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A Hamiltonian approach to stabilization of nonholonomic mechanical systems

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Abstract

A simple procedure is provided to write the equations of motion of controlled mechanical systems with constraints as controlled Hamiltonian equations with respect to a "Poisson" bracket which does not necessarily satisfy the Jacobi-identity. Based on the Hamiltonian form a stabilization procedure is proposed.

1 Introduction

In a recent paper we have shown that (uncontrolled) mechanical systems with classical constraints can be written as Hamiltonian equations of motion with respect to a generalized type of Poisson bracket, and with respect to a Hamiltonian which is obtained by restricting the internal energy to the constrained state space. This bracket does not necessarily satisfy the Jacobi-identity, which is one of the defining properties of a true Poisson bracket. In fact, the Jacobi-identity is satisfied if and only if the constraints are holonomic. This work was motivated by a paper of Bates & Sniatycki [3] on the Hamiltonian formulation of nonholonomic systems, as well as by our previous work on the Hamiltonian formulation of non-resistive physical systems by network modelling [8], [9].

In the present paper we extend this set-up to *controlled* nonholonomic mechanical systems. Furthermore we show how the Hamiltonian form of the equations (the Jacobi-identity being satisfied or not) may be used for stabilization purposes. Indeed we show how the stabilization procedure for standard Hamiltonian control systems as proposed in [14], [11], see also [5], can be extended to this case. These considerations were very much motivated by the papers [2], [4] on stabilization of controlled nonholonomic systems. We close with our treatment of two well-known simple examples of nonholonomic systems, discussed before in [2].

2 The Hamiltonian formulation of systems with constraints

Let Q be an n-dimensional configuration manifold with local coordinates $q = (q_1, \cdots, q_n)$. Consider a smooth Lagrangian function $L : TQ \to \mathbf{R}$, denoted by $L(q, \dot{q})$, satisfying throughout the usual regularity condition

$$det \left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right] \neq 0 \tag{1}$$

(This is e.g. satisfied if L equals kinetic energy with positive definite generalized mass matrix minus potential energy.) Classical constraints are given in local coordinates as

$$A^T(q)\dot{q} = 0 \tag{2}$$

with $A(q)$ a $k \times n$ matrix, $k \leq n$, with entries depending smoothly on q. Throughout we assume that $A(q)$ has rank equal to k everywhere. The constraints (2) determine a k-dimensional *distribution* D on Q, given in every point $q_0 \in Q$ as

$$D(q_0) = ker A^T(q_0) \tag{3}$$

The constraints (2) are called *holonomic* if the distribution D is involutive, i.e. for any two vectorfields X, Y on Q

$$X \in D, Y \in D \Rightarrow [X, Y] \in D \tag{4}$$

with $[X, Y]$ the Lie-bracket, defined in local coordinates q as $[X, Y](q) = \frac{\partial Y}{\partial q}(q)X(q) - \frac{\partial X}{\partial q}(q)Y(q)$, with $\frac{\partial Y}{\partial q}, \frac{\partial X}{\partial q}$ the Jacobian matrices. In this case we may find, by Frobenius' theorem, local coordinates $\bar{q} = (\bar{q}_1, \cdots, \bar{q}_n)$ such that the constraints (2) are expressed as

$$\dot{\bar{q}}_{n-k+1} = \cdots = \dot{\bar{q}}_n = 0, \tag{5}$$

or equivalently $\bar{q}_{n-k+1} = c_{n-k+1}, \cdots \bar{q}_n = c_n$ for certain constants $c_{n-k+1}, \cdots, c_n$ determined by the initial conditions, and we may eliminate the coordinates $\bar{q}_{n-k+1}, \cdots, \bar{q}_n$. The constraints (2) are called *nonholonomic* if D is *not* involutive, implying that we can *not* use the above elimination procedure.

The equations of motion for the mechanical system on Q with Lagrangian $L(q, \dot{q})$ and constraints (2) are given as (see e.g. [10], [13], [1])

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \; - \; \frac{\partial L}{\partial q} &= A(q)\lambda + B(q)u, \\ A^T(q)\dot{q} \; &= \; 0, \;\; \lambda \in \mathbf{R}^k, \;\; u \in \mathbf{R}^m \end{aligned} \tag{6}$$

where $B(q)u$ are the external forces (controls) applied to the system, with $B(q)$ an $n \times m$ matrix with entries depending smoothly on x. Here $\frac{\partial L}{\partial q}$ denotes the column vector $\left(\frac{\partial L}{\partial q_1}, \cdots, \frac{\partial L}{\partial q_n}\right)^T$, and similarly for $\frac{\partial L}{\partial \dot{q}}$ and subsequent expressions. The *constraint forces* $A(q(t))\lambda(t)$ are determined by the requirement that the constraints $A^T(q(t))\dot{q}(t) = 0$ have to be satisfied for all t.

Defining in the usual way the Hamiltonian $H(q, p)$ by the Legendre transformation

$$H(q,p) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q}), \quad p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{7}$$

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the constrained Euler-Lagrange equations (6) transform into the constrained Hamiltonian equations on T^*Q

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p}(q,p)\\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)u\\ A^T(q)\frac{\partial H}{\partial p}(q,p) &= A^T(q)\dot{q} = 0\end{aligned} \tag{8}$$

An intrinsic definition of the constrained Hamiltonian equations may be given as follows. The cotangent bundle T^*Q is equipped with its canonical Poisson bracket $\{,\}$, in natural coordinates $(q,p) = (q_1,\cdots,q_n,p_1,\cdots,p_n)$ for T^*Q expressed as (with F and G smooth functions on T^*Q)

$$\begin{aligned}\{F,G\}(q,p) &= \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)(q,p) =\\ &= \left(\frac{\partial F^T}{\partial q}\ \frac{\partial F^T}{\partial p}\right) J \begin{pmatrix}\frac{\partial G}{\partial q}\\ \frac{\partial G}{\partial p}\end{pmatrix}, \quad J = \begin{pmatrix}0_n & I_n\\ -I_n & 0_n\end{pmatrix},\end{aligned} \tag{9}$$

with J the standard Poisson structure matrix. Recall that for any smooth function $H : T^*Q \to \mathbf{R}$ its Hamiltonian vectorfield X_H on T^*Q is defined in the local coordinates (q,p) as

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = J\begin{pmatrix}\frac{\partial H}{\partial q}\\ \frac{\partial H}{\partial p}\end{pmatrix} = \begin{pmatrix}-\frac{\partial H}{\partial p}\\ \frac{\partial H}{\partial q}\end{pmatrix} \tag{10}$$

Similarly, for any one-form α on T^*Q we may define the "Hamiltonian" vectorfield Z_α as

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = J\begin{pmatrix}\alpha_1(q,p)\\ \vdots\\ \alpha_n(q,p)\end{pmatrix} \tag{11}$$

where $(\alpha_1(q,p),\cdots,\alpha_n(q,p))$ is the local coordinate expression of the one-form α. (Note that $Z_{dH} = X_H$.) Now the columns of $A(q)$ define in local coordinates k one-forms $\alpha^1,\cdots,\alpha^k$ on Q. Similarly, the columns of $B(q)$ define m one-forms $\beta^1,\cdots,\beta^m$ on Q. Since any one-form on Q may be also regarded as a one-form on T^*Q, we can thus define the vectorfields $Z_{\alpha^1},\cdots,Z_{\alpha^k},Z_{\beta^1},\cdots,Z_{\beta^m}$ on T^*Q. It can now be readily seen that a coordinate-free description of the first part of (8) is given as (see also [2])

$$\dot{x} = X_H(x) + a(x)\lambda + b(x)u, \quad x \in T^*Q, \tag{12}$$

where $a(x)$ is the matrix with columns $Z_{\alpha^1},\cdots,Z_{\alpha^k}$, and $b(x)$ is the matrix with columns $Z_{\beta^1},\cdots,Z_{\beta^m}$.

The Lagrange multipliers λ may be computed by differentiating $A^T(q)\frac{\partial H}{\partial p}(q,p) = 0$ along (8), i.e.

$$\begin{aligned}&\left[\frac{\partial}{\partial q}\left(A^T(q)\frac{\partial H}{\partial p}(q,p)\right)\right]^T \frac{\partial H}{\partial p}(q,p) + A^T(q)\frac{\partial^2 H}{\partial p^2}(q,p)\cdot\\ &\left[-\frac{\partial H}{\partial q}(q,p) + B(q)u\right] + A^T(q)\frac{\partial^2 H}{\partial p^2}(q,p)A(q)\lambda = 0\end{aligned} \tag{13}$$

with $\frac{\partial^2 H}{\partial p^2}$ the Hessian matrix with respect to p. This equation may be solved for λ (as function of q,p,u) as long as

$$det A^T(q)\frac{\partial^2 H}{\partial p^2}(q,p)A(q) \neq 0, \quad (q,p) \in T^*Q, \tag{14}$$

which condition is obviously satisfied because of our standing assumptions (1) and rank $A(q) = k$. Expressing λ as a function of (q,p,u) and substituting in (8) then leads to the dynamical equations of motion on the constrained state space

$$\mathfrak{X}_r = \{(q,p) \in T^*Q | A^T(q)\frac{\partial H}{\partial p}(q,p) = 0\} \tag{15}$$

As shown in [15] a much more efficient and insightful way of obtaining the equations of motion on $\mathfrak{X}_r$ is however the following. Since rank $A(q) = k$, there exists locally a smooth $n \times (n-k)$ matrix $S(q)$ of rank $n-k$ such that

$$A^T(q)S(q) = 0 \tag{16}$$

(Equivalently, $S(q)$ is such that $D(q) = Im S(q)$.) Now define $\tilde{p} = (\tilde{p}^1,\tilde{p}^2) = (\tilde{p}_1,\cdots,\tilde{p}_{n-k},\tilde{p}_{n-k+1},\cdots,\tilde{p}_n)$ as

$$\begin{aligned}\tilde{p}^1 &:= S^T(q)p, \quad \tilde{p}^1 \in \mathbf{R}^{n-k}\\ \tilde{p}^2 &:= A^T(q)p, \quad \tilde{p}^2 \in \mathbf{R}^k\end{aligned} \tag{17}$$

It immediately follows from (16) that $(q,p) \mapsto (q,\tilde{p}^1,\tilde{p}^2)$ is a coordinate transformation. The constrained Hamiltonian dynamics (8) in the new coordinates $(q,\tilde{p}^1,\tilde{p}^2)$ take the following form. In the new coordinates $(q,\tilde{p})$ the Poisson structure matrix transforms from (9) into

$$\tilde{J}(q,\tilde{p}) = \begin{pmatrix}(\{q_i,q_j\})_{i,j} & (\{q_i,\tilde{p}_j\})_{i,j}\\ (\{\tilde{p}_j,q_i\})_{i,j} & (\{\tilde{p}_i,\tilde{p}_j\})_{i,j}\end{pmatrix}, \quad i,j = 1,\cdot\cdot,n \tag{18}$$

and the constrained Hamiltonian dynamics (8) transform into

$$\begin{aligned}\begin{bmatrix}\dot{q}\\ \dot{\tilde{p}}^1\\ \dot{\tilde{p}}^2\end{bmatrix} &= \tilde{J}(q,\tilde{p})\begin{bmatrix}\frac{\partial \tilde{H}}{\partial q}\\ \frac{\partial \tilde{H}}{\partial \tilde{p}^1}\\ \frac{\partial \tilde{H}}{\partial \tilde{p}^2}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \tilde{A}(q)\end{bmatrix}\lambda + \begin{bmatrix}0\\ B_r(q)\\ \tilde{B}(q)\end{bmatrix}u\\ \tilde{A}(q)\frac{\partial \tilde{H}}{\partial \tilde{p}^2} &= 0\end{aligned} \tag{19}$$

with $\tilde{A}(q) := A^T(q)A(q)$ an invertible matrix, and $\tilde{H}(q,\tilde{p})$ the Hamiltonian $H(q,p)$ expressed in the new coordinates $q,\tilde{p}$. Now truncate the transformed Poisson structure matrix $\tilde{J}$ in (18) by leaving out the last k columns and last k rows, and let $\tilde{p}$ satisfy the constraint equation $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$. This defines a $(2n-k)\times(2n-k)$ skew-symmetric matrix J_r on $\mathfrak{X}_r$. An explicit expression for J_r is obtained as follows [15]. Denote the i-th column of $S(q)$ by $S_i(q)$, then

$$J_r = \begin{pmatrix}0_n & S(q)\\ -S^T(q) & \left(-p^T[S_i,S_j](q)\right)_{i,j=1,\cdots,n-k}\end{pmatrix} \tag{20}$$

where p is expressed as function of $q,\tilde{p}$, with $\tilde{p}$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$. Note that rank $J_r = 2(n-k)$ everywhere on $\mathfrak{X}_r$. Furthermore, define the reduced Hamiltonian $H_r : \mathfrak{X}_r \to \mathbf{R}$ as $\tilde{H}(q,\tilde{p})$ with $\tilde{p}$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$.

Clearly, $(q,\tilde{p}^1)$ serve as local coordinates for $\mathfrak{X}_r$. It immediately follows from (19) by disregarding the last equations involving λ and noting that $\frac{\partial \tilde{H}}{\partial \tilde{p}^2}(q,\tilde{p})=0$ that the dynamics on $\mathfrak{X}_r$ in coordinates $(q,\tilde{p}^1)$ are described as

$$\begin{pmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \end{pmatrix} = J_r(q,\tilde{p}^1)\begin{pmatrix} \frac{\partial H_r}{\partial q}(q,\tilde{p}^1) \\ \frac{\partial H_r}{\partial \tilde{p}^1}(q,\tilde{p}^1) \end{pmatrix} + \begin{pmatrix} 0 \\ B_r(q) \end{pmatrix} u \qquad (21)$$

These equations are in Hamiltonian format! Indeed, the matrix J_r defines a *bracket* $\{,\}_r$ on $\mathfrak{X}_r$ by setting

$$\{F_r,G_r\}_r(q,\tilde{p}^1) := \left(\frac{\partial F_r^T}{\partial q}\frac{\partial F_r^T}{\partial \tilde{p}^1}\right) J_r(q,\tilde{p}^1)\begin{pmatrix} \frac{\partial G_r}{\partial q} \\ \frac{\partial G_r}{\partial \tilde{p}^1}\end{pmatrix} \qquad (22)$$

for any two smooth functions $F_r, G_r : \mathfrak{X}_r \to \mathbf{R}$. Clearly, this bracket satisfies the first two defining properties of a *Poisson* bracket (see e.g. [6],[12], [7]),

$$\begin{array}{llll} (i) & \{F_r,G_r\}_r & = & -\{G_r,F_r\}_r \\ & & & \textit{(skew-symmetry)} \\ (ii) & \{F_r,G_rH_r\}_r & = & \{F_r,G_r\}_rH_r + G_r\{F_r,H_r\}_r \\ & & & \textit{(Leibniz' rule)} \end{array} \qquad (23)$$

for every $F_r, G_r, H_r : \mathfrak{X}_r \to \mathbf{R}$. However, for $\{,\}_r$ to be a Poisson bracket also the following property

$$\begin{array}{ll} (iii) & \{F_r,\{G_r,H_r\}_r\}_r + \{G_r,\{H_r,F_r\}_r\}_r \\ & +\{H_r,\{F_r,G_r\}_r\}_r = 0 \textit{(Jacobi-identity)} \end{array} \qquad (24)$$

needs to be satisfied. If (24) is satisfied then (21) with $u=0$ ***defines a generalized Hamiltonian system with respect to a Poisson bracket***, see e.g. [12], [7], [8]. In this case local coordinates $(\bar{q},\bar{p},\bar{s})$ for $\mathfrak{X}_r$ may be found such that the system for $u=0$ takes the form [6], [12], [8]

$$\begin{array}{ll} \dot{\bar{q}} = \frac{\partial H_r}{\partial \bar{p}}, \dot{\bar{p}} = -\frac{\partial H_r}{\partial \bar{q}}, & \bar{q},\bar{p} \in \mathbf{R}^{n-k} \\ \dot{\bar{s}} = 0, & \bar{s} \in \mathbf{R}^k. \end{array} \qquad (25)$$

However, in [15] it has been shown that $\{,\}_r$ satisfies the Jacobi-identity (and thus is a true Poisson bracket) *if and only if* the constraints $A^T(q)\dot{q}=0$ are holonomic! This underscores the difficulties of nonholonomic constraints. On the other hand, even if the Jacobi-identity is not satisfied (as in the case for nonholonomic systems), the (pseudo-)Hamiltonian format (21) may still be useful, as we wish to indicate in the next section.

Note that our approach is not unrelated to the approach taken in [4]. Here the Lagrange multipliers λ in the Euler-Lagrange equations (6) are eliminated by premultiplying the equations (6) by the matrix $S^T(q)$, and it is shown that the thus reduced equations can be written as a set of first-order differential equations in q and $\eta \in \mathbf{R}^{n-k}$ with $\dot{q}=S(q)\eta$ parametrizing the admissible velocities $\dot{q}$. This can be regarded as the "Lagrangian counterpart" of our Hamiltonian approach.

3 Stabilization

We note that the dynamics (21) are energy preserving. In fact, by skew-symmetry of J_r we immediately obtain

$$\frac{d}{dt}H_r = \frac{\partial H_r}{\partial \tilde{p}^1}(q,\tilde{p}^1)B_r(q)u \qquad (26)$$

with $\frac{d}{dt}$ denoting differentiation along (21). Suppose now that $(q_0,\tilde{p}_0^1)$ is a stationary point of the Hamiltonian H_r, i.e. $\frac{\partial H_r}{\partial q}(q_0,\tilde{p}_0^1)=0$, $\frac{\partial H_r}{\partial \tilde{p}^1}(q_0,\tilde{p}_0^1)=0$, implying that $(q_0,\tilde{p}_0^1)$ is an equilibrium of the uncontrolled constrained dynamics $(u=0)$

$$\begin{pmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \end{pmatrix} = J_r(q,\tilde{p}^1)\begin{pmatrix} \frac{\partial H_r}{\partial q}(q,\tilde{p}^1) \\ \frac{\partial H_r}{\partial \tilde{p}^1}(q,\tilde{p}^1) \end{pmatrix} \qquad (27)$$

If H_r happens to have a strict minimum in $(q_0,\tilde{p}_0^1)$, then it follows from (26) with $u=0$ that $(q_0,\tilde{p}_0^1)$ is a Lyapunov stable equilibrium of (27). On the other hand, as in the case of ordinary Hamiltonian control systems (see e.g. [14], [11]), equation (26) suggests for improved stabilization the smooth state feedback

$$u = -B_r^T(q)\frac{\partial H_r^T}{\partial \tilde{p}^1}(q,\tilde{p}^1) \qquad (28)$$

which results in the monotonous energy decrease

$$\frac{d}{dt}H_r = -\frac{\partial H_r}{\partial \tilde{p}^1}(q,\tilde{p}^1)B_r(q)B_r^T(q)\frac{\partial H_r^T}{\partial \tilde{p}^1}(q,\tilde{p}^1) \leq 0 \qquad (29)$$

(Note that (28) can be written as $u=-y$, with y the conjugated effort corresponding to the generalized flow u [8].) If H_r has a strict minimum in $(q_0,\tilde{p}_0^1)$, then $(q_0,\tilde{p}_0^1)$ will thus be at least a Lyapunov stable equilibrium of the closed-loop system (21), (28), and moreover the trajectories will converge to the largest invariant (with respect to (27)) set contained in

$$\{(q,\tilde{p}^1) \in \mathfrak{X}_r \mid \frac{\partial H_r}{\partial \tilde{p}^1}(q,\tilde{p}^1)B_r(q)=0\} \qquad (30)$$

However it can be shown, as in [2], [4], that (21) does not satisfy Brockett's necessary condition, and thus *cannot* be *asymptotically* stabilized by a smooth state feedback. Hence this largest invariant set will be always larger than the singleton $\{(q_0,\tilde{p}_0^1)\}$.

If H_r does not have a strict minimum in $(q_0,\tilde{p}_0^1)$ then, as in the case of ordinary Hamiltonian control systems ([14], [11]), we may try to *shape* by preliminary feedback the internal energy H_r, if possible, to a function which *does* have a strict minimum in $(q_0,\tilde{p}_0^1)$. Indeed, let H_r be of the form, as usually encountered in applications,

$$H_r(q,\tilde{p}^1) = V(q) + \frac{1}{2}(\tilde{p}^1)^T G(q)\tilde{p}^1, \quad G(q) > 0 \qquad (31)$$

(potential energy plus kinetic energy). Necessarily $\tilde{p}_0^1=0$, and $\frac{\partial V}{\partial q}(q_0)=0$. Now consider the equation

$$-S^T(q)\frac{\partial \bar{V}}{\partial q}(q) = B_r(q)u \qquad (32)$$

For every smooth function $\bar{V}$ such that $S^T(q)\frac{\partial \bar{V}}{\partial q}(q) \in Im\ B_r(q)$, for all q, we can determine a smooth feedback $u = \tilde{u}(q)$ which solves (32). Application of the feedback $u = \tilde{u}(q) + v$, with v the new control variables, will result in a modified system (instead of (21))

$$\begin{pmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \end{pmatrix} = J_r(q, \tilde{p}^1) \begin{pmatrix} \frac{\partial \bar{H}_r}{\partial q}(q, \tilde{p}^1) \\ \frac{\partial \bar{H}_r}{\partial p}(q, \tilde{p}^1) \end{pmatrix} + \begin{pmatrix} 0 \\ B_r(q) \end{pmatrix} v \quad (33)$$

with $\bar{H}_r(q, \tilde{p}^1) = H_r(q, \tilde{p}^1) + \bar{V}(q)$. (This results from the special form of J_r given in (20).) If it is possible to find in this manner a function $\bar{V}$ such that $V + \bar{V}$ has a strict minimum in q_0, then $\bar{H}_r$ will have a strict minimum in $(q_0, 0)$, and thus the additional feedback (28), with u replaced by v, will further stabilize the system. The resulting combined feedback is then given as

$$u = \tilde{u}(q) - B_r^T(q)\frac{\partial H_r^T}{\partial \tilde{p}^1}(q, \tilde{p}^1) \quad (34)$$

with $\tilde{u}(q)$ solving (32).

The treatment of [2], [4] corresponds to the special case that $B_r(q)$ has rank $m = n - k$. In this case equation (32) is solvable for every function $\bar{V}(q)$, and thus the potential energy can be shaped in an arbitrary fashion. Therefore, for H_r given by (31), every point $(q_0, 0) \in \mathfrak{X}_r$ can be rendered a Lyapunov stable equilibrium by a feedback (34). Note furthermore that in this case the largest invariant (with respect to (27)) set contained in (30) is actually given as

$$\{(q, 0) \in \mathfrak{X}_r \mid S^T(q)\frac{\partial (V + \bar{V})}{\partial q}(q) = 0\} \quad (35)$$

(as follows from the form of J_r given in (20)), where $\bar{V}$ is taken such that $V + \bar{V}$ has a strict minimum in q_0. A similar result has been obtained before in [4] (in the reduced Lagrangian framework) using a different Lyapunov function, and a different feedback control based on this. The main difference is that in our approach the Lyapunov function $\bar{H}_r$ is directly based on the internal energy of the constrained dynamics, and consequently that u given in (34) has a direct physical interpretation. Furthermore, contrary to [4], we consider the stabilization problem for arbitrary B_r and an arbitrary number of controls.

We now treat within our approach two examples of nonholonomic control systems, both of which have been studied before in [2].

Example 3.1 *(Knife edge)* Consider the control of a knife edge moving in point contact on a plane surface. The constrained Lagrangian equations are given as (all numerical constants are set to unity)

$$\begin{aligned} \ddot{x} &= \lambda \sin\varphi + u_1 \cos\varphi \\ \ddot{y} &= -\lambda\cos\varphi + u_1 \sin\varphi \\ \ddot{\varphi} &= u_2 \end{aligned} \quad (36)$$

with (x, y) Cartesian coordinates of the contact point, φ the heading angle of the knife-edge, u_1 the control in the direction of the heading angle, and u_2 the control torque about the vertical axis. The nonholonomic constraint is

$$\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0 \quad (37)$$

The total energy H is given as $\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_\varphi^2$, with p_x, p_y, p_φ the corresponding generalized momenta. The constraint (37) can be written as $p_x\sin\varphi - p_y\cos\varphi = 0$. Define as in (17) new coordinates

$$\begin{aligned} p_1 &= p_\varphi \\ p_2 &= p_x\cos\varphi + p_y\sin\varphi \\ p_3 &= p_x\sin\varphi - p_y\cos\varphi \end{aligned} \quad (38)$$

Then $(\varphi, x, y, p_1, p_2)$ are coordinates for $\mathfrak{X}_r$, and the dynamics (21) is computed as

$$\begin{bmatrix} \dot{\varphi} \\ \dot{x} \\ \dot{y} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cos\varphi \\ 0 & 0 & 0 & 0 & \sin\varphi \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\cos\varphi & -\sin\varphi & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_r}{\partial \varphi} \\ \frac{\partial H_r}{\partial x} \\ \frac{\partial H_r}{\partial y} \\ \frac{\partial H_r}{\partial p_1} \\ \frac{\partial H_r}{\partial p_2} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (39)$$

with $H_r(\varphi, x, y, p_1, p_2) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$. Take $\bar{V}(\varphi, x, y) = \frac{1}{2}\varphi^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2$, then the preliminary feedback $\tilde{u}(\varphi, x, y)$ is determined by (see (32))

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\varphi & -\sin\varphi \end{bmatrix} \begin{bmatrix} \varphi \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tilde{u} \quad (40)$$

and the resulting combined feedback (34) is

$$\begin{aligned} u_1 &= -x\cos\varphi - y\sin\varphi \quad -p_2 \\ u_2 &= -\varphi \quad -p_1 \end{aligned} \quad (41)$$

The trajectories will converge to the invariant set $\varphi = 0$, $x = 0$, $p_1 = 0$, $p_2 = 0$. A different $\bar{V}$, however, will generally yield a different invariant set. □

Example 3.2 *(Rolling vertical wheel)* Let x, y be the Cartesian coordinates of the point of contact of the wheel with the plane, φ denotes heading angle, and θ rotation angle. With all constants set to unity, the Lagrangian equations of motion are

$$\begin{aligned} \ddot{x} &= \lambda_1 \\ \ddot{y} &= \lambda_2 \\ \ddot{\theta} &= -\lambda_1\cos\varphi - \lambda_2\sin\varphi + u_1 \\ \ddot{\varphi} &= u_2 \end{aligned} \quad (42)$$

with u_1 the control torque about the rolling axis and u_2 the control torque about the vertical axis. The nonholonomic constraints are (rolling without slipping)

$$\dot{x} = \dot{\theta}\cos\varphi, \quad \dot{y} = \dot{\theta}\sin\varphi \quad (43)$$

The total energy H is $\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_\theta^2 + \frac{1}{2}p_\varphi^2$, and the constraints can thus be rewritten as $p_x = p_\theta\cos\varphi, p_y = p_\theta\sin\varphi$. Define according to (17) new coordinates

$$\begin{aligned} p_1 &= p_\varphi \\ p_2 &= p_\theta + p_x \cos\varphi + p_y \sin\varphi \\ p_3 &= p_x - p_\theta \cos\varphi \\ p_4 &= p_y - p_\theta \sin\varphi \end{aligned} \tag{44}$$

Then $(x, y, \theta, \varphi, p_1, p_2)$ are coordinates for $\mathfrak{X}_r$, and the dynamics (21) is computed as (see also [15])

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\varphi} \\ \dot{p}_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} & & & & 0 & \cos\varphi \\ & 0_4 & & & 0 & \sin\varphi \\ & & & & 0 & 1 \\ & & & & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -\cos\varphi & -\sin\varphi & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_r}{\partial x} \\ \frac{\partial H_r}{\partial y} \\ \frac{\partial H_r}{\partial \theta} \\ \frac{\partial H_r}{\partial \varphi} \\ \frac{\partial H_r}{\partial p_1} \\ \frac{\partial H_r}{\partial p_2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{45}$$

with $H_r = \frac{1}{2}p_1^2 + \frac{1}{4}p_2^2$. A feedback (34) can be computed as in the preceeding example. Note that Example 2 (as well as Example 1) can be easily generalized to a knife edge or rolling wheel on *any* surface. This corresponds to adding a potential energy to H (and to H_r). Furthermore, in both examples one control torque instead of two control torques can be considered. □

4 Conclusions

We have shown, as an extension to [15], that the equations of motion of controlled mechanical systems with constraints may be directly formulated as Hamiltonian equations of motion with respect to a bracket which for nonholonomic constraints does not satisfy the Jacobi-identity, and with respect to a reduced Hamiltonian which is obtained by restricting the total energy to the constrained state space.

Like for ordinary Hamiltonian control systems a stabilization procedure has been proposed, based on the use of the reduced Hamiltonian as a candidate Lyapunov function. However, since Brockett's necessary condition is not satisfied, this will only result in Lyapunov *stability*, whereas asymptotic convergence is to a non-trivial invariant set. The main challenge is to investigate how the Hamiltonian structure may be used for *asymptotic* stabilization, in which case discontinuous or time-varying feedback is needed ([2], [4]).

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