Solution to Problem 73-17: A Hadamard-type bound on the coefficients of a determinant of polynomials

Citation for published version (APA):
Lossers, O. P. (1974). Solution to Problem 73-17: A Hadamard-type bound on the coefficients of a determinant of polynomials. *SIAM Review*, *16*(3), 394-395. https://doi.org/10.1137/1016065, https://doi.org/10.1137/1016064

DOI:
10.1137/1016065
10.1137/1016064

Document status and date:
Published: 01/01/1974

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

where M is $r \times r$ and nonsingular. In what follows, it will be assumed without loss of generality that A is already partitioned as above, for if $RAC = BG$ is a nonnegative rank factorization of RAC, then $A = (R^TB)(GC^T)$ is a nonnegative rank factorization of A.

If P is a real matrix with r columns, then define $\mathscr{G}(P) = \{Px | x \geqq 0\}$ and $\mathscr{C}(P) = \{x \geqq 0 | Px \geqq 0\}$. Note that $\mathscr{G}(P)$ and $\mathscr{C}(P)$ are polyhedral cones. A cone $\mathscr{S}$ will be said to be *solid* if there exists a nonsingular matrix N of rank r such that $\mathscr{G}(N) \subseteq \mathscr{S}$, and *simplicial* if there exists such an N for which equality holds.

THEOREM. *If A is an $m \times n$ nonnegative matrix of rank r where r satisfies $0 < r < \min\{m, n\}$ and A is partitioned as stated previously, then A has a nonnegative rank factorization if and only if there exists a simplicial cone $\mathscr{S}$ satisfying $\mathscr{G}([M, MQ]) \subseteq \mathscr{S} \subseteq \mathscr{C}(P)$.*

Proof. If such an $\mathscr{S}$ exists, then $\mathscr{S} = \mathscr{G}(X)$, where X is $r \times r$ and nonsingular. Thus $X \geqq 0$ and $PX \geqq 0$. Since $\mathscr{G}([M, MQ]) \subseteq \mathscr{S}$, there exist nonnegative matrices Y and Z such that $M = XY$ and $MQ = XZ$. Then A has the nonnegative rank factorization

$$A = \begin{pmatrix} X \\ PX \end{pmatrix} \cdot (Y, Z).$$

Conversely, suppose

$$A = \begin{pmatrix} X \\ W \end{pmatrix} \cdot (Y, Z)$$

is a nonnegative rank factorization of A, where X and Y are $r \times r$. Since M is nonsingular, then so are the nonnegative matrices X and Y. Let $\mathscr{S} = \mathscr{G}(X)$. Then $\mathscr{S}$ is simplicial and also $\mathscr{S} \subseteq \mathscr{C}(P)$ since $PX = W$ and W is nonnegative. Since $M = XY$ and $MQ = XZ$, $\mathscr{G}([M, MQ]) \subseteq \mathscr{S}$.

COROLLARY. *Every rank* 1 *and rank* 2 *nonnegative matrix has a nonnegative rank factorization.*

Proof. Every solid polyhedral cone in R^1 and R^2 is simplicial.

In light of the corollary, the minimum dimensions and rank possible for a nonnegative matrix A having no nonnegative rank factorization are 4×4 and 3, respectively. The following is such a matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Proof. $P = [-1, 1, 1]$. $\mathscr{C}(P)$ is not simplicial, while $\mathscr{G}([M, MQ]) = \mathscr{C}(P)$.

A. BEN-ISRAEL (The Technion, Haifa, Israel) also gave a similar counterexample.

Problem 73-17, *A Hadamard-Type Bound on the Coefficients of a Determinant of Polynomials*, by A. J. GOLDSTEIN and R. L. GRAHAM (Bell Telephone Laboratories).

If $A = (a_{ij})$ is an $n \times n$ matrix, a classical inequality of Hadamard [2, p. 253] asserts that

$$|\det A| \leqq \left(\prod_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} \equiv H(A).$$

In recent studies on coefficient growth in greatest common divisor algorithms for polynomials, W. S. Brown [1] was led to inquire about possible analogues of this inequality for the case in which the entries of the matrix are *polynomials*.

Let $A(x) = (A_{ij}(x))$ be a matrix whose elements are polynomials and let $a_0, a_1, \cdots$ be the coefficients of the polynomial representation of $\det A(x)$. If $W = (w_{ij})$, where w_{ij} denotes the sum of the absolute values of the coefficients of $A_{ij}(x)$, then show that

$$\left(\sum |a_k|^2\right)^{1/2} \leqq H(W).$$

REFERENCES

[1] W. S. Brown, *On Euclid's algorithm and the computation of polynomial greatest common divisors*, J. Assoc. Comput. Mach., 18 (1971), pp. 478–504.

[2] F. R. Gantmacher, *The Theory of Matrices*, vol. 1, Chelsea, New York, 1960.

Solution by O. P. Lossers (Technological University, Eindhoven, the Netherlands).

Since $|A_{kl}(e^{it})| \leqq w_{kl}$, it follows from Hadamard's inequality that

$$|\det A(e^{it})|^2 \leqq \prod_{k=1}^{n} \sum_{l=1}^{n} |A_{kl}(e^{it})|^2 \leqq \prod_{k=1}^{n} \sum_{l=1}^{n} w_{kl}^2 = (H(W))^2.$$

However,

$$\frac{1}{2\pi}\int_0^{2\pi} |\det A(e^{it})|^2\,dt = \frac{1}{2\pi}\int_0^{2\pi} \left(\left(\sum a_k e^{ikt}\right)\left(\sum \bar{a}_l e^{-ilt}\right)\right) dt = \sum |a_k|^2.$$

Hence

$$\sum |a_k|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\det A(e^{it})|^2\,dt \leqq \frac{1}{2\pi}\int_0^{2\pi} (H(W))^2\,dt = (H(W))^2.$$

Also solved by A. A. Jagers (Technische Hogeschool Twente, Enschede, the Netherlands) and the proposers.