}X$ using the first projection of $T \times_K Y_{\Sigma_0}$ and axiom (b) in 12.1. Note that the first projection also gives a closed embedding of $f^{-1}(\bar{y})$ into \mathbb{T} and hence the special fibre $f^{-1}(z)$ is isomorphic to the special fibre of the closure of $y^{-1}X$ in \mathbb{T} as claimed.

Proposition 12.8 *Let Σ be a Γ -admissible tropical fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let z be an F -rational point of \mathcal{Y}_Σ for a field F . If we use the first projection to identify the following fibres of f with closed subschemes of \mathbb{T}_F as in Remark 12.7, then we have $f^{-1}(sz) = s^{-1} \cdot f^{-1}(z)$ for all $s \in \mathbb{T}(F)$.*

Proof: Note that F is either an extension of K or of the residue field \tilde{K} . It is enough to consider the case F/\tilde{K} as we may deduce the case F/K from the previous one by using the trivial valuation on K . There is a valued field (L, w) extending (K, v) such that the residue field \tilde{L} contains F . Since all the objects are defined over F , it is enough to show $f^{-1}(sz) = s^{-1} \cdot f^{-1}(z)$ over \tilde{L} . Let $t \in \mathbb{T}(L^\circ)$ be a lift of s , i.e. $\pi(t) = s$. By enlarging L , we may assume that $z = \pi(y)$ for some $y \in T(L)$ (see Proposition 11.5). Using Remark 12.7, we see that the special fibre of the closure of $(ty)^{-1}(X)$ is equal to $f^{-1}(sz)$. On the other hand, multiplication with t^{-1} induces an automorphism of \mathbb{T} and hence is compatible with taking closures. This automorphism is given on the special fibre by multiplication with s^{-1} and hence we get the claim. \square

Corollary 12.9 *Let Σ be a Γ -admissible tropical fan in $N_{\mathbb{R}} \times \mathbb{R}_+$. Suppose that $\omega, \nu \in \text{relint}(\sigma)$ for some $\sigma \in \Sigma$, then we have $\text{in}_\omega(X) = \text{in}_{\omega'}(X)$.*

Proof: This follows immediately from the orbit correspondence (Proposition 8.8), Remark 12.7 and Proposition 12.8. \square

Proposition 12.10 *Let σ be a cone of the Γ -admissible tropical fan Σ in $N_{\mathbb{R}} \times \mathbb{R}_+$. For every $\omega \in \sigma$ and every $\omega' = \omega + \Delta\omega \in \text{relint}(\sigma)$, we have*

$$\text{in}_{\omega'}(X) = \text{in}_{\Delta\omega}(\text{in}_\omega(X)),$$

where the initial degeneration at $\Delta\omega$ is with respect to the trivial valuation.

Proof: We have seen in Corollary 10.12 that the identity holds in a neighbourhood of ω . Now the claim follows from Corollary 12.9. \square

13 Tropical multiplicities

In this section, X is a closed subscheme of T and we will define a tropical multiplicity function on the tropical variety $\text{Trop}_v(X)$. It will be used to define $\text{Trop}_v(X)$ as a tropical cycle, i.e. a weighted polyhedral complex satisfying the balancing condition. This appeared first in Speyer's thesis [Spe]. The balancing condition relies on the description of the Chow cohomology of a toric variety given by Fulton–Sturmfels [FS]. This is very implicit in the presentation here as we reduce the claim to the case of the trivial valuation where the balancing condition of $\text{Trop}_v(X)$ is a result of Sturmfels and Tevelev based on [FS]. Further references: [AR], [BPR], [ST].

Definition 13.1 A point ω of $\text{Trop}_v(X)$ is called *regular* if there is a polytope $\sigma \subset \text{Trop}_v(X)$ such that $\text{relint}(\sigma)$ is a neighbourhood of ω in $\text{Trop}_v(X)$.

Proposition 13.2 *A point ω of $\text{Trop}_v(X)$ is regular if and only if 0 is regular in $\text{Trop}_0(\text{in}_\omega(X))$.*

Proof: This follows immediately from Proposition 10.15. \square

13.3 For $\omega \in N_{\mathbb{R}}$, we have seen that $\text{in}_\omega(X)$ is a closed subscheme of the special fibre of \mathbb{T} defined over a field extension of the residue field and it is well-defined up to multiplication with elements $g \in \mathbb{T}$ which are rational over a possibly larger field extension. Let F be an algebraically closed field extension over which $\text{in}_\omega(X)$ is defined. Then the irreducible components of $\text{in}_\omega(X)$ over F are also irreducible components over every field extension of F and hence the following definition makes sense.

Definition 13.4 The *tropical multiplicity* $m(\omega, X)$ of $\omega \in N_{\mathbb{R}}$ is defined as the sum of the multiplicities of $\text{in}_{\omega}(X)$ in its irreducible components over the algebraically closed field F . For a cycle $Z = \sum m_Y Y$ of T with prime components Y , we define the tropical multiplicity of Z in ω by $m(\omega, Z) := \sum_Y m_Y m(\omega, Y)$.

We have defined the initial degeneration as an equivalence class of closed subschemes up to multiplication by torus elements over an extension of the residue field (see 5.4). In the next result, we form the cycle of an initial degeneration. This means that we consider cycles up to the obvious linear extension of the above equivalence relation from prime components to all cycles. Note that the following result is a special case of [OP], Theorem 4.4.5. Here, we give a different proof using intersection theory with Cartier divisors. We have to deal with the fact that the models are usually non-noetherian and hence we cannot use algebraic intersection theory, but there is an analytic replacement introduced in [Gub1].

Lemma 13.5 *Let $\text{cyc}(X) = \sum_Y m_Y Y$ be the representation of the cycle associated to X as a sum of its irreducible components Y counted with multiplicities. Then we have*

$$\text{cyc}(\text{in}_{\omega}(X)) = \sum_Y m_Y \text{cyc}(\text{in}_{\omega}(Y)).$$

Proof: By base change, we may assume that v is non-trivial and that K is an algebraically closed complete field such that all occurring initial degenerations are defined over \tilde{K} . Moreover, we may suppose that $\omega = \text{trop}_v(t)$ for some $t \in T(K)$. Replacing X by $t^{-1}X$, we may assume $t = e$ and $\omega = 0$. Then $\text{in}_{\omega}(X)$ is the special fibre \mathcal{X}_s of the closure \mathcal{X} of X in \mathbb{T} .

By Proposition 6.7, we have $\mathcal{X} = \text{Spec}(A)$ for a flat K° -algebra of finite type. Let us choose a non-zero $\nu \in K^{\circ\circ}$. We have seen in 4.13 that the ν -adic completion \hat{A} of A is a flat K° -algebra which is topologically of finite type, i.e. $\hat{\mathcal{X}} := \text{Spf}(\hat{A})$ is an admissible formal affine scheme over K° in the theory of Raynaud, Bosch and Lütkebohmert (see [BL], §1). Its generic fibre is defined as the Berkovich spectrum $\mathcal{M}(\hat{\mathcal{A}})$ of the strictly affinoid algebra $\hat{\mathcal{A}} := \hat{A} \otimes_{K^{\circ}} K$ and it is equal to the affinoid subdomain X° of X^{an} from 4.9. If T° is the formal affinoid torus constructed in the same way from T , then we have $X^{\circ} = X^{\text{an}} \cap T^{\circ}$.

Using that $\hat{\mathcal{A}}$ is a noetherian algebra, we have a theory of cycles and Cartier divisors on X° (see [Gub1], §2, for details). Hence we have a cycle decomposition $\text{cyc}(X^{\circ}) = \sum_{W \in S} m_W W$ for a finite set S of prime cycles of X° . If Y is an irreducible component of X , then the GAGA principle shows that Y° is a closed reduced analytic subvariety of X° , but Y° is not necessarily irreducible. Hence we have $\text{cyc}(Y^{\circ}) = \sum_{W \in S_Y} m_W W$ for a subset S_Y of S . It is clear that S is the disjoint union of the sets S_Y . By [Gub1], Proposition 6.3, we have $m_Y = m_W$ for all $W \in S_Y$. Moreover, it is obvious that $S_Y \neq \emptyset$ if and only if Y meets the affinoid torus T° and the latter is equivalent to $\text{in}_0(Y) \neq \emptyset$. By [Gub2], Lemma 4.5, we have

$$\text{cyc}(\mathcal{X}_s) = \sum_{W \in S} m_W \text{cyc}(\overline{W}_s) \tag{13}$$

where \overline{W}_s is the special fibre of the closure \overline{W} of W in $\hat{\mathcal{X}}$. Similarly, we get

$$\text{cyc}(\overline{Y}_s) = \sum_{W \in S_Y} m_W \text{cyc}(\overline{W}_s) \tag{14}$$

where \overline{Y} is the closure of Y in \mathcal{X} . Using (13), (14) and the above facts, we get

$$\text{cyc}(\text{in}_0(X)) = \sum_Y \sum_{W \in S_Y} m_Y \text{cyc}(\overline{W}_s) = \sum_Y m_Y \text{cyc}(\overline{Y}_s) = \sum_Y m_Y \text{cyc}(\text{in}_0(Y))$$

proving the claim. \square

Proposition 13.6 *Tropical multiplicities have the following properties:*

- (a) *They are invariant under base change of X or Z to valued field (L, w) extending (K, v) .*
- (b) *The tropical multiplicity $m(\omega, Z)$ is linear in the cycle Z .*
- (c) *For the cycle $\text{cyc}(X)$ associated to X , we have $m(\omega, X) = m(\omega, \text{cyc}(X))$.*

Proof: Property (a) follows from Proposition 5.5 and (b) is obvious. Finally, (c) follows from Lemma 13.5. \square

The following result shows that we may compute tropical multiplicities locally over the trivially valued residue field.

Proposition 13.7 *For $\omega_0 \in N_{\mathbb{R}}$, there is a neighbourhood Ω of ω_0 in $N_{\mathbb{R}}$ such that $m(\omega, X) = m(\omega - \omega_0, \text{in}_{\omega_0}(X))$ for all $\omega \in \Omega$.*

Proof: This follows from Corollary 10.12. \square

We have now the setup to generalize the following result of Sturmfels–Tevelev, which was given in the case of trivial valuations.

Theorem 13.8 *The restriction of the tropical multiplicity function $m(\cdot, X)$ to the open subset of regular points in $\text{Trop}_v(X)$ is locally constant.*

Proof: By Proposition 13.7, we reduce to the case of a trivially valued base field. By Proposition 13.6, we may assume that base field is algebraically closed and that X is an irreducible subvariety. Then the claim follows from [ST], Corollaries 3.8 and 3.15. \square

13.9 A Γ -rational polyhedral complex \mathcal{C} in $N_{\mathbb{R}}$ is called *of pure dimension d* if every maximal $\sigma \in \mathcal{C}$ has dimension d . Such a complex is called *weighted* if it is endowed with a *multiplicity function* m which maps every d -dimensional $\sigma \in \mathcal{C}$ to a number $m_{\sigma} \in \mathbb{Z}$.

A polyhedron $\sigma \in \mathcal{C}$ generates an affine space in $N_{\mathbb{R}}$ which is a translate of a linear space \mathbb{L}_{σ} . By Γ -rationality of σ , the vector space \mathbb{L}_{σ} is defined over \mathbb{Q} and $N_{\sigma} := \mathbb{L}_{\sigma} \cap N$ is a lattice in \mathbb{L}_{σ} .

We say that a weighted Γ -rational polyhedral complex \mathcal{C} in $N_{\mathbb{R}}$ of pure dimension d satisfies the *balancing condition* if for every $d - 1$ -dimensional polyhedron ν , we have

$$\sum_{\sigma \supset \nu} m_{\sigma} n_{\sigma, \nu} \in N_{\nu},$$

where σ ranges over all d -dimensional polyhedra of \mathcal{C} containing ν , and $n_{\sigma, \nu}$ is any representative of the generator of the 1-dimensional lattice N_{σ}/N_{ν} pointing in the direction of σ .

A weighted Γ -rational polyhedral complex \mathcal{C} in $N_{\mathbb{R}}$ of pure dimension d is called a *tropical cycle* if it satisfies the balancing condition. We identify tropical cycles if there is a common Γ -rational subdivision of both complexes for which the multiplicities coincide. This allows us to add tropical cycles. In general, we define a Γ -rational tropical cycle \mathcal{C} in $N_{\mathbb{R}}$ as a formal sum $\mathcal{C} = \sum_{j=0}^n \mathcal{C}_j$, where \mathcal{C}_j is a tropical cycle in $N_{\mathbb{R}}$ of pure dimension j . For details about tropical cycles, we refer to [AR].

13.10 We suppose that X is a pure-dimensional closed subscheme of T and we set $d := \dim(X)$. Let \mathcal{C} be any Γ -rational polyhedral complex with support equal to $\text{Trop}_v(X)$. By Theorem 10.14, we know that such complexes exist. Theorem

3.3 shows that \mathcal{C} is of pure dimension d . Note that the relative interior of a d -dimensional polyhedron $\sigma \in \mathcal{C}$ is contained in the regular part of $\text{Trop}_v(X)$. By Theorem 13.8, the multiplicity function $m(\cdot, X)$ is constant on $\text{relint}(\sigma)$ and this constant is denoted by m_σ . We call m_σ the *tropical multiplicity* of σ .

Theorem 13.11 *Under the hypothesis of 13.10, the complex \mathcal{C} endowed with the tropical multiplicities is a Γ -rational tropical cycle of pure dimension d .*

Proof: The balancing condition is a local condition in any $\omega \in N_{\mathbb{R}}$. By Propositions 10.15 and 13.7, it is enough to check the balancing condition for $\text{in}_\omega(X)$ in a neighbourhood of 0. Hence we have reduced the claim to the case of trivial valuation. Again, we may assume that the base field is algebraically closed and that X is an irreducible subvariety. This case is proved in [ST], Corollary 3.8. \square

Remark 13.12 It follows from Theorem 13.8 that the tropical cycle from Theorem 13.11 does not depend on the choice of the complex \mathcal{C} from 13.10. We conclude that $\text{Trop}_v(X)$ is canonically a tropical cycle which we denote also by $\text{Trop}_v(X)$.

If X is any closed subscheme of T , then we define $\text{Trop}_v(X)$ by linearity in its irreducible components, i.e. we set $\text{Trop}_v(X) := \sum_Y m_Y \text{Trop}_v(Y)$ as a tropical cycle, where m_Y is the multiplicity of X in the irreducible component Y . This is a tropical cycle in $N_{\mathbb{R}}$ with support equal to the set-theoretic tropical variety of X . By Proposition 13.6, this agrees with the above construction in the pure dimensional case.

If $Z = \sum_Y m_Y Y$ is any cycle on X with prime components Y , then we define the *tropical cycle associated to Z* by $\text{Trop}_v(Z) := \sum_Y m_Y \text{Trop}_v(Y)$, where we use the induced reduced structure on every Y .

Proposition 13.13 *Let X be any closed subscheme of T and let $\omega \in N_{\mathbb{R}}$. Replacing the polyhedra in the tropical cycle $\text{Trop}_v(X)$ by its local cones in ω and using the same tropical multiplicities, we get a tropical cycle in $N_{\mathbb{R}}$ which is equal to the tropical cycle $\text{Trop}_0(\text{in}_\omega(X))$ with respect to the trivial absolute value 0.*

Proof: This follows from Proposition 13.7 and Proposition 10.15. \square

13.14 Let T' be another split torus over K with lattice N' of one-parameter-subgroups. Let $\varphi : T \rightarrow T'$ be a homomorphism of split tori over K . Then there is a unique map $\text{Trop}_v(\varphi) : N_{\mathbb{R}} \rightarrow N'_{\mathbb{R}}$ such that $\text{Trop}_v(\varphi) \circ \text{trop}_v = \text{trop}_v \circ \varphi$. This follows from surjectivity of the tropicalization map trop_v . The fact that homomorphisms of tori are given by characters implies that $\text{Trop}_v(\varphi)$ is a linear map defined over \mathbb{Z} .

13.15 The push-forward of a cycle Z on X with respect to the homomorphism φ is a cycle $\varphi_*(Z)$ on the closure X' of $\varphi(X)$ defined in the following way: If Z is a prime cycle and Z' is the closure of $\varphi(Z)$, then

$$\varphi_*(Z) := \begin{cases} [K(Z) : K(Z')]Z', & \text{if } [K(Z) : K(Z')] < \infty, \\ 0, & \text{if } [K(Z) : K(Z')] = \infty. \end{cases}$$

In general, $\varphi_*(Z)$ is defined by linearity in its prime components.

Usually, the push-forward of cycles is defined with respect to proper morphisms. This could be easily obtained by using tropical compactifications as in Section 12, but as we are not interested in compatibility with rational equivalence of cycles, this plays no role here.

13.16 We will explain how the linear map $f := \text{Trop}_v(\varphi) : N_{\mathbb{R}} \rightarrow N'_{\mathbb{R}}$ induces a push-forward map of tropical cycles. For details, we refer to [AR], §7. Let \mathcal{C} be a tropical cycle in $N_{\mathbb{R}}$ of pure dimension d . After a subdivision of \mathcal{C} , we may assume that

$$f_*(\mathcal{C}) := \{f(\sigma) \mid \sigma \text{ is a face of } \nu \in \Sigma \text{ with } \dim(f(\nu)) = d\}$$

is a (d -dimensional Γ -rational) polyhedral complex in $N'_{\mathbb{R}}$. We define the multiplicity of a d -dimensional $f(\sigma) \in f_*(\mathcal{C})$ by

$$m_{f(\sigma)} := \sum_{\nu \subset f^{-1}(f(\sigma))} [N_{\nu} : N_{f(\sigma)}] m_{\nu},$$

where ν ranges over all d -dimensional $\nu \in \mathcal{C}$ contained in $f^{-1}(f(\sigma))$. Endowed with these multiplicities, we get a weighted polyhedral complex which is a tropical cycle in $N'_{\mathbb{R}}$. It might happen that $f_*(\mathcal{C})$ is empty, then we get the tropical zero-cycle.

The following result is the Sturmfels–Tevelev multiplicity formula (see [ST]). It was generalized to the case of non-trivial valuations in [BPR], Corollary 8.4 and Appendix A.

Theorem 13.17 *Let $\varphi : T \rightarrow T'$ be a homomorphism of split tori over K and let Z be cycle on T . Then we have*

$$\text{Trop}_v(\varphi)_*(\text{Trop}_v(Z)) = \text{Trop}_v(\varphi_*(Z))$$

as an identity of tropical cycles.

Proof: By base change, we may assume that K is an algebraically closed field with a complete non-trivial valuation. Using linearity of the identity in the prime components of Z , we may assume that $X = Z$ is an integral closed subscheme of T . If $\dim(X') < \dim(X)$ for $X' := \varphi(X)$, then $\varphi_*(X) = 0$ by definition. Since $\text{Trop}_v(X)$ is a polyhedral complex of pure dimension $d := \dim(X)$ and since $\varphi_*(\text{Trop}_v(X))$ is a tropical cycle supported in $\text{Trop}_v(X')$ which is of lower dimension, we conclude $\varphi_*(\text{Trop}_v(X)) = 0$ as well. So we may assume that φ induces a generically finite map $X \rightarrow X'$ and then we may deduce the claim from [BPR], Corollary 8.4. \square

14 Proper intersection with orbits

As usual, (K, v) is a valued field which serves as a ground field. Let Σ be a Γ -admissible fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ with associated toric scheme \mathcal{B}_{Σ} over K° . Let X be a closed subscheme of the dense torus T with closure \mathcal{X} in \mathcal{B}_{Σ} . We have seen in Proposition 12.6 that \mathcal{X} intersects the orbits of \mathcal{B}_{Σ} properly in case of a tropical fan. In this section, we will generalize this result and we prove that this property is a purely combinatorial property of the fan Σ . I am very grateful to Sam Payne for explaining me some of the arguments for this nice result.

14.1 Let $\sigma \in \Sigma_1$ and $\tau = \text{relint}(\sigma)$. Then σ generates an affine space in $N_{\mathbb{R}}$ which is a translate of a linear space \mathbb{L}_{σ} . By Γ -rationality of σ , the vector space \mathbb{L}_{σ} is defined over \mathbb{Q} . Then $N_{\sigma} := N \cap \mathbb{L}_{\sigma}$ and $N(\sigma) := N/N_{\sigma}$ are free abelian groups of finite rank with quotient homomorphism $\pi_{\sigma} : N \rightarrow N(\sigma)$. Dually, we have $M(\sigma) := \mathbb{L}_{\sigma}^{\perp} \cap M = \text{Hom}(N(\sigma), \mathbb{Z})$.

For $S \subset N_{\mathbb{R}}$, we define the *local cone of S at τ* by

$$\text{LC}_{\tau}(S) := \bigcup_{\omega \in \tau} \text{LC}_{\omega}(S)$$

using the local cones at points from A.6. If S is a polyhedron containing τ , then we have $\text{LC}_{\tau}(S) = \text{LC}_{\omega}(S)$ for any $\omega \in \tau$.

14.2 We recall from 7.9 that τ corresponds to an orbit $Z = Z_\tau$ of the special fibre of \mathcal{B}_Σ . By choosing a base point $z_0 \in Z(K)$, Proposition 7.15 shows that Z may be identified with the torus $\text{Spec}(\tilde{K}[M(\sigma)_\tau])$ for the sublattice $M(\sigma)_\tau := \{u \in M(\sigma) \mid \langle u, \omega \rangle \in \Gamma \ \forall \omega \in \tau\}$ of finite index in $M(\sigma)$. We get tropical varieties of closed subschemes of Z with respect to the trivial valuation which do not depend on the choice of the base point z_0 .

Proposition 14.3 *Using the notions from above, we have*

$$\text{Trop}_0(\mathcal{X} \cap Z_\tau) = \pi_\sigma(\text{LC}_\tau(\text{Trop}_v(X))).$$

Proof: By base change and Lemma 11.2, we may assume that Γ is divisible and hence we have $M(\sigma) = M(\sigma)_\tau$. We assume first that Σ has a tropical subfan for X . If $\tau \cap \text{Trop}_v(X)$ is empty, then Tevelev's Lemma 11.6 shows that both sides of the claim are empty. By Proposition 12.5, we have $\text{Trop}_v(X) = |\Sigma_1|$ and so we may assume that $\tau \subset \text{Trop}_v(X)$. We choose $\omega \in \tau \cap N_\Gamma$. By translation, we may assume that $\omega = 0$ and therefore the affine toric scheme \mathcal{U}_ω from 6.11 is just the split torus \mathbb{T} over K° . We conclude that $\text{in}_\omega(X)$ is the special fibre of the closure of X in \mathcal{U}_ω . To identify Z_τ with $T(\sigma) := \text{Spec}(\tilde{K}[M(\sigma)])$, we choose the base point z_0 of Z_τ as the reduction of the unit element in $T(K)$. Then the canonical quotient homomorphism $q : \mathbb{T}_{\tilde{K}} \rightarrow T(\sigma)$ of tori over \tilde{K} maps $\text{in}_\omega(X)$ into $\mathcal{X} \cap Z_\tau$. Since τ is an open face of a tropical subfan of Σ , the proof of Proposition 12.6 and Remark 12.7 show that $\text{in}_\omega(X) = q^{-1}(\mathcal{X} \cap Z_\tau)$ holds set theoretically. This yields

$$\pi_\sigma(\text{Trop}_0(\text{in}_\omega(X))) = \text{Trop}_0(\mathcal{X} \cap Z_\tau). \quad (15)$$

By Proposition 10.15, we have $\text{Trop}_0(\text{in}_\omega(X)) = \text{LC}_\omega(\text{Trop}_v(X))$. Since $\text{Trop}_v(X)$ is a finite union of polyhedra which either contain τ or are disjoint from τ , we get $\text{LC}_\omega(\text{Trop}_v(X)) = \text{LC}_\tau(\text{Trop}_v(X))$. Inserting these facts into (15), we get the claim.

Now we prove the claim in the case of an arbitrary Γ -admissible fan Σ . The affine toric scheme \mathcal{U}_σ associated to the closure σ of τ is an open subset of \mathcal{B}_Σ containing Z_τ . The claim in the proposition depends only on \mathcal{U}_σ and hence we may change Σ outside of σ . So we may assume that Σ_1 is a complete Γ -rational fan containing σ . By Theorem 12.3 and Proposition 12.4, there is a Γ -admissible fan Σ' which is a subdivision of Σ and which has a tropical subfan. Then we have a canonical \mathbb{T} -equivariant morphism $\varphi : \mathcal{B}_{\Sigma'} \rightarrow \mathcal{B}_\Sigma$ of \mathbb{T} -toric schemes over K° . It follows from Proposition 11.10 that φ is closed and surjective. For the closure \mathcal{X}' of X in $\mathcal{B}_{\Sigma'}$, we conclude that $\varphi(\mathcal{X}') = \mathcal{X}$. Relevant for our purposes is that τ has a subdivision into open faces τ_1, \dots, τ_r of Σ' . The orbit correspondence in Proposition 8.8 leads to the partition of $\varphi^{-1}(Z_\tau)$ into the orbits $Z_{\tau_1}, \dots, Z_{\tau_r}$. We conclude that $\mathcal{X} \cap Z_\tau$ is the union of the sets $\varphi(\mathcal{X}' \cap Z_{\tau_i})$. Let σ_i be the closure of τ_i and let $\pi_i : N(\sigma_i) \rightarrow N(\sigma)$ be the canonical homomorphism. Then we get

$$\text{Trop}_0(\mathcal{X} \cap Z_\tau) = \bigcup_{i=1}^r \text{Trop}_0(\varphi(\mathcal{X}' \cap Z_{\tau_i})) = \bigcup_{i=1}^r \pi_i(\text{Trop}_0(\mathcal{X}' \cap Z_{\tau_i})). \quad (16)$$

Using the special case above, we have

$$\text{Trop}_0(\mathcal{X}' \cap Z_{\tau_i}) = \pi_{\sigma_i}(\text{LC}_{\tau_i}(\text{Trop}_v(X))).$$

Inserting this in (16) and using $\pi_i \circ \pi_{\sigma_i} = \pi_\sigma$, we get the claim. \square

For simplicity, we assume now that the closed subscheme X of T is of pure dimension d . By the Bieri–Groves Theorem 3.3, there is a finite set S of d -dimensional Γ -rational polyhedra in $N_{\mathbb{R}}$ such that $\text{Trop}_v(X) = \bigcup_{\Delta \in S} \Delta$.

Corollary 14.4 *Under the hypothesis above, we have*

$$\dim(\mathcal{X} \cap Z_\tau) = d - \inf\{\dim(\Delta \cap \tau) \mid \Delta \in S, \Delta \cap \tau \neq \emptyset\}.$$

Proof: By Tevelev's Lemma 11.6, $\mathcal{X} \cap Z_\tau$ is empty if and only if no $\Delta \in S$ intersects τ . We see that the claim holds in this special case as the dimension of the empty set is defined as $-\infty$ and the infimum over an empty set is ∞ . Proposition 14.3 shows that we have

$$\mathrm{Trop}_0(\mathcal{X} \cap Z_\tau) = \bigcup_{\Delta \in S} \pi_\sigma(\mathrm{LC}_\tau(\Delta)). \quad (17)$$

For $\Delta \in S$ with $\Delta \cap \tau \neq \emptyset$, we have $\dim(\pi_\sigma(\mathrm{LC}_\tau(\Delta))) = d - \dim(\Delta \cap \tau)$. Using this in (17), we get the claim. \square

Remark 14.5 We recall from 7.9 that the open faces τ of Σ_0 correspond to the orbits Z_τ contained in the generic fibre Y_{Σ_0} of \mathcal{Y}_Σ . If we use a decomposition $\mathrm{Trop}_0(X) = \bigcup_{\Delta \in S} \Delta$ into d -dimensional rational cones Δ in $N_{\mathbb{R}}$ (see Remark 3.4), then Corollary 14.4 holds also for these orbits. This follows immediately from Corollary 14.4 replacing v by the trivial valuation. Then the generic fibre is equal to the special fibre.

14.6 Let X be a closed subscheme of T of pure dimension d with closure \mathcal{X} in \mathcal{Y}_Σ and let Z_τ be the orbit of \mathcal{Y}_Σ induced by the open face τ of Σ_1 (resp. Σ_0). We say that \mathcal{X} *intersects Z_τ properly* if $\dim(\mathcal{X} \cap Z_\tau) = d - \dim(\tau)$. We emphasize that in this case, $\mathcal{X} \cap Z_\tau$ is not empty. Note that \mathcal{Y}_Σ is a noetherian topological space and one can easily show that \mathcal{X} intersects Z_τ properly if and only if every irreducible component of $\mathcal{X} \cap Z_\tau$ has codimension in \mathcal{X} equal to $\mathrm{codim}(Z_\tau, \mathcal{Y}_\Sigma)$.

The following result was shown to me by Sam Payne.

Proposition 14.7 *If \mathcal{X} intersects Z_τ properly, then $\mathcal{X} \cap Z_\tau$ is pure dimensional.*

Proof: We may assume that $\tau \in \Sigma_1$ and hence Z_τ is contained in the special fibre of \mathcal{Y}_Σ . Indeed, the case $\tau \in \Sigma_0$ follows as usual from this replacing v by the trivial valuation. We choose a vertex ω of $\bar{\tau}$. By Proposition 7.15, the irreducible component Y_ω of \mathcal{X}_s corresponding to ω is a toric variety over \tilde{K} associated to the fan $\mathrm{LC}_\omega(\Sigma_1) := \{\mathrm{LC}_\omega(\Delta) \mid \Delta \in \Sigma_1\}$. We note that Z_τ is also an orbit of Y_ω and we have $\mathrm{codim}(Z_\tau, Y_\omega) = \dim(\tau)$. For every $z \in Z_\tau$, there is a neighbourhood U of z in Y_ω such that $Z_\tau \cap U$ is set theoretically the intersection of $\mathrm{codim}(Z_\tau, Y_\omega)$ effective Cartier divisors. This is clear if Z_τ is a point and the general situation is obtained from this by pull back. We conclude that every irreducible component of $\mathcal{X} \cap Z_\tau$ has dimension at least $d - \dim(\tau)$. \square

Proposition 14.8 *Let τ be an open face of Σ_1 . If \mathcal{X} intersects Z_τ properly, then $\tau \subset \mathrm{Trop}_v(X)$.*

Proof: Assuming that \mathcal{X} intersects Z_τ properly, Corollary 14.4 shows that $\dim(\Delta \cap \tau) = \dim(\tau)$ for any d -dimensional polyhedron $\Delta \subset \mathrm{Trop}_v(X)$ with $\Delta \cap \tau \neq \emptyset$. By Theorem 10.14, there is a complete Γ -rational polyhedral complex \mathcal{D} in $N_{\mathbb{R}}$ with a subcomplex \mathcal{C} such that $\mathrm{Trop}_v(X) = |\mathcal{C}|$. We consider the collection \mathcal{E} of all $\sigma \in \mathcal{D}$ with $\dim(\mathrm{relint}(\sigma) \cap \tau) = \dim(\tau)$. We note that $(\sigma \cap \tau)_{\sigma \in \mathcal{E}}$ is a covering of the open face τ . By assumption, $\mathcal{X} \cap Z_\tau$ is non-empty and hence $\tau \cap \mathrm{Trop}_v(X) \neq \emptyset$ by Tevelev's Lemma 11.6. The Bieri–Groves theorem yields a d -dimensional polyhedron $\Delta \in \mathcal{C}$ with $\Delta \cap \tau \neq \emptyset$. Using an appropriate closed face of Δ , we get the existence of a polyhedron $\sigma \in \mathcal{C} \cap \mathcal{E}$. Let $\sigma' \in \mathcal{E}$ such that

$\nu := \sigma \cap \sigma' \cap \tau$ has dimension equal to $\dim(\tau) - 1$. Since τ may be covered by using successively such neighboring $\sigma' \cap \tau$, it is enough to show that $\sigma' \cap \tau \subset \text{Trop}_v(X)$. Note that ν is obtained by intersecting τ with a proper closed face of σ . Since σ is a closed face of a d -dimensional polyhedron $\Delta \in \mathcal{C}$, there is a closed face ρ of Δ with $\dim(\rho) = d - 1$ which contains ν but not σ . We have $\dim(\text{relint}(\sigma) \cap \tau) = \dim(\tau)$ and hence $\Delta \cap \tau = \sigma \cap \tau$ contains $\rho \cap \tau$ as a proper subset. Since ν is of codimension 1 in $\sigma \cap \tau$, we get $\nu = \rho \cap \tau$. We choose a hyperplane H in $N_{\mathbb{R}}$ which contains ρ but not Δ . By the balancing condition in Theorem 13.11, there is a d -dimensional polyhedron $\Delta' \in \mathcal{C}$ with closed face ρ on the other side of H than Δ . We conclude that $\Delta' \cap \tau \subset \sigma' \cap \tau$. Using the first remark in the proof, we get $\dim(\Delta' \cap \tau) = \dim(\tau)$ and hence $\Delta' \cap \tau = \sigma' \cap \tau$. This proves $\sigma' \cap \tau \subset \text{Trop}_v(X)$. \square

To deal with orbits in the special fibre and in the generic fibre simultaneously, one has to use the Γ -admissible cone Σ in $N_{\mathbb{R}} \times \mathbb{R}_+$ and the tropical cone $\text{Trop}_W(X)$ of X in $N_{\mathbb{R}} \times \mathbb{R}_+$ (see Definition 8.3).

Theorem 14.9 *Let Σ be a Γ -admissible fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let X be a closed subscheme of T of pure dimension d . Then the following properties are equivalent for the closure \mathcal{X} of X in the toric scheme \mathcal{B}_{Σ} :*

- (a) *The special fibre \mathcal{X}_s is non-empty, proper over \tilde{K} , and \mathcal{X} intersects all the orbits of \mathcal{B}_{Σ} properly.*
- (b) *The support of Σ is equal to the tropical cone $\text{Trop}_W(X)$.*

If the value group Γ is divisible or discrete in \mathbb{R} , then (a) and (b) are also equivalent to the condition that \mathcal{X} is a proper scheme over K° which intersects all the orbits properly.

Proof: We assume that (a) holds. By Proposition 14.8, the assumption that \mathcal{X} intersects all orbits properly yields that $|\Sigma_1|$ is contained in $\text{Trop}_v(X)$. If we replace v by the trivial valuation, then the same argument shows that $|\Sigma_0| \subset \text{Trop}_0(X)$. Since $\text{Trop}_W(X)$ is the closed cone in $N_{\mathbb{R}} \times \mathbb{R}_+$ generated by $\text{Trop}_v(X) \times \{1\}$ (see Proposition 8.4), we conclude that $|\Sigma| \subset \text{Trop}_W(X)$. On the other hand, \mathcal{X}_s is a non-empty proper scheme over \tilde{K} and hence $\text{Trop}_v(X)$ is contained in $|\Sigma_1|$ by Proposition 11.11. We conclude that (a) yields (b).

Now we suppose that (b) holds. Then Σ_1 is a Γ -rational complex with support equal to $\text{Trop}_v(X)$. We choose an open face τ of Σ_1 with corresponding orbit Z_τ in the special fibre of \mathcal{B}_{Σ} . For any d -dimensional polyhedron $\Delta \in \Sigma_1$, either $\Delta \cap \tau$ is empty or τ . From Corollary 14.4, we deduce that \mathcal{X} intersects Z_τ properly. Using $\text{Trop}_0(X) = |\Sigma_0|$, Remark 14.5 shows that \mathcal{X} intersects the orbits in the generic fibre of \mathcal{B}_{Σ} properly. It follows from Proposition 11.11 that \mathcal{X}_s is a non-empty proper scheme over \tilde{K} .

If Γ is divisible or discrete in \mathbb{R} , then the last claim follows immediately from Proposition 11.12. \square

Remark 14.10 We have seen in Proposition 12.5 that every tropical fan satisfies the equivalent properties (a) and (b) of Theorem 14.9. However, the converse does not hold as it was shown by Dustin Cartwright [Car] giving a counterexample in the case of curves.

A Convex geometry

In this appendix, we collect the notation used from convex geometry. We denote by Γ a subgroup of \mathbb{R} . We consider a free abelian group M of rank n with dual

$N := \text{Hom}(M, \mathbb{Z})$ and the corresponding real vector spaces $V := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $W := \text{Hom}(V, \mathbb{R}) = N \otimes_{\mathbb{Z}} \mathbb{R}$. The natural duality between $u \in V$ and $\omega \in W$ is denoted by $\langle u, \omega \rangle$. References: [Roc], [McM].

A.1 A *polyhedron* Δ in W is an intersection of finitely many closed half-spaces $\{\omega \in W \mid \langle u_i, \omega \rangle \geq c_i\}$. We say that Δ is Γ -*rational* if we may choose all $u_i \in M$ and all $c_i \in \Gamma$. If $\Gamma = \mathbb{Q}$, then we say that Δ is *rational*. A *closed face* of Δ is either Δ itself or has the form $H \cap \Delta$ where H is the boundary of a closed half-space containing Δ . An *open face* of Δ is a closed face without all its properly contained closed faces. We denote by $\text{relint}(\Delta)$ the unique open face of Δ which is dense in Δ .

A.2 A bounded polyhedron is called a *polytope*. This is equivalent to be the convex hull of finitely many points. Let $G := \{\lambda \in \mathbb{R} \mid \exists m \in \mathbb{N} \setminus \{0\}, m\lambda \in \Gamma\}$ be the divisible hull of Γ in \mathbb{R} . Simple linear algebra shows that a polytope is Γ -rational if and only if all vertices are G -rational and the edges have rational slopes. Similarly, a polyhedron is Γ -rational if and only if every closed face spans an affine subspace which is a translate of a rational linear subspace by a G -rational vector.

A.3 A *polyhedral complex* \mathcal{C} in W is a finite set of polyhedra such that

- (a) $\Delta \in \mathcal{C} \Rightarrow$ all closed faces of Δ are in \mathcal{C} ;
- (b) $\Delta, \sigma \in \mathcal{C} \Rightarrow \Delta \cap \sigma$ is either empty or a closed face of Δ and σ .

The polyhedral complex is called Γ -*rational* if every $\Delta \in \mathcal{C}$ is Γ -rational. The *support* of \mathcal{C} is defined as

$$|\mathcal{C}| := \bigcup_{\Delta \in \mathcal{C}} \Delta.$$

The polyhedral complex \mathcal{C} is called *complete* if $|\mathcal{C}| = W$.

A.4 A polyhedral complex \mathcal{D} *subdivides* the polyhedral complex \mathcal{C} if they have the same support and if every polyhedron Δ of \mathcal{D} is contained in a polyhedron of \mathcal{C} . In this case, we say that \mathcal{D} is a *subdivision* of \mathcal{C} .

A.5 A *cone* σ in W is centered at 0, i.e. it is characterized by $\mathbb{R}_+\sigma = \sigma$. Its *dual* is defined by

$$\check{\sigma} := \{u \in V \mid \langle u, \omega \rangle \geq 0 \ \forall \omega \in \sigma\}.$$

A *fan* is a polyhedral complex consisting of polyhedral cones.

A.6 The *local cone* $\text{LC}_\omega(S)$ of $S \subset W$ at ω is defined by

$$\text{LC}_\omega(S) := \{\omega' \in W \mid \omega + [0, \varepsilon)\omega' \subset S \text{ for some } \varepsilon > 0\}.$$

A.7 The *recession cone* of a polyhedron Δ is defined by

$$\text{rec}(\Delta) := \{\omega \in W \mid \omega + \Delta \subset \Delta\}.$$

By the Minkowski–Weil theorem, the recession cone is the unique convex polyhedral cone σ such that $\Delta = \sigma + \rho$ for a polytope ρ of W . If Δ is Γ -rational, then $\text{rec}(\Delta)$ is a rational convex polyhedral cone.

A.8 A polyhedron Δ is called *pointed* if it does not contain an affine line. Note that Δ is a pointed polyhedron if and only if $\text{rec}(\Delta)$ has the origin 0 as a vertex. This explains the terminology. A *pointed polyhedral complex* is a polyhedral complex consisting of pointed polyhedra.

A.9 We say that $f : W \rightarrow \mathbb{R} \cup \{\infty\}$ is a *proper polyhedral function* if the *epigraph* $\text{epi}(f) := \{(\omega, s) \in W \times \mathbb{R} \mid f(\omega) \leq s\}$ is a non-empty polyhedron. Then the faces of the polyhedron $\text{epi}(f)$ contained in the graph of f form a polyhedral complex in $W \times \mathbb{R}$ called the *graph complex*. The projection of the graph complex onto W gives a polyhedral complex in W . Such a complex is called a *coherent polyhedral complex* in W .

Note that f is a proper polyhedral function if and only if there is a non-empty polyhedron Σ in W and a function $f_\Sigma : \Sigma \rightarrow \mathbb{R}$ with the following properties:

- (a) f_Σ is continuous and piecewise affine;
- (b) f_Σ is a convex function in the usual sense, i.e.

$$f_\Sigma(r\omega + s\omega') \leq rf_\Sigma(\omega) + sf_\Sigma(\omega') \quad (18)$$

for $\omega, \omega' \in \Sigma$ and $r, s \in [0, 1]$ with $r + s = 1$.

- (c) f agrees with f_Σ on Σ and $f = \infty$ outside of Σ .

We call Σ the *domain of f* . The *domains of linearity* are the maximal subsets of W where f_Σ is affine. They are just the maximal dimensional polyhedra Δ from the coherent polyhedral complex corresponding to f . On such a Δ , we have

$$f(\omega) = c_\Delta + \langle u_\Delta, \omega \rangle$$

for some $u_\Delta \in V$ and $c_\Delta \in \mathbb{R}$. We call u_Δ the *peg* of Δ .

A.10 Let f be a proper polyhedral function with associated coherent polyhedral complex \mathcal{C} . Then the *conjugate* of f is the proper polyhedral function $f^* : V \rightarrow \mathbb{R} \cup \{\infty\}$ given by

$$f^*(u) := \sup\{\langle u, \omega \rangle - f(\omega) \mid \omega \in W\}.$$

We have $f^{**} = f$.

A.11 Let f be a proper polyhedral function on W with associated coherent polyhedral complex \mathcal{C} . The coherent polyhedral complex in V associated to f^* is called the *dual complex \mathcal{C}^f* of \mathcal{C} . The duality is a bijective order reversing correspondence $\sigma \mapsto \sigma^f$ between polyhedra of \mathcal{C} and polyhedra of \mathcal{C}^f given by

$$\sigma^f = \{u \in V \mid f^*(u) = \langle u, \omega \rangle - f(\omega) \forall \omega \in \sigma\}$$

and we have

$$\dim(\sigma) + \dim(\sigma^f) = n.$$

This follows from [McM], Theorem 7.1 and its proof. Note also that \mathcal{C}^f is complete if and only if the support of \mathcal{C} is bounded (which is then a polytope).

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Walter Gubler, Universität Regensburg, Fakultät für Mathematik, Universitätsstrasse 31, D-93040 Regensburg, walter.gubler@mathematik.uni-regensburg.de