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Research Article

A Hardy Inequality with Remainder Terms in the Heisenberg Group and the Weighted Eigenvalue Problem

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Based on properties of vector fields, we prove Hardy inequalities with remainder terms in the Heisenberg group and a compact embedding in weighted Sobolev spaces. The best constants in Hardy inequalities are determined. Then we discuss the existence of solutions for the nonlinear eigenvalue problems in the Heisenberg group with weights for the p-sub-Laplacian. The asymptotic behaviour, simplicity, and isolation of the first eigenvalue are also considered.

1. Introduction

Let

$$L_{p,\mu}u = -\Delta_{H,p}u - \mu\psi_p \frac{|u|^{p-2}u}{d^p}, \quad 0 \le \mu \le \left(\frac{Q-p}{p}\right)^p, \tag{1.1}$$

be the Hardy operator on the Heisenberg group. We consider the following weighted eigenvalue problem with a singular weight:

$$\begin{aligned} L_{p,\mu}u &= \lambda f(\xi)|u|^{p-2}u, \quad \text{in } \Omega \subset \mathbb{H}^n, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $1 < p < Q = 2n+2$, $\lambda \in \mathbb{R}$, $f(\xi) \in \mathcal{F}_p := \{f : \Omega \to \mathbb{R}^+ \mid \lim_{d(\xi)\to 0}(d^p(\xi)f(\xi)/(\psi_p(\xi)) = 0,\ f(\xi) \in L^\infty_{\text{loc}}(\Omega \setminus \{0\})\}$, Ω is a bounded domain in the Heisenberg group, and the definitions of $d(\xi)$ and $\psi_p(\xi)$; see below. We investigate the weak solution of (1.2) and the asymptotic behavior of the first eigenvalue for different singular weights as μ increases to $((Q-p)/p)^p$. Furthermore, we show that the first eigenvalue is simple and isolated, as well as the eigenfunctions corresponding to other eigenvalues change sign. Our proof is mainly based on a Hardy inequality with remainder terms. It is established by the vector field method and an elementary integral inequality. In addition, we show that the constants appearing in Hardy inequality are the best. Then we conclude a compact embedding in the weighted Sobolev space.

The main difficulty to study the properties of the first eigenvalue is the lack of regularity of the weak solutions of the p-sub-Laplacian in the Heisenberg group. Let us note that the C^α regularity for the weak solutions of the p-subelliptic operators formed by the vector field satisfying Hörmander's condition was given in [1] and the $C^{1,\alpha}$ regularity of the weak solutions of the p-sub-Laplacian $\Delta_{H,p}$ in the Heisenberg group for p near 2 was proved in [2]. To obtain results here, we employ the Picone identity and Harnack inequality to avoid effectively the use of the regularity.

The eigenvalue problems in the Euclidean space have been studied by many authors. We refer to [3–11]. These results depend usually on Hardy inequalities or improved Hardy inequalities (see [4, 12–14]).

Let us recall some elementary facts on the Heisenberg group (e.g., see [15]). Let $\mathbb{H}^n$ be a Heisenberg group endowed with the group law

$$\xi \circ \xi' = \left(x+x', y+y', t+t'+2\sum_{i=1}^{n}(x_i y_i' - x_i' y_i)\right), \tag{1.3}$$

where $\xi = (z,t) = (x,y,t) = (x_1,x_2,\ldots,x_n,y_1,\ldots,y_n,t)$, $z = (x,y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $t \in \mathbb{R}$, $n \geq 1$; $\xi' = (x',y',t') \in \mathbb{R}^{2n+1}$. This group multiplication endows $\mathbb{H}^n$ with a structure of nilpotent Lie group. A family of dilations on $\mathbb{H}^n$ is defined as

$$\delta_\tau(x,y,t) = (\tau x, \tau y, \tau^2 t), \quad \tau > 0. \tag{1.4}$$

The homogeneous dimension with respect to dilations is $Q = 2n+2$. The left invariant vector fields on the Heisenberg group have the form

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1,2,\ldots,n. \tag{1.5}$$

We denote the horizontal gradient by $\nabla_H = (X_1,\ldots,X_n,Y_1,\ldots,Y_n)$, and write $\operatorname{div}_H(v_1,v_2,\ldots,v_{2n}) = \sum_{i=1}^{n}(X_i v_i + Y_i v_{n+i})$. Hence, the sub-Laplacian Δ_H and the p-sub-Laplacian $\Delta_{H,p}$ are expressed by

$$\begin{gathered}\Delta_H = \sum_{i=1}^{n} X_i^2 + Y_i^2 = \nabla_H \cdot \nabla_H, \\ \Delta_{H,p} u = \nabla_H\left(\left|\nabla_H u\right|^{p-2} \nabla_H u\right) = \operatorname{div}_H\left(\left|\nabla_H u\right|^{p-2} \nabla_H u\right), \quad p > 1,\end{gathered} \tag{1.6}$$

respectively.

The distance function is

$$d(\xi,\xi') = \left\{\left[(x-x')^2 + (y-y')^2\right]^2 + \left[t-t'-2(x\cdot y' - x'\cdot y)\right]^2\right\}^{1/4}, \quad \text{for } \xi,\xi' \in \mathbb{H}^n. \tag{1.7}$$

If $\xi' = 0$, we denote

$$d(\xi) = d(\xi,0) = \left(|z|^4 + t^2\right)^{1/4}, \quad \text{with } |z| = \left(x^2 + y^2\right)^{1/2}. \tag{1.8}$$

Note that $d(\xi)$ is usually called the homogeneous norm.

For $d = d(\xi)$, it is easy to calculate

$$\nabla_H d = \frac{1}{d^3}\begin{pmatrix} |z|^2 x + yt \\ |z|^2 y - xt \end{pmatrix}, \qquad |\nabla_H d|^p = \frac{|z|^p}{d^p} = \psi_p, \qquad \Delta_{H,p} d = \psi_p \frac{Q-1}{d}. \tag{1.9}$$

Denote by $B_H(R) = \{\xi \in \mathbb{H}^n \mid d(\xi) < R\}$ the ball of radius R centered at the origin. Let $\Omega_1 = B_H(R_2) \backslash B_H(R_1)$ with $0 \le R_1 < R_2 \le \infty$ and $u(\xi) = v(d(\xi)) \in C^2(\Omega_1)$ be a radial function with respect to $d(\xi)$. Then

$$\Delta_{H,p} u = \psi_p |v'|^{p-2}\left[(p-1)v'' + \frac{Q-1}{d}v'\right]. \tag{1.10}$$

Let us recall the change of polar coordinates $(x,y,t) \to (\rho,\theta,\theta_1,\ldots,\theta_{2n-1})$ in [16]. If $u(\xi) = \psi_p(\xi)v(d(\xi))$, then

$$\int_{\Omega 1} u(\xi)d\xi = s_H \int_{R_1}^{R_2} \rho^{Q-1} v(\rho)d\rho, \tag{1.11}$$

where $s_H = \omega_n \int_0^\pi (\sin\theta)^{n-1+p/2} d\theta$, ω_n is the $2n$-Lebesgue measure of the unitary Euclidean sphere in $\mathbb{R}^{2n}$.

The Sobolev space in $\mathbb{H}^n$ is written by $D^{1,p}(\Omega) = \{u : \Omega \to \mathbb{R}; u, |\nabla_H u| \in L^p(\Omega)\}$. $D_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{D_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla_H u|^p d\xi\right)^{1/p}$.

In the sequel, we denote by c, c_1, C, and so forth some positive constants usually except special narrating.

This paper is organized as follows. In Section 2, we prove the Hardy inequality with remainder terms by the vector field method in the Heisenberg group. In Section 3, we discuss the optimality of the constants in the inequalities which is of its independent interest. In Section 4, we show some useful properties concerning the Hardy operator (1.1), and then check the existence of solutions of the eigenvalue problem (1.2) ($1 < p < Q$) and the asymptotic behavior of the first eigenvalue as μ increases to $((Q-p)/p)^p$. In Section 5, we study the simplicity and isolation of the first eigenvalue.

2. The Hardy inequality with remainder terms

D'Ambrosio in [17] has proved a Hardy inequality in the bounded domain $\Omega \subset \mathbb{H}^n$: let $p > 1$ and $p \neq Q$. For any $u \in D_0^{1,p}(\Omega, |z|^p/d^{2p})$, it holds that

$$C_{Q,p} \int_\Omega \psi_p \frac{|u|^p}{d^p} d\xi \le \int_\Omega |\nabla_H u|^p d\xi, \tag{2.1}$$

where $C_{Q,p} = |(Q-p)/p|^p$. Moreover, if $0 \in \Omega$, then the constant $C_{Q,p}$ is best. In this section, we give the Hardy inequality with remainder terms on Ω, based on the careful choice of a suitable vector field and an elementary integral inequality. Note that we also require that $0 \in \Omega$.

Theorem 2.1. *Let $u \in D_0^{1,p}(\Omega / \{0\})$. Then*

(1) *if $p \neq Q$ and there exists a positive constant M_0 such that* $\sup_{\xi\in\Omega} d(\xi)e^{1/M_0} := R_0 < \infty$, *then for any $R \geq R_0$,*

$$\int_\Omega |\nabla_H u|^p d\xi \geq \left|\frac{Q-p}{p}\right|^p \int_\Omega \psi_p \frac{|u|^p}{d^p} d\xi + \frac{p-1}{2p}\left|\frac{Q-p}{p}\right|^{p-2} \int_\Omega \psi_p \frac{|u|^p}{d^p}\left(\ln\left(\frac{R}{d}\right)\right)^{-2} d\xi; \tag{2.2}$$

moreover, if $2 \leq p < Q$, then choose $\sup_{\xi\in\Omega} d(\xi) = R_0$*;*

(2) *if $p = Q$ and there exits M_0 such that* $\sup_{\xi\in\Omega} d(\xi)e^{1/M_0} < R$, *then*

$$\int_\Omega |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \psi_p \frac{|u|^p}{(d\ln(R/d))^p} d\xi. \tag{2.3}$$

Before we prove the theorem, let us recall that

$$\Gamma(d(\xi)) = \begin{cases} d(\xi)^{(p-Q)/(p-1)} & \text{if } p \neq Q, \\ -\ln d(\xi) & \text{if } p = Q \end{cases} \tag{2.4}$$

is the solution of $\Delta_{H,p}$ at the origin, that is, $\Delta_{H,p}\Gamma(d(\xi)) = 0$ on $\Omega \setminus \{0\}$. Equation (2.4) is useful in our proof. For convenience, write $\mathcal{B}(s) = -1/\ln(s)$, $s \in (0,1)$, and $A = (Q-p)/p$. Thus, for some positive constant $M > 0$,

$$0 \leq \mathcal{B}\left(\frac{d(\xi)}{R}\right) \leq M, \quad \sup_{\xi\in\Omega} d(\xi) < R, \quad \xi \in \Omega. \tag{2.5}$$

Furthermore,

$$\nabla_H \mathcal{B}^\gamma\left(\frac{d}{R}\right) = \gamma\frac{\mathcal{B}^{\gamma+1}(d/R)\nabla_H d}{d}, \quad \frac{d\mathcal{B}^\gamma(\rho/R)}{d\rho} = \gamma\frac{\mathcal{B}^{\gamma+1}(\rho/R)}{\rho} \quad \forall \gamma \in \mathbb{R}, \tag{2.6}$$

$$\int_a^b \frac{\mathcal{B}^{\gamma+1}(s)}{s} ds = \frac{1}{\gamma}[\mathcal{B}^\gamma(b) - \mathcal{B}^\gamma(a)]. \tag{2.7}$$

Proof. Let T be a C^1 vector field on Ω and let it be specified later. For any $u \in C_0^\infty(\Omega \setminus \{0\})$, we use Hölder's inequality and Young's inequality to get

$$\begin{aligned}\int_\Omega (\operatorname{div}_H \mathrm{T})|u|^p d\xi &= -p\int_\Omega \langle \mathrm{T}, \nabla_H u\rangle |u|^{p-2}u\, d\xi \\ &\leq p\left(\int_\Omega |\nabla_H u|^p d\xi\right)^{1/p}\left(\int_\Omega |\mathrm{T}|^{p/(p-1)}|u|^p d\xi\right)^{(p-1)/p} \\ &\leq \int_\Omega |\nabla_H u|^p d\xi + (p-1)\int_\Omega |\mathrm{T}|^{p/(p-1)}|u|^p d\xi.\end{aligned} \tag{2.8}$$

Thus, the following elementary integral inequality:

$$\int_\Omega |\nabla_H u|^p d\xi \geq \int_\Omega [\mathrm{div}_H \mathrm{T} - (p-1)|\mathrm{T}|^{p/(p-1)}]|u|^p d\xi \tag{2.9}$$

holds.

(1) Let a be a free parameter to be chosen later. Denote

$$I_1(\mathcal{B}) = 1 + \frac{p-1}{pA}\mathcal{B}\left(\frac{d}{R}\right) + a\mathcal{B}^2\left(\frac{d}{R}\right), \qquad I_2(\mathcal{B}) = \frac{p-1}{pA}\mathcal{B}^2\left(\frac{d}{R}\right) + 2a\mathcal{B}^3\left(\frac{d}{R}\right), \tag{2.10}$$

and pick $\mathrm{T}(d) = A|A|^{p-2}(|\nabla_H d|^{p-2}\nabla_H d/d^{p-1})I_1$. An immediate computation shows

$$\begin{aligned}
&\mathrm{div}_H\left(A|A|^{p-2}\frac{|\nabla_H d|^{p-2}\nabla_H d}{d^{p-1}}\right) \\
&\quad = A|A|^{p-2}\frac{d\Delta_{H,p}d - (p-1)|\nabla_H d|^p}{d^p} \\
&\quad = A|A|^{p-2}\frac{(Q-1-p+1)|\nabla_H d|^p}{d^p} = p|A|^p\frac{|\nabla_H d|^p}{d^p}.
\end{aligned} \tag{2.11}$$

By (2.6),

$$\begin{aligned}
\mathrm{div}_H \mathrm{T} &= p|A|^p\frac{|\nabla_H d|^p}{d^p}I_1 + A|A|^{p-2}\frac{|\nabla_H d|^{p-2}\nabla_H d}{d^{p-1}} \\
&\qquad \times \frac{\nabla_H d}{d}\left[\frac{p-1}{pA}\mathcal{B}^2\left(\frac{d}{R}\right) + 2a\mathcal{B}^3\left(\frac{d}{R}\right)\right] \\
&= p|A|^p\frac{|\nabla_H d|^p}{d^p}I_1 + A|A|^{p-2}\frac{|\nabla_H d|^p}{d^p}I_2, \\
\mathrm{div}_H \mathrm{T} - (p-1)|\mathrm{T}|^{p/(p-1)} &= p|A|^p\frac{|\nabla_H d|^p}{d^p}I_1 + A|A|^{p-2}\frac{|\nabla_H d|^p}{d^p}I_2 \\
&\qquad - (p-1)|A|^p\frac{|\nabla_H d|^p}{d^p}I_1^{p/(p-1)} \\
&= |A|^p\frac{|\nabla_H d|^p}{d^p}\left(pI_1 + \frac{1}{A}I_2 - (p-1)I_1^{p/(p-1)}\right).
\end{aligned} \tag{2.12}$$

We claim

$$\mathrm{div}_H \mathrm{T} - (p-1)|\mathrm{T}|^{p/(p-1)} \geq |A|^p\frac{|\nabla_H d|^p}{d^p}\left(1 + \frac{p-1}{2pA^2}\mathcal{B}^2\left(\frac{d}{R}\right)\right). \tag{2.13}$$

In fact, arguing as in the proof of [13, Theorem 4.1], we set

$$f(s) := pI_1(s) + \frac{1}{A}I_2(s) - (p-1)I_1^{p/(p-1)}(s) \tag{2.14}$$

and $M = M(R) := \sup_{\xi\in\Omega}\mathcal{B}(d(\xi)/R)$, and distinguish three cases

(i)

$$1 < p < 2 < Q, \qquad a > \frac{(2-p)(p-1)}{6p^2A^2}, \tag{2.15}$$

(ii)

$$2 \le p < Q, \qquad a = 0, \tag{2.16}$$

(iii)

$$p > Q, \qquad a < \frac{(2-p)(p-1)}{6p^2A^2} < 0. \tag{2.17}$$

It yields that

$$f(s) \ge 1 + \frac{p-1}{2pA^2}s^2, \quad 0 \le s \le M, \tag{2.18}$$

(see [13]) and then follows (2.13). Hence (2.2) is proved.

(2) If $p = Q$, then we choose the vector field $\mathrm{T}(d) = ((p-1)/p)^{p-1}(|\nabla_H d|^{p-2}\nabla_H d/d^{p-1})\mathcal{B}^{p-1}(d/R)$. It gives

$$\begin{aligned}\mathrm{div}_H\mathrm{T} &= \left(\frac{p-1}{p}\right)^{p-1}\left\{\frac{[Q-1-(p-1)]\,|\nabla_H d|^p}{d^p}\mathcal{B}^{p-1}\left(\frac{d}{R}\right) + (p-1)\mathcal{B}^p\left(\frac{d}{R}\right)\frac{|\nabla_H d|^p}{d^p}\right\}\\ &= p\left(\frac{p-1}{p}\right)^p\mathcal{B}^p\left(\frac{d}{R}\right)\frac{|\nabla_H d|^p}{d^p},\end{aligned} \tag{2.19}$$

and hence

$$\mathrm{div}_H\mathrm{T} - (p-1)|\mathrm{T}|^{p/(p-1)} = \left(\frac{p-1}{p}\right)^p\mathcal{B}^p\left(\frac{d}{R}\right)\frac{|\nabla_H d|^p}{d^p}. \tag{2.20}$$

Combining (2.20) with (2.9) follows (2.3). □

Remark 2.2. The domain Ω in (2.9) may be bounded or unbounded. In addition, if we select that $\mathrm{T}(d) = A|A|^{p-2}(|\nabla_H d|^{p-2}\nabla_H d/d^{p-1})$, then

$$\mathrm{div}_H\mathrm{T} - (p-1)|\mathrm{T}|^{p/(p-1)} = p|A|^p\frac{|\nabla_H d|^p}{d^p} - (p-1)|A|^p\frac{|\nabla_H d|^p}{d^p} = |A|^p\frac{|\nabla_H d|^p}{d^p}. \tag{2.21}$$

Hence, from (2.9) we conclude (2.1) on the bounded domain Ω and on $\mathbb{H}^n$ (see [15]), respectively.

We will prove in next section that the constants in (2.2) and (2.3) are best.

Now, we state the Poincaré inequality proved in [17].

Lemma 2.3. *Let Ω be a subset of $\mathbb{H}^n$ bounded in x_1 direction, that is, there exists $R > 0$ such that $0 < r = |x_1| \le R$ for $\xi = (x_1, x_2, \ldots, x_n, y_1, \ldots, y_n, t) \in \Omega$. Then for any $u \in D_0^{1,p}(\Omega)$, then*

$$c\int_{\Omega} |u|^p d\xi \le \int_{\Omega} |\nabla_H u|^p d\xi, \tag{2.22}$$

where $c = ((p-1)/pR)^p$.

Using (2.9) by choosing $\mathrm{T} = -((p-1)/p)^{p-1}(\nabla_H r/r^{p-1})$ immediately provides a different proof to (2.22).

In the following, we describe a compactness result by using (2.1) and (2.22).

Theorem 2.4. *Suppose $p \neq Q$ and $f(\xi) \in \mathcal{F}_p$. Then there exists a positive constant $C_{f,Q,p}$ such that*

$$C_{f,Q,p}\int_{\Omega} f(\xi)|u|^p d\xi \le \int_{\Omega} |\nabla_H u|^p d\xi, \tag{2.23}$$

and the embedding $D_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f\,d\xi)$ is compact.

Proof. Since $f(\xi) \in \mathcal{F}_p$, we have that for any $\epsilon > 0$, there exist $\delta > 0$ and $C_\delta > 0$ such that

$$\sup_{B_H(\delta)\subseteq\Omega} \frac{d^p}{\psi_p} f(\xi) \le \epsilon, \qquad f(\xi)|_{\Omega\setminus B_H(\delta)} \le C_\delta. \tag{2.24}$$

By (2.1) and (2.22), it follows

$$\begin{aligned}\int_{\Omega} f(\xi)|u|^p d\xi &= \int_{B_H(\delta)} f|u|^p d\xi + \int_{\Omega\setminus B_H(\delta)} f|u|^p d\xi \\ &\le \epsilon\int_{B_H(\delta)} \psi_p \frac{|u|^p}{d^p} d\xi + C_\delta \int_{\Omega\setminus B_H(\delta)} |u|^p d\xi \le C_{f,Q,p}^{-1}\int_{\Omega} |\nabla_H u|^p d\xi,\end{aligned} \tag{2.25}$$

then (2.23) is obtained.

Now, we prove the compactness. Let $\{u_m\} \subseteq D_0^{1,p}(\Omega)$ be a bounded sequence. By reflexivity of the space $D_0^{1,p}(\Omega)$ and the Sobolev embedding for vector fields (see [18]), it yields

$$\begin{aligned} u_{m_j} &\rightharpoonup u \text{ weakly in } D_0^{1,p}(\Omega), \\ u_{m_j} &\longrightarrow u \text{ strongly in } L^p(\Omega)\end{aligned} \tag{2.26}$$

for a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ as $j \to \infty$. Write $C_\delta = \|f\|_{L^\infty(\Omega\setminus B_H(\delta))}$. From (2.1),

$$\begin{aligned}\int_{\Omega} f|u_{m_j} - u|^p d\xi &= \int_{B_H(\delta)} f|u_{m_j} - u|^p d\xi + \int_{\Omega\setminus B_H(\delta)} f|u_{m_j} - u|^p d\xi \\ &\le \epsilon\int_{B_H(\delta)} \psi_p \frac{|u_{m_j} - u|^p}{d^p} d\xi + C_\delta\int_{\Omega\setminus B_H(\delta)} |u_{m_j} - u|^p d\xi \\ &\le \epsilon C_{Q,p}^{-1}\int_{\Omega} |\nabla_H(u_{m_j} - u)|^p d\xi + C_\delta\int_{\Omega} |u_{m_j} - u|^p d\xi.\end{aligned} \tag{2.27}$$

Since $\{u_m\} \subseteq D_0^{1,p}(\Omega)$ is bounded, we have

$$\int_\Omega f|u_{m_j} - u|^p d\xi \leq \epsilon M + C_\delta \int_\Omega |u_{m_j} - u|^p d\xi, \tag{2.28}$$

where $M > 0$ is a constant depending on Q and p. By (2.26),

$$\lim_{j\to\infty} \int_\Omega f|u_{m_j} - u|^p d\xi \leq \epsilon M. \tag{2.29}$$

As ϵ is arbitrary, $\lim_{j\to\infty}\int_\Omega f|u_{m_j} - u|^p d\xi = 0$. Hence $D_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f\,d\xi)$ is compact. □

Remark 2.5. The class of the functions $f(\xi) \in \mathcal{F}_p$ has lower-order singularity than $d^{-p}(\xi)$ at the origin. The examples of such functions are

(a) any bounded function,

(b) in a small neighborhood of 0, $f(\xi) = \psi_p(\xi)/d^\beta(\xi)$, $0 < \beta < p$,

(c) $f(\xi) = \psi_p(\xi)/d^p(\xi)(\ln(1/d(\xi)))^2$ in a small neighborhood of 0.

3. Proof of best constants in (2.2) and (2.3)

In this section, we prove that the constants appearing in Theorem 2.1 are the best. To do this, we need two lemmas. First we introduce some notations.

For some fixed small $\delta > 0$, let the test function $\varphi(\xi) \in C_0^\infty(\Omega)$ satisfy $0 \leq \varphi \leq 1$ and

$$\varphi(\xi) = \begin{cases} 1 & \text{if } \xi \in B_H\left(0, \dfrac{\delta}{2}\right), \\ 0 & \text{if } \xi \in \Omega \setminus B_H(0,\delta), \end{cases} \tag{3.1}$$

with $|\nabla_H \varphi| < 2|\nabla_H d|/d$. Let $\epsilon > 0$ small enough, and define

$$\begin{aligned} V_\epsilon(\xi) &= \varphi(\xi)\varpi_\epsilon, \quad \text{with } \varpi_\epsilon = d^{-A+\epsilon}\mathcal{B}^{-\kappa}\left(\frac{d}{R}\right), \quad \frac{1}{p} < \kappa < \frac{2}{p}, \\ J_\gamma(\epsilon) &= \int_\Omega \varphi^p(\xi)\frac{|\nabla_H d|^p}{d^{Q-p\epsilon}}\mathcal{B}^{-\gamma}\left(\frac{d}{R}\right)d\xi, \quad \gamma \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Lemma 3.1. *For $\epsilon > 0$ small, it holds*

(i) $c\epsilon^{-1-\gamma} \leq J_\gamma(\epsilon) \leq C\epsilon^{-1-\gamma}$, $\gamma > -1$,

(ii) $J_\gamma(\epsilon) = (p\epsilon/(\gamma+1))J_{\gamma+1}(\epsilon) + O_\epsilon(1)$, $\gamma > -1$,

(iii) $J_\gamma(\epsilon) = O_\epsilon(1)$, $\gamma < -1$.

Proof. By the change of polar coordinates (1.11) and $0 \leq \varphi \leq 1$, we have

$$\begin{aligned} J_\gamma(\epsilon) &\leq \int_{B_H(\delta)} \frac{|\nabla_H d|^p}{d^{Q-p\epsilon}}\mathcal{B}^{-\gamma}\left(\frac{d}{R}\right)d\xi = s_H\int_{\rho<\delta}\rho^{-Q+p\epsilon}\mathcal{B}^{-\gamma}\left(\frac{\rho}{R}\right)\rho^{Q-1}d\rho \\ &= s_H\int_{\rho<\delta}\rho^{-1+p\epsilon}\mathcal{B}^{-\gamma}\left(\frac{\rho}{R}\right)d\rho. \end{aligned} \tag{3.3}$$

By (2.7) we know that for $\gamma < -1$ the right-hand side of (3.3) has a finite limit, hence (iii) follows from $\epsilon \to 0$.

To show (i), we set $\rho = R\tau^{1/\epsilon}$. Thus, $d\rho = (1/\epsilon)R\tau^{1/\epsilon-1}d\tau$, $\mathcal{B}^{-\gamma}(\tau^{1/\epsilon}) = \epsilon^{-\gamma}\mathcal{B}^{-\gamma}(\tau)$, and

$$\begin{aligned} J_\gamma(\epsilon) &\leq s_H \int_{\rho<\delta} \rho^{-1+p\epsilon}\mathcal{B}^{-\gamma}\left(\frac{\rho}{R}\right)d\rho = s_H \int_0^{(\delta/R)^\epsilon} \left(R\tau^{1/\epsilon}\right)^{-1+p\epsilon}\mathcal{B}^{-\gamma}\left(\frac{R\tau^{1/\epsilon}}{R}\right)\frac{1}{\epsilon}R\tau^{1/\epsilon-1}d\tau \\ &= s_H R^{p\epsilon}\epsilon^{-1-\gamma}\int_0^{(\delta/R)^\epsilon} \tau^{p-1}\mathcal{B}^{-\gamma}(\tau)d\tau. \end{aligned} \tag{3.4}$$

It follows the right-hand side of (i). Using the fact that $\varphi = 1$ in $B_H(\delta/2)$,

$$J_\gamma(\epsilon) \geq \int_{B_H(\delta/2)} \frac{|\nabla_H d|^p}{d^{Q-p\epsilon}}\mathcal{B}^{-\gamma}\left(\frac{d}{R}\right)d\xi = s_H R^{p\epsilon}\epsilon^{-1-\gamma}\int_0^{(\delta/2R)^\epsilon} \tau^{p-1}\mathcal{B}^{-\gamma}(\tau)d\tau, \tag{3.5}$$

and the left-hand side of (i) is proved.

Now we prove (ii). Let $\Omega_\eta := \{\xi \in \Omega \mid d(\xi) > \eta\}$, $\eta > 0$, be small and note the boundary term

$$-\int_{d=\eta}\left(\frac{\varphi^p|\nabla_H d|^{p-2}\nabla_H d}{d^{Q-1-p\epsilon}}\right)\mathcal{B}^{-\gamma-1}\left(\frac{d}{R}\right)\nabla_H d\cdot\vec{n}dS \longrightarrow 0 \quad \text{as } \eta \longrightarrow 0. \tag{3.6}$$

From (2.6),

$$\begin{aligned} &\int_\Omega \operatorname{div}_H\left(\frac{\varphi^p|\nabla_H d|^{p-2}\nabla_H d}{d^{Q-1-p\epsilon}}\right)\mathcal{B}^{-\gamma-1}\left(\frac{d}{R}\right)d\xi \\ &\quad = -\int_\Omega \frac{\varphi^p|\nabla_H d|^{p-2}}{d^{Q-1-p\epsilon}}\left\langle\nabla_H d, \nabla_H\mathcal{B}^{-\gamma-1}\left(\frac{d}{R}\right)\right\rangle d\xi \\ &\quad = (\gamma+1)\int_\Omega \frac{\varphi^p|\nabla_H d|^p}{d^{Q-p\epsilon}}\mathcal{B}^{-\gamma}\left(\frac{d}{R}\right)d\xi = (\gamma+1)J_\gamma(\epsilon). \end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned} &\int_\Omega \operatorname{div}_H\left(\frac{\varphi^p|\nabla_H d|^{p-2}\nabla_H d}{d^{Q-1-p\epsilon}}\right)\mathcal{B}^{-\gamma-1}\left(\frac{d}{R}\right)d\xi \\ &\quad = p\int_\Omega \varphi^{p-1}\frac{|\nabla_H d|^{p-2}\langle\nabla_H d, \nabla_H\varphi\rangle}{d^{Q-1-p\epsilon}}\mathcal{B}^{-\gamma-1}\left(\frac{d}{R}\right)d\xi \\ &\qquad + (1-Q+p\epsilon+Q-1)\int_\Omega \varphi^p\frac{|\nabla_H d|^p}{d^{Q-p\epsilon}}\mathcal{B}^{-\gamma-1}\left(\frac{d}{R}\right)d\xi \\ &\quad = p\int_\Omega \varphi^{p-1}\frac{|\nabla_H d|^{p-2}\langle\nabla_H d, \nabla_H\varphi\rangle}{d^{Q-1-p\epsilon}}\mathcal{B}^{-\gamma-1}\left(\frac{d}{R}\right)d\xi + p\epsilon J_{\gamma+1}(\epsilon). \end{aligned} \tag{3.8}$$

We claim that $p\int_\Omega \varphi^{p-1}(|\nabla_H d|^{p-2}\langle\nabla_H d,\nabla_H\varphi\rangle/d^{Q-1-p\epsilon})\mathcal{B}^{-\gamma-1}(d/R)d\xi = O_\epsilon(1)$. In fact, by (3.1) and (1.11),

$$\begin{aligned}\int_\Omega \varphi^{p-1}\frac{|\nabla_H d|^{p-2}\langle\nabla_H d,\nabla_H\varphi\rangle}{d^{Q-1-p\epsilon}}\mathcal{B}^{-\gamma-1}\left(\frac{d}{R}\right)d\xi &\\ \le 2\int_{B_H(\delta)}\frac{|\nabla_H d|^p}{d^{Q-p\epsilon}}\mathcal{B}^{-\gamma-1}\left(\frac{d}{R}\right)d\xi &\\ \le 2s_H\int_{B_H(\delta)}\rho^{-Q+p\epsilon}\mathcal{B}^{-\gamma-1}\left(\frac{\rho}{R}\right)\rho^{Q-1}d\rho &\\ = 2s_H\int_{B_H(\delta)}\rho^{p\epsilon-1}\mathcal{B}^{-\gamma-1}\left(\frac{\rho}{R}\right)d\rho. &\end{aligned} \tag{3.9}$$

Using the estimate (i) follows that $\int_{B_H(\delta)}\rho^{p\epsilon-1}\mathcal{B}^{-\gamma-1}(\rho/R)d\rho = O_\epsilon(1)$. Combining (3.7) with (3.8) gives

$$(\gamma+1)J_\gamma(\epsilon) = p\epsilon J_{\gamma+1}(\epsilon)+O_\epsilon(1). \tag{3.10}$$

This allows us to conclude (ii). □

We next estimate the quantity

$$I[V_\epsilon] = \int_\Omega |\nabla_H V_\epsilon|^p d\xi - |A|^p\int_\Omega |\nabla_H d|^p\frac{|V_\epsilon|^p}{d^p}d\xi. \tag{3.11}$$

Lemma 3.2. *As $\epsilon\to 0$, it holds*

(i) $I(V_\epsilon) \le (\kappa(p-1)/2)|A|^{p-2}J_{p\kappa-2}(\epsilon)+O_\epsilon(1)$;

(ii) $\int_{B_H(\delta)}|\nabla_H V_\epsilon|^p d\xi \le |A|^p J_{p\kappa}(\epsilon)+O_\epsilon(\epsilon^{1-p\kappa})$.

Proof. By the definition of $V_\epsilon(\xi)$, we see $\nabla_H V_\epsilon(\xi) = \varphi(\xi)\nabla_H\varpi_\epsilon + \varpi_\epsilon\nabla_H\varphi$. Using the elementary inequality

$$|a+b|^p \le |a|^p + c_p\left(|a|^{p-1}|b|+|b|^p\right), \quad a,b\in\mathbb{R}^{2n},\ p>1, \tag{3.12}$$

one has

$$\begin{aligned}|\nabla_H V_\epsilon|^p &\le \varphi^p|\nabla_H\varpi_\epsilon|^p + c_p\left(|\nabla_H\varphi|\varpi_\epsilon\varphi^{p-1}|\nabla_H\varpi_\epsilon|^{p-1}+|\nabla_H\varphi|^p|\varpi_\epsilon|^p\right)\\ &\le \varphi^p|\nabla_H\varpi_\epsilon|^p + c_p\left(\frac{|2\nabla_H d|}{d}\varpi_\epsilon\varphi^{p-1}|\nabla_H\varpi_\epsilon|^{p-1}+\left(\frac{|2\nabla_H d|}{d}\right)^p|\varpi_\epsilon|^p\right).\end{aligned} \tag{3.13}$$

Since $\nabla_H \varpi_\epsilon = -d^{-A+\epsilon-1}\mathcal{B}^{-\kappa}(d/R)(A-\epsilon+\kappa\mathcal{B}(d/R))\nabla_H d$, it follows

$$\begin{aligned}\int_\Omega |\nabla_H V_\epsilon|^p d\xi \le & \int_{B_H(\delta)} |\nabla_H d|^p \varphi^p d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\left|A-\left(\epsilon-\kappa\mathcal{B}\left(\frac{d}{R}\right)\right)\right|^p d\xi \\ & + 2c_p\int_{B_H(\delta)} |\nabla_H d|^p \varphi^{p-1} d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\left|A-\left(\epsilon-\kappa\mathcal{B}\left(\frac{d}{R}\right)\right)\right|^{p-1} d\xi \\ & + 2^p c_p\int_{B_H(\delta)} |\nabla_H d|^p d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)d\xi := \Pi_A+\Pi_1+\Pi_2.\end{aligned} \tag{3.14}$$

We claim that

$$\Pi_1, \Pi_2 = O_\epsilon(1). \tag{3.15}$$

Indeed, since $|A-(\epsilon-\kappa\mathcal{B}(d/R))|$ is bounded, using (3.1) we get

$$\begin{aligned}\Pi_1 &\le C\int_{B_H(\delta)} |\nabla_H d|^p \varphi^{p-1} d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\left|A-\left(\epsilon-\kappa\mathcal{B}\left(\frac{d}{R}\right)\right)\right|^{p-1} d\xi \\ &\le C\int_{B_H(\delta)} |\nabla_H d|^p d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)d\xi, \\ \Pi_2 &\le C\int_{B_H(\delta)} |\nabla_H d|^p d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)d\xi.\end{aligned} \tag{3.16}$$

By (i) of Lemma 3.1, it derives Π_1, $\Pi_2 = O_\epsilon(1)$, as $\epsilon \to 0$.

From (3.14), (3.15) and the definition of $J_\gamma(\epsilon)$, it clearly shows

$$I[V_\epsilon] = \int_{B_H(\delta)} |\nabla_H V_\epsilon|^p d\xi - |A|^p J_{p\kappa}(\epsilon) \le \Pi_A - |A|^p J_{p\kappa}(\epsilon) + O_\epsilon(1) = \Pi_3 + O_\epsilon(1), \tag{3.17}$$

where $\Pi_3 = \int_{B_H(\delta)} |\nabla_H d|^p \varphi^p d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}(d/R)(|A-(\epsilon-\kappa\mathcal{B}(d/R))|^p - |A|^p)d\xi$. For simplicity, denote $\zeta = \epsilon - \kappa\mathcal{B}(d/R)$. Since ζ is small compared to A, we use Taylor's expansion to yield

$$|A-\zeta|^p - |A|^p \le -pA|A|^{p-2}\zeta + \frac{p(p-1)}{2}|A|^{p-2}\zeta^2 + C|\zeta|^3. \tag{3.18}$$

Thus, we can estimate Π_3 by

$$\Pi_3 \le \Pi_{31} + \Pi_{32} + \Pi_{33}, \tag{3.19}$$

where

$$\begin{aligned}\Pi_{31} &= -pA|A|^{p-2}\int_{B_H(\delta)} |\nabla_H d|^p \varphi^p d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\left(\epsilon-\kappa\mathcal{B}\left(\frac{d}{R}\right)\right)d\xi, \\ \Pi_{32} &= \frac{p(p-1)}{2}|A|^{p-2}\int_{B_H(\delta)} |\nabla_H d|^p \varphi^p d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\left(\epsilon-\kappa\mathcal{B}\left(\frac{d}{R}\right)\right)^2 d\xi, \\ \Pi_{33} &= C\int_{B_H(\delta)} |\nabla_H d|^p \varphi^p d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\left|\epsilon-\kappa\mathcal{B}\left(\frac{d}{R}\right)\right|^3 d\xi.\end{aligned} \tag{3.20}$$

We will show that

$$\Pi_{31}, \Pi_{33} = O_\epsilon(1), \quad \text{as } \epsilon \longrightarrow 0. \tag{3.21}$$

In fact, using (ii) of Lemma 3.1 with $\gamma = -1 + p\kappa$ yields

$$\begin{aligned}\Pi_{31} &= -pA|A|^{p-2}(\epsilon J_{p\kappa}(\epsilon) - \kappa J_{p\kappa-1}(\epsilon)) \\ &= -pA|A|^{p-2}(\epsilon J_{p\kappa}(\epsilon) - \epsilon J_{p\kappa}(\epsilon) + O_\epsilon(1)) = O_\epsilon(1).\end{aligned} \tag{3.22}$$

Recalling $(a-b)^3 \le (|a|+|b|)^3 \le c(|a|^3+|b|^3)$, we obtain

$$\Pi_{33} \le c\epsilon^3 J_{p\kappa}(\epsilon) + cJ_{p\kappa-3}(\epsilon) \quad \text{for } \epsilon > 0. \tag{3.23}$$

From (i) and (iii) in Lemma 3.1 and the fact that $1 < p\kappa < 2$ it follows $\Pi_{33} = O_\epsilon(1)$. Using (ii) of Lemma 3.1 twice (pick $\gamma = p\kappa - 1 > -1$ and $\gamma = p\kappa - 2 > -1$, resp.), we conclude that

$$\begin{aligned}\Pi_{32} &= \frac{p(p-1)}{2}|A|^{p-2}\int_{B_H(\delta)} |\nabla_H d|^p \varphi^p d^{-Q+p\epsilon}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\left(\epsilon^2 - 2\epsilon\kappa\mathcal{B}\left(\frac{d}{R}\right) + \kappa^2\mathcal{B}^2\left(\frac{d}{R}\right)\right)d\xi \\ &= \frac{p(p-1)}{2}|A|^{p-2}\left[\epsilon^2 J_{p\kappa}(\epsilon) - 2\epsilon\kappa J_{p\kappa-1}(\epsilon) + \kappa^2\left(\frac{p\kappa-1}{p\kappa} + \frac{1}{p\kappa}\right)J_{p\kappa-2}(\epsilon)\right] \\ &= \frac{p(p-1)}{2}|A|^{p-2}\Bigg(\epsilon^2 J_{p\kappa}(\epsilon) - \epsilon\kappa\frac{p\epsilon}{p\kappa}J_{p\kappa}(\epsilon) - \epsilon\kappa J_{p\kappa-1}(\epsilon) \\ &\qquad + \frac{\kappa^2(p\kappa-1)}{p\kappa}\frac{p\epsilon}{p\kappa-1}J_{p\kappa-1}(\epsilon) + \frac{\kappa}{p}J_{p\kappa-2}(\epsilon) + O_\epsilon(1)\Bigg) \\ &= \frac{\kappa(p-1)}{2}|A|^{p-2}J_{p\kappa-2}(\epsilon) + O_\epsilon(1).\end{aligned} \tag{3.24}$$

In virtue of (3.17), (3.19), (3.21), and (3.24) we deduce (i) of Lemma 3.2. By (3.17), (3.24), and (i) of Lemma 3.2,

$$\int_{B_H(\delta)} |\nabla_H V_\epsilon|^p d\xi = I[V_\epsilon] + |A|^p J_{p\kappa}(\epsilon) \le |A|^p J_{p\kappa}(\epsilon) + \frac{\kappa(p-1)}{2}|A|^{p-2}J_{p\kappa-2}(\epsilon) + O_\epsilon(1). \tag{3.25}$$

Hence (ii) of Lemma 3.2 follows from (i) in Lemma 3.1. It completes the proof. □

We are now ready to give the proof of the best constants in Theorem 2.1.

Theorem 3.3. *Let $0 \in \Omega$ be a bounded domain in $\mathbb{H}^n$ and $p \ne Q$. Suppose that for some constants $B > 0$, $D \ge 0$, and $\iota > 0$, the following inequality holds for any $u(\xi) \in C_0^\infty(\Omega \setminus \{0\})$:*

$$\int_\Omega |\nabla_H u|^p d\xi \ge B\int_\Omega \psi_p \frac{|u|^p}{d^p}d\xi + D\int_\Omega \psi_p\frac{|u|^p}{d^p}\mathcal{B}^\iota\left(\frac{d}{R}\right)d\xi. \tag{3.26}$$

Then,

(i) $B \leq |A|^p$;

(ii) *if* $B = |A|^p$ *and* $D > 0$, *then* $\iota \geq 2$;

(iii) *if* $B = |A|^p$ *and* $\iota = 2$, *then* $D \leq ((p-1)/2p)|A|^{p-2}$.

Proof. Choose $u(\xi) = V_\epsilon(\xi)$.

(i) By (ii) of Lemma 3.2, we have

$$\begin{aligned} B &\leq \frac{\int_{B_H(\delta)} |\nabla_H V_\epsilon|^p d\xi}{\int_{B_H(\delta)} \psi_p(|V_\epsilon|^p/d^p)d\xi} \leq \frac{|A|^p J_{p\kappa}(\epsilon) + O_\epsilon(\epsilon^{1-p\kappa})}{\int_{B_H(\delta)} \psi_p(|\varphi d^{-A+\epsilon}\mathcal{B}^{-\kappa}(d/R)|^p/d^p)d\xi} \\ &= \frac{|A|^p(1+c\epsilon^2)J_{p\kappa}(\epsilon) + O_\epsilon(1)}{J_{p\kappa}(\epsilon)}. \end{aligned} \tag{3.27}$$

Note that $J_{p\kappa}(\epsilon) \to \infty$, as $\epsilon \to 0$, so $B \leq |A|^p$.

(ii) Set $B = |A|^p$ and assume by contradiction that $\iota < 2$. Since $p\kappa - \iota > -1$, using (i) of Lemma 3.2 and (i) of Lemma 3.1 leads to

$$\begin{aligned} 0 < D &\leq \frac{I(V_\epsilon)}{\int_\Omega \psi_p(|V_\epsilon|^p/d^p)\mathcal{B}^\iota(d/R)d\xi} = \frac{I(V_\epsilon)}{J_{p\kappa-\iota}(\epsilon)} \leq \frac{(\kappa(p-1)/2)|A|^{p-2}J_{p\kappa-2}(\epsilon) + O_\epsilon(1)}{J_{p\kappa-\iota}(\epsilon)} \\ &\leq \frac{C\epsilon^{1-p\kappa}}{c\epsilon^{-1-p\kappa+\iota}} = C\epsilon^{2-\iota} \longrightarrow 0, \quad \text{as } \epsilon \longrightarrow 0, \end{aligned} \tag{3.28}$$

which is a contradiction. Hence $\iota \geq 2$.

(iii) If $B = |A|^p$ and $\iota = 2$, then by (i) of Lemma 3.2,

$$D \leq \frac{I(V_\epsilon)}{J_{p\kappa-2}(\epsilon)} \leq \frac{(\kappa(p-1)/2)|A|^{p-2}J_{p\kappa-2}(\epsilon) + O_\epsilon(1)}{J_{p\kappa-2}(\epsilon)}. \tag{3.29}$$

The assumption $\kappa > 1/p$ implies $J_{p\kappa-2}(\epsilon) \to \infty$, as $\epsilon \to 0$. Hence, by (i) of Lemma 3.1 we conclude that $D \leq (\kappa(p-1)/2)|A|^{p-2}$, as $\epsilon \to 0$. Then letting $\kappa \to 1/p$, the proof is finished. □

Theorem 3.4. *Set* $0 \in \Omega$ *and* $p = Q$. *Suppose that there exist some constants* $D \geq 0$ *and* $\iota > 0$ *such that the following inequality holds for all* $u(\xi) \in C_0^\infty(\Omega \setminus \{0\})$:

$$\int_\Omega |\nabla_H u|^p d\xi \geq D \int_\Omega \psi_p \frac{|u|^p}{d^p} \mathcal{B}^\iota\left(\frac{d}{R}\right) d\xi. \tag{3.30}$$

Then,

(i) *if* $D > 0$, *then* $\iota \geq p$;

(ii) *if* $\iota = p$, *then* $D \leq ((p-1)/p)^p$.

Proof. The proof is essentially similar to one of Theorem 3.3. Let the test function φ be as before (see (3.1)). For $\epsilon > 0$, $\kappa > (p-1)/p$, define $V_\epsilon = \varphi \varpi_\epsilon$ with $\varpi_\epsilon = d^\epsilon(-\ln(d/R))^\kappa$.

Using (3.12) yields

$$\begin{aligned}\int_{\Omega}|\nabla_H V_\epsilon|^p d\xi &= \int_{B_H(\delta)}|\varphi\nabla_H\varpi_\epsilon+\varpi_\epsilon\nabla_H\varphi|^p d\xi \\ &\le \int_{B_H(\delta)}\varphi^p|\nabla_H\varpi_\epsilon|^p d\xi + c_p\int_{B_H(\delta)}\varphi^{p-1}\varpi_\epsilon|\nabla_H\varpi_\epsilon|^{p-1}|\nabla_H\varphi|d\xi \\ &\quad + c_p\int_{B_H(\delta)}\varpi_\epsilon^p|\nabla_H\varphi|^p d\xi := \Pi_4+\Pi_5+\Pi_6.\end{aligned} \tag{3.31}$$

Arguing as in the proof of previous theorem and letting $\epsilon\to 0$, we have

$$\Pi_5,\ \Pi_6 = O_\epsilon(1). \tag{3.32}$$

Denoting by c_i^p the coefficients of binormial expansion, we get

$$\begin{aligned}|\nabla_H\varpi_\epsilon|^p &= |\nabla_H d|^p d^{p\epsilon-p}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\left|\epsilon-\kappa\mathcal{B}\left(\frac{d}{R}\right)\right|^p \\ &\le |\nabla_H d|^p d^{p\epsilon-p}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\left(\epsilon+\kappa\mathcal{B}\left(\frac{d}{R}\right)\right)^p \\ &= |\nabla_H d|^p d^{p\epsilon-p}\mathcal{B}^{-p\kappa}\left(\frac{d}{R}\right)\Sigma_{i=0}^p c_i^p\epsilon^{p-i}\kappa^i\mathcal{B}^i\left(\frac{d}{R}\right).\end{aligned} \tag{3.33}$$

Hence,

$$\Pi_4 \le \Sigma_{i=0}^p c_i^p\epsilon^{p-i}\kappa^i J_{p\kappa-i}(\epsilon), \tag{3.34}$$

where $J_\gamma(\epsilon)=\int_\Omega|\nabla_H d|^p\varphi^p d^{p\epsilon-p}\mathcal{B}^{-\gamma}(d/R)$. By (ii) of Lemma 3.1 and the induction argument it holds

$$\epsilon^{p-i}J_{p\kappa-i}(\epsilon)=\left(\kappa-\frac{i}{p}\right)\left(\kappa-\frac{i+1}{p}\right)\cdots\left(\kappa-\frac{p-1}{p}\right)J_{p\kappa-p}(\epsilon)+O_\epsilon(1),\quad i=0,1,\dots,p-1. \tag{3.35}$$

Now (i) of Lemma 3.1 and the assumption $\kappa>(p-1)/p$ imply that $J_{p\kappa-p}(\epsilon)\to\infty$, as $\epsilon\to 0$, and

$$\begin{aligned}D &\le \limsup_{\epsilon\to 0}\frac{\int_\Omega|\nabla_H V_\epsilon|^p d\xi}{\int_\Omega\psi_p(|V_\epsilon|^p/d^p)\mathcal{B}^p(d/R)d\xi} \\ &\le \limsup_{\epsilon\to 0}\frac{[\kappa^p+\Sigma_{i=0}^{p-1}c_i^p\kappa^i(\kappa-i/p)(\kappa-(i+1)/p)\cdots(\kappa-(p-1)/p)]J_{p\kappa-p}(\epsilon)+O_\epsilon(1)}{J_{p\kappa-p}(\epsilon)} \\ &= \left[\kappa^p+\Sigma_{i=0}^{p-1}c_i^p\kappa^i\left(\kappa-\frac{i}{p}\right)\left(\kappa-\frac{i+1}{p}\right)\cdots\left(\kappa-\frac{p-1}{p}\right)\right].\end{aligned} \tag{3.36}$$

The last expression converges to $((p-1)/p)^p$ as $\kappa\to(p-1)/p$. The proof is over. □

4. The weighted eigenvalue problem

This section is devoted to the problem (1.2) by using the Hardy inequality with remainder terms.

We begin with some properties concerning the Hardy operator (1.1).

Lemma 4.1. *Suppose that $u(\xi) \in D_0^{1,p}(\Omega)$ and $p \neq Q$. Then*

(1) *$L_{p,\mu}$ is a positive operator if $\mu \leq C_{Q,p}$; in particular, if $\mu = C_{Q,p}$, then $v(\xi) = d^{(p-Q)/p}(\xi)$ is a solution of $L_{p,\mu}u = 0$;*

(2) *$L_{p,\mu}$ is unbounded from below if $\mu > C_{Q,p}$.*

Proof. (1) It is obvious from (2.1) that $L_{p,\mu}$ is a positive operator.

We now suppose that $\mu = C_{Q,p}$ and verify that $v = d^{(p-Q)/p}$ satisfies $L_{p,\mu}u = 0$. For the purpose, set $v_\epsilon = d^{(p-Q)/p+\epsilon} \in D_0^{1,p}(\Omega)$ and $A = (Q-p)/p$. Since

$$v'_\epsilon = (\epsilon - A)d^{-A-1+\epsilon}, \qquad v''_\epsilon = (\epsilon - A)(\epsilon - A - 1)d^{-A-2+\epsilon}, \tag{4.1}$$

it yields from (1.10) and (4.1) that

$$\begin{aligned}
-\Delta_{H,p}v_\epsilon &= -\psi_p\,|\,v'_\epsilon\,|^{p-2}\left[(p-1)v''_\epsilon + \frac{Q-1}{d}v'_\epsilon\right]\\
&= -\psi_p\,|\,(\epsilon - A)d^{-A-1+\epsilon}\,|^{p-2}\Big[(p-1)(\epsilon - A)(\epsilon - A - 1)d^{-A-2+\epsilon}\\
&\qquad\qquad + \frac{Q-1}{d}(\epsilon - A)d^{-A-1+\epsilon}\Big]\\
&= -\psi_p(\epsilon - A)|\epsilon - A|^{p-2}d^{(-A-1+\epsilon)(p-2)+(-A-2+\epsilon)}[(p-1)(\epsilon - A - 1) + Q - 1]\\
&= -\psi_p[(\epsilon - A)(p-1) + Q - p](\epsilon - A)|\epsilon - A|^{p-2}d^{(-A+\epsilon)(p-1)-p}\\
&= -[(\epsilon - A)(p-1) + pA](\epsilon - A)|\epsilon - A|^{p-2}\psi_p\frac{v_\epsilon^{p-1}}{d^p}.
\end{aligned} \tag{4.2}$$

Letting $\epsilon \to 0$, the conclusion follows.

(2) By the density argument, we select $\phi(\xi) \in C_0^\infty(\Omega)$, $\|\phi\|_{L^p} = 1$, such that $C_{Q,p} = \int_\Omega |\nabla_H\phi|^p / (\int_\Omega (|z|^p/d^p)(|\phi|^p/d^p))$. Using the best constant of the inequality (2.1), one has

$$\langle L_{p,\mu}\phi, \phi\rangle = \int_\Omega |\,\nabla_H\phi\,|^p - \mu\int_\Omega \frac{|z|^p}{d^p}\frac{|\phi|^p}{d^p} < \int_\Omega |\,\nabla_H\phi\,|^p - C_{Q,p}\int_\Omega \frac{|z|^p}{d^p}\frac{|\phi|^p}{d^p} = 0. \tag{4.3}$$

Denote $u_\tau(x,y,t) = \tau^{Q/p}\phi(\delta_\tau(x,y,t))$ and $\xi = (x,y,t)$. Thus,

$$||u_\tau(x,y,t)||_{L^p}^p = \int_\Omega |\,\tau^{Q/p}\phi(\delta_\tau(x,y,t))\,|^p d\xi = \int_\Omega |\,\phi(\delta_\tau(x,y,t))\,|^p \tau^Q d\xi = 1, \tag{4.4}$$

and $\langle L_{p,\mu}u_\tau, u_\tau\rangle = \tau^p\langle L_{p,\mu}\phi, \phi\rangle < 0$. This concludes the result. □

In order to prove the main result (Theorem 4.6 below) we need the following two preliminary lemmas.

Lemma 4.2. *Let* $\{g_m\} \subset L^p(\Omega)(1 \le p < \infty)$ *be such that as* $m \to \infty$,

$$\begin{aligned} g_m &\rightharpoonup g \quad \text{weakly in } L^p(\Omega), \\ g_m &\longrightarrow g \quad \text{a.e. in } \Omega. \end{aligned} \tag{4.5}$$

Then,

$$\lim_{m\to\infty} [||g_m||^p_{L^p(\Omega)} - ||g_m - g||^p_{L^p(\Omega)}] = \|g\|^p_{L^p(\Omega)}. \tag{4.6}$$

The proof is similar to one in the Euclidean space (see [19, Chapter 1, Section 4]. We omit it here.

Lemma 4.3. *Suppose that* $\{u_m\} \subset D_0^{1,p}(\Omega)(1 \le p < \infty)$ *satisfies*

$$\begin{aligned} u_m &\rightharpoonup u \quad \text{weakly in } D_0^{1,p}(\Omega), \\ u_m &\longrightarrow u \quad \text{strongly in } L^p_{loc}(\Omega), \end{aligned} \tag{4.7}$$

as $m \to \infty$, *and*

$$-\Delta_{H,p} u_m = f_m + g_m, \quad \text{in } \mathcal{D}'(\Omega), \tag{4.8}$$

where $f_m \to f$ *strongly in* $D^{-1,p'}(\Omega)(p' = p/(p-1))$, g_m *is bounded in* $M(\Omega)$ *(the space of Radon measures), that is,*

$$|\langle g_m, \varphi \rangle| \le C_K \|\varphi\|_{L^\infty} \tag{4.9}$$

for all $\varphi \in \mathcal{D}(\Omega)$ *with* $\operatorname{supp}(\varphi) \subset K$, *where* C_K *is a constant which depends on the compact set* K. *Then there exists a subsequence* $\{u_{m_j}\}$ *of* $\{u_m\}$ *such that*

$$u_{m_j} \longrightarrow u \quad \text{strongly in } D_0^{1,q}(\Omega),\ \forall q < p. \tag{4.10}$$

Its proof is similar to one of [20, Theorem 2.1].

Lemma 4.4. *Let* $p \neq Q$ *and*

$$\mathfrak{I}_p := \left\{ f : \Omega \longrightarrow \mathbb{R}^+ \mid \lim_{d(\xi)\to 0} \frac{d^p(\xi)}{\psi_p(\xi)} f(\xi) \left(\ln \frac{1}{d(\xi)} \right)^2 < \infty,\ f(\xi) \in L^\infty_{loc}(\Omega \setminus \{0\}) \right\}. \tag{4.11}$$

(i) *If* $f(\xi) \in \mathfrak{I}_p$, *then there exists* $\lambda(f) > 0$ *such that for all* $u \in D_0^{1,p}(\Omega \setminus \{0\})$, *the following holds:*

$$\int_\Omega |\nabla_H u|^p d\xi \ge \left| \frac{Q-p}{p} \right|^p \int_\Omega \psi_p(\xi) \frac{|u|^p}{d^p} d\xi + \lambda(f) \int_\Omega |u|^p f(\xi) d\xi. \tag{4.12}$$

(ii) *If* $f(\xi) \notin \mathfrak{I}_p$ *and* $(d^p(\xi)/\psi_p(\xi)) f(\xi)(\ln 1/d(\xi))^2 \to \infty$ *as* $d(\xi) \to 0$, *then (4.12) is not true.*

Proof. (i) If $f(\xi) \in \mathfrak{I}_p$, then

$$\lim_{\epsilon\to 0} \sup_{\xi\in B_H(\epsilon)} \frac{d^p(\xi)}{\psi_p} f(\xi)\left(\ln \frac{1}{d(\xi)}\right)^2 < \infty. \tag{4.13}$$

Without any loss of generality we assume that $R = 1$ in (2.2). For the sufficiently small $\epsilon > 0$, we have

$$f(\xi) < \frac{C\psi_p}{d^p(\ln 1/d)^2}, \quad \text{in } B_H(\epsilon). \tag{4.14}$$

Outside $B_H(\epsilon)$, $f(\xi)$ is also bounded. Hence there exists $C(f) > 0$ such that

$$\int_\Omega |u|^p f(\xi)d\xi \le C(f) \int_\Omega \psi_p \frac{|u|^p}{d^p(\ln 1/d)^2} d\xi, \quad \text{on } \Omega. \tag{4.15}$$

Taking $\lambda(f) = C(f)^{-1}((p-1)/2p)|A|^{p-2} > 0$, (4.12) follows from (2.2).

(ii) We write $f(\xi) = \psi_p h(\xi)/d^p(\xi)(\ln 1/d(\xi))^2$, where $h(\xi)\to\infty$ as $d(\xi)\to 0$. Then, for the sufficiently small $\epsilon > 0$, we select $u(\xi) = V_\epsilon(\xi) = \varphi(\xi)d^{-A+\epsilon}(\xi)\mathcal{B}^{-\kappa}(d(\xi)/R)$ and get from (i) of Lemma 3.2,

$$\begin{aligned} 0 < \lambda(f) &\le \frac{I(V_\epsilon)}{\int_{B_H(\delta)} |V_\epsilon|^p f(\xi)d\xi} \le \frac{I(V_\epsilon)}{\int_{B_H(\delta)} \psi_p(|V_\epsilon|^p h(\xi)/d^p(\ln 1/d)^2)d\xi} \\ &\le \frac{I(V_\epsilon)}{K_\delta \int_{B_H(\delta)} \psi_p(|V_\epsilon|^p/d^p(\ln 1/d)^2)d\xi} \le \frac{(\kappa(p-1)/2)|A|^{p-2}J_{p\kappa-2}(\epsilon) + O_\epsilon(1)}{K_\delta J_{p\kappa-2}(\epsilon)} \\ &\le \frac{c\epsilon^{1-p\kappa}}{cK_\delta \epsilon^{1-p\kappa}} = \frac{C}{K_\delta} \longrightarrow 0, \quad \text{as } \delta,\epsilon \longrightarrow 0, \end{aligned} \tag{4.16}$$

where $K_\delta = \inf_{\xi\in B_H(\delta)} h(\xi)$. The impossibility shows that (4.12) cannot hold for $f(\xi) \notin \mathfrak{I}_p$. □

Definition 4.5. Let $\lambda \in \mathbb{R}$, $u \in D_0^{1,p}(\Omega)$, and $u \not\equiv 0$. Call that (λ, u) is a weak solution of (1.2) if

$$\int_\Omega |\nabla_H u|^{p-2}\langle \nabla_H u, \nabla_H \varphi\rangle d\xi - \mu \int_\Omega \frac{\psi_p}{d^p}|u|^{p-2}u\varphi\, d\xi = \lambda \int_\Omega f(\xi)|u|^{p-2}u\varphi\, d\xi \tag{4.17}$$

for any $\varphi \in C_0^\infty(\Omega)$. In this case, we call that u is the eigenfunction of problem (1.2) associated to the eigenvalue λ.

THEOREM 4.6. *Suppose that* $1 < p < Q$, $0 \le \mu < ((Q-p)/p)^p$, *and* $f(\xi) \in \mathcal{F}_p$. *The problem (1.2) admits a positive weak solution* $u \in D_0^{1,p}(\Omega)$, *corresponding to the first eigenvalue* $\lambda = \lambda_\mu^1(f) > 0$. *Moreover, as* μ *increases to* $((Q-p)/p)^p$, $\lambda_\mu^1(f)\to\lambda(f) \ge 0$ *for* $f(\xi) \in \mathcal{F}_p$ *and the limit* $\lambda(f) > 0$ *for* $f(\xi) \in \mathfrak{I}_p$. *If* $f(\xi) \notin \mathfrak{I}_p$ *and* $\psi_p^{-1}(\xi)d^p(\xi)f(\xi)(\ln 1/d(\xi))^2\to\infty$, *as* $d(\xi)\to 0$, *then the limit* $\lambda(f) = 0$.

Proof. We define

$$J_\mu(u) := \int_\Omega |\nabla_H u|^p - \mu \int_\Omega \psi_p \frac{|u|^p}{d^p}. \tag{4.18}$$

Obviously, J_μ is continuous and Gâteaux differentiable on $D_0^{1,p}(\Omega)$. By (2.1),

$$J_\mu(u) \geq \int_\Omega |\nabla_H u|^p - \frac{\mu}{C_{Q,p}} \int_\Omega |\nabla_H u|^p = \left(1 - \frac{\mu}{C_{Q,p}}\right) \int_\Omega |\nabla_H u|^p \tag{4.19}$$

for $0 \leq \mu < ((Q-p)/p)^p$. Note that $C_{Q,p} = ((Q-p)/p)^p$ for $1 < p < Q$. Hence J_μ is coercive in $D_0^{1,p}(\Omega)$. We minimize the function $J_\mu(u)$ over the mainfold $\mathcal{M} = \{u \in D_0^{1,p}(\Omega) \mid \int_\Omega |u|^p f(\xi) d\xi = 1\}$ and let λ_μ^1 be the infimum. It is clear that $\lambda_\mu^1 > 0$ from Lemma 4.1. Now, we choose a special minimizing sequence $\{u_m\} \subset \mathcal{M}$ with $\int_\Omega |u_m|^p f(\xi) d\xi = 1$, and $J_\mu(u_m) \to \lambda_\mu^1$ and $J'_\mu(u_m) \to 0$ strongly in $D_0^{-1,p'}(\Omega)$, when the component of $J'_\mu(u_m)$ is restricted to $\mathcal{M}$. The coercivity of J_μ implies that $\{u_m\}$ is bounded and then there exists a subsequence, still denoted by $\{u_m\}$, such that

$$\begin{aligned} u_m &\rightharpoonup u \quad \text{weakly in } D_0^{1,p}(\Omega), \\ u_m &\rightharpoonup u \quad \text{weakly in } L^p(\Omega, \psi_p d^{-p}), \\ u_m &\longrightarrow u \quad \text{strongly in } L^p(\Omega), \end{aligned} \tag{4.20}$$

as $m \to \infty$. By Theorem 2.4 in Section 2 we know that $D_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega, f d\xi)$, and it follows that $\mathcal{M}$ is weakly closed and hence $u \in \mathcal{M}$. Moreover, u_m satisfies

$$-\Delta_{H,p} u_m - \psi_p \frac{\mu}{d^p} |u_m|^{p-2} u_m = \lambda_m |u_m|^{p-2} u_m f + f_m, \quad \text{in } \mathcal{D}'(\Omega), \tag{4.21}$$

where $f_m \to 0$ strongly in $D^{-1,p'}(\Omega)$ and $\lambda_m \to \lambda$, as $m \to \infty$. Letting $g_m = \psi_p(\mu/d^p)|u_m|^{p-2}u_m + \lambda_m |u_m|^{p-2} u_m f$, we check easily that g_m is bounded in $\mathrm{M}(\Omega)$ and conclude almost everywhere convergence of $\nabla_H u_m$ to $\nabla_H u$ in Ω by Lemma 4.3, and

$$\begin{aligned} J_\mu(u_m) &= \|\nabla_H u_m\|_{L^p(\Omega)}^p - \mu \|u_m\|_{L^p(\Omega,\psi_p d^{-p})}^p \\ &= \|\nabla_H (u_m - u)\|_{L^p(\Omega)}^p - \mu \|u_m - u\|_{L^p(\Omega,\psi_p d^{-p})}^p + \|\nabla_H u\|_{L^p(\Omega)}^p - \mu \|u\|_{L^p(\Omega,\psi_p d^{-p})}^p + o(1) \\ &\geq (C_{Q,p} - \mu) \|u_m - u\|_{L^p(\Omega,\psi_p d^{-p})}^p + \lambda_\mu^1 + o(1), \end{aligned} \tag{4.22}$$

by applying Lemma 4.2 to u_m and $\nabla_H u_m$, where $o(1) \to 0$ as $m \to \infty$. Thus $C_{Q,p} > \mu$, $\|u_m - u\|_{L^p(\Omega,\psi_p d^{-p})}^p \to 0$, and $\|\nabla_H(u_m - u)\|_{L^p(\Omega)}^p \to 0$ as $m \to \infty$. It shows that $J_\mu(u) = \lambda_\mu^1$ and $\lambda = \lambda_\mu^1$. Since $J_\mu(|u|) = J_\mu(u)$, we can take $u > 0$ in Ω. By Lemma 4.3, u is a distribution solution of (1.2) and since $u \in D_0^{1,p}(\Omega)$, it is a weak solution to eigenvalue problem (1.2) corresponding to $\lambda = \lambda_\mu^1$. Moreover, if $f(\xi) \in \mathfrak{I}_p$, then by Lemma 4.4,

$$\lambda_\mu^1(f) \longrightarrow \lambda(f) = \inf_{u \in D^{1,p}(\Omega \setminus \{0\})} \frac{\int_\Omega (|\nabla_H u|^p - C_{Q,p} \psi_p (|u|^p / d^p)) d\xi}{\int_\Omega |u|^p f d\xi} > 0, \tag{4.23}$$

as μ increases to $((Q-p)/p)^p$. When $f(\xi) \notin \mathfrak{I}_p$, using Lemma 4.4 again, it follows that (4.12) is not true and hence $\lambda(f)=0$. This completes the proof. □

Remark 4.7. The set $\mathcal{M}$ is a C^1 manifold in $D_0^{1,p}(\Omega)$. By Ljusternik-Schnirelman critical point theory on C^1 manifold, there exists a sequence $\{\lambda_m\}$ of eigenvalues of (1.2), that is, writing $\Gamma_m = \{A \subset \mathcal{G} \mid A \text{ is symmetric, compact, and } \gamma(A) \geq m\}$, where $\gamma(A)$ is the Krasnoselski's genus of A (see [19]), then for any integer $m>0$,

$$\lambda_m = \inf_{A\in\Gamma_m} \sup_{u\in A} J_\mu(u) \tag{4.24}$$

is an eigenvalue of (1.2). Moreover, $\lim_{m\to\infty}\lambda_m \to \infty$.

5. Simplicity and isolation for the first eigenvalue

This section is to consider the simplicity and isolation for the first eigenvalue. We always assume that f satisfies the conditions in Theorem 4.6. From the previous results we know clearly that the first eigenvalue is

$$\lambda_\mu^1 = \lambda_\mu^1(f) = \inf\left\{J_\mu(u) \mid u \in D_0^{1,p}(\Omega\setminus\{0\}),\ \int_\Omega |u|^p f(\xi)d\xi = 1\right\}. \tag{5.1}$$

In what follows we need the Picone identity proved in [15].

Proposition 5.1 (Picone identity). *For differentiable functions $u \geq 0$, $v>0$ on $\Omega \subset \mathbb{H}^n$, with Ω a bounded or unbounded domain in $\mathbb{H}^n$, then*

$$L(u,v) = R(u,v) \geq 0, \tag{5.2}$$

with

$$\begin{aligned} L(u,v) &= |\nabla_H u|^p + (p-1)\frac{u^p}{v^p}|\nabla_H v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla_H v|^{p-2}\nabla_H u\cdot\nabla_H v, \\ R(u,v) &= |\nabla_H u|^p - |\nabla_H v|^{p-2}\nabla_H\left(\frac{u^p}{v^{p-1}}\right)\cdot\nabla_H v \end{aligned} \tag{5.3}$$

for $p>1$. Moreover, $L(u,v)=0$ a.e. on Ω if and only if $\nabla_H(u/v)=0$ a.e. on Ω.

Theorem 5.2. (i) *λ_μ^1 is simple, that is, the positive eigenfunction corresponding to λ_μ^1 is unique up to a constant multiple.*

(ii) *λ_μ^1 is unique, that is, if $v \geq 0$ is an eigenfunction associated with an eigenvalue λ with $\int_\Omega f(\xi)|v|^p d\xi = 1$, then $\lambda = \lambda_\mu^1$.*

(iii) *Every eigenfunction corresponding to the eigenvalue λ $(0<\lambda\neq\lambda_\mu^1)$ changes sign in Ω.*

Proof. (i) Let $u>0$ and $v>0$ be two eigenfunctions corresponding to λ_μ^1 in $\mathcal{M}$. For sufficiently small $\varepsilon>0$, set $\phi = u^p/(v+\varepsilon)^{p-1} \in D_0^{1,p}(\Omega)$. Then

$$\int_\Omega |\nabla_H v|^{p-2}\langle\nabla_H v, \nabla_H\phi\rangle d\xi = \mu\int_\Omega \psi_p \frac{u^{p-1}}{d^p}\phi d\xi + \lambda_\mu^1\int_\Omega f(\xi)v^{p-1}\phi d\xi. \tag{5.4}$$

Using Proposition 5.1 and (5.4),

$$\begin{aligned}
0 &\le \int_\Omega L(u,v+\varepsilon) = \int_\Omega R(u,v+\varepsilon)\\
&= \int_\Omega |\nabla_H u|^p d\xi - \int_\Omega |\nabla_H v|^{p-2}\left\langle \nabla_H\left(\frac{u^p}{(v+\varepsilon)^{p-1}}\right), \nabla_H v\right\rangle d\xi\\
&= \int_\Omega\left(\mu\frac{\psi_p}{d^p}+\lambda_\mu^1 f(\xi)\right)u^p d\xi - \int_\Omega\left(\mu\frac{\psi_p}{d^p}+\lambda_\mu^1 f(\xi)\right)v^{p-1}\frac{u^p}{(v+\varepsilon)^{p-1}}d\xi\\
&= \int_\Omega\left(\mu\frac{\psi_p}{d^p}+\lambda_\mu^1 f(\xi)\right)u^p\left(1-\frac{v^{p-1}}{(v+\varepsilon)^{p-1}}\right)d\xi.
\end{aligned} \tag{5.5}$$

The right-hand side of (5.5) tends to zero when $\varepsilon\to 0$. It follows that $L(u,v) = 0$ and by Proposition 5.1 there exists a constant c such that $u = cv$.

(ii) Let $u > 0$ and $v > 0$ be eigenfunctions corresponding to λ_μ^1 and λ, respectively. Similarly to (5.5), we have

$$\begin{aligned}
&\int_\Omega\left(\mu\frac{\psi_p}{d^p}+\lambda_\mu^1 f(\xi)\right)u^p d\xi - \int_\Omega\left(\mu\frac{\psi_p}{d^p}+\lambda f(\xi)\right)v^{p-1}\frac{u^p}{(v+\varepsilon)^{p-1}}d\xi\\
&\quad= \int_\Omega \mu\frac{\psi_p}{d^p}\left(1-\frac{v^{p-1}}{(v+\varepsilon)^{p-1}}\right)u^p d\xi + \int_\Omega f(\xi)u^p\left(\lambda_\mu^1-\lambda\frac{v^{p-1}}{(v+\varepsilon)^{p-1}}\right)d\xi \ge 0.
\end{aligned} \tag{5.6}$$

Letting $\varepsilon\to 0$ shows that $(\lambda_\mu^1-\lambda)\int_\Omega f(\xi)u^p d\xi \ge 0$, which is impossible for $\lambda > \lambda_\mu^1$. Hence $\lambda = \lambda_\mu^1$.

(iii) With the same treatment as in (5.6) we get

$$(\lambda_\mu^1-\lambda)\int_\Omega f(\xi)u^p d\xi \ge 0. \tag{5.7}$$

Noting that $\int_\Omega f(\xi)u^p d\xi > 0$ and $\lambda > \lambda_\mu^1$ leads to a contradiction. So v must change sign in Ω. □

Lemma 5.3. *If $u \in D_0^{1,p}(\Omega)$ is a nonnegative weak solution of (1.2), then either $u(\xi) \equiv 0$ or $u(\xi) > 0$ for all $\xi \in \Omega$.*

Proof. For any $R > r$, $B_H(0,R) \supset B_H(0,r)$, let $u \in D_0^{1,p}(\Omega)$ be a nonnegative weak solution of (1.2). In virtue of Harnack's inequality (see [1]), there exists a constant $C_R > 0$ such that

$$\sup_{B_H(0,R)}\{u(\xi)\} \le C_R \inf_{B_H(0,R)}\{u(\xi)\}. \tag{5.8}$$

This implies $u \equiv 0$ or $u > 0$ in Ω. □

Theorem 5.4. *Every eigenfunction u_1 corresponding to λ_μ^1 does not change sign in Ω: either $u_1 > 0$ or $u_1 < 0$.*

Proof. From the proof of existence of the first eigenvalue we see that there exists a positive eigenfunction, that is, if v is an eigenfunction, then $u_1 = |v|$ is a solution of the minimization problem and also an eigenfunction. Thus, from Lemma 5.3 it follows that $|v| > 0$ and then u_1 has constant sign. □

Lemma 5.5. *For $u \in C(\Omega \setminus \{0\}) \cap D_0^{1,p}(\Omega)$, let $\mathcal{N}$ be a component of $\{\xi \in \Omega \mid u(\xi) > 0\}$. Then $u|_{\mathcal{N}} \in D_0^{1,p}(\mathcal{N})$.*

Proof. Let $u_m \in C(\Omega \setminus \{0\}) \cap D_0^{1,p}(\Omega)$ be such that $u_m \to u$ in $D_0^{1,p}(\Omega)$. Therefore, $u_m^+ \to u^+$ in $D_0^{1,p}(\Omega)$. Set $v_m(\xi) = \min\{u_m(\xi), u(\xi)\}$, and let $\varphi_R(\xi) \in C(\Omega)$ be a cutoff function such that

$$\varphi_R(\xi) = \begin{cases} 0 & \text{if } d(\xi) \le \dfrac{R}{2}, \\ 1 & \text{if } d(\xi) \ge R, \end{cases} \tag{5.9}$$

with $|\nabla_H \varphi_R| \le C|\nabla_H d|/d(\xi)$, for some positive constant C. Now, consider the sequence $\omega_m(\xi) = \varphi_R(\xi) v_m(\xi)|_{\mathcal{N}}$. Since $\varphi_R(\xi) v_m(\xi) \in C(\Omega)$, we claim that $\omega_m \in C(\mathcal{N})$ and $\omega_m = 0$ on the boundary $\partial\mathcal{N}$. In fact, if $\xi \in \partial\mathcal{N}$ and $\xi = 0$, then $\varphi_R = 0$, and so $\omega_m = 0$. If $\xi \in \partial\mathcal{N} \cap \Omega$ and $\xi \ne 0$, then $u = 0$ (since u is continuous except at $\{0\}$), and hence $v_m = 0$. If $\xi \in \partial\Omega$, then $u_m = 0$ and so $v_m = 0$. Therefore, $\omega_m = 0$ on $\partial\mathcal{N}$, and $\omega_m \in D_0^{1,p}(\mathcal{N})$. Noting

$$\begin{aligned} \int_\Omega \left| \nabla_H \omega_m - \nabla_H(\varphi_R u) \right|^p d\xi &= \int_{\mathcal{N}} \left| \varphi_R \nabla_H v_m + v_m \nabla_H \varphi_R - \varphi_R \nabla_H u - u \nabla_H \varphi_R \right|^p d\xi \\ &\le \|\varphi_R(\nabla_H v_m - \nabla_H u)\|_{L^p(\mathcal{N})}^p + \|\nabla_H \varphi_R (v_m - u)\|_{L^p(\mathcal{N})}^p, \end{aligned} \tag{5.10}$$

it is obvious that $\int_\Omega |\nabla_H \omega_m - \nabla_H(\varphi_R u)|^p d\xi \to 0$, as $m \to \infty$. That is $\omega_m \to \varphi_R u|_{\mathcal{N}}$ in $D_0^{1,p}(\mathcal{N})$. By (2.1),

$$\begin{aligned} &\int_{\mathcal{N}} \left| u \nabla_H \varphi_R + \varphi_R \nabla_H u - \nabla_H u \right|^p d\xi \\ &\quad \le \int_{\mathcal{N}} \left| \varphi_R \nabla_H u - \nabla_H u \right|^p d\xi + \int_{\mathcal{N} \cap \{R/2 < d < R\}} \left| u \nabla_H \varphi_R \right|^p d\xi \\ &\quad \le \int_{\mathcal{N}} \left| \varphi_R \nabla_H u - \nabla_H u \right|^p d\xi + C^p \int_{\mathcal{N} \cap \{R/2 < d < R\}} \psi_p \frac{|u|^p}{d^p} d\xi \\ &\quad \le \int_{\mathcal{N}} \left| \varphi_R \nabla_H u - \nabla_H u \right|^p d\xi + C_1 \int_{\mathcal{N} \cap \{R/2 < d < R\}} \left| \nabla_H u \right|^p d\xi, \end{aligned} \tag{5.11}$$

which approaches 0, as $R \to 0$. Hence $u|_{\mathcal{N}} \in D_0^{1,p}(\mathcal{N})$. □

Theorem 5.6. *The eigenvalue λ_μ^1 is isolated in the spectrum, that is, there exists $\delta > 0$ such that there is no other eigenvalues of (1.2) in the interval $(\lambda_\mu^1, \lambda_\mu^1 + \delta)$. Moreover, if v is an eigenfunction corresponding to the eigenvalue $\lambda \ne \lambda_\mu^1$ and $\mathcal{N}$ is a nodal domain of v, then*

$$\left(C\lambda \|f\|_{L^\infty}\right)^{-Q/p} \le |\mathcal{N}|, \tag{5.12}$$

where C is a constant depending only on Q and p.

Proof. Let u_1 be the eigenfunction corresponding to the eigenvalue λ_μ^1. Let $\{\lambda_m\}$ be a sequence of eigenvalues such that $\lambda_m > \lambda_\mu^1$ and $\lambda_m \searrow \lambda_\mu^1$, and the corresponding eigenfunctions $u_m \to u_1$ with $\int_\Omega f(\xi)|u_m|^p d\xi = 1$, that is, λ_m and u_m satisfy

$$L_{p,\mu}u_m = \lambda_m f(\xi)\,|\,u_m\,|^{p-2}u_m. \tag{5.13}$$

Since

$$0 < \int_\Omega |\,\nabla_H u_m\,|^p d\xi - \mu\int_\Omega \frac{\psi_p}{d^p}\,|\,u_m\,|^p d\xi = \lambda_m \int_\Omega f(\xi)\,|\,u_m\,|^p d\xi = \lambda_m, \tag{5.14}$$

it follows that u_m is bounded. By Lemma 4.3, there exists a subsequence (still denoted by $\{u_m\}$) of $\{u_m\}$ such that $u_m \rightharpoonup u$ weakly in $D_0^{1,p}(\Omega)$, $u_m \to u$ strongly in $L^p(\Omega)$ and $\nabla_H u_m \to \nabla_H u$ a.e in Ω. Letting $m \to \infty$ in (5.13) yields

$$L_{p,\mu}u = \lambda_\mu^1 f(\xi)|u|^{p-2}u. \tag{5.15}$$

Therefore, $u = \pm u_1$. Using (iii) of Theorem 5.2 we see that u_m changes sign. For convenience, we assume that $u = +u_1$. Then

$$|\,\{\xi \in \Omega \mid u_m < 0\}\,| \longrightarrow 0. \tag{5.16}$$

Now, we check (5.13) with $u_m = u_m^-$,

$$\int_\Omega |\,\nabla_H u_m^-\,|^p d\xi - \mu\int_\Omega \frac{\psi_p}{d^p}\,|\,u_m^-\,|^p d\xi = \lambda_m \int_\Omega f(\xi)\,|\,u_m^-\,|^p d\xi. \tag{5.17}$$

Using the Hardy inequality and Sobolev inequality yields

$$\begin{aligned}
\left(1 - \frac{\mu}{C_{Q,p}}\right)\int_{\Omega^-} |\,\nabla_H u_m\,|^p d\xi
&\le \int_{\Omega^-} |\,\nabla_H u_m\,|^p d\xi - \mu\int_{\Omega^-} \frac{\psi_p}{d^p}\,|\,u_m\,|^p d\xi \\
&= \lambda_m \int_{\Omega^-} f(\xi)\,|\,u_m\,|^p d\xi \\
&\le \lambda_m \|f\|_{L^\infty} \int_{\Omega^-} |\,u_m\,|^p d\xi \\
&\le c_1 \|f\|_{L^\infty} \|u_m\|_{D^{1,p}}^p\,|\,\Omega_m^-\,|^{p/Q}, \\
|\,\Omega_m^-\,| \ge (c_2\|f\|_{L^\infty})^{Q/p}, \qquad \Omega_m^- &= \{\xi \in \Omega \mid u_m < 0\}.
\end{aligned} \tag{5.18}$$

It contradicts with (5.16). Hence, there is no other eigenvalue of (1.2) in $(\lambda_\mu^1, \lambda_\mu^1 + \delta)$ for $\delta > 0$.

Next, we prove (5.12). Assume $v > 0$ in $\mathcal{N}$ (the case $v < 0$ being treated similarly). In view of Lemma 5.5, we have $v|_{\mathcal{N}} \in D_0^{1,p}(\mathcal{N})$. Define the function

$$\omega(\xi) = \begin{cases} v(\xi) & \text{if } \xi \in \mathcal{N}, \\ 0 & \text{if } \xi \in \Omega \setminus \mathcal{N}. \end{cases} \tag{5.19}$$

Clearly, $\omega(\xi) \in D_0^{1,p}(\Omega)$. Taking ω as a test function in (4.17) satisfied by v and arguing as in (5.18), we have

$$\left(1 - \frac{\mu}{C_{Q,p}}\right) \|v\|^p_{D^{1,p}(\mathcal{N})} \le \lambda \|f\|_{L^\infty} \int_{\mathcal{N}} |v|^p d\xi \le \lambda \widetilde{C} \|f\|_{L^\infty} \|v\|^p_{D^{1,p}(\mathcal{N})} |\mathcal{N}|^{p/Q} \tag{5.20}$$

for some constant $\widetilde{C} = C(Q,p)$ and hence (5.12) holds. □

Corollary 5.7. *Each eigenfunction has a finite number of nodal domains.*

Proof. Let $\mathcal{N}_j$ be a nodal domain of an eigenfunction associated to some positive eigenvalue λ. It follows from (5.12) that

$$|\Omega| \ge \sum_j |\mathcal{N}_j| \ge (C\lambda \|f\|_{L^\infty})^{-Q/p} \sum_j 1. \tag{5.21}$$

The result is proved. □

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