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A hereditarily indecomposable $\mathscr{L}_\infty$-space that solves the scalar-plus-compact problem

by

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1. Introduction

The question of whether there exists a Banach space X on which every bounded linear operator is a compact perturbation of a scalar multiple of the identity has become known as the "scalar-plus-compact problem". It is mentioned by Lindenstrauss as Question 1 in his 1976 list of open problems in Banach space theory [32]. Lindenstrauss remarks that, by the main theorem of [10] or [33], every operator on a space of this type has a proper non-trivial invariant subspace. Related questions go further back: for instance, Thorp [42] asks whether the space of compact operators $\mathcal{K}(X;Y)$ can ever be a proper complemented subspace of $\mathcal{L}(X;Y)$. On the Gowers–Maurey space $\mathfrak{X}_{\mathrm{gm}}$ [23], every operator is a *strictly singular* perturbation of a scalar, and other hereditarily indecomposable (HI) spaces also have this property. Indeed it seemed for a time that $\mathfrak{X}_{\mathrm{gm}}$ might already solve the scalar-plus-compact problem. However, after Gowers [22] had shown that there is a strictly singular, non-compact operator from a subspace of $\mathfrak{X}_{\mathrm{gm}}$ to $\mathfrak{X}_{\mathrm{gm}}$, Androulakis and Schlumprecht [4] showed that such an operator can be defined on the whole of $\mathfrak{X}_{\mathrm{gm}}$. Gasparis [20] has done the same for the Argyros–Deliyanni space $\mathfrak{X}_{\mathrm{ad}}$ of [5].

In the present paper, we solve the scalar-plus-compact problem by combining techniques that are familiar from other HI constructions with an additional ingredient, the Bourgain–Delbaen method for constructing special $\mathscr{L}_\infty$-spaces [12]. The initial motivation for combining these two constructions was to exhibit a hereditarily indecomposable predual of ℓ_1; such a space is, in some sense, the extreme example of a known phenomenon—that the HI property does not pass from a space to its dual [19], [9], [6]. Serendipitously, it turned out that the additional structure was just what we needed to show that strictly singular operators are compact. It is interesting, perhaps, to note that the Schur property of ℓ_1 does not play a role in our proof and, indeed, we have no general result to say that an HI predual of ℓ_1 necessarily has the scalar-plus-compact property. We use, in an essential way, the specific structure of the BD construction, which embeds into our space some very explicit finite-dimensional ℓ_∞-spaces. As well as the (now) classical machinery of HI constructions—a space of Schlumprecht type (cf. [40]), Maurey–Rosenthal coding (cf. [35]) and rapidly increasing sequences based on ℓ_1-averages—we add the possibility of splitting an arbitrary vector into pieces of comparable norm, while staying in one of these ℓ_∞^n's. This allows us to introduce two additional classes of rapidly increasing sequences, and these in turn lead to the stronger result about operators.

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2. Background

2.1. Notation

We use standard notation: if A is any set, $\ell_\infty(A)$ is the space of all bounded (real-valued) functions on A, equipped with the supremum norm $\|\cdot\|_\infty$ and $\ell_1(A)$ is the space of all absolutely summable functions on A, equipped with the norm $\|x\|_1=\sum_{a\in A}|x(a)|$. The *support* of a function x is the set of all a such that $x(a)\neq 0$; $c_{00}(A)$ is the space of functions of finite support. We shall write ℓ_p for the space $\ell_p(\mathbb{N})$, where $\mathbb{N}$ is the set $\{1,2,3,\dots\}$ of positive integers, and ℓ_p^n for $\ell_p(\{1,2,\dots,n\})$. Even when we are dealing with these sequence spaces we shall use function notation $x(m)$, rather than subscript notation, for the mth coordinate of the vector x.

When x and y are in $c_{00}(A)$ (and more generally when the sum exists) we shall write $\langle y,x\rangle$ for $\sum_{a\in A}x(a)y(a)$. If we are thinking of y as a functional acting on x (rather than vice versa) we shall usually choose a notation involving a star, denoting y by f^*, or something of this kind. In particular, e_a and e_a^* are two notations for the same unit vector in $c_{00}(A)$ (given by $e_a(a')=\delta_{a,a'}$), to be employed depending on whether we are thinking of it as a unit vector or as the evaluation functional $x\mapsto\langle e_a^*,x\rangle=x(a)$. We apologize to those readers who may find this kind of notation somewhat babyish.

We say that (finitely or infinitely many) vectors $z_1,z_2,\ldots$ in c_{00} are *successive*, or that $(z_i)_{i\in\mathbb{N}}$ is a *block-sequence*, if $\max\operatorname{supp} z_i<\min\operatorname{supp} z_{i+1}$ for all i. In a Banach space X we say that vectors y_j are *successive linear combinations*, or that $(y_j)_{j\in\mathbb{N}}$ is a *block sequence* of a basic sequence $(x_i)_{i\in\mathbb{N}}$ if there exist $0=q_1<q_2<\ldots$ such that, for all $j\geqslant 1$, y_j is in the linear span $[x_i:q_{j-1}<i\leqslant q_j]$. If we may arrange that $y_j\in[x_i:q_{j-1}<i<q_j]$, we say that $(y_j)_{j\in\mathbb{N}}$ is a *skipped block sequence*. More generally, if X has a Schauder decomposition $X=\bigoplus_{n\in\mathbb{N}}F_n$ we say that $(y_j)_{j\in\mathbb{N}}$ is a block sequence (resp. a skipped block sequence) with respect to $(F_n)_{n\in\mathbb{N}}$ if there exist $0=q_0<q_1<\ldots$ such that y_j is in $\bigoplus_{q_{j-1}<n\leqslant q_j}F_n$ (resp. $\bigoplus_{q_{j-1}<n<q_j}F_n$). A *block subspace* is the closed subspace generated by a block sequence.

2.2. Hereditary indecomposability

A Banach space X is *indecomposable* if there do not exist infinite-dimensional closed subspaces Y and Z of X with $X=Y\oplus Z$, and is *hereditarily indecomposable* (HI) if every closed subspace is indecomposable. The following useful criterion, like so much else in this area, goes back to the original paper of Gowers and Maurey [23].

PROPOSITION 2.1. *Let X be an infinite-dimensional Banach space. Then X is* HI *if and only if, for every pair Y,Z of infinite-dimensional subspaces, and every $\varepsilon>0$, there exist $y\in Y$ and $z\in Z$ with $\|y+z\|>1$ and $\|y-z\|<\varepsilon$. If X has a finite-dimensional decomposition $(F_n)_{n\in\mathbb{N}}$ it is enough that the above should hold for block subspaces.*

We shall make use of the following well-known blocking lemma, the first part of which can be found as Lemma 1 in [34]. The proof of the second part is very similar, and, as Maurey remarks, both can be traced back to R. C. James [29].

LEMMA 2.2. *Let $n\geqslant 2$ be an integer, let $\varepsilon\in(0,1)$ be a real number and let N be an integer that can be written as $N=n^k$ for some $k\geqslant 1$. Let $(x_i)_{i=1}^N$ be a sequence of vectors in the unit sphere of a Banach space X.*

(i) *If*

$$\Big\|\sum_{i=1}^{N}\pm x_i\Big\|\geqslant(n-\varepsilon)^k$$

for all choices of signs $\pm$, then there is a block sequence $y_1,y_2,\ldots,y_n\in[x_i:1\leqslant i\leqslant N]$ which is $(1-\varepsilon)^{-1}$-equivalent to the unit-vector basis of ℓ_1^n.

(ii) *If*

$$\left\|\sum_{i=1}^{N} \pm x_i\right\| \leqslant (1+\varepsilon)^k$$

for all choices of signs $\pm$, *then there is a block sequence* $y_1, y_2, ..., y_n \in [x_i : 1 \leqslant i \leqslant N]$ *which is* $(1+\varepsilon)$*-equivalent to the unit-vector basis of* ℓ_∞^n.

2.3. $\mathscr{L}_\infty$-spaces

A separable Banach space X is an $\mathscr{L}_{\infty,\lambda}$-space if there is an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite-dimensional subspaces of X such that the union $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X and, for each n, F_n is λ-isomorphic to $\ell_\infty^{\dim F_n}$. It has long been known that if a separable $\mathscr{L}_\infty$-space X has no subspace isomorphic to ℓ_1, then the dual space X^* is necessarily isomorphic to ℓ_1. (This follows from [26, Corollary 8], and can also be deduced easily from results in [2], [25] and [31].) It is, of course, an immediate consequence that the dual of a separable, hereditarily indecomposable $\mathscr{L}_\infty$-space is isomorphic to ℓ_1.

The Bourgain–Delbaen spaces $X_{a,b}$, which inspired the construction given in this paper, were the first examples of $\mathscr{L}_\infty$-spaces not containing c_0.

Further results in this direction are given in [13].

2.4. Mixed Tsirelson spaces

All existing hereditarily indecomposable have, somewhere at the heart of them, a space of Schlumprecht type; rather than working with the original space of [40], we find it convenient to look at a different mixed Tsirelson space. We recall some notation and terminology from [8]. Let $(l_j)_{j\in\mathbb{N}}$ be a sequence of positive integers and let $(\theta_j)_{j\in\mathbb{N}}$ be a sequence of real numbers with $0<\theta_j<1$. We define $W[(\mathscr{A}_{l_j}, \theta_j)_{j\in\mathbb{N}}]$ to be the smallest subset W of c_{00} with the following properties:

(1) $\pm e_k^* \in W$ for all $k \in \mathbb{N}$;

(2) whenever $f_1^*, f_2^*, ..., f_n^* \in W$ are successive vectors,

$$\theta_j \sum_{i=1}^{n} f_i^* \in W,$$

provided $n \leqslant l_j$.

We say that an element f^* of W is of *type* 0 if $f^* = \pm e_k^*$ for some k and of *type* 1 otherwise; an element of type 1 is said to have *weight* θ_j if

$$f^* = \theta_j \sum_{i=1}^{n} f_i^*$$

for a suitable sequence $(f_i^*)_{i=1}^n$ of successive elements of W. It is possible for a given f^* to have more than one weight (but this does not cause problems). The reader who is familiar with [8] should note that our W is what would there be denoted W_0'. It is worth noting that if $f^*\in W$ and, for all i, $g^*(i)$ is either 0 or $\pm f^*(i)$, then $g^*\in W$.

The *mixed Tsirelson space* $T[(\mathscr{A}_{l_j},\theta_j)_{j\in\mathbb{N}}]$ is defined to be the completion of c_{00} with respect to the norm

$$\|x\|=\sup\{\langle f^*,x\rangle : f^*\in W[(\mathscr{A}_{l_j},\theta_j)_{j\in\mathbb{N}}]\}.$$

We may also characterize the norm of this space implicitly as being the smallest function $x\mapsto\|x\|$ satisfying

$$\|x\|=\max\left\{\|x\|_\infty, \sup\theta_j\sum_{i=1}^{l_j}\|x\chi_{E_i}\|\right\},$$

where the supremum is taken over all j and all sequences of finite subsets

$$E_1<E_2<\ldots<E_{l_j}.$$

Schlumprecht's original space is the result of taking $l_j=j$ and $\theta_j=1/\log_2(j+1)$.

In the rest of the paper we shall choose to work with two sequences of natural numbers $(m_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$. We require m_j to grow quite fast, and n_j to grow even faster. The precise requirements are as follows.

Assumption 2.3. We assume that $(m_j,n_j)_{j\in\mathbb{N}}$ satisfy the following:

(1) $m_1\geqslant 4$;
(2) $m_{j+1}\geqslant m_j^2$;
(3) $n_1\geqslant m_1^2$;
(4) $n_{j+1}\geqslant(16n_j)^{\log_2 m_{j+1}}=m_{j+1}^2(4n_j)^{\log_2 m_{j+1}}$.

A straightforward way to achieve this is to assume that $(m_j,n_j)_{j\in\mathbb{N}}$ is some subsequence of the sequence $(2^{2^j},2^{2^{j^2+1}})_{j\in\mathbb{N}}$. From now on, whenever m_j and n_j appear, we shall assume that we are dealing with sequences satisfying Assumption 2.3.

As in [8], it will be important to have good upper norm estimates for some special vectors in a certain mixed Tsirelson space and we shall need a minor modification of Lemma II.9 of that work. For convenience in our calculations we work with the space $T[(\mathscr{A}_{4n_j},m_j^{-1})_{j\in\mathbb{N}}]$, though our later applications $3n_j$ rather than $4n_j$ would suffice.

LEMMA 2.4. *For $g^*\in c_{00}$ and $r\in\mathbb{N}$, let $M_r(g^*)$ be the number of values of i for which $|g^*(i)|>4^{-r}$. If $g^*\in W[(\mathscr{A}_{4n_j},m_j^{-1})_{j\neq j_0}]$ and $r\leqslant\log_2 m_{j_0}$ then $M_r(g^*)\leqslant(4n_{j_0-1})^{r-1}$. In particular,*

$$\#\{i\in\mathbb{N}:|g^*(i)|>m_{j_0}^{-2}\}\leqslant(4n_{j_0-1})^{\log_2 m_{j_0}-1}.$$

For any $f\in W[(\mathscr{A}_{4n_j}, m_j^{-1})_{j\in\mathbb{N}}]$, *we have*

$$\#\{i\in\mathbb{N} : |f^*(i)| > m_{j_0}^{-1}\} \leqslant (4n_{j_0-1})^{\log_2 m_{j_0} - 1}.$$

Proof. We proceed by induction on the size of the support of g^*, noting initially that if this support is a singleton then $M_r(g^*)\leqslant 1$ for all r, so that there is nothing to prove. Otherwise, for some $h\neq j_0$, some $n\leqslant 4n_h$ and some sequence of successive vectors $g_k^*\in W[(\mathscr{A}_{4n_j}, m_j^{-1})_{j\neq j_0}]$, we have

$$g^* = m_h^{-1}\sum_{k=1}^{n} g_k^*.$$

If $h>j_0$ then $|g^*(i)|\leqslant m_h^{-1}\leqslant m_{j_0}^{-2}$ for all i, and so $M_r(g^*)=0$ for all $r\leqslant \log_2 m_{j_0}$.

If $h<j_0$ and $|g^*(i)|>4^{-r}$ then for some k we have $m_h^{-1}|g_k^*(i)|>4^{-r}$, which implies that $|g_k^*(i)|>4^{-r+1}$. Thus

$$M_r(g^*) \leqslant \sum_{k=1}^{n} M_{r-1}(g_k^*) \leqslant n(4n_{j_0-1})^{r-2} \leqslant 4n_h(4n_{j_0-1})^{r-2} \leqslant (4n_{j_0-1})^{r-1},$$

by our inductive hypothesis applied to the g_k^*'s, which have support smaller than that of g^*.

If we now consider an arbitrary $f^*\in W[(\mathscr{A}_{4n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ we may define g^* by setting

$$g^*(i) = \begin{cases} f^*(i), & \text{if } |f^*(i)| > m_{j_0}^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

We note that $g^*\in W[(\mathscr{A}_{4n_j}, m_j^{-1})_{j\in\mathbb{N}}]$, by a remark we made earlier. Since the values of $|g^*(i)|$ are all either 0 or greater than $m_{j_0}^{-1}$, g^* is in fact in $W[(\mathscr{A}_{4n_j}, m_j^{-1})_{j<j_0}]$ and

$$\begin{aligned}\#\{i\in\mathbb{N} : |f^*(i)| > m_{j_0}^{-1}\} &= \#\{i\in\mathbb{N} : |g^*(i)| > m_{j_0}^{-1}\} \\ &\leqslant \#\{i\in\mathbb{N} : |g^*(i)| > m_{j_0}^{-2}\} \leqslant (4n_{j_0-1})^{\log_2 m_{j_0} - 1}. \qquad \square\end{aligned}$$

PROPOSITION 2.5. *If* $j_0\in\mathbb{N}$ *and* $f^*\in W[(\mathscr{A}_{4n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ *is an element of weight* m_h^{-1}, *then*

$$\left|\left\langle f^*, n_{j_0}^{-1}\sum_{l=1}^{n_{j_0}} e_l\right\rangle\right| \leqslant \begin{cases} 2m_h^{-1}m_{j_0}^{-1}, & \text{if } h<j_0, \\ m_h^{-1}, & \text{if } h\geqslant j_0. \end{cases}$$

In particular, the norm of $n_{j_0}^{-1}\sum_{l=1}^{n_{j_0}} e_l$ *in* $T[(\mathscr{A}_{4n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ *is exactly* $m_{j_0}^{-1}$.

If we make the additional assumption that $f^*\in W[(\mathscr{A}_{4n_j}, m_j^{-1})_{j\neq j_0}]$ *then*

$$\left|\left\langle f^*, n_{j_0}^{-1}\sum_{l=1}^{n_{j_0}} e_l\right\rangle\right| \leqslant \begin{cases} 2m_h^{-1}m_{j_0}^{-2}, & \text{if } h<j_0, \\ m_h^{-1}, & \text{if } h>j_0. \end{cases}$$

In particular, the norm of $n_{j_0}^{-1}\sum_{l=1}^{n_{j_0}} e_l$ *in* $T[(\mathscr{A}_{4n_j}, m_j^{-1})_{j\neq j_0}]$ *is at most* $m_{j_0}^{-2}$.

Proof. We write

$$f^* = m_h^{-1} \sum_{k=1}^{n} f_k^*,$$

where $n \leqslant 4n_h$ and $f_1^*, \ldots, f_n^*$ are successive elements of $W[(\mathscr{A}_{4n_j}, m_j^{-1})_j]$. If $h \geqslant j_0$ then $\|f^*\|_\infty \leqslant m_h^{-1}$, which immediately yields

$$\left|\left\langle f^*, n_{j_0}^{-1} \sum_{l=1}^{n_{j_0}} e_l \right\rangle\right| \leqslant m_h^{-1}.$$

Assuming now that $h < j_0$, we may estimate as follows:

$$\begin{aligned}\left|\left\langle f^*, n_{j_0}^{-1} \sum_{l=1}^{n_{j_0}} e_l \right\rangle\right| &\leqslant m_h^{-1} m_{j_0}^{-1} + n_{j_0}^{-1} \sum_{i:|f^*(i)| > m_h^{-1} m_{j_0}^{-1}} |f^*(i)| \\ &\leqslant m_h^{-1} m_{j_0}^{-1} + m_h^{-1} n_{j_0}^{-1} \sum_{k=1}^{n} \sum_{i:|f_k^*(i)| > m_{j_0}^{-1}} |f_k^*(i)|\end{aligned}$$

(because the supports of the f_k^* are disjoint)

$$\leqslant m_h^{-1} m_{j_0}^{-1} + m_h^{-1} n_{j_0}^{-1} \sum_{k=1}^{n} \#\{i : |f_k^*(i)| > m_{j_0}^{-1}\}$$

(because $\|f_k^*\|_\infty \leqslant 1$ for all k)

$$\leqslant m_h^{-1} m_{j_0}^{-1} + m_h^{-1} n_{j_0}^{-1} 4n_h (4n_{j_0-1})^{\log_2 m_{j_0} - 1}$$

(by Lemma 2.4)

$$\begin{aligned}&\leqslant m_h^{-1} m_{j_0}^{-1} + m_h^{-1} n_{j_0}^{-1} (4n_{j_0-1})^{\log_2 m_{j_0}} \\ &\leqslant m_h^{-1} m_{j_0}^{-1} + m_h^{-1} m_{j_0}^{-2} \\ &\leqslant 2m_h^{-1} m_{j_0}^{-1}\end{aligned}$$

(by Assumption 2.3 (4)).

If we make the additional assumption that $f^* \in W[(\mathscr{A}_{4n_j}, m_j^{-1})_{j \neq j_0}]$ an analogous calculation shows that

$$\begin{aligned}\left|\left\langle f^*, n_{j_0}^{-1} \sum_{l=1}^{n_{j_0}} e_l \right\rangle\right| &\leqslant m_h^{-1} m_{j_0}^{-2} + m_h^{-1} n_{j_0}^{-1} \sum_{k=1}^{n} \#\{i : |f_k^*(i)| > m_{j_0}^{-2}\} \\ &\leqslant m_h^{-1} m_{j_0}^{-2} + m_h^{-1} n_{j_0}^{-1} 4n_h (4n_{j_0-1})^{\log_2 m_{j_0} - 1}\end{aligned}$$

(by Lemma 2.4 applied to $f_k^* \in W[(\mathscr{A}_{4n_j}, m_j^{-1})_{j \neq j_0}]$)

$$\leqslant m_h^{-1} m_{j_0}^{-1} + 2m_h^{-1} n_{j_0}^{-2}$$

(as before). □

3. The general Bourgain–Delbaen construction

In this section we shall present a generalization of the Bourgain–Delbaen construction of separable $\mathscr{L}_\infty$-spaces. Our approach is slightly different from that of [11] and [12], but the mathematical essentials are the same. We choose to set things out in some detail partly because we believe our approach yields new insights into the original BD construction, and partly because the calculations presented here are a good introduction to the notation and methods we use later. It is perhaps worth emphasizing here that BD constructions are very different from the majority of constructions that occur in Banach space theory. Normally we start with the unit vectors in the space c_{00} and complete with respect to some (possibly exotic) norm. The only norms that occur in a BD construction are the usual norms of ℓ_∞ and ℓ_1. What we construct here are exotic *vectors* in ℓ_∞ whose closed linear span is the space we want.

The idea will be to introduce a particular kind of (conditional) basis for the space ℓ_1 and to study the subspace X of ℓ_∞ spanned by the biorthogonal elements. Since ℓ_1 is then in a natural way a subspace of (and in some cases the whole of) X^*, we shall be thinking of elements of ℓ_1 as functionals and, in accordance with the convention explained earlier, denote them by b^*, c^* and so on. In our initial discussion we shall consider the space $\ell_1(\mathbb{N})$ (which we shall later replace by $\ell_1(\Gamma)$, with Γ being a certain countable set better adapted to our needs).

Definition 3.1. We shall say that a basic sequence $(d_n^*)_{n\in\mathbb{N}}$ in $\ell_1(\mathbb{N})$ is a *triangular basis* if $\operatorname{supp} d_n^* \subseteq \{1,2,...,n\}$, for all n. We thus have

$$d_n^* = \sum_{m=1}^{n} a_{n,m} e_m^*,$$

where, by linear independence, we necessarily have $a_{n,n} \neq 0$. Notice that the linear span $[d_1^*, d_2^*, ..., d_n^*]$ is the same as $[e_1^*, e_2^*, ..., e_n^*]$, that is to say, the space ℓ_1^n, regarded as a subspace of $\ell_1(\mathbb{N})$ in the usual way. So, in particular, the basic sequence $(d_n^*)_{n\in\mathbb{N}}$ is indeed a basis for the whole of ℓ_1. The biorthogonal sequence in ℓ_∞ will be denoted $(d_n)_{n\in\mathbb{N}}$; it is a weak* basis for ℓ_∞ and a basis for its closed linear span, which will be our space X.

Proposition 3.2. *If $(d_n^*)_{n\in\mathbb{N}}$ is a triangular basis for $\ell_1(\mathbb{N})$, with basis constant M, then the closed linear span $X=[d_n:n\in\mathbb{N}]$ is an $\mathscr{L}_{\infty,M}$-space. If $(d_n^*)_{n\in\mathbb{N}}$ is boundedly complete, or equivalently $(d_n)_{n\in\mathbb{N}}$ is shrinking, then X^* may be identified with $\ell_1(\mathbb{N})$, where $\|g^*\|_{X^*} \leqslant \|g^*\|_1 \leqslant M\|g^*\|_{X^*}$.*

Proof. In accordance with our "star" notation, let us write P_n^* for the basis projections $\ell_1 \to \ell_1$ associated with the basis $(d_n^*)_{n\in\mathbb{N}}$. Thus

$$P_n^*(d_m^*) = \begin{cases} d_m^*, & \text{if } m \leqslant n, \\ 0, & \text{otherwise;} \end{cases}$$

because $e_m^* \in \ell_1^n = [d_1^*, ..., d_n^*]$, we also have $P_n^* e_m^* = e_m^*$ when $m \leqslant n$. If we modify P_n^* by taking the codomain to be the image $\operatorname{im} P_n^* = \ell_1^n$, rather than the whole of ℓ_1, what we have is a quotient operator, which we shall denote by q_n, of norm at most M. The dual of this quotient operator is an isomorphic embedding $i_n: \ell_\infty^n \to \ell_\infty(\mathbb{N})$, also of norm at most M. If $m \leqslant n$ and $u \in \ell_\infty^n$, we have

$$(i_n u)(m) = \langle e_m^*, i_n u \rangle = \langle q_n e_m^*, u \rangle = \langle e_m^*, u \rangle = u(m).$$

So i_n is an *extension operator* $\ell_\infty^n \to \ell_\infty(\mathbb{N})$ and we have

$$\|u\|_\infty \leqslant \|i_n u\|_\infty \leqslant M \|u\|_\infty$$

for all $u \in \ell_\infty^n$. In particular, the image of i_n, which is exactly $[d_1, ..., d_n]$ is M-isomorphic to ℓ_∞^n, which implies that X is an $\mathscr{L}_{\infty,M}$-space.

In the case where $(d_n^*)_{n\in\mathbb{N}}$ is a boundedly complete basis of ℓ_1, X^* may be identified with ℓ_1 by a standard result about bases. Moreover, for $g^* \in \ell_1$, we have

$$\|g^*\|_{X^*} = \sup\{\langle g^*, x\rangle : x \in X \text{ and } \|x\|_\infty \leqslant 1\} \leqslant \|g^*\|_1.$$

On the other hand, if g^* has finite support, say $\operatorname{supp} g^* \subseteq \{1, 2, ..., n\}$, we can choose $u \in \ell_\infty^n$ with $\|u\| = 1$ and $\langle g^*, u\rangle = \|g^*\|_1$. The extension $x = i_n(u)$ is now in X and satisfies

$$\|x\| \leqslant M \quad \text{and} \quad \langle g^*, x\rangle = \|g^*\|_1.$$

Thus $\|g^*\|_1 \leqslant M \|g^*\|_{X^*}$. □

We shall say that $(d_n^*)_n$ is a *unit-triangular basis* of $\ell_1(\mathbb{N})$ if it is a triangular basis and the non-zero scalars $a_{n,n}$ are all equal to 1. We can thus write

$$d_n^* = e_n^* - c_n^*,$$

where $c_1^* = 0$ and $\operatorname{supp} c_n^* \subset \{1, 2, ..., n-1\}$ for $n \geqslant 2$. The clever part of the Bourgain–Delbaen construction is to find a method of choosing the c_n^* in such a way that $(d_n^*)_{n\in\mathbb{N}}$ is indeed a basic sequence. The idea is to proceed recursively assuming that, for some $n \geqslant 1$, we already have a unit-triangular basis $(d_m^*)_{m=1}^n$ of ℓ_1^n. The value of $P_r^* b^*$ is thus already determined when $1 \leqslant r \leqslant n$ and $b^* \in \ell_1^n$.

Definition 3.3. In the set-up described above, we shall say that an element c^* of ℓ_1^n is a BD-*functional* (with respect to the triangular basis $(d_m^*)_{m=1}^n$) if there exist real numbers $\alpha\in[0,1]$ and $\beta\in\left[0,\frac{1}{2}\right)$ such that we can express c^* in one of the following forms:

(0) αe_j^* with $1\leqslant j\leqslant n$,

(1) $\beta(I-P_k^*)b^*$ with $0\leqslant k<n$ and $b^*\in\operatorname{ball}\ell_1^n\cap[e_j^*:k<j\leqslant n]$,

(2) $\alpha e_j^*+\beta(I-P_k^*)b^*$ with $1\leqslant j\leqslant k<n$ and $b^*\in\operatorname{ball}\ell_1^n\cap[e_j^*:k<j\leqslant n]$.

We shall refer to such a functional c^* as being of *type* 0, 1 or 2, respectively. The non-negative constant β will be called the *weight* of the functional c^*. We may regard a type-0 functional as being *of weight zero*. Indeed, both (0) and (1) are "almost" special cases of (2), with β (resp. α) equal to 0. In the construction presented in this paper, we do not use functionals of type 0, and the constant α in case (2) is always equal to 1. However, it may be worth stating the following theorem in full generality.

THEOREM 3.4. ([11], [12]) *Let θ be a real number with $0<\theta<\frac{1}{2}$ and let $d_n^*=e_n^*-c_n^*$ in ℓ_1 be such that $c_1^*=0$ while, for each n, $c_{n+1}^*\in\ell_1^n$ is a* BD-*functional of weight at most θ with respect to $(d_m^*)_{m=1}^n$. Then $(d_n^*)_{n\in\mathbb{N}}$ is a triangular basis of ℓ_1, with basis constant at most $M=1/(1-2\theta)$. The subspace $X=[d_n:n\in\mathbb{N}]$ of ℓ_∞ is thus an $\mathscr{L}_{\infty,M}$-space.*

Proof. Despite the disguise, this is essentially the same argument as in the original papers of Bourgain and Delbaen. What we need to show is that P_m^* is a bounded operator, with $\|P_m^*\|\leqslant M$ for all m. Because we are working on the space ℓ_1, it is enough to show that $\|P_m^*e_n^*\|\leqslant M$ for every m and n.

First, if $n\leqslant m$, then $P_m^*e_n^*=e_n^*$, so there is nothing to prove. Now let us assume that $n\geqslant m$ and make the inductive hypothesis that $\|P_k^*e_j^*\|\leqslant M$ for all $k\leqslant m$ and all $j\leqslant n$; we need to look at $P_m^*e_{n+1}^*$. We use the fact that

$$e_{n+1}^*=d_{n+1}^*+c_{n+1}^*,$$

with $c_{n+1}^*\in\ell_1^n$ being a BD-functional. We shall consider a functional of type (2), which presents the most difficulty. We thus have

$$c_{n+1}^*=\alpha e_j^*+\beta(I-P_k^*)b^*,$$

where $1\leqslant j\leqslant k<n$ and α, β and b^* are as in Definition 3.3, and $\beta\leqslant\theta$ by our hypothesis. Now, because $n+1>m$, we have $P_m^*d_{n+1}^*=0$, so

$$P_m^*e_{n+1}^*=\alpha P_m^*e_j^*+\beta(P_m^*-P_{m\wedge k}^*)b^*.$$

If $k\geqslant m$ the second term vanishes, so that

$$\|P_m^*e_n^*\|=\alpha\|P_m^*e_j^*\|\leqslant\|P_m^*e_j^*\|,$$

which is at most M by our inductive hypothesis.

If, on the other hand, $k<m$, we certainly have $j<m$ so that $P_m^* e_j^* = e_j^*$, leading to the estimate

$$\|P_m^* e_{n+1}^*\| \leqslant \alpha \|e_j^*\| + \beta \|P_m^* b^*\| + \beta \|P_k^* b^*\|.$$

Now b^* is a convex combination of functionals $\pm e_l^*$ with $l \leqslant n$, and our inductive hypothesis is applicable to all of these. We thus obtain

$$\|P_m^* e_{n+1}^*\| \leqslant \alpha + 2M\beta \leqslant 1 + 2M\theta = M,$$

by the definition of $M = 1/(1-2\theta)$ and the assumption that $0 \leqslant \beta \leqslant \theta$. □

The $\mathscr{L}_\infty$-spaces of Bourgain and Delbaen, and those we construct in the present paper are of the above type. However, the "cuts" k that occur in the definition of BD-functionals are restricted to lie in a certain subset of $\mathbb{N}$, and thus naturally dividing the coordinate set $\mathbb{N}$ into successive intervals. As in [27], it will be convenient to replace the set $\mathbb{N}$ with a different countable set Γ having a structure that reflects this decomposition. This will also enable us later to use a notation in which an element $\gamma \in \Gamma$ automatically codes the BD-functional associated with it.

THEOREM 3.5. *Let $(\Delta_q)_{q\in\mathbb{N}}$ be a disjoint sequence of non-empty finite sets, with $\#\Delta_1 = 1$; write*

$$\Gamma_q = \bigcup_{p=1}^{q} \Delta_p \quad \text{and} \quad \Gamma = \bigcup_{p\in\mathbb{N}} \Delta_p.$$

Assume that there exists $\theta < \frac{1}{2}$ and a mapping τ defined on $\Gamma \setminus \Delta_1$, assigning to each $\gamma \in \Delta_{q+1}$ a tuple of one of the following forms:

(0) (α, ξ) *with* $0 \leqslant \alpha \leqslant 1$ *and* $\xi \in \Gamma_q$;

(1) (p, β, b^*) *with* $0 \leqslant p < q$, $0 < \beta \leqslant \theta$ *and* $b^* \in \operatorname{ball} \ell_1(\Gamma_q \setminus \Gamma_p)$;

(2) $(\alpha, \xi, p, \beta, b^*)$ *with* $0 < \alpha \leqslant 1$, $1 \leqslant p < q$, $\xi \in \Gamma_p$, $0 < \beta \leqslant \theta$ *and* $b^* \in \operatorname{ball} \ell_1(\Gamma_q \setminus \Gamma_p)$.

Then there exist $d_\gamma^* = e_\gamma^* - c_\gamma^* \in \ell_1(\Gamma)$ and projections $P_{(0,q]}^*$ on $\ell_1(\Gamma)$ uniquely determined by the following properties:

(A) $P_{(0,q]}^* d_\gamma^* = \begin{cases} d_\gamma^*, & \text{if } \gamma \in \Gamma_q, \\ 0, & \text{if } \gamma \in \Gamma \setminus \Gamma_q. \end{cases}$

(B) $c_\gamma^* = \begin{cases} 0, & \text{if } \gamma \in \Delta_1, \\ \alpha e_\xi^*, & \text{if } \tau(\gamma) = (\alpha, \xi), \\ \beta (I - P_{(0,p]}^*) b^*, & \text{if } \tau(\gamma) = (p, \beta, b^*), \\ \alpha e_\xi^* + \beta (I - P_{(0,p]}^*) b^*, & \text{if } \tau(\gamma) = (\alpha, \xi, p, \beta, b^*). \end{cases}$

Furthermore, the family $(d^*_\gamma)_{\gamma\in\Gamma}$ *is a basis for* $\ell_1(\Gamma)$ *with basis constant at most* $M=(1-2\theta)^{-1}$. *The norm of each projection* $P^*_{(0,q]}$ *is at most* M. *The biorthogonal vectors* d_γ *generate an* $\mathscr{L}_{\infty,M}$*-subspace* $X(\Gamma,\tau)$ *of* $\ell_\infty(\Gamma)$. *For each* q *and each* $u\in\ell_\infty(\Gamma_q)$, *there is a unique* $i_q(u)\in[d_\gamma:\gamma\in\Gamma_q]$ *whose restriction to* Γ_q *is* u; *the extension operator* $i_q:\ell_\infty(\Gamma_q)\to X(\Gamma,\tau)$ *has norm at most* M. *The subspaces* $M_q=[d_\gamma:\gamma\in\Delta_q]=i_q[\ell_\infty(\Delta_q)]$ *form a finite-dimensional decomposition* (FDD) *for* X; *if this* FDD *is shrinking then* X^* *is naturally isomorphic to* $\ell_1(\Gamma)$.

Proof. We shall show that, with a suitable identification of Γ with $\mathbb{N}$, this theorem is just a special case of Theorem 3.4. Let $k_p=\#\Gamma_p$ and let $n\mapsto\gamma(n):\mathbb{N}\to\Gamma$ be a bijection with the property that $\Delta_1=\{\gamma(1)\}$, while, for each $q\geqslant 2$, $\Delta_q=\{\gamma(n):k_{q-1}<n\leqslant k_q\}$. There is a natural isometry $J:\ell_1(\mathbb{N})\to\ell_1(\Gamma)$ satisfying $J(e^*_n)=e^*_{\gamma(n)}$. It is straightforward to check that if $d^*_n=J^{-1}(d^*_{\gamma(n)})=e^*_n-c^*_n$, then the hypotheses of Theorem 3.4 are satisfied. (The cuts k that occur in the BD-functionals c^*_n are all of the form $k=k_p$.) All the assertions in the present theorem are now immediate consequences. The projections $P^*_{(0,q]}$ whose existence is claimed here are given by $P^*_{(0,q]}=JP^*_{k_q}J^{-1}$, where P^*_n is the basis projection of Theorem 3.4. When ordered as $(d_{\gamma(n)})_{n\in\mathbb{N}}$, the vectors d_γ form a basis of their closed linear span, which is an $\mathscr{L}_{\infty,M}$-space. The extension operator, which (by abuse of notation) we here denote by i_q, corresponds to the i_{k_q} that occurs in the proof of Proposition 3.2. The assertions about the subspaces $M_q=[d_{\gamma(n)}:k_{q-1}<n\leqslant k_q]$ follow from the fact that $(d_{\gamma(n)})_{n\in\mathbb{N}}$ is a basis. □

We now make a few observations about the space $X=X(\Gamma,\tau)$ and the functions d_γ, taking the opportunity to introduce notation that will be used in the rest of the paper. We have seen that for each $\gamma\in\Delta_{n+1}$ the functional d^*_γ has support contained in $\Gamma_n\cup\{\gamma\}$. Using biorthogonality, we see that d_γ is supported by $\{\gamma\}\cup(\Gamma\setminus\Gamma_{n+1})$. It should be noted that we should not expect the support of d_γ to be finite; in fact, in all interesting cases, we have $X\cap c_0(\Gamma)=\{0\}$.

As noted above, the subspaces $M_n=[d_\gamma:\gamma\in\Delta_n]$ form a finite-dimensional decomposition for X. For each interval $I\subseteq\mathbb{N}$ we define the projection $P_I:X\to\bigoplus_{n\in I}M_n$ in the natural way; this is consistent with our use of $P^*_{(0,n]}$ in Theorem 3.5. Most of our arguments will involve sequences of vectors that are block sequences with respect to this FDD. Since we are using the word "support" to refer to the set of γ where a given function is non-zero, we need some other terminology for the set of n such that x has a non-zero component in M_n. We define the *range* of x, denoted $\operatorname{ran} x$, to be the smallest interval $I\subseteq\mathbb{N}$ such that $x\in\bigoplus_{n\in I}M_n$. It is worth noting that if $\operatorname{ran} x=(p,q]$ then we can write $x=i_q(u)$, where $u=x|_{\Gamma_q}\in\ell_\infty(\Gamma_q)$ satisfies $\Gamma_p\cap\operatorname{supp} u=\varnothing$.

4. Construction of $\mathfrak{B}_{\mathbf{mT}}$ and $\mathfrak{X}_{\mathbf{K}}$

We now set about constructing specific BD spaces which will be modelled on mixed Tsirelson spaces, in rather the same way that the original spaces of Bourgain and Delbaen have been found to be modelled on ℓ_p. We shall adopt a notation in which elements γ of Δ_{n+1} automatically code the corresponding BD-functionals. This will allow us to write $X(\Gamma)$ rather than $X(\Gamma,\tau)$ for the resulting $\mathscr{L}_\infty$-space. To be more precise, an element γ of Δ_{n+1} will be a tuple of one of the forms:

(1) $\gamma=(n+1,\beta,b^*)$, in which case $\tau(\gamma)=(0,\beta,b^*)$;

(2) $\gamma=(n+1,\xi,\beta,b^*)$, in which case $\tau(\gamma)=(1,\xi,\operatorname{rank}\xi,\beta,b^*)$.

In each case, the first coordinate of γ tells us what the *rank* of γ is, that is to say to which set Δ_{n+1} it belongs, while the remaining coordinates specify the corresponding BD-functional.

It will be observed that BD-functionals of type 0 (in the terminology of Definition 3.3) do not arise in this construction and that the p in the definition of a type-1 functional is always 0. In the definition of a type-2 functional, the scalar α that occurs is always 1 and p equals $\operatorname{rank}\xi$. We shall make the further restriction that the weight β must be of the form m_j^{-1}, where the sequences $(m_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$ satisfy Assumption 2.3. We shall say that the element γ has *weight* m_j^{-1}. Notice that here (in contrast to §2.4) the weight is unambiguously defined. In the case of a type-2 element $\gamma=(n+1,\xi,m_j^{-1},b^*)$, we shall insist that ξ be of the same weight m_j^{-1} as γ.

To ensure that our sets Δ_{n+1} are finite, we shall admit into Δ_{n+1} only elements of weight m_j with $j\leqslant n+1$. A further restriction involves a recursively-defined function which we call age. For a type-1 element $\gamma=(n+1,\beta,b^*)$ we define $\operatorname{age}\gamma=1$. For a type-2 element $\gamma=(n+1,\xi,m_j^{-1},b^*)$, we define $\operatorname{age}\gamma=1+\operatorname{age}\xi$, and further restrict the elements of Δ_{n+1} by insisting that the age of an element of weight m_j^{-1} may not exceed n_j. Finally, we shall restrict the functionals b^* that occur in an element of Δ_{n+1} by requiring them to lie in some finite subset B_n of $\ell_1(\Gamma_n)$. It is convenient to fix an increasing sequence of natural numbers $(N_n)_{n\in\mathbb{N}}$ and take $B_{p,n}$ to be the set of all linear combinations

$$b^* = \sum_{\eta\in\Gamma_n\setminus\Gamma_p} a_\eta e_\eta^*,$$

where $\sum_{\eta\in\Gamma_n\setminus\Gamma_p}|a_\eta|\leqslant 1$ and each a_η is a rational with denominator dividing $N_n!$. We may suppose the N_n are chosen in such a way that $B_{p,n}$ is a 2^{-n}-net in the unit ball of $\ell_1(\Gamma_n\setminus\Gamma_p)$. The above restrictions may be summarized as follows.

Assumption 4.1.

$$\Delta_{n+1} \subseteq \bigcup_{j=1}^{n+1} \{(n+1, m_j^{-1}, b^*) : b^* \in B_{0,n}\}$$
$$\cup \bigcup_{p=1}^{n-1} \bigcup_{j=1}^{p} \{(n+1, \xi, m_j^{-1}, b^*) : \xi \in \Delta_p, \text{weight}\, \xi = m_j^{-1}, \text{age}\, \xi < n_j, b^* \in B_{p,n}\}.$$

We shall also assume that Δ_{n+1} contains a rich supply of elements of "even weight", more exactly of weight m_j^{-1} with j even.

Assumption 4.2.

$$\Delta_{n+1} \supseteq \bigcup_{j=1}^{\lfloor (n+1)/2 \rfloor} \{(n+1, m_{2j}^{-1}, b^*) : b^* \in B_{0,n}\}$$
$$\cup \bigcup_{p=1}^{n-1} \bigcup_{j=1}^{\lfloor p/2 \rfloor} \{(n+1, \xi, m_{2j}^{-1}, b^*) : \xi \in \Delta_p, \text{weight}\, \xi = m_{2j}^{-1}, \text{age}\, \xi < n_{2j}, b^* \in B_{p,n}\}.$$

For our main construction, there are additional restrictions on the elements with "odd weight" m_{2j-1}^{-1}. However, there is some interest already in the space we obtain without making such restrictions. We denote this space $\mathfrak{B}_{\mathrm{mT}}$; it is an isomorphic predual of ℓ_1 which is unconditionally saturated but contains no copy of c_0 or ℓ_p. An analogous space $\mathfrak{B}_T$, modelled on the standard Tsirelson space, rather than a mixed Tsirelson space, was constructed a few years ago by Haydon [28].

Definition 4.3. We define $\mathfrak{B}_{\mathrm{mT}} = \mathfrak{B}_{\mathrm{mT}}[(m_j, n_j)_{j \in \mathbb{N}}]$ to be the space $X(\Gamma)$, where $\Gamma = \Gamma^{\max}$ is defined by the recursion $\Delta_1 = \{1\}$ and

$$\Delta_{n+1} = \bigcup_{j=1}^{n+1} \{(n+1, m_j^{-1}, b^*) : b^* \in B_{0,n}\}$$
$$\cup \bigcup_{p=1}^{n-1} \bigcup_{j=1}^{p} \{(n+1, \xi, m_j^{-1}, b^*) : \xi \in \Delta_p, \text{weight}\, \xi = m_j^{-1}, \text{age}\, \xi < n_j, b^* \in B_{p,n}\}.$$

The extra constraints that we place on "odd-weight" elements in order to obtain hereditary indecomposability will involve a coding function that will produce the analogues of the "special functionals" that occur in [23] and other HI constructions. In our case, all we need is an injective function $\sigma : \Gamma \to \mathbb{N}$ satisfying $\sigma(\gamma) > \text{rank}\, \gamma$ for all γ. This may easily be included in our recursive construction of Γ. We then insist that a type-1 element of odd weight must have the form

$$(n+1, m_{2j-1}^{-1}, e_\eta^*)$$

with rank $\eta \leqslant n$ and weight $\eta = m_{4i-2}^{-1} < n_{2j-1}^{-2}$, while a type-2 element of odd weight must be

$$(n+1, \xi, m_{2j-1}^{-1}, e_\eta^*)$$

with rank $\xi <$ rank $\eta \leqslant n$ and weight $\eta = m_{4\sigma(\xi)}^{-1}$.

Definition 4.4. We define $\mathfrak{X}_{\mathrm{K}} = \mathfrak{X}_{\mathrm{K}}[(m_j, n_j)_{j\in\mathbb{N}}]$ to be the space $X(\Gamma)$, where $\Gamma = \Gamma^{\mathrm{K}}$ is defined by the recursion $\Delta_1 = \{1\}$ and

$$\begin{aligned}
\Delta_{n+1} = & \bigcup_{j=1}^{\lfloor (n+1)/2 \rfloor} \{(n+1, m_{2j}^{-1}, b^*) : b^* \in B_{0,n}\} \\
& \cup \bigcup_{p=1}^{n-1} \bigcup_{j=1}^{\lfloor p/2 \rfloor} \{(n+1, \xi, m_{2j}^{-1}, b^*) : \xi \in \Delta_p, \text{weight}\, \xi = m_{2j}^{-1}, \text{age}\, \xi < n_{2j}, b^* \in B_{p,n}\} \\
& \cup \bigcup_{j=1}^{\lfloor (n+2)/2 \rfloor} \{(n+1, m_{2j-1}^{-1}, e_\eta^*) : \eta \in \Gamma_n \text{ and weight}\, \eta = m_{4i-2}^{-1} < n_{2j-1}^{-2}\} \\
& \cup \bigcup_{p=1}^{n-1} \bigcup_{j=1}^{\lfloor (p+1)/2 \rfloor} \{(n+1, \xi, m_{2j-1}^{-1}, e_\eta^*) : \xi \in \Delta_p, \text{weight}\, \xi = m_{2j-1}^{-1}, \\
& \qquad \text{age}\, \xi < n_{2j-1}, \eta \in \Gamma_n \setminus \Gamma_p, \text{weight}\, \eta = m_{4\sigma(\xi)}^{-1}\}.
\end{aligned}$$

Although the structure of the space $X(\Gamma)$ is most easily understood in terms of the basis $(d_\gamma)_{\gamma\in\Gamma}$ and the biorthogonal functionals d_γ^*, it is with the evaluation functionals e_γ^* that we have to deal in order to estimate norms. The recursive definition of the functionals d_γ^* can be unpicked to yield the following proposition.

PROPOSITION 4.5. *Assume that the set Γ satisfies Assumption* 4.1. *Let n be a positive integer and let γ be an element of Δ_{n+1} of weight m_j^{-1} and age $a \leqslant n_j$. Then there exist natural numbers $0 = p_0 < p_1 < \ldots < p_a = n+1$, elements $\xi_1, \ldots, \xi_a = \gamma$ of weight m_j^{-1} with $\xi_r \in \Delta_{p_r}$ and functionals $b_r^* \in \text{ball}\, \ell_1(\Gamma_{p_r-1} \setminus \Gamma_{p_{r-1}})$ such that*

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + m_j^{-1} \sum_{r=1}^{a} P_{(p_{r-1},\infty)}^* b_r^* = \sum_{r=1}^{a} d_{\xi_r}^* + m_j^{-1} \sum_{r=1}^{a} P_{(p_{r-1},p_r)}^* b_r^*.$$

If $1 \leqslant t < a$ then we have

$$e_\gamma^* = e_{\xi_t}^* + \sum_{r=t+1}^{a} d_{\xi_r}^* + m_j^{-1} \sum_{r=t+1}^{a} P_{(p_{r-1},\infty)}^* b_r^*.$$

Proof. Given Assumption 4.1, this is an easy induction on the age a of γ. If $a=1$ then γ has the form $(n+1, m_j^{-1}, b^*)$ and

$$e_\gamma^* = d_\gamma^* + c_\gamma^*,$$

where c^*_γ is the type-1 BD-functional

$$c^*_\gamma = m_j^{-1} P^*_{(0,\infty)} b^* = m_j^{-1} b^*,$$

with $b^* \in B_{0,n} \subset \mathrm{ball}\,\ell_1(\Gamma_n)$. So, if we set $p_0=0$, $p_1=n+1$, $b^*_1=b^*$ and $\xi_1=\gamma$, we have

$$e^*_\gamma = d^*_{\xi_1} + m_j^{-1} P^*_{(p_0,\infty)} b^*_1 = d^*_{\xi_1} + m_j^{-1} P^*_{(p_0,p_1)} b^*_1.$$

If $a>1$ then γ has the form $(n+1, \xi, m_j^{-1}, b^*)$ and c^*_γ is the type-2 BD-functional

$$c^*_\gamma = e^*_\xi + m_j^{-1} P^*_{(p,\infty)} b^*.$$

If we apply our inductive hypothesis to the element ξ of weight m_j^{-1}, rank p and age $a-1$, we obtain the desired expressions for e^*_γ. □

We shall refer to the first identity presented in the above proposition as the *evaluation analysis* of γ and shall use it repeatedly in norm estimations. The form of the second term in the evaluation analysis, involving a sum weighted by m_j^{-1}, indicates that there is going to be a connection with mixed Tsirelson spaces; the first term, involving functionals d^*_ξ, with no weighting, can cause inconvenience in some of our calculations, but is an inevitable feature of the BD construction. The data $(p_r, b^*_r, \xi_r)_{r=1}^a$ will be called the *analysis* of γ. We note that if $1 \leqslant s \leqslant a$ the analysis of ξ_s is just $(p_r, b^*_r, \xi_r)_{r=1}^s$.

With the definition still readily at hand, this is a convenient moment to record an important "tree-like" property of odd-weight elements of Γ^{K}, even though we shall not be exploiting these special elements until later on.

Lemma 4.6. *Let γ and γ' be two elements of Γ^{K} both of weight m_{2h-1}^{-1} and of ages $a \geqslant a'$, respectively. Let $(p_i, e^*_{\eta_i}, \xi_i)_{i=1}^a$, resp. $(p'_i, e^*_{\eta'_i}, \xi'_i)_{i=1}^{a'}$, be the analysis of γ, resp. γ'. Then there exists l with $1 \leqslant l \leqslant a'$ such that $\xi'_i = \xi_i$ when $i<l$, while* $\mathrm{weight}\,\eta_j \neq \mathrm{weight}\,\eta'_i$ *for all j when $l < i \leqslant a'$.*

Proof. If $\mathrm{weight}\,\eta'_i \neq \mathrm{weight}\,\eta_j$ for all $i \geqslant 2$ and all j, then there is nothing to prove (we may take $l=1$). Otherwise, let $2 \leqslant l \leqslant a$ be maximal subject to the existence of j such that $\mathrm{weight}\,\eta_j = \mathrm{weight}\,\eta'_l$. Now this weight is exactly $m_{4\sigma(\xi'_{l-1})}$, which means that j cannot be 1 (because the weight of η_1 has the form m_{4k-2}). Thus $\sigma(\xi'_{l-1}) = \sigma(\xi_{j-1})$, which implies that $\xi'_{l-1} = \xi_{j-1}$ by the injectivity of σ. Since $l-1 = \mathrm{age}\,\xi'_{l-1}$ and $j-1 = \mathrm{age}\,\xi_{j-1}$, we deduce that $j=l$. Moreover, since the elements ξ_i with $i<l-1$ are determined by ξ_{l-1}, we have $\xi_i = \xi'_i$ for $i<l$. □

In the remainder of this section, and in the next one, we shall be dealing with a space $X = X(\Gamma)$ and shall be making the Assumptions 4.1 and 4.2. Our results thus apply both to $\mathfrak{B}_{\mathrm{mT}}$ and $\mathfrak{X}_{\mathrm{K}}$. We note that, since the weights m_j^{-1} are all at most $\frac{1}{4}$, the constant M in Theorem 3.5 may be taken to be 2. This leads to the following norm estimates for the extension operators i_n and for the projections P_I associated with the FDD $(M_n)_{n\in\mathbb{N}}$:

$$\|i_n\| = \|P_{(0,n]}\| \leqslant 2, \quad \|P_{(n,\infty)}\| \leqslant 3, \quad \|P_{(m,n]}\| \leqslant 4, \quad \|d_\xi^*\| = \|P^*_{[\operatorname{rank}\xi,\infty)} e_\xi^*\| \leqslant 3.$$

Assumption 4.2 enables us to write down a kind of converse to Proposition 4.5 which will lead to our first norm estimate.

PROPOSITION 4.7. *Let j and $a\leqslant n_{2j}$ be positive integers, let $0=p_0<p_1<...<p_a$ be natural numbers with $p_1\geqslant 2j$ and let b_r^* be functionals in B_{p_{r-1},p_r-1} for $1\leqslant r\leqslant a$. Then there are elements $\xi_r\in\Gamma_{p_r}$ such that the analysis of $\gamma=\xi_a$ is $(p_r, b_r^*, \xi_r)_{r=1}^a$.*

Proof. This is another easy induction on a. For $a=1$, Assumption 4.2 and the hypothesis that $p_1\geqslant 2j$ guarantee that the tuple $\xi_1=(p_1, m_{2j}^{-1}, b_1^*)$ is in Γ_{p_1}. We continue recursively, setting $\xi_{r+1}=(p_{r+1}, \xi_r, m_{2j}^{-1}, b_{r+1}^*)$. □

PROPOSITION 4.8. *Let $(x_r)_{r=1}^a$ be a skipped block sequence (with respect to the* FDD *$(M_n)_{n\in\mathbb{N}}$) in X. If j is a positive integer such that $a\leqslant n_{2j}$ and $2j<\min\operatorname{ran} x_2$, then there exists an element γ of weight m_{2j}^{-1} satisfying*

$$\sum_{r=1}^a x_r(\gamma) \geqslant \frac{1}{2} m_{2j}^{-1} \sum_{r=1}^a \|x_r\|.$$

Hence

$$\left\|\sum_{r=1}^a x_r\right\| \geqslant \frac{1}{2} m_{2j}^{-1} \sum_{r=1}^a \|x_r\|.$$

Proof. Let $p_0=0$, and choose $p_1, p_2, ..., p_a$ in such a way that $\operatorname{ran} x_r\subseteq (p_{r-1}, p_r)$. Thus we have $x_r=i_{p_r-1}(u_r)$, where the element $u_r=x_r|_{\Gamma_{p_r-1}}$ has support disjoint from $\Gamma_{p_{r-1}}$. Since $\|i_n\|\leqslant 2$ for all n, we have $\|u_r\|\geqslant\frac{1}{2}\|x_r\|$, and so there exist $\eta_r\in\Gamma_{p_r-1}\setminus\Gamma_{p_{r-1}}$ with

$$|u_r(\eta_r)| \geqslant \tfrac{1}{2}\|x_r\|.$$

The functional $b_r^*=\pm e_{\eta_r}^*$ is certainly in B_{p_{r-1},p_r-1} and with a suitable choice of sign we may arrange that

$$\langle b_r^*, x_r\rangle = |u_r(\eta_r)| \geqslant \tfrac{1}{2}\|x_r\|.$$

By Proposition 4.7, there is an element γ of Δ_{p_a} whose analysis is $(p_r, b_r^*, \xi_r)_{r=1}^a$. We shall use the evaluation analysis to calculate

$$\sum_{s=1}^a x_s(\gamma) = \left\langle e_\gamma^*, \sum_{s=1}^a x_s\right\rangle.$$

For any r and s, $x_s \in [d_\xi : p_{s-1} < \operatorname{rank} \xi < p_s]$, while $\operatorname{rank} \xi_r = p_r$, whence

$$\langle d^*_{\xi_r}, x_s \rangle = 0 \quad \text{for all } r \text{ and } s,$$

while

$$\langle P^*_{(p_{r-1},p_r)} b^*_r, x_s \rangle = \langle b^*_r, P_{(p_{r-1},p_r)} x_s \rangle = 0,$$

for all $r \neq s$. In the case $r=s$ we have

$$\langle P^*_{(p_{r-1},p_r)} b^*_r, x_r \rangle = \langle b^*_r, P_{(p_{r-1},p_r)} x_r \rangle = \langle b^*_r, x_r \rangle.$$

The evaluation analysis thus simplifies to yield

$$\sum_{r=1}^{a} x_r(\gamma) = m_{2j}^{-1} \sum_{r=1}^{a} \langle b^*_r, x_r \rangle \geqslant \frac{1}{2} m_{2j}^{-1} \sum_{r=1}^{a} \|x_r\|. \qquad \square$$

The lower estimate we have just obtained indicates that there is a close connection between our space X and mixed Tsirelson spaces of the kind considered in §2.4. With a bit more work, one can show that a normalized skipped-block sequence in X dominates the unit vector basis of $T[(\mathscr{A}_{n_{2j}}, m_{2j}^{-1})_{j\in\mathbb{N}}]$. We shall not need this more precise result in the present work.

5. Rapidly increasing sequences

We continue to work with the space $X=X(\Gamma)$, introduced in the previous section. We saw in the last section that skipped block sequences admit useful mixed Tsirelson lower estimates provided Assumption 4.2 holds. We now pass to a class of block sequences that admit upper estimates of a similar kind provided Assumption 4.1 is true. The following definition is a variant of something that is familiar from other HI constructions.

Definition 5.1. Let I be an interval in $\mathbb{N}$ and let $(x_k)_{k\in I}$ be a block sequence (with respect to the FDD $(M_n)_{n\in\mathbb{N}}$). We say that $(x_k)_{k\in I}$ is a *rapidly increasing sequence*, or RIS, if there exists a constant C such that the following hold:

(1) $\|x_k\| \leqslant C$ for all $k \in I$,

and there is an increasing sequence $(j_k)_{k\in I}$ such that, for all k,

(2) $j_{k+1} > \max \operatorname{ran} x_k$,

(3) $|x_k(\gamma)| \leqslant C m_i^{-1}$ whenever $\operatorname{weight} \gamma = m_i^{-1}$ and $i < j_k$.

If we need to be specific about the constant, we shall refer to a sequence satisfying the above conditions as a C-RIS.

LEMMA 5.2. *Let $(x_k)_{k\in I}$ be a C-RIS and let $(j_k)_{k\in I}$ be an increasing sequence of natural numbers as in the definition. If $\gamma\in\Gamma$ and* $\operatorname{weight}\gamma=m_i^{-1}$ *then, for any natural s,*

$$|\langle e_\gamma^*, P_{(s,\infty)}x_k\rangle| \leqslant \begin{cases} 6Cm_i^{-1}, & \text{if } i<j_k,\\ 3Cm_i^{-1}, & \text{if } i\geqslant j_{k+1}.\end{cases}$$

Proof. We first consider the case where $i\geqslant j_{k+1}$, noting that this implies that $i>\max\operatorname{ran} x_k$ by RIS condition (2). As in Proposition 4.5, we may write down the evaluation analysis of γ as

$$e_\gamma^* = \sum_{r=1}^{\alpha} d_{\xi_r}^* + m_i^{-1}\sum_{r=1}^{\alpha} b_r^*\circ P_{(p_{r-1},\infty)},$$

where $0=p_0<...<p_{\alpha-1}$, and b_r^* is a norm-1 element of $\ell_1(\Gamma)$, supported by $\Gamma_{p_r-1}\setminus\Gamma_{p_{r-1}}$, while ξ_r is of rank p_r and weight m_i^{-1}. Since Δ_q contains no elements of weight m_i^{-1} unless $q\geqslant i$, we must have $p_1\geqslant i$. Thus $p_1>\max\operatorname{ran} x_k$, from which it follows that

$$P_{(p_r,\infty)}\circ P_{(s,\infty)}x_k = P_{(s\vee p_r,\infty)}x_k = 0$$

for all $r\geqslant 1$. For the same reason, we also have

$$\langle d_{\xi_r}^*, P_{(s,\infty)}x_k\rangle = \langle P_{[p_r,\infty)}^* e_{\xi_r}^*, P_{(s,\infty)}x_k\rangle = \langle e_{\xi_r}^*, P_{(s,\infty)}P_{[p_r,\infty)}x_k\rangle = 0$$

for all r. We are left with

$$|\langle e_\gamma^*, P_{(s,\infty)}x_k\rangle| = m_i^{-1}|\langle b_1^*, P_{(s,\infty)}x_k\rangle| \leqslant m_i^{-1}\|P_{(s,\infty)}\|\,\|x_k\| \leqslant 3Cm_i^{-1}.$$

In the case where $i<j_k$, we again use the evaluation analysis, but need to be more careful about the value of s. Since we shall need this argument again, we state it as a separate lemma. Clearly, what we need here is an immediate consequence if we put $\delta=Cm_i^{-1}$. □

LEMMA 5.3. *Let i be a positive integer and suppose that $x\in X$ has the property that $\|x\|\leqslant C$ and $|x(\xi)|\leqslant\delta$ whenever* $\operatorname{weight}\xi=m_i^{-1}$. *Then for any s and any γ of weight m_i^{-1} we have*

$$|\langle e_\gamma^*, P_{(s,\infty)}x\rangle| \leqslant 2\delta+4Cm_i^{-1}.$$

Proof. As before we consider the evaluation analysis

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + m_i^{-1}\sum_{r=1}^{a} b_r^*\circ P_{(p_{r-1},\infty)}.$$

If $s \geqslant p_a$ then $P^*_{(s,\infty)} e^*_\gamma = 0$. If $0 < s < p_1$, by applying $P^*_{(0,s]}$ to each of the terms in the evaluation analysis, we see that

$$P^*_{(s,\infty)} e^*_\gamma = e^*_\gamma - P^*_{(0,s]} e^*_\gamma = e^*_\gamma - m_i^{-1} P^*_{(0,s]} b^*_1,$$

which leads to

$$|\langle e^*_\gamma, P_{(s,\infty)} x\rangle| \leqslant \delta + m_i^{-1} \|b^*_1\| \, \|P_{(0,s]}\| \, \|x\| \leqslant \delta + 2Cm_i^{-1},$$

by our assumptions.

In the remaining case, there is some t with $1 \leqslant t < a$ such that $p_t \leqslant s$ while $p_{t+1} > s$. As in Proposition 4.5, we may rewrite the evaluation analysis of γ as

$$e^*_\gamma = e^*_{\xi_t} + \sum_{r=t+1}^{a} d^*_{\xi_r} + m_i^{-1} \sum_{r=t+1}^{a} P^*_{(p_{r-1},\infty)}(b^*_r),$$

which gives us

$$P^*_{(s,\infty)} e^*_\gamma = e^*_\gamma - P^*_{(0,s]}(e^*_\gamma) = e^*_\gamma - e^*_{\xi_t} - m_i^{-1} P^*_{(p_t,s]}(b^*_{t+1}).$$

When we recall that weight ξ_t = weight γ, this yields

$$|\langle e^*_\gamma, P_{(s,\infty)} x\rangle| \leqslant 2\delta + m_i^{-1} \|P^*_{(p_t,s]}\| \, \|x\| \leqslant 4Cm_i^{-1},$$

as above. □

Proposition 5.4. (Basic inequality) *Let $(x_k)_{k\in I}$ be a C-RIS, let λ_k be real numbers, let s be a natural number and let γ be an element of Γ. Then there exist $k_0 \in I$ and a functional $g^* \in W[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ such that*

(1) *either* $g^* = 0$, *or* $\mathrm{weight}(g^*) = \mathrm{weight}(\gamma)$ *and* $\operatorname{supp} g^* \subseteq \{k \in I : k > k_0\}$;

(2)
$$\left|\left\langle e^*_\gamma, P_{(s,\infty)} \sum_{k\in I} \lambda_k x_k \right\rangle\right| \leqslant 4C|\lambda_{k_0}| + 6C\left\langle g^*, \sum_{k\in I} |\lambda_k| e_k \right\rangle.$$

Moreover, if j_0 is such that

$$\left|\left\langle e^*_\xi, \sum_{k\in J} \lambda_k x_k \right\rangle\right| \leqslant C \max_{k\in J} |\lambda_k|,$$

for all subintervals J of I and all $\xi \in \Gamma$ of weight $m_{j_0}^{-1}$, then we may choose g^* to be in $W[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$.

Proof. We proceed by induction on the rank of γ, noting that if γ is of rank 1 then we have $P^*_{(s,\infty)}e^*_\gamma=0$ whenever $s\geqslant 1$, so that

$$\left\langle e^*_\gamma, P_{(s,\infty)}\sum_{k\in I}\lambda_k x_k\right\rangle = \begin{cases} 0, & \text{if } s\geqslant 1,\\ \lambda_1 x_1(\gamma), & \text{if } s=0.\end{cases}$$

Thus $k_0=1$ and $g^*=0$ have the desired property.

Now consider an element γ of rank greater than 1, of age a and of weight m_h^{-1}. Taking $(j_k)_{k\in\mathbb{N}}$ to be a sequence as in the definition of RIS, we shall suppose that there is some $l\in I$ such that $j_l\leqslant h<j_{l+1}$. (The cases where $h<j_k$ for all $k\in I$ and where $h\geqslant j_{k+1}$ for all $k\in I$ are simpler.)

We split the summation over k into three parts as follows:

$$\left\langle e^*_\gamma, P_{(s,\infty)}\sum_{k\in I}\lambda_k x_k\right\rangle = \sum_{\substack{k\in I\\ k<l}}\lambda_k\langle e^*_\gamma, P_{(s,\infty)}x_k\rangle + \langle e^*_\gamma, P_{(s,\infty)}\lambda_l x_l\rangle + \left\langle e^*_\gamma, P_{(s,\infty)}\sum_{\substack{k\in I\\ k>l}}\lambda_k x_k\right\rangle,$$

and estimate the three terms separately.

When $k<l$ we have $h\geqslant j_l\geqslant j_{k+1}$ so that

$$|\langle e^*_\gamma, P_{(s,\infty)}\lambda_k x_k\rangle|\leqslant 3Cm_h^{-1}|\lambda_k|\leqslant 3Cm_{j_k}^{-1}|\lambda_k|,$$

by Lemma 5.2. Using the inequality $m_j\geqslant 4^j$, which follows from Assumption 2.3 (2), we obtain

$$\left|\sum_{\substack{k\in I\\ k<l}}\lambda_k\langle e^*_\gamma, P_{(s,\infty)}x_k\rangle\right|\leqslant 3C\sum_{k<l}m_{j_k}^{-1}|\lambda_k|\leqslant 3C\sum_{j=1}^{\infty}m_j^{-1}\max_{k<l}|\lambda_k|\leqslant C\max_{k<l}|\lambda_k|.$$

For the second term, we have the immediate estimate

$$\left|\left\langle e^*_\gamma, P_{(s,\infty)}\lambda_l x_l\right\rangle\right|\leqslant \|P_{(s,\infty)}\|\,|\lambda_l|\,\|x_l\|\leqslant 3C|\lambda_l|.$$

Thus, putting the first two terms together, we have

$$\left|\left\langle e^*_\gamma, P_{(s,\infty)}\sum_{k\leqslant l}\lambda_k x_k\right\rangle\right|\leqslant C\max_{k<l}|\lambda_k|+3C|\lambda_l|\leqslant 4C|\lambda_{k_0}| \tag{5.1}$$

for a suitably chosen $k_0\leqslant l$.

We now have to estimate the last term

$$\left|\left\langle e^*_\gamma, \sum_{k\in I'}\lambda_k x'_k\right\rangle\right|,$$

where $I'=\{k\in I : k>l\}$ and $x'_k=P_{(s,\infty)}x_k$. We shall use the evaluation analysis of γ,

$$e^*_\gamma=\sum_{r=1}^{a} d^*_{\xi_r}+m_h^{-1}\sum_{r=1}^{a} b^*_r\circ P_{(p_{r-1},\infty)}.$$

Let $I'_0=\{k\in I' : \operatorname{ran} x'_k \text{ contains } p_r \text{ for some } r\}$, noting first that $\#I'_0\leqslant a$ and secondly that for $k\in I'\setminus I'_0$ the interval $\operatorname{ran} x'_k$ meets (p_{r-1},p_r) for at most one value of r. If, for $r\geqslant 1$, we set $I'_r=\{k\in I'\setminus I'_0 : \operatorname{ran} x_k \text{ meets } (p_{r-1},p_r) \text{ but no other } (p_{r'-1},p_{r'})\}$, then each I'_r is a subinterval of I' and we have

$$\langle e^*_\gamma, x'_k\rangle=m_h^{-1}\langle b^*_r, P_{(p_{r-1},\infty)}x'_k\rangle=m_h^{-1}\langle b^*_r, P_{(s\vee p_{r-1},\infty)}x_k\rangle$$

if $k\in I'_r$ $(r\geqslant 1)$, while

$$\langle e^*_\gamma, x'_k\rangle=0$$

if $k\in I'\setminus\bigcup_{r=0}^{a} I'_r$. Thus,

$$\left\langle e^*_\gamma,\sum_{k\in I'}\lambda_k x'_k\right\rangle=\left\langle e^*_\gamma,\sum_{k\in I'_0}\lambda_k x'_k\right\rangle+m_h^{-1}\sum_{r=1}^{a}\left\langle b^*_r, P_{(s\vee p_{r-1},\infty)}\sum_{k\in I'_r}\lambda_k x_k\right\rangle.$$

Applying Lemma 5.2, we see that, for any $k\in I'$,

$$|\langle e^*_\gamma, x'_k\rangle|=|\langle e^*_\gamma, P_{(s,\infty)}(x_k)\rangle|\leqslant 6Cm_h^{-1}.$$

Applying this inequality just to the $k\in I'_0$, we deduce that

$$\left|\left\langle e^*_\gamma,\sum_{k\in I'}\lambda_k x'_k\right\rangle\right|\leqslant 6Cm_h^{-1}\sum_{k\in I'_0}|\lambda_k|+m_h^{-1}\left|\sum_{r=1}^{a}\left\langle b^*_r, P_{(s\vee p_{r-1},\infty)}\sum_{k\in I'_r}\lambda_k x_k\right\rangle\right|. \tag{5.2}$$

Now, for each $r\geqslant 1$, the functional b^*_r is a convex combination of functionals $\pm e^*_\eta$ with $p_{r-1}<\operatorname{rank}\eta<p_r$, so we may choose η_r to be such an η with

$$\left|\left\langle b^*_r, P_{(s\vee p_{r-1},\infty)}\sum_{k\in I'_r}\lambda_k x'_k\right\rangle\right|\leqslant\left|\left\langle e^*_{\eta_r}, P_{(s\vee p_{r-1},\infty)}\sum_{k\in I'_r}\lambda_k x'_k\right\rangle\right|.$$

For each r, we may apply our inductive hypothesis to the element $\eta_r\in\Gamma$, the RIS $(x_k)_{k\in I'_r}$ and $s'=s\vee p_{r-1}$ obtaining $k_r\in I'_r$ and $g^*_r\in W[(\mathscr{A}_{3n_j},m_j^{-1})_{j\in\mathbb{N}}]$ supported on $\{k\in I'_r : k>k_r\}$ satisfying

$$\left|\left\langle e^*_{\eta_r}, P_{(s\vee p_{r-1},\infty)}\sum_{k\in I'_r}\lambda_k x_k\right\rangle\right|\leqslant 4C|\lambda_{k_r}|+6C\left\langle g^*_r,\sum_{k\in I'_r}|\lambda_k|e_k\right\rangle. \tag{5.3}$$

We now define g^* by setting

$$g^* = m_h^{-1}\bigg(\sum_{k\in I_0'} e_k^* + \sum_{r=1}^{a}(e_{k_r}^* + g_r^*)\bigg).$$

This is a sum, weighted by m_h^{-1}, of at most $3n_h$ functionals in $W[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$, supported by disjoint intervals, and is hence itself in $W[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$. Putting together (5.1), (5.2) and (5.3), we finally obtain

$$\begin{aligned}
\bigg|\bigg\langle e_\gamma^*, P_{(s,\infty)}\sum_{k\in I}\lambda_k x_k\bigg\rangle\bigg| &\leqslant 4C|\lambda_{k_0}| + 6Cm_h^{-1}\sum_{k\in I_0'}|\lambda_k| + m_h^{-1}\bigg|\sum_{r=1}^{a}\bigg\langle b_r^*, \sum_{k\in I_r'}\lambda_k x_k'\bigg\rangle\bigg| \\
&\leqslant 4C|\lambda_{k_0}| + 6Cm_h^{-1}\sum_{k\in I_0'}|\lambda_k| + m_h^{-1}\bigg|\sum_{r=1}^{a}\bigg\langle e_{\eta_r}^*, P_{(s,\infty)}\sum_{k\in I_r'}\lambda_k x_k\bigg\rangle\bigg| \\
&\leqslant 4C|\lambda_{k_0}| + 6Cm_h^{-1}\bigg(\sum_{k\in I_0'}|\lambda_k| + \sum_{r=1}^{a}\bigg(|\lambda_{k_r}| + \bigg\langle g_r^*, \sum_{k\in I_r'}|\lambda_k| e_k\bigg\rangle\bigg)\bigg) \\
&\leqslant 4C|\lambda_{k_0}| + 6C\bigg\langle g^*, \sum_{k\in I'}|\lambda_k| e_k\bigg\rangle.
\end{aligned}$$

If j_0 satisfies the additional condition set out in the statement of the theorem, we proceed by the same induction. The base case certainly presents no problem. When we pass to the inductive step, considering γ with weight $\gamma = m_h^{-1}$, it is necessary to consider separately the cases $h \neq j_0$ and $h = j_0$. In the first of these cases, the proof proceeds unchanged, since we may assume inductively that the g_r^* introduced above have been chosen to lie in $W[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$.

Finally, if $h = j_0$ we have a simple way to estimate

$$\bigg\langle e_\gamma^*, P_{(s,\infty)}\sum_{k\in I}\lambda_k x_k\bigg\rangle.$$

Indeed there is at most one value of k, say l, for which s is in $\operatorname{ran} x_k$ and $P_{(s,\infty)}x_k = 0$ for $k < l$. If we set $J = \{k\in I : k > l\}$ we then have

$$\bigg|\bigg\langle e_\gamma^*, P_{(s,\infty)}\sum_{k\in I}\lambda_k x_k\bigg\rangle\bigg| \leqslant |\lambda_l|\,\|P_{(s,\infty)}\|\,\|x_l\| + \bigg|\bigg\langle e_\gamma^*, \sum_{k\in J}\lambda_k x_k\bigg\rangle\bigg|.$$

By our usual estimate $\|P_{(s,\infty)}\| \leqslant 3$ and the assumed additional condition, this is at most $4C|\lambda_{k_0}|$ for some $l \leqslant k_0 \in I$. We can then take $g^* = 0$. □

Corollary 5.5. *Any* RIS *is dominated by the unit vector basis of*

$$T[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}].$$

More precisely, if $(x_k)_{k\in I}$ *is a* C-RIS *then, for any real* λ_k, *we have*

$$\left\|\sum_{k\in I}\lambda_k x_k\right\| \leqslant 10C\left\|\sum_{k\in I}\lambda_k e_k\right\|,$$

where the norm on the right-hand side is taken in $T[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$.

As well as this domination result, we shall need the following more precise lemma.

Proposition 5.6. *Let* $(x_k)_{k=1}^{n_{j_0}}$ *be a* C-RIS. *Then*

(1) *For every* $\gamma\in\Gamma$ *with* weight $\gamma = m_h^{-1}$ *we have*

$$\left|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k(\gamma)\right| \leqslant \begin{cases} 16Cm_{j_0}^{-1}m_h^{-1}, & \text{if } h<j_0,\\ 4Cn_{j_0}^{-1}+6Cm_h^{-1}, & \text{if } h\geqslant j_0.\end{cases}$$

In particular, if $h>j_0$, *we have*

$$\left|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k(\gamma)\right| \leqslant 10Cm_{j_0}^{-2},$$

and also

$$\left\|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k\right\| \leqslant 10Cm_{j_0}^{-1}.$$

(2) *If* λ_k, $1\leqslant k\leqslant n_{j_0}$, *are scalars with* $|\lambda_k|\leqslant 1$ *and having the property that*

$$\left|\sum_{k\in J}\lambda_k x_k(\gamma)\right| \leqslant C\max_{k\in J}|\lambda_k|$$

for every γ *of weight* $m_{j_0}^{-1}$ *and every interval* $J\subseteq\{1,2,...,n_{j_0}\}$, *then*

$$\left\|n_{j_0}^{-1}\sum_{k=1}^{j_0}\lambda_k x_k\right\| \leqslant 10Cm_{j_0}^{-2}.$$

Proof. This is a direct application of the basic inequality, with all the coefficients λ_k equal to $n_{j_0}^{-1}$. Indeed, for (1) there exists $g^*\in W[(\mathscr{A}_{3n_j}, m_j^{-1})_{k\in\mathbb{N}}]$ (either zero or of weight m_h^{-1}) such that

$$\left|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k(\gamma)\right| \leqslant 4Cn_{j_0}^{-1}+6C\left\langle g^*, n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} e_k\right\rangle.$$

Using Proposition 2.5 to estimate the term involving g^*, we obtain

$$\left|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k(\gamma)\right| \leqslant \begin{cases} 4Cn_{j_0}^{-1}+12Cm_{j_0}^{-1}m_h^{-1}, & \text{if } h<j_0,\\ 4Cn_{j_0}^{-1}+6Cm_h^{-1}, & \text{if } h\geqslant j_0.\end{cases}$$

The formulae given in (1) follow easily when we note that n_{j_0} is larger than $m_{j_0}^2$.

If the scalars λ_k satisfy the additional condition, then the g^* whose existence is guaranteed by the basic inequality may be taken to be in $W[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$ so that the second part of Proposition 2.5 may be applied, yielding

$$\left|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k(\gamma)\right| \leqslant \begin{cases} 4Cn_{j_0}^{-1}+12Cm_{j_0}^{-2}m_h^{-1}, & \text{if } h<j_0,\\ 4Cn_{j_0}^{-1}+6Cm_h^{-1}, & \text{if } h> j_0.\end{cases}$$

This leads easily to the claimed estimate for

$$\left\|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} \lambda_k x_k\right\|. \qquad \square$$

It turns out that in our space there are three useful types of RIS. One of these is based on an idea that will be familiar from other constructions, that of introducing long ℓ_1-averages. We defer our discussion of this construction until a later section. We shall deal first with the other two types of RIS, which involve the $\mathscr{L}_\infty$ structure of our space, and provide the extra tool that we eventually use to solve the scalar-plus-compact problem.

We have already remarked that the *support* of an element of X is not of great interest—indeed the support of any non-zero element of X is an infinite set, and contains elements γ of Γ of all possible weights. There is, however, a related notion which is of much use. Recall that a vector x whose range is contained in the interval $(p,q]$ can be expressed as $i_q(u)$, where $u\in\ell_\infty(\Gamma_q)$ and $\operatorname{supp}(u)\subseteq\Gamma_q\setminus\Gamma_p$. It turns out that the support of u contains a lot of information about x. We shall refer to $\operatorname{supp}(u)$ as the *local support*. A formal (and unambiguous) definition may be formulated as follows.

Definition 5.7. Let x be a vector in $\bigoplus_{n\in\mathbb{N}} M_n$ and let $q=\max\operatorname{ran} x$; thus x may be expressed as $i_q(u)$ with $u=x|_{\Gamma_q}$. The subset $\operatorname{supp} u=\{\gamma\in\Gamma_q: x(\gamma)\neq 0\}$ is defined to be the *local support* of x.

The following easy lemma uses an idea that has already occurred in Lemma 5.2.

LEMMA 5.8. *Let $\gamma\in\Gamma$ be of weight m_h^{-1} and assume that* $\operatorname{weight}(\xi)\neq m_h^{-1}$ *for all ξ in the local support of x. Then $|x(\gamma)|\leqslant 3m_h^{-1}\|x\|$.*

Proof. Let $q = \max \operatorname{ran} x$ so that $x = i_q(x|_{\Gamma_q})$ and, by hypothesis, weight $\xi \neq m_h^{-1}$ whenever $\xi \in \Gamma_q$ and $x(\xi) \neq 0$. If rank $\gamma \leqslant q$ we thus have $x(\gamma) = 0$ and there is nothing to prove. Otherwise we consider the evaluation analysis of γ,

$$e_\gamma^* = \sum_{r=1}^a d_{\xi_r}^* + m_h^{-1} \sum_{r=1}^a b_r^* \circ P_{(p_{r-1},\infty)},$$

and let t be chosen minimal subject to $p_t = \operatorname{rank} \xi_t > q$. (Since $\gamma = \xi_a$ such a t certainly exists.) As in Lemma 4.5, we may rewrite the evaluation analysis as

$$e_\gamma^* = e_{\xi_t}^* + \sum_{r=t+1}^a d_{\xi_r}^* + m_h^{-1} \sum_{r=t+1}^a b_r^* \circ P_{(p_{r-1},\infty)}.$$

For $r > t$ we have $p_{r-1} > \max \operatorname{ran} x$, whence $P_{(p_{r-1},\infty)} x = 0$, and similarly $\langle d_{\xi_r}^*, x \rangle = 0$ for all $r \geqslant t$. It follows that

$$x(\gamma) = \langle e_{\xi_t}^*, x \rangle = \begin{cases} m_h^{-1} \langle b_t^*, P_{(p_{t-1},\infty)} x \rangle + \langle e_{\xi_{t-1}}^*, x \rangle, & \text{if } t > 1, \\ m_h^{-1} \langle b_1^*, x \rangle, & \text{if } t = 1. \end{cases}$$

Since, in the first of the above cases, we have rank $\xi_{t-1} \leqslant q$ and weight $\xi_{t-1} = m_h^{-1}$, which imply $\langle e_{\xi_{t-1}}^*, x \rangle = 0$, we deduce that in both cases

$$|x(\gamma)| = m_h^{-1} |\langle b_s^*, P_{(p_{s-1},\infty)} x \rangle| \leqslant 3 m_h^{-1} \|x\|. \qquad \square$$

We can now introduce two classes of block sequences, characterized by the weights of the elements of the local support.

Definition 5.9. We shall say that a block sequence $(x_k)_{k\in\mathbb{N}}$ in X has *bounded local weight* if there exists some j_1 such that weight $\gamma \geqslant m_{j_1}^{-1}$ for all γ in the local support of x_k, and all values of k. We shall say that $(x_k)_{k\in\mathbb{N}}$ has *rapidly decreasing local weight* if, for each k and each γ in the local support of x_{k+1}, we have weight $\gamma < m_{i_k}^{-1}$, where $i_k = \max \operatorname{ran} x_k$.

PROPOSITION 5.10. *Let $(x_k)_{k\in\mathbb{N}}$ be a bounded block sequence. If either $(x_k)_{k\in\mathbb{N}}$ has bounded local weight, or $(x_k)_{k\in\mathbb{N}}$ has rapidly decreasing local weight, then the sequence $(x_k)_{k\in\mathbb{N}}$ is an* RIS.

Proof. We start with the case of rapidly decreasing local weight and let $m_{j_k}^{-1}$ be the maximum weight of an element γ in the local support of x_k. By hypothesis, $j_{k+1} > \max \operatorname{supp} x_k$ so that RIS condition (2) is satisfied. Also, if $h < j_k$ and γ is of weight m_h then $|x_k(\gamma)| \leqslant 3 m_h^{-1} \|x_k\|$ by Lemma 5.8. So $(x_k)_{k\in\mathbb{N}}$ is a C-RIS with $C = 3 \sup_k \|x_k\|$.

Now let us suppose that weight $\gamma \geqslant m_{j_1}^{-1}$ for all γ in the local support of x_k and all k. For $k \geqslant 2$ define $j_k = 1 + \max \operatorname{supp} x_{k-1}$, thus ensuring that RIS condition (2) is satisfied. If weight $\gamma = m_h^{-1}$, where $h < j_k$, there are two possibilities: if $h > j_1$ then $|x_k(\gamma)| \leqslant 3m_h^{-1}\|x_k\|$ by Lemma 5.8; if $h \leqslant j_1$ then $|x_k(\gamma)| \leqslant \|x_k\| \leqslant m_h^{-1} m_{j_1} \|x_k\|$. Thus $(x_k)_{k\in\mathbb{N}}$ is a C-RIS, where C is the (possibly quite large) constant $m_{j_1} \sup_k \|x_k\|$. □

Proposition 5.11. *Let Y be any Banach space and $T: X(\Gamma) \to Y$ be a bounded linear operator. If $\|T(x_k)\| \to 0$ for every* RIS *$(x_k)_{k\in\mathbb{N}}$ in $X(\Gamma)$ then $\|T(x_k)\| \to 0$ for every bounded block sequence in $X(\Gamma)$.*

Proof. It is enough to consider a bounded block sequence $(x_k)_{k\in\mathbb{N}}$ and show that there is a subsequence $(x'_j)_{j\in\mathbb{N}}$ such that $\|T(x'_j)\| \to 0$. We may write $x_k = i_{q_k}(u_k)$ with $u_k = x_k|_{\Gamma_{q_k}}$ supported by $\Gamma_{q_k} \setminus \Gamma_{q_{k-1}}$. For each k and each $N \in \mathbb{N}$, we split u_k as $v_k^N + w_k^N$, where, for $\gamma \in \Gamma_{q_k}$,

$$v_k^N(\gamma) = \begin{cases} u_k(\gamma), & \text{if weight}\,\gamma \geqslant m_N^{-1}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad w_k^N(\gamma) = \begin{cases} u_k(\gamma), & \text{if weight}\,\gamma < m_N^{-1}, \\ 0, & \text{otherwise,} \end{cases}$$

and set

$$y_k^N = i_{q_k}(v_k^N) \quad \text{and} \quad z_k^N = i_{q_k}(w_k^N).$$

We notice that $\|y_k^N\| \leqslant \|i_{q_k}\| \, \|v_k^N\| \leqslant 2\|x_k\|$, with a similar estimate for $\|z_k^N\|$, so that the sequences $(y_k^N)_{k\in\mathbb{N}}$ and $(z_k^N)_{k\in\mathbb{N}}$ are bounded. We note also that weight $\gamma \geqslant m_N^{-1}$ for all γ in the local support of y_k^N and weight $\gamma < m_N^{-1}$ for all γ in the local support of z_k^N.

So, for each N, the sequence $(y_k^N)_{k\in\mathbb{N}}$ has bounded local weight and is thus an RIS, by Proposition 5.10. By hypothesis, $\|T(y_k^N)\| \to 0$ for each N. Hence we can choose a sequence $(k_N)_{N\in\mathbb{N}}$ tending to ∞ such that $\|T(y_{k_N}^N)\| \to 0$. If we put $N_1 = 1$ and then, recursively, set $N_{j+1} = q_{k_{N_j}}$, it is easy to see that the sequence $\big(z_{k_{N_j}}^{N_j}\big)_{j\in\mathbb{N}}$ has rapidly decreasing local weight. Thus this sequence is an RIS and we hence have $\|T(z_{k_{N_j}}^{N_j})\| \to 0$. Since $x_{k_{N_j}} = y_{k_{N_j}}^{N_j} + z_{k_{N_j}}^{N_j}$, setting $x'_j = x_{k_{N_j}}$ we have found a subsequence $(x'_j)_{j\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ with $\|T(x'_j)\| \to 0$. □

The above proposition will play an important role in proving compactness of operators, but in the meantime we shall use it to give our promised proof that the dual of X is ℓ_1. There is an alternative approach using ℓ_1-averages.

Proposition 5.12. *The dual of $X(\Gamma)$ is $\ell_1(\Gamma)$.*

Proof. As we have already noted in Theorem 3.5 it is enough to show that the FDD $(M_n)_{n\in\mathbb{N}}$ is shrinking, that is to say, that every bounded block sequence in X is weakly null. So let φ be an element of X^*. By the upper estimate of Proposition 5.6 we see that

$\varphi(x_k)\to 0$ for every RIS $(x_k)_{k\in\mathbb{N}}$. Now Proposition 5.11, applied with $T=\varphi$, shows that $\varphi(x_k)\to 0$ for every bounded block sequence $(x_k)_{k\in\mathbb{N}}$. □

Remark. We can see that $X(\Gamma)$ has many reflexive subspaces. Indeed, suppose that $(q_n)_{n\in\mathbb{N}}$ is an increasing sequence of natural numbers, and that, for each n, F_n is a finite-dimensional subspace of $\bigoplus_{q_n<k<q_{n+1}} M_k$. Then $(F_n)_{n\in\mathbb{N}}$ is an FDD for the subspace

$$W=\overline{\bigoplus_{n\in\mathbb{N}} F_n},$$

and $(F_n)_{n\in\mathbb{N}}$ is shrinking because $(M_n)_{n\in\mathbb{N}}$ is. But $(F_n)_{n\in\mathbb{N}}$ is also boundedly complete, by the lower estimate of Proposition 4.8. Thus W is reflexive. We shall see later in §9 that W^* is hereditarily indecomposable whenever W is a subspace of this type in the space $\mathfrak{X}_{\mathrm{K}}$.

6. Exact pairs and dependent sequences

In the first part of this section, we shall still only be using Assumptions 4.1 and 4.2, so that our results will apply when X is either of the spaces $\mathfrak{B}_{\mathrm{mT}}$ and $\mathfrak{X}_{\mathrm{K}}$. The special properties of the second of these spaces will come into play only from Definition 6.3 onwards.

Definition 6.1. Let $C>0$ and let $\varepsilon\in\{0,1\}$. A pair $(x,\eta)\in X\times\Gamma$ is said to be a (C,j,ε)-*exact pair* if

(1) $|\langle d_\xi^*, x\rangle|\leqslant Cm_j^{-1}$ for all $\xi\in\Gamma$;

(2) weight $\eta=m_j^{-1}$,

(3) $\|x\|\leqslant C$ and $x(\eta)=\varepsilon$;

(4) for every element η' of Γ with weight $\eta'=m_i^{-1}\neq m_j^{-1}$, we have

$$|x(\eta')|\leqslant\begin{cases} Cm_i^{-1}, & \text{if } i<j,\\ Cm_j^{-1}, & \text{if } i>j.\end{cases}$$

Remark. It is an immediate consequence of Lemma 5.3 that a (C,j,ε)-exact pair also satisfies the estimates

$$|\langle e_{\eta'}^*, P_{(s,\infty)}x\rangle|\leqslant\begin{cases} 6Cm_i^{-1}, & \text{if } i<j,\\ 6Cm_j^{-1}, & \text{if } i>j,\end{cases}$$

for elements η' of Γ with weight $\eta'=m_i^{-1}\neq m_j^{-1}$.

It will be seen that these estimates, as well as those in the definition, have much in common with those of Lemma 5.2. Our first task is to show how we can construct exact pairs, starting from an RIS. For the moment, we are only concerned with the case $\varepsilon=0$; the case $\varepsilon=1$ will become important when we come to the HI property in §8.

LEMMA 6.2. *Let* $(x_k)_{k=1}^{n_{2j}}$ *be a skipped-block* C-RIS, *and let* $q_0<q_1<...<q_{n_{2j}}$ *be natural numbers such that* $\operatorname{ran} x_k \subseteq (q_{k-1}, q_k)$ *for all* k. *Let* z *denote the weighted sum*

$$z = m_{2j} n_{2j}^{-1} \sum_{k=1}^{n_{2j}} x_k.$$

For each k *let* b_k^* *be an element of* $B_{q_{k-1}, q_k - 1}$ *with* $\langle b_k^*, x_k\rangle=0$. *Then there exist* $\zeta_i \in \Delta_{q_i}$, $1\leqslant i\leqslant n_{2j}$, *such that the element* $\eta=\zeta_{n_{2j}}$ *has analysis* $(q_i, b_i^*, \zeta_i)_{i=1}^{n_{2j}}$ *and the pair* (z, η) *is a* $(16C, 2j, 0)$-*exact pair.*

Proof. For any $\xi\in\Gamma$, $\langle d_\xi^*, x_k\rangle$ is non-zero for at most one value of k, and so

$$|\langle d_\xi^*, z\rangle| = m_{2j} n_{2j}^{-1} |\langle d_\xi^*, x_k\rangle| \leqslant m_{2j} n_{2j}^{-1} \|d_\xi^*\| \, \|x_k\| \leqslant C m_{2j}^{-1},$$

which is condition (1) in the definition of a $(C, 2j, 0)$-exact pair. We next note that, by Assumption 4.2, there exist $\zeta_i \in \Delta_{q_i}$, $1\leqslant i\leqslant n_{2j}$, given by

$$\zeta_1 = (q_1, m_{2j}^{-1}, b_1^*) \quad \text{and} \quad \zeta_k = (q_k, \zeta_{k-1}, m_{2j}^{-1}, b_k^*), \ \text{ for } 2 \leqslant k \leqslant n_{2j}.$$

The element $\eta=\zeta_{2j}$ has weight m_{2j}^{-1} and analysis $(q_i, b_i^*, \zeta_i)_{i=1}^{n_{2j}}$; its evaluation analysis is

$$e_\eta^* = \sum_{k=1}^{n_{2j}} d_{\zeta_k}^* + m_{2_j}^{-1} \sum_{k=1}^{n_{2j}} P_{(q_{k-1}, q_k)}^*(b_k^*).$$

Since $\operatorname{ran} x_k \subseteq (q_{k-1}, q_k)$ for all k, we see that

$$z(\eta) = \langle e_\eta^*, z\rangle = n_{2j}^{-1} \sum_{k=1}^{n_{2j}} \langle b_k^*, x_k\rangle = 0.$$

Moreover, $\|z\|\leqslant 10C$ by Proposition 5.6 (1). So condition (2) holds.

We now have to consider an element η' of Γ with weight $\eta'=m_h^{-1}\neq m_{2j}$, in order to establish condition (4). We shall use the fact that $(x_k)_{k=1}^{n_{2j}}$ is a C-RIS and apply Proposition 5.6 (2), with $j_0=2j$:

$$|x(\eta')| = m_{2j} n_{2j}^{-1} \left| \sum_{k=1}^{n_{2j}} x_k(\eta') \right| \leqslant \begin{cases} 16 C m_h^{-1}, & \text{if } h < 2j, \\ 4 C m_{2j} n_{2j}^{-1} + 6 C m_{2j} m_h^{-1} < 10 C m_{2j}^{-1}, & \text{if } h > 2j. \end{cases} \qquad \square$$

We are finally ready to make use of the special conditions governing "odd-weight" elements of Γ. We need to consider a special type of rapidly increasing sequences whose members belong to exact pairs.

Definition 6.3. Consider the space $\mathfrak{X}_{\mathrm{K}}=X(\Gamma)$, where $\Gamma=\Gamma^{\mathrm{K}}$ is given in Definition 4.4. We say that a sequence $(x_i)_{i=1}^{n_{2j_0-1}}$ is a $(C, 2j_0-1, \varepsilon)$-*dependent sequence* if there exist $0=p_0<p_1<...<p_{n_{2j_0-1}}$, together with $\eta_i\in\Gamma_{p_i-1}\setminus\Gamma_{p_{i-1}}$ and $\xi_i\in\Delta_{p_i}$, $1\leqslant i\leqslant n_{2j_0-1}$, such that

(1) for each k, $\operatorname{ran} x_k\subseteq(p_{k-1}, p_k)$;

(2) the element $\xi=\xi_{n_{2j_0-1}}\in\Delta_{p_{2j_0-1}}$ has weight $m_{2j_0-1}^{-1}$ and analysis $(p_i, e_{\eta_i}^*, \xi_i)_{i=1}^{n_{2j_0-1}}$;

(3) (x_1, η_1) is a $(C, 4j_1-2, \varepsilon)$-exact pair;

(4) for each $2\leqslant i\leqslant n_{2j_0-1}$, (x_i, η_i) is a $(C, 4j_i, \varepsilon)$-exact pair, with $\operatorname{ran} x_i\subseteq(p_{i-1}, p_i)$.

We notice that, because of the special odd-weight conditions in Definition 4.4, we necessarily have $m_{4j_1-2}^{-1}=\operatorname{weight}\eta_1<n_{2j_0-1}^{-2}$, and $\operatorname{weight}\eta_{i+1}=m_{4j_{i+1}}^{-1}$, where $j_{i+1}=\sigma(\xi_i)$ for $1\leqslant i<n_{2j_0-1}$.

LEMMA 6.4. *A $(C, 2j_0-1, \varepsilon)$-dependent sequence in $\mathfrak{X}_{\mathrm{K}}$ is a C-RIS.*

Proof. We shall show that the RIS conditions are satisfied for the sequence of integers $(j_i')_{i\in\mathbb{N}}$, where $j_1'=4j_1-2$ and $j_i'=4j_i$ for $i\geqslant2$. For each $i\geqslant1$ we have $\max\operatorname{ran} x_i<p_i$ and $j_{i+1}'=4\sigma(\xi_i)>\operatorname{rank}\xi_i=p_i$. This establishes condition (2) in the definition of RIS. Condition (3) follows from the definition of a C-exact pair. □

LEMMA 6.5. *Let $(x_i)_{i=1}^{n_{2j_0-1}}$ be a $(C, 2j_0-1, 0)$-dependent sequence in $\mathfrak{X}_{\mathrm{K}}$ and let J be a sub-interval of $[1, n_{2j_0-1}]$. For any $\gamma'\in\Gamma$ of weight m_{2j_0-1} we have*

$$\left|\sum_{i\in J} x_i(\gamma')\right|\leqslant 3C.$$

Proof. Let ξ_i, η_i, p_i and j_i be as in the definition of a dependent sequence and let γ denote $\xi_{n_{2j_0-1}}$, an element of weight m_{2j_0-1}. Let $(p_i', e_{\eta_i'}^*, \xi_i')_{i=1}^{a'}$ be the analysis of γ' and let the weight of η_i' be

$$\operatorname{weight}\eta_i'=\begin{cases} m_{4j_1'-2}, & \text{if } i=1,\\ m_{4j_i'}, & \text{if } 2\leqslant i\leqslant a'.\end{cases}$$

We note that $a'\leqslant n_{2j_0-1}$ because γ' is of weight m_{2j_0-1}. We may apply the tree-like property of Lemma 4.6 deducing that there exists $1\leqslant l\leqslant a'$ such that $(p_i', \eta_i', \xi_i')=(p_i, \eta_i, \xi_i)$ for $i<l$ while $j_k\neq j_i'$ for all $i\leqslant a'$ when $k>l$. Since $\operatorname{ran} x_k\subseteq(p_{k-1}, p_k)$ and $(p_{k-1}, p_k)=(p_{k-1}', p_k')$ for $k<l$,

$$x_k(\gamma')=x_k(\gamma)=m_{2j_0-1}^{-1}\langle P_{p_{k-1},\infty)}^* e_{\eta_k}^*, x_k\rangle=m_{2j_0-1}^{-1}x_k(\eta_k)=0$$

for all such k.

We may now estimate as follows:

$$\begin{aligned}\Big|\sum_{k\in J} x_k(\gamma')\Big| &\leqslant \Big|\sum_{\substack{k\in J\\ k<l}} x_k(\gamma')\Big| + |x_l(\gamma')| + \sum_{\substack{k\in J\\ k>l}} |x_i(\gamma')| \\ &\leqslant \|x_l\| + \sum_{\substack{k\in J\\ k>l}} \sum_{i\leqslant a'} |\langle d^*_{\xi_i}, x_k\rangle + m^{-1}_{2j_0-1}\langle e^*_{\eta'_i}, P_{(p'_{i-1},\infty)}x_k\rangle| \\ &\leqslant C + n^2_{2j_0-1} \max_{\substack{k\in J\\ k>l\\ i\leqslant a'}} |\langle d^*_{\xi'_i}, x_k\rangle + m^{-1}_{2j_0-1}\langle e^*_{\eta'_i}, P_{(p'_{i-1},\infty)}x_k\rangle|.\end{aligned}$$

Now we know that, provided $k>l$, weight $\eta_k \neq$ weight η'_i for all i, so by the definition of an exact pair (Definition 6.1), and the remark following it, we have

$$\begin{aligned}|\langle d^*_{\xi'_i}, x_k\rangle + m^{-1}_{2j_0-1} P_{(p_{i-1},\infty)} x_k(\eta'_i)| &\leqslant C \,\mathrm{weight}\,\eta_k + 6Cm^{-1}_{2j_0-1} \max\{\mathrm{weight}\,\eta'_i, \mathrm{weight}\,\eta_k\} \\ &\leqslant 2C \max\{\mathrm{weight}\,\eta_1, \mathrm{weight}\,\eta'_1\} \\ &= 2C \max\{m^{-1}_{4j_1-2}, m^{-1}_{4j'_1-2}\} \\ &\leqslant 2Cn^{-2}_{2j_0-1},\end{aligned}$$

using the fact that m_{4j_1-2} and $m_{4j'_1-2}$ are both at least $n^2_{2j_0-1}$. We now deduce the inequality $\big|\sum_{i\in J} x_i(\gamma')\big| \leqslant 3C$ as required. □

We finish with the main inequality of this section, which shows that averages of 0-dependent sequences satisfy a much stronger upper norm estimate than given by Proposition 5.4 for a general RIS.

PROPOSITION 6.6. *Let $(x_i)_{i=1}^{n_{2j-1}}$ be a $(C, 2j_0-1, 0)$-dependent sequence in $\mathfrak{X}_{\mathrm{K}}$. Then*

$$\Big\|n^{-1}_{2j_0-1} \sum_{i=1}^{n_{2j_0-1}} x_i\Big\| \leqslant 30Cm^{-2}_{2j_0-1}.$$

Proof. We apply the second part of Lemma 5.6, with $\lambda_i = 1$ and with $2j_0-1$ playing the role of j_0. Lemma 6.5 shows that the extra hypothesis of the second part of Lemma 5.6 is indeed satisfied, provided we replace C by $3C$. We deduce the claimed inequality. □

7. Bounded linear operators on $\mathfrak{X}_{\mathrm{K}}$

We shall now show that $\mathfrak{X}_{\mathrm{K}}$ has the scalar-plus-compact property. For technical reasons it will be convenient in the first few results to work with elements of $\mathfrak{X}_{\mathrm{K}}$ all of whose coordinates are rational, that is to say with elements of $\mathfrak{X}_\mathrm{K}\cap\mathbb{Q}^\Gamma$. Since (as may be readily checked) each d_ξ is in $\mathfrak{X}_\mathrm{K}\cap\mathbb{Q}^\Gamma$, as are all rational linear combinations of these, we see that $\mathfrak{X}_\mathrm{K}\cap\mathbb{Q}^\Gamma$ is dense in $\mathfrak{X}_\mathrm{K}$.

LEMMA 7.1. *Let $m<n$ be natural numbers and let $x\in\mathfrak{X}_\mathrm{K}\cap\mathbb{Q}^\Gamma$ and $y\in\mathfrak{X}_\mathrm{K}$ be such that* $\operatorname{ran} x$ *and* $\operatorname{ran} y$ *are both contained in* $(m,n]$. *Suppose that* $\operatorname{dist}(y,\mathbb{R}x)>\delta$. *Then there is* $b^*\in\operatorname{ball}\ell_1(\Gamma_n\setminus\Gamma_m)$, *with rational coordinates, such that* $b^*(x)=0$ *and* $b^*(y)>\frac{1}{2}\delta$.

Proof. Let $u,v\in\ell_\infty(\Gamma_n\setminus\Gamma_m)$ be the restrictions of x and y, respectively. Then $x=i_nu$ and $y=i_nv$, and so, for any scalar λ,

$$\|y-\lambda x\|\leqslant\|i_n\|\,\|v-\lambda u\|.$$

Hence $\operatorname{dist}(v,\mathbb{R}u)>\frac{1}{2}\delta$ and so, by the Hahn–Banach theorem in the finite-dimensional space $\ell_\infty(\Gamma_n\setminus\Gamma_m)$, there exists $a^*\in\operatorname{ball}\ell_1(\Gamma_n\setminus\Gamma_m)$ with $a^*(u)=0$ and $a^*(v)>\frac{1}{2}\delta$. Since x has rational coordinates, our vector u is in $\mathbb{Q}^{\Gamma_n\setminus\Gamma_m}$. It follows that we can approximate a^* arbitrarily well with $b^*\in\mathbb{Q}^{\Gamma_n\setminus\Gamma_m}$ retaining the condition $b^*(u)=0$. □

LEMMA 7.2. *Let T be a bounded linear operator on $\mathfrak{X}_\mathrm{K}$, let $(x_i)_{i\in\mathbb{N}}$ be a C-*RIS *in $\mathfrak{X}_\mathrm{K}\cap\mathbb{Q}^\Gamma$ and assume that* $\operatorname{dist}(Tx_i,\mathbb{R}x_i)>\delta>0$ *for all i. Then, for all $j,p\in\mathbb{N}$, there exist $z\in[x_i:i\in\mathbb{N}]$, $q>p$ and $\eta\in\Delta_q$ such that*

(1) *(z,η) is a $(16C,2j,0)$-exact pair;*
(2) $(Tz)(\eta)>\frac{7}{16}\delta$;
(3) $\|(I-P_{(p,q)})Tz\|<m_{2j}^{-1}\delta$;
(4) $\langle P^*_{(p,q]}e^*_\eta,Tz\rangle>\frac{3}{8}\delta$.

Proof. Since the sequence $(Tx_i)_{i\in\mathbb{N}}$ is weakly null, we may, by taking a subsequence if necessary, assume that there are $p<q_0<q_1<...$ such that $\operatorname{ran} x_i\subseteq(q_{i-1},q_i)$ for all $i\geqslant1$, and

$$\|(I-P_{(q_{i-1},q_i)})Tx_i\|<\tfrac{1}{5}m_{2j}^{-2}\delta\leqslant\tfrac{1}{80}m_{2j}^{-1}\delta\leqslant\tfrac{1}{1280}\delta.$$

It certainly follows from this that

$$\operatorname{dist}(P_{(q_{i-1},q_i)}Tx_i,\mathbb{R}x_i)>\tfrac{1279}{1280}\delta.$$

We may apply Lemma 7.1 to obtain $b_i^*\in\operatorname{ball}\ell_1(\Gamma_{q_i-1}\setminus\Gamma_{q_{i-1}})$, with rational coordinates, satisfying

$$\langle b_i^*,x_i\rangle=0\quad\text{and}\quad\langle b_i^*,P_{(q_{i-1},q_i)}Tx_i\rangle>\tfrac{1279}{2560}\delta.$$

Taking a further subsequence if necessary (and redefining the q_i), we may assume that the coordinates of b_i^* have denominators dividing $N_{q_i-1}!$, so that $b_i^*\in B_{q_{i-1},q_i-1}$, and we may also assume that $q_1\geqslant2j$. We are thus in a position to apply Lemma 6.2, getting elements ξ_i of weight m_{2j}^{-1} in Δ_{q_i} such that the element $\eta=\xi_{n_{2j}}$ of $\Delta_{q_{n_{2j}}}$ has evaluation analysis

$$e_\eta^* = \sum_{i=1}^{n_{2j}} d_{\xi_i}^* + m_{2j}^{-1} \sum_{i=1}^{n_{2j}} P_{(q_{i-1},q_i)}^* b_i^*,$$

and such that (z,η) is a $(16C,2j,0)$-exact pair, where z denotes the weighted average

$$z = m_{2j} n_{2j}^{-1} \sum_{i=1}^{n_{2j}} x_i.$$

We next need to estimate $(Tz)(\eta)$. For each k, we have

$$\|(I-P_{(q_{k-1},q_k)})Tx_k\| < \tfrac{1}{80} m_{2j}^{-1}\delta,$$

so that

$$(Tx_k)(\eta) \geqslant \langle e_\eta^*, P_{(q_{k-1},q_k)}Tx_k\rangle - \tfrac{1}{80} m_{2j}^{-1}\delta = m_{2j}^{-1}\langle b_k^*, P_{(q_{k-1},q_k)}Tx_k\rangle - \tfrac{1}{80} m_{2j}^{-1}\delta > \tfrac{1247}{2560} m_{2j}^{-1}\delta.$$

It follows that

$$(Tz)(\eta) = n_{2j}^{-1} m_{2j} \sum_{k=1}^{n_{2j}} (Tx_k)(\eta) > \frac{7}{16}\delta.$$

For inequality (3), in which we are taking $q=q_{n_{2j}}$, we note that $p<q_{k-1}<q_k\leqslant q$ for all k, so that

$$\begin{aligned}\|(I-P_{(p,q]})Tx_k\| &= \|(P_{(0,p]}+P_{(q,\infty)})Tx_k\| \\ &= \|(P_{(0,p]}+P_{(q,\infty)})(I-P_{(q_{k-1},q_k)})Tx_k\| \\ &\leqslant 5\|(I-P_{(q_{k-1},q_k)})Tx_k\| \\ &< m_{2j}^{-2}\delta,\end{aligned}$$

using our usual estimates for norms of FDD projections. The inequality for the weighted average z follows at once. Inequality (4) follows from (2) and (3), since

$$\langle P_{(p,q]}^* e_\eta^*, Tz\rangle \geqslant (Tz)(\eta) - \|(I-P_{(p,q]})Tz\| > \tfrac{7}{16}\delta - m_{2j}^{-1}\delta \geqslant \tfrac{3}{8}\delta. \qquad \square$$

PROPOSITION 7.3. *Let T be a bounded linear operator on $\mathfrak{X}_{\mathrm{K}}$ and let $(x_i)_{i\in\mathbb{N}}$ be an* RIS *in $\mathfrak{X}_{\mathrm{K}}$. Then $\mathrm{dist}(Tx_i,\mathbb{R}x_i)\to 0$ as $i\to\infty$.*

Proof. By density, it will be enough to prove the result for an RIS in $\mathfrak{X}_{\mathrm{K}}\cap\mathbb{Q}^\Gamma$. Suppose, if possible, that $(x_i)_{i\in\mathbb{N}}$ is a C-RIS in this subspace, with $\mathrm{dist}(Tx_i,\mathbb{R}x_i)>\delta>0$ for all i. The idea is to obtain a 0-dependent sequence by making repeated applications of Lemma 7.2.

We start by choosing j_0 such that

$$m_{2j_0-1} > 1920C\|T\|\delta^{-1}$$

and j_1 such that $m_{4j_1-2} > n^2_{2j_0-1}$. Taking $p=p_0=0$ and $j=2j_1-1$ in Lemma 7.2, we can find q_1 and a $(16C, 4j_1-2, 0)$-exact pair (z_1, η_1) with $\operatorname{rank}\eta_1 = q_1$, $(Tz_1)(\eta_1) > \frac{3}{8}\delta$ and $\|(I-P_{(0,q_1]})(Tz_1)\| < m^{-1}_{4j_1-2}\delta$. Let $p_1 = q_1+1$ and let ξ_1 be the special type-1 element of Δ_{p_1} given by $\xi_1 = (p_1, m_{2j_0-1}, e^*_{\eta_1})$.

Now, recursively for $2 \leqslant i \leqslant n_{2j_0-1}$, define $j_i = \sigma(\xi_{i-1})$, and use the lemma again to choose q_i and a $(16C, 4j_i, 0)$-exact pair (z_i, η_i) with $\operatorname{rank}\eta_i = q_i$, $\operatorname{ran} z_i \subseteq (p_{i-1}, q_i]$, $\langle P^*_{(p_{i-1},q_i]} e^*_{\eta_i}, Tz_i\rangle > \frac{3}{8}\delta$ and $\|(I-P_{(p_i,q_i]})(Tz_i)\| < m^{-1}_{4j_i}\delta$. We now define $p_i = q_i+1$ and let ξ_i be the type-2 element $(p_i, \xi_{i-1}, m^{-1}_{2j_0-1}, e^*_{\eta_i})$ of Δ_{p_i}.

It is clear that we have constructed a $(16C, 2j_0-1, 0)$-dependent sequence $(z_i)_{i=1}^{n_{2j_0-1}}$.

By the estimate of Proposition 6.6, we have

$$\|z\| \leqslant 30\cdot 16Cm^{-2}_{2j_0-1}$$

for the average

$$z = n^{-1}_{2j_0-1}\sum_{i=1}^{n_{2j_0-1}} z_i.$$

However, let us consider the element $\gamma = \xi_{n_{2j_0-1}}$ of $\Delta_{p_{n_{2j_0-1}}}$, which has evaluation analysis

$$e^*_\gamma = \sum_{i=1}^{n_{2j_0-1}} d^*_{\xi_i} + m^{-1}_{2j_0-1}\sum_{i=1}^{n_{2j_0-1}} P^*_{(p_{i-1},p_i)} e^*_{\eta_i}.$$

Noting that $p_k = q_k+1$ for $k \geqslant 1$, and that $m_{4j_i} > m_{4j_1-2} > n^2_{2j_0-1}$, we may estimate $(Tz)(\gamma)$ as follows

$$\begin{aligned}
(Tz)(\gamma) &= n^{-1}_{2j_0-1}\sum_{k=1}^{n_{2j_0-1}} (Tx_k)(\gamma)\\
&\geqslant n^{-1}_{2j_0-1}\sum_{k=1}^{n_{2j_0-1}} (\langle P^*_{(p_{k-1},p_k)} e^*_\gamma, Tx_k\rangle - \|(I-P_{(p_{k-1},q_k]})(Tx_k)\|)\\
&\geqslant n^{-1}_{2j_0-1}\sum_{k=1}^{n_{2j_0-1}} (m^{-1}_{2j_0-1}\langle P^*_{(p_{k-1},p_k)} e^*_{\eta_k}, Tx_k\rangle - m^{-1}_{4j_1-2}\delta)\\
&\geqslant \delta n^{-1}_{2j_0-1}\sum_{k=1}^{n_{2j_0-1}} \left(\frac{3}{8} m^{-1}_{2j_0-1} - n^{-2}_{2j_0-1}\right)\\
&> \frac{m^{-1}_{2j_0-1}\delta}{4}.
\end{aligned}$$

So

$$\|Tz\| \geqslant \frac{\delta m_{2j_0-1}^{-1}}{4} > \frac{\delta m_{2j_0-1}\|z\|}{4\cdot 30\cdot 16C},$$

which is a contradiction because

$$\frac{C^{-1}\delta m_{2j_0-1}}{1920} > \|T\|$$

by our original choice of j_0. □

THEOREM 7.4. *Let T be a bounded linear operator on $\mathfrak{X}_{\mathrm{K}}$. Then there exists a scalar λ such that $T-\lambda I$ is compact.*

Proof. We start by considering a normalized RIS $(x_i)_{i\in\mathbb{N}}$ in $\mathfrak{X}_{\mathrm{K}}$. By Proposition 7.3 there exist scalars λ_i such that $\|Tx_i-\lambda_i x_i\|\to 0$. We claim that λ_i necessarily tends to some limit λ. Indeed, if not, by passing to a subsequence, we may suppose that $|\lambda_{i+1}-\lambda_i|>\delta$ for all i. Now the sequence $(y_i)_{i\in\mathbb{N}}$ where $y_i=x_{2i-1}+x_{2i}$ is again an RIS, so that there exist μ_i with $\|Ty_i-\mu_i y_i\|\to 0$ by Proposition 7.3 again. We thus have

$$\begin{aligned}&\|(\lambda_{2i}-\mu_i)x_{2i}+(\lambda_{2i-1}-\mu_i)x_{2i-1}\| \\ &\qquad\qquad \leqslant \|Tx_{2i}-\lambda_{2i}x_{2i}\|+\|Tx_{2i-1}-\lambda_{2i-1}x_{2i-1}\|+\|Ty_i-\mu_i y_i\| \to 0.\end{aligned}$$

Since the RIS $(x_i)_{i\in\mathbb{N}}$ is a block sequence, there exist l_i such that $P_{(0,l_i]}y_i=x_{2i-1}$ and $P_{(l_i,\infty)}y_i=x_{2i}$. Using the assumption that the sequence $(x_i)_{i\in\mathbb{N}}$ is normalized, we now have

$$|\lambda_{2i-1}-\mu_i| = \|(\lambda_{2i-1}-\mu_i)x_{2i-1}\| \leqslant \|P_{(0,l_i]}\|\,\|(\lambda_{2i}-\mu_i)x_{2i}+(\lambda_{2i-1}-\mu_i)x_{2i-1}\|,$$

with a similar estimate for $|\lambda_{2i}-\mu_i|$. Each of these sequences thus tends to 0, so that $\lambda_{2i}-\lambda_{2i-1}$ also tends to 0, contrary to our assumption.

We now show that the scalar λ is the same for all rapidly increasing sequences. Indeed, if $(x_i)_{i\in\mathbb{N}}$ and $(x_i')_{i\in\mathbb{N}}$ are RIS, with $\|Tx_i-\lambda x_i\|\to 0$ and $\|Tx_i'-\lambda' x_i'\|\to 0$, we may find $i_1<i_2<...$ such that the sequence $(y_k)_{k\in\mathbb{N}}$ defined by

$$y_k = \begin{cases} x_{i_k}, & \text{if } k \text{ is odd,} \\ x_{i_k}', & \text{if } k \text{ is even,}\end{cases}$$

is again an RIS. By the first part of the proof we must have $\lambda=\lambda'$.

We have now obtained λ such that $\|(T-\lambda I)x_i\|\to 0$ for every RIS. By Proposition 5.11, we deduce that $\|(T-\lambda I)x_i\|\to 0$ for every bounded block sequence in $\mathfrak{X}_{\mathrm{K}}$. This, of course, implies that $T-\lambda I$ is compact. □

8. ℓ_1-averages and the HI property

Definition 8.1. An element x of X will be called a C-ℓ_1^n *average* if there exists a block sequence $(x_i)_{k=1}^n$ in X such that $x=n^{-1}\sum_{k=1}^n x_k$ and $\|x_k\|\leqslant C$ for all k. We say that x is a *normalized* C-ℓ_1^n average if, in addition, $\|x\|=1$.

A standard argument (cf. [8, §II.22]) using the lower estimate of Lemma 4.8 and Lemma 2.2 leads to the following result.

LEMMA 8.2. *Let Z be any block subspace of X. For any n and $C>1$, Z contains a normalized C-ℓ_1^n average.*

Proof. Write $C=(1-\varepsilon)^{-1}$ and choose an integer l with $n(1-\varepsilon/n)^l<1$; next choose j sufficiently large as to ensure that $n_{2j}>(2m_{2j})^l$ (by Assumption 2.3 (4) any $j\geqslant\frac{1}{2}l$ will achieve this in some comfort); finally let k be minimal subject to

$$m_{2j}<(1-\varepsilon/n)^{-k}.$$

Since $\frac{1}{2}(1-\varepsilon/n)^{-k}\leqslant(1-\varepsilon/n)^{-k+1}\leqslant m_{2j}$ we have

$$n_{2j}>(2m_{2j})^l\geqslant(1-\varepsilon/n)^{-kl}>n^k.$$

If $(x_i)_{i\in\mathbb{N}}$ is any normalized skipped-block sequence in Z, we can apply Lemma 4.8 to see that

$$\left\|\sum_{i=1}^{n^k}x_i\right\|\geqslant\frac{1}{2}m_{2j}^{-1}n^k\geqslant(n-\varepsilon)^k.$$

It now follows from Lemma 2.2 that there are normalized successive linear combinations $y_1,...,y_n$ of $(x_i)_{i\in\mathbb{N}}$ such that

$$\left\|\sum_{i=1}^{n}a_iy_i\right\|\geqslant(1-\varepsilon)\sum_{i=1}^{n}|a_i|$$

for all real a_i. In particular, there is a normalized C-ℓ_1^n average. □

LEMMA 8.3. *Let x be a C-$\ell_1^{n_j}$ average. For all $\gamma\in\Gamma$ we have $|\langle d_\gamma^*,x\rangle|\leqslant 3Cn_j^{-1}$. If γ is of weight m_i with $i<j$ then $|x(\gamma)|\leqslant 2Cm_i^{-1}$.*

Proof. Let $x=n_j^{-1}\sum_{k=1}^{n_j}x_k$, as in the definition of a C-ℓ_1^n average. For any γ there is some k such that $\langle d_\gamma^*,x\rangle=n_j^{-1}\langle d_\gamma^*,x_k\rangle$. Thus

$$|\langle d_\gamma^*,x\rangle|\leqslant n_j^{-1}\|d_\gamma^*\|\,\|x_k\|\leqslant 3Cn_j^{-1}.$$

Let us now consider the case where weight $\gamma = m_i$, with $i<j$. From the evaluation analysis,

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + m_i^{-1} \sum_{r=1}^{a} b_r^* \circ P_{(p_{r-1},\infty)},$$

it follows that

$$|x(\gamma)| \leqslant \sum_{r=1}^{a} |\langle d_{\xi_r}^*, x\rangle| + m_i^{-1} \sum_{r=1}^{a} \|P_{(p_{r-1},p_r)} x\|. \tag{8.1}$$

By what we have already observed, we have

$$\sum_{r=1}^{a} |\langle d_{\xi_r}^*, x\rangle| \leqslant 3Can_j^{-1}. \tag{8.2}$$

To estimate the second term in (8.1) we follow the argument in [8, p. 33], letting I_r (resp. J_r) be the set of k such that $\operatorname{ran} x_k$ is contained in (resp. meets) the interval (p_{r-1}, p_r). We have $\#J_r \leqslant \#I_r + 2$ and $\sum_{r=1}^{a} \#I_r \leqslant n_j$. Moreover, for each r, we have $P_{(p_{r-1},p_r)} x_k = x_k$ if $k \in I_r$, while $P_{(p_{r-1},p_r)} x_k = 0$ if $k \notin J_r$ and

$$\|P_{(p_{r-1},p_r)} x_k\| \leqslant 4\|x_k\| \leqslant 4C, \quad \text{if } k \in J_r \setminus I_r.$$

It follows that

$$\|P_{(p_{r-1},p_r]} x\| \leqslant n_j^{-1}(C\#I_r + 8C) \leqslant Cn_j^{-1}(\#I_r + 8).$$

Summing over r leads us to

$$\sum_{r=1}^{a} \|P_{(p_{r-1},p_r]} x\| \leqslant Cn_j^{-1}(n_j + 8a). \tag{8.3}$$

Combining our inequalities, and using the fact that $a \leqslant n_i$, we obtain

$$|x(\gamma)| \leqslant 3Can_j^{-1} + m_i^{-1} n_j^{-1}(Cn_j + 8Ca) \leqslant Cm_i^{-1} + 5Cn_i n_j^{-1} < 2Cm_i^{-1}. \qquad \square$$

LEMMA 8.4. *Let I be an interval in $\mathbb{N}$, let $(x_k)_{k\in I}$ be a block sequence in X and let $(j_k)_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers. Suppose that, for each k, x_k is a C-$\ell_1^{n_{j_k}}$-average and that $j_{k+1} > \max \operatorname{ran} x_k$. Then $(x_k)_{k\in I}$ is a $2C$-*RIS.

Proof. We just have to prove RIS condition (3) and this is an immediate consequence of Lemma 8.3. $\square$

COROLLARY 8.5. *Let Z be a block subspace of X, and let $C>2$ be a real number. Then X contains a normalized C-*RIS.

Proof. This is immediate from Lemmas 8.2 and 8.4. $\square$

We shall next need modifications of some lemmas established in the previous section, here adapted so as to deal with $(C,2j,1)$-exact pairs and $(C,2j-1,1)$-dependent sequences, rather than with the "$\varepsilon=0$" case.

LEMMA 8.6. *Let j be a positive integer and let $(x_k)_{k=1}^{n_{2j}}$ be a skipped-block C-RIS, such that $\min\operatorname{ran} x_2\geqslant 2j$ and $\|x_k\|\geqslant 1$ for all k. Then there exists $\theta\in\mathbb{R}$, with $|\theta|\leqslant 2$, and there exists $\gamma\in\Gamma$, such that (x,γ) is a $(32C,2j,1)$-exact pair, where x is the weighted sum*

$$x=\theta m_{2j} n_{2j}^{-1}\sum_{k=1}^{n_{2j}} x_k.$$

Proof. We may apply the construction of Lemma 4.8 to obtain an element γ of Γ of weight m_{2j} such that

$$n_{2j}^{-1}\sum_{k=1}^{n_{2j}} x_k(\gamma)\geqslant \tfrac{1}{2} m_{2j}^{-1}.$$

For a suitably chosen $\theta\in\mathbb{R}$ with $0<\theta\leqslant 2$ we have $x(\gamma)=1$, where

$$x=\theta m_{2j} n_{2j}^{-1}\sum_{k=1}^{n_{2j}} x_k.$$

Proposition 5.6 (1) implies that $\|x\|\leqslant 20$. We thus have condition (2) in the definition of an exact pair.

Conditions (1) and (3) are established by the same arguments as were used in Lemma 6.2, the constant θ resulting in the change from $16C$ to $32C$. □

Using Lemma 8.5, we now immediately obtain the following consequence.

LEMMA 8.7. *If Z is a block subspace of X then for all $j\in\mathbb{N}$ there exists a $(65,2j,1)$-exact pair (x,η) with $x\in Z$.*

The proof of the following lemma is sufficiently close to that of Lemma 6.5 for us to omit it.

LEMMA 8.8. *Let $(x_i)_{i=1}^{n_{2j_0-1}}$ be a $(C,2j_0-1,1)$-dependent sequence in $\mathfrak{X}_{\mathrm{K}}$ and let J be a subinterval of $[1,n_{2j_0-1}]$. For any $\gamma'\in\Gamma$ of weight m_{2j_0-1} we have*

$$\left|\sum_{i\in J}(-1)^i x_i(\gamma')\right|\leqslant 4C.$$

LEMMA 8.9. *Let $(x_i)_{i=1}^{n_{2j-1}}$ be a $(C,2j_0-1,1)$-dependent sequence in $\mathfrak{X}_{\mathrm{K}}$. Then*

$$\left\|n_{2j_0-1}^{-1}\sum_{i=1}^{n_{2j_0-1}} x_i\right\|\geqslant m_{2j_0-1}^{-1},\quad \textit{but}\quad \left\|n_{2j_0-1}^{-1}\sum_{i=1}^{n_{2j_0-1}}(-1)^i x_i\right\|\leqslant 40C m_{2j_0-1}^{-2}.$$

Proof. Using the notation of Definition 6.3 it is easy to show, by induction on a, as in Lemma 4.8, that

$$\sum_{i=1}^{a} x_i(\xi_a) = m_{2j_0-1}^{-1} a,$$

whence we immediately obtain

$$\left\| n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} x_i \right\| \geqslant n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} x_i(\xi_{n_{2j_0-1}}) = m_{2j_0-1}^{-1}.$$

To estimate

$$\left\| n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} (-1)^i x_i \right\|,$$

we consider any $\gamma \in \Gamma$ and apply the second part of Proposition 5.6, with $\lambda_i = (-1)^n$ and with $2j_0-1$ playing the role of j_0. Lemma 8.8 shows that the extra hypothesis of the second part of Lemma 5.6 is indeed satisfied, provided we replace C by $4C$. We deduce that

$$\left\| n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} (-1)^i x_i \right\| \leqslant 40 C m_{2j_0-1}^{-2},$$

as claimed. □

LEMMA 8.10. *Let Y and Z be block subspaces of $\mathfrak{X}_{\mathrm{K}}$. Then, for each $\varepsilon > 0$, there exist $y \in Y$ and $z \in Z$ with $\|y-z\| < \varepsilon \|y+z\|$.*

Proof. We start by choosing j_0 and j_1 with $m_{2j_0-1} > 2600\varepsilon^{-1}$ and $m_{4j_1-2} > n_{2j_0-1}^2$. Next we use Lemma 8.7 to choose a $(65, m_{4j_1-2}, 1)$-exact pair (x_1, η_1) with $x_1 \in Y$. Now, for some $p_1 > \operatorname{rank} \eta_1 \vee \max \operatorname{ran} x_1$, we define $\xi_1 \in \Delta_{p_1}$ to be $(p_1, m_{2j_0-1}, e_{\eta_1}^*)$.

We now set $j_2 = \sigma(\xi_1)$ and choose a $(65, m_{4j_2}, 1)$-exact pair (x_2, η_2) with $x_2 \in Z$ and $\min \operatorname{ran} x_2 > p_1$. We pick $p_2 > \operatorname{rank} \eta_2 \vee \max \operatorname{ran} x_2$ and take ξ_2 to be the element $(p_2, \xi_1, m_{2j_0-1}^{-1}, e_{\eta_2}^*)$ of Δ_{p_2}. Notice that this tuple is indeed in Δ_{p_2} because we have ensured that $\operatorname{weight} \eta_2 = m_{4\sigma(\xi_1)}^{-1}$.

Continuing in this way, we obtain a $(65, 2j_0-1, 1)$-dependent sequence $(x_i)_{i \in \mathbb{N}}$ such that $x_i \in Y$ when i is odd and $x_i \in Z$ when i is even. We define

$$y = \sum_{i \text{ odd}} x_i \quad \text{and} \quad z = \sum_{i \text{ even}} x_i,$$

and observe that, by Lemma 8.9,

$$\|y+z\| = \left\| \sum_{i=1}^{n_{2j_0-1}} x_i \right\| \geqslant n_{2j_0-1} m_{2j_0-1}^{-1},$$

while

$$\|y-z\| = \left\| \sum_{i=1}^{n_{2j_0-1}} (-1)^i x_i \right\| \leqslant 40\cdot 65 n_{2j_0-1} m_{2j_0-1}^{-2}. \qquad \square$$

The previous and Proposition 2.1 now yield the following theorem.

THEOREM 8.11. *The space* $\mathfrak{X}_{\mathrm{K}}$ *is hereditarily indecomposable.*

9. Reflexive subspaces with HI duals

We devote this section to a proof that $\mathfrak{X}_{\mathrm{K}}$ is saturated with reflexive HI subspaces having HI duals. The proof involves reworking much of the construction of §6 in the context of a subspace of $\mathfrak{X}_{\mathrm{K}}$ and its dual. By standard blocking arguments, it is enough to prove the following theorem.

THEOREM 9.1. *Let* $L=\{l_0, l_1, l_2, \ldots\}$ *be a set of natural numbers with* $l_{n-1}+1<l_n$, *and for each* $n\geqslant 1$ *let* F_n *be a subspace of the finite-dimensional space*

$$P_{(l_{n-1},l_n)}\mathfrak{X}_{\mathrm{K}} = \bigoplus_{l_{n-1}<k<l_n} M_k.$$

Then the subspace

$$W = \overline{\bigoplus_{n\in\mathbb{N}} F_n}$$

of $\mathfrak{X}_{\mathrm{K}}$ *is reflexive and has* HI *dual.*

We note in passing the following corollary, which gives an indication of the "very conditional" nature of the basis of ℓ_1 that we have constructed. For the purposes of the statement we briefly abandon the "Γ notation" and revert to the notation of Definition 3.1 and Theorem 3.4.

COROLLARY 9.2. *There exist a basis* $(d_n^*)_{n\in\mathbb{N}}$ *of* ℓ_1 *and natural numbers* $k_1<k_2<\ldots$ *with the property that the quotient* $\ell_1/[d_n^*:n\in M]$ *is hereditarily indecomposable whenever* M *is a subset of* $\mathbb{N}$ *such that both* M *and* $\mathbb{N}\setminus M$ *contain infinitely many of the intervals* $(k_p, k_{p+1}]$.

The rest of this section will be devoted to the proof of Theorem 9.1. We have already remarked at the end of §5 that the subspace W defined in the statement of the theorem is reflexive. The subspaces F_n form a finite-dimensional decomposition of W, the corresponding FDD projections being $Q_{(m,n]}=P_{(l_m,l_n]}|_W=P_{(l_m,l_n)}|_W$, when $0\leqslant m<n$. The dual space W^* has a dual FDD $(F_n^*)_{n\in\mathbb{N}}$ and corresponding projections $Q^*_{(m,n]}$. We shall establish hereditary indecomposability of W^* via the criterion Proposition 2.1. We

write R for the quotient mapping $\mathfrak{X}_{\mathrm{K}}^*=\ell_1\to W^*$ and observe that if $f_n^*\in F_n^*$ for $1\leqslant n\leqslant N$ then the norm of $f^*=\sum_{n=1}^N f_n^*$ in W^* is given by

$$\|f^*\|_{W^*}=\inf\{\|g^*\|: g^*\in\mathfrak{X}_{\mathrm{K}}^* \text{ and } Rg^*=f^*\}.$$

LEMMA 9.3. *If* $f^*\in\operatorname{im} Q^*_{(M,N]}=\bigoplus_{M<n\leqslant N} F_n^*\subset W^*$ *then there exists* $k^*\in\mathfrak{X}_{\mathrm{K}}^*=\ell_1(\Gamma)$ *with* $\operatorname{supp} k^*\subseteq\Gamma_{l_N-1}\setminus\Gamma_{l_M}$, $\|k^*\|_1\leqslant 4\|f^*\|$ *and* $RP^*_{(l_M,l_N)}k^*=RP^*_{(l_M,\infty)}k^*=f^*$.

Proof. We extend f^* by the Hahn–Banach theorem to obtain $g^*\in\mathfrak{X}_{\mathrm{K}}^*=\ell_1(\Gamma)$ with $Rg^*=f^*$ and $\|g^*\|_{\mathfrak{X}_{\mathrm{K}}^*}=\|f^*\|_{W^*}$. We set $h^*=P_{(0,l_N)}g^*\in\ell_1(\Gamma_{l_N-1})$ and $k^*=h^*\chi_{\Gamma_{l_N-1}\setminus\Gamma_{l_M}}$, noting that

$$\|k^*\|_1\leqslant\|h^*\|_1\leqslant 2\|g^*\|_1\leqslant 4\|g^*\|_{\mathfrak{X}_{\mathrm{K}}^*}=4\|f^*\|.$$

To check that $RP^*_{(l_M,l_N)}k^*=RP^*_{(l_M,\infty)}k^*=f^*$, we first note that

$$P^*_{(l_M,\infty)}k^*=P^*_{(l_M,\infty)}h^*,$$

because $P^*_{(l_M,\infty)}l^*=0$ whenever $\operatorname{supp} l^*\subseteq\Gamma_{l_M}$. Since both k^* and h^* are supported by Γ_{l_N-1}, we have

$$P^*_{(l_M,l_N)}k^*=P^*_{(l_M,\infty)}P^*_{(0,l_N)}k^*=P^*_{(l_M,\infty)}k^*=P^*_{(l_M,\infty)}h^*=P^*_{(l_M,\infty)}P^*_{(0,l_N)}h^*=P^*_{(l_M,l_N)}g^*.$$

It follows that

$$R^*P^*_{(l_M,l_N)}k^*=R^*P^*_{(l_M,l_N)}g^*=g^*\circ P_{(l_M,l_N)}|_W=g^*\circ Q_{(M,N]}=f^*. \qquad\square$$

LEMMA 9.4. *Let* $j\geqslant 1$, $1\leqslant a\leqslant n_{2j}$ *and* $M\leqslant M_0<M_1<\ldots<M_a$ *be natural numbers, with* $2j\leqslant M_1$. *For each* $i\leqslant a$, *let* f_i^* *be in* $\operatorname{ball}\bigoplus_{M_{i-1}<n\leqslant M_i} F_n^*$ *and write* $f^*=\sum_{i=1}^a f_i^*$. *Then there exists* $\gamma\in\Gamma$ *with* $P^*_{(0,l_M]}e^*_\gamma=0$ *and* $\|4m_{2j}R(e^*_\gamma)-f^*\|\leqslant 2^{-l_M+3}$; *in particular* $\|f^*\|_{W^*}\leqslant 5m_{2j}$.

Proof. By Lemma 9.3, there exist $k_i^*\in\ell_1(\Gamma_{l_{M_i-1}}\setminus\Gamma_{l_{M_{i-1}}})$ such that $\|k_i^*\|_1\leqslant 4$ and $R(P^*_{(l_{M_{i-1}},l_{M_i})}k_i^*)=f_i^*$. Since $B_{l_{M_{i-1}},l_{M_i-1}}$ is an ε-net in $\operatorname{ball}\ell_1(\Gamma_{l_{M_i}-1}\setminus\Gamma_{l_{M_{i-1}}})$, with $\varepsilon=2^{-l_{M_i}+1}\leqslant 2^{-l_M-2i+1}$, we can choose $b_i^*\in B_{l_{M_{i-1}},l_{M_i-1}}$ such that $\|h_i^*-4b_i^*\|_1\leqslant 2^{-l_M-2i+3}$.

Now write $p_i=l_{M_i}$ for $1\leqslant i\leqslant a$ and apply the construction of Proposition 4.7 to obtain $\gamma\in\Delta_{p_a}$ with evaluation analysis

$$e^*_\gamma=\sum_{i=1}^a d^*_{\xi_i}+m_{2j}^{-1}\sum_{i=1}^a P^*_{(p_{i-1},\infty)}b_i^*.$$

Since $\operatorname{rank}\xi_i=p_i\in L$ for all i, we have $Rd^*_{\xi_i}=0$ and so

$$\begin{aligned}\|f^*-4m_{2j}R(e^*_\gamma)\|&=\left\|\sum_{i=1}^a(f_i^*-4RP^*_{(p_{i-1},\infty)}b_i^*)\right\|\leqslant\sum_{i=1}^a\|RP^*_{(p_{i-1},\infty]}h_i^*-4RP^*_{(p_{i-1},\infty)}b_i^*\|\\&\leqslant 3\sum_{i=1}^a\|h_i^*-4b_i^*\|\leqslant 3\sum_{i=1}^\infty 2^{-l_M-2i+3}=2^{-l_M+3}.\end{aligned}$$

It follows that $\|f^*\|\leqslant\|4m_{2j}R(e^*_\gamma)\|+8\leqslant 5m_{2j}$. $\square$

Lemma 9.5. *Let Y be any block subspace of W^* and let n and M be positive integers. For every $C>1$ there exists a $4C$-ℓ_1^n-average $w\in W$, with $Q_{(0,M]}w=0$, and a functional $g^*\in \operatorname{ball} Y$ with $Q^*_{(0,M]}g^*=0$ and $\langle g^*, w\rangle \geqslant 1$.*

Proof. The proof is a dualized version of Lemma 8.2. We suppose, without loss of generality, that $C<2$ and choose l and j such that $C^l>n$ and $n_{2j}>(10m_{2j})^l$; we take k minimal subject to $C^k>5m_{2j}$ noting that

$$n_{2j} > (10m_{2j})^l \geqslant (2C^{k-1})^l \geqslant C^{kl} > n^k.$$

Now take $(f_i^*)_{i=1}^{n^k}$ to be a normalized block sequence in $Y\cap \ker Q^*_{(0,M]}$; we may apply Lemma 9.4 to obtain

$$\left\|\sum_{i=1}^{n^k} \pm f_i^*\right\| \leqslant 5m_{2j} < C^k.$$

So by part (ii) of Lemma 2.2 (with $C=1+\varepsilon$) there are successive linear combinations $g_1^*, ..., g_n^*$ such that $\|g_i^*\|\geqslant C^{-1}$ for all i, while

$$\left\|\sum_{i=1}^{n} \pm g_i^*\right\| \leqslant 1$$

for all choices of sign. Since $(g_i^*)_{i\in\mathbb{N}}$ is a block sequence in $\ker Q^*_{(0,M]}$, we can choose $M\leqslant N_0<N_1<...<N_n$ such that $Q^*_{(N_{i-1},N_i]}g_i^*=g_i^*$. Now we choose, for all i, an element w_i of W such that $\|w_i\|\leqslant C$ and $\langle g_i^*, w_i\rangle=1$. If we set $w_i'=Q_{(N_{i-1},N_i]}w_i$ then we have $\|w_i'\|\leqslant 4C$ and $\langle g_i^*, w_i'\rangle=\langle g_i^*, w_i\rangle=1$, while $\langle g_i^*, w_h'\rangle=0$ when $h\neq i$. The element

$$w = n^{-1}\sum_{i=1}^{n} w_i'$$

is thus a $4C$-ℓ_1^n average, with $Q_{(0,p]}w=0$, and satisfies $\langle g^*, w\rangle=1$, where

$$g^* = \sum_{i=1}^{n} g_i^* \in \operatorname{ball} Y. \qquad \square$$

Lemma 9.6. *Let Y be any block subspace of W^*, and N and j be positive integers. Then there exists a $(1280, 2j, 1)$-exact pair (z,γ) with $z\in W$, $Q_{(0,N]}z=0$, $P^*_{(0,l_N]}e^*_\gamma=0$ and $\operatorname{dist}(Re^*_\gamma, Y)<2^{-l_N}$.*

Proof. We may assume that $l_N\geqslant 7$. By repeated applications of Lemma 9.5, we construct natural numbers $N\leqslant M_0<M_1<...$ and $j_1<j_2<...$, elements $w_i=Q_{(M_{i-1},M_i]}w_i$ of W, and functionals $g_i^*=Q^*_{(M_{i-1},M_i]}g_i^*\in \operatorname{ball} Y$ such that

(1) w_i is a 5-$\ell_1^{n_{j_i}}$-average;

(2) $\langle g_i^*, w_i\rangle\geqslant 1$;

(3) $j_{i+1}>M_i$.

It follows from Lemma 8.4 that $(w_i)_{i\in\mathbb{N}}$ is a 10-RIS.

Writing $g^*=\sum_{i=1}^{n_{2j}} g_i^*$ and applying Lemma 9.4, we find γ of weight m_{2j}^{-1} such that $\|4m_{2j}R(e_\gamma^*)-g^*\|\leqslant 2^{-l_N+3}$. We thus have

$$\operatorname{dist}(Re_\gamma^*, Y) \leqslant \left\|Re_\gamma^* - \tfrac{1}{4}m_{2j}^{-1}g^*\right\| \leqslant 2^{-l_N+1}m_{2j}^{-1} < 2^{-l_N},$$

and

$$4m_{2j}\sum_{i=1}^{n_{2j}} w_i(\gamma) \geqslant \sum_{i=1}^{n_{2j}}(\langle g^*, w_i\rangle - 2^{-l_N+3}\|w_i\|) \geqslant \sum_{i=1}^{n_{2j}}(1-5\cdot 2^{-4}) \geqslant \frac{n_{2j}}{2}.$$

We now set $z=\theta m_{2j}n_{2j}^{-1}\sum_{i=1}^{n_{2j}} w_i$, where θ is chosen so that $z(\gamma)=1$; by the above inequality $0<\theta\leqslant 8$.

To estimate $\|z\|$ and $|z(\gamma')|$ when weight $\gamma'=m_h^{-1}\neq m_{2j}^{-1}$, we return to Lemma 5.6 deducing that

$$\|z\| \leqslant 100\theta \quad \text{and} \quad |z(\gamma')| \leqslant \begin{cases} 160\theta m_h^{-1}, & \text{if } h<2j, \\ 100\theta m_{2j}^{-1}, & \text{if } h>2j. \end{cases}$$

So (z,γ) is certainly a $(1280, 2j, 1)$-exact pair. □

LEMMA 9.7. *Let Y_1 and Y_2 be block subspaces of W^* and let j_0 be a natural number. Then there exists a sequence $(x_i)_{i=1}^{n_{2j_0-1}}$ in W, together with natural numbers $0=p_0<p_1<...<p_{n_{2j_0-1}}$, and elements $\eta_i\in\Gamma_{p_i-1}\setminus\Gamma_{p_{i-1}}$ and $\xi_i\in\Delta_{p_i}$, $1\leqslant i\leqslant n_{2j_0-1}$, satisfying the conditions* (1)–(4) *of Definition* 6.3 *with $C=1280$ and $\varepsilon=1$, and such that, for all $i\geqslant 1$, the following additional properties hold*:

(5) $\operatorname{rank}\xi_i=p_i\in L$;

(6) $P^*_{(p_{i-1},p_i]}e^*_{\eta_i}=e^*_{\eta_i}$ *and* $P_{(p_{i-1},p_i]}(x_i)=x_i$;

(7) $\operatorname{dist}(Re^*_{\eta_i}, Y_k)<2^{-p_{i-1}}$, *where $k=1$ for odd i and $k=2$ for even i.*

Proof. We start by choosing j_1 such that $m_{4j_1-2}>n^2_{2j_0-1}$ and then apply Lemma 9.6 to obtain a $(1280, 4j_1-2, 1)$-exact pair (x_1,η_1) with $x_1\in W$. Set $p_1=l_{N_1}$, where N_1 is large enough to ensure that $P_{(0,p_1)}x_1=Q_{(0,N_1]}x_1=x_1$, $\operatorname{rank}\eta_1<p_1$ and $2^{p_1}>2n_{2j_0-1}$. Let $\xi_1=(p_1, m^{-1}_{2j_0-1}, \eta_1)\in\Delta_{p_1}$.

Continuing recursively, if for some $i<n_{2j_0-1}$, we have defined $\xi_i\in\Delta_{p_i}$, where $p_i=l_{N_i}$, we set $j_{i+1}=\sigma(\xi_i)$ and apply Lemma 9.6 to get a $(1280, 4j_{i+1}, 1)$-exact pair (x_{i+1},η_{i+1}) with $x_{i+1}\in W$, $Q_{(0,N_i]}x_{i+1}=P_{(0,p_i]}x_{i+1}=0$, $P^*_{(0,p_i]}e^*_{\eta_{i+1}}=0$ and $\operatorname{dist}(R^*e^*_{\eta_{i+1}}, Y_k)<2^{-p_i}$, where k depends on the parity of $i+1$. We now take N_{i+1} large enough, set $p_{i+1}=l_{N_{i+1}}$ and define $\xi_{i+1}=(p_{i+1},\xi_i, m^{-1}_{2j_0-1}, \eta_{i+1})\in\Delta_{p_{i+1}}$. □

We are now ready to finish the proof of the theorem. We consider any two infinite-dimensional subspaces Y_1 and Y_2 of W^* and apply Lemma 9.7 obtaining a dependent sequence satisfying properties (1)–(7). By property (7) we may choose, for each i, an element y_i^* of Y_k with

$$\|y_i^* - Re_{\eta_i}^*\| < 2^{-p_i}.$$

We set

$$y^* = m_{2j_0-1}^{-1} \sum_{i \text{ odd}} y_i^* \in Y_1 \quad \text{and} \quad z^* = m_{2j_0-1}^{-1} \sum_{i \text{ even}} y_i^* \in Y_2.$$

If γ is the element $\xi_{n_{2j_0-1}}$, then the evaluation analysis of γ is

$$e_\gamma^* = \sum_{i=1}^{n_{2j_0-1}} d_{\xi_i}^* + m_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} P_{(p_{i-1},\infty)}^* e_{\eta_i}^* = \sum_{i=1}^{n_{2j_0-1}} d_{\xi_i}^* + m_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} e_{\eta_i}^*,$$

because $P_{(0,p_{i-1}]}^* e_{\eta_i}^* = 0$. Since $\operatorname{rank} \xi_i = p_i \in L$ for all i, we have

$$Re_\gamma^* = m_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} Re_{\eta_i}^*,$$

which leads to

$$\|y^* + z^*\| \leqslant 1 + \left\| m_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} Re_{\eta_i}^* \right\| = 1 + \|Re_\gamma^*\| \leqslant 2.$$

We shall prove that $\|y^* - z^*\|$ is very large by estimating $\langle y^* - z^*, x\rangle$, where x is the average

$$x = n_{2j_0-1}^{-1} \sum_{k=1}^{n_{2j_0-1}} (-1)^k x_k,$$

about which we know from Lemma 8.9 that

$$\|x\| \leqslant 40 \cdot 1280 m_{2j_0-1}^{-2}.$$

By (7) and the definition of a 1-exact pair, we have

$$\langle e_{\eta_i}^*, x_k\rangle = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k, \end{cases}$$

so that

$$\begin{aligned} \langle y^* - z^*, x\rangle &= n_{2j_0-1}^{-1} m_{2j_0-1}^{-1} \sum_{i,k=1}^{n_{2j_0-1}} \langle y_i^*, x_k\rangle \geqslant n_{2j_0-1}^{-1} m_{2j_0-1}^{-1} \sum_{i,k=1}^{n_{2j_0-1}} (\langle e_{\eta_i}^*, x_k\rangle - 2^{-p_i}) \\ &\geqslant m_{2j_0-1}^{-1}(1 - n_{2j_0-1} 2^{-p_1}) \geqslant \tfrac{1}{2} m_{2j_0-1}^{-1}, \end{aligned}$$

the last step following from our choice of p_1 with $2^{p_1} > 2n_{2j_0-1}$.

We can now deduce that

$$\|y^* - z^*\| \geqslant \frac{m_{2j_0-1}}{102400}.$$

We have shown that the subspaces Y_1 and Y_2 of W^* contain elements y^* and z^*, with $\|y^* + z^*\| \leqslant 2$ and $\|y^* - z^*\|$ arbitrarily large. By Proposition 2.1, we have established hereditary indecomposability of W^*.

10. Concluding remarks

10.1. Operators on subspaces of $\mathfrak{X}_{\mathrm{K}}$

If we are looking at a bounded linear operator $T: Y \to \mathfrak{X}_{\mathrm{K}}$ defined only on a subspace Y of $\mathfrak{X}_{\mathrm{K}}$, rather than on the whole space, then, as in other HI constructions, the arguments of the preceding section can be used to show that T can be expressed as $\lambda I_Y + S$ with S strictly singular. However, as we shall now see, in this case the perturbation need not be compact.

PROPOSITION 10.1. *There exists a subspace Y of $\mathfrak{X}_{\mathrm{K}}$ and a strictly singular, non-compact operator T from Y into $\mathfrak{X}_{\mathrm{K}}$. In fact, for a suitably chosen Y, we may choose T mapping Y into itself.*

Proof. By a theorem of Androulakis, Odell, Schlumprecht and Tomczak–Jaegermann [3], in order to find Y and a strictly singular, non-compact $T: Y \to \mathfrak{X}_{\mathrm{K}}$, it is enough to exhibit normalized sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ in $\mathfrak{X}_{\mathrm{K}}$ such that $(y_i)_{i\in\mathbb{N}}$ has a spreading model equivalent to the usual ℓ_1-basis, while $(x_i)_{i\in\mathbb{N}}$ has a spreading model that is not equivalent to that basis. For $(x_i)_{i\in\mathbb{N}}$ we may take any normalized RIS; indeed, by Proposition 5.4, the spreading model associated with any RIS is dominated by the unit vector basis of the mixed Tsirelson space $T[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$, and so is not equivalent to the ℓ_1-basis. For $(y_i)_{i\in\mathbb{N}}$ we may take a specific sequence, setting

$$y_n = \sum_{\xi\in\Delta_n} d_\xi.$$

The result we need is a lemma about norms of linear combinations of these vectors.

LEMMA 10.2. *Let F be a finite set of natural numbers with $\min F \geqslant j$ and $\#F < 2n_{2j}$. Then, for all real scalars a_n,*

$$\Bigl\|\sum_{n\in F} a_n y_n\Bigr\| \geqslant \frac{1}{4}\sum_{n\in F}|a_n|.$$

Proof. Without loss of generality, we may assume that

$$\sum_{n\in F} a_n^+ \geqslant \frac{1}{2}\sum_{n\in F}|a_n|$$

and we may choose $p_1, p_2, ..., p_r$ in F, with $p_{i+1} > p_i + 1$, $r \leqslant p_{2j}$ and

$$\sum_{i=1}^{r} a_{p_i} \geqslant \frac{1}{4}\sum_{n\in F}|a_n|.$$

Since $p_1 \geqslant \min F \geqslant 2j$, we have that Δ_{p_1} does contain type-1 elements of the form $(p_1, m_{2j}^{-1}, \pm e_{\eta_1}^*)$, with $\eta_1 \in \Gamma_{p_1-1}$. We take ξ_1 to be such an element, and continue recursively, for $1 \leqslant i < r$, taking η_{i+1} to be any element of Δ_{p_i+1} and ξ_{i+1} to be the type-2 element $(p_{i+1}, \xi_i, m_{2j}^{-1}, \pm e_{\eta_{i+1}}^*)$ of $\Delta_{p_{i+1}}$. If $\gamma = \xi_r$, then the evaluation analysis of Γ is

$$e_\gamma^* = \sum_{i=1}^r d_{\xi_i}^* + m_{2j}^{-1} \sum_{i=1}^r \pm P_{(p_{i-1}, p_i)}^* e_{\eta_i}^*.$$

If we write $y = \sum_{n \in F} a_n y_n$, we have $\langle d_{\xi_i}^*, y \rangle = a_{n_i}$ for each i, so that

$$e_\gamma^*(y) = \sum_{i=1}^r a_{p_i} + m_{2j}^{-1} \sum_{i=1}^r \pm P_{(p_{i-1}, p_i)}^* e_{\eta_i}^*(y).$$

We have not until now been explicit about how the signs $\pm$ were chosen, but it is now clear that this may be done in such a way that $e_\gamma^*(y) \geqslant \sum_{i=1}^r a_{p_i} \geqslant \frac{1}{4} \sum_{n \in F} |a_n|$. □

It is now clear that the theorem of Androulakis et al. may be applied. In order to get the refined version where T takes Y into itself, it is enough to look a little more closely at the proof given in [3]. It turns out that we may take $(y_i)_{i \in \mathbb{N}}$ as above and Y to be the closed linear span $[y_i : i \in \mathbb{N}]$. It may be shown that, for any RIS $(x_i)_{i \in \mathbb{N}}$, the mapping $y_i \mapsto x_i$ extends to a bounded linear operator from Y to $\mathfrak{X}_{\mathrm{K}}$. Since Y, like all other infinite-dimensional subspaces, contains an RIS, we may choose the x_i to lie in Y. □

10.2. Very incomparable Banach spaces

The original spaces $X_{a,b}$ of Bourgain and Delbaen provided, for the first time, a continuum of non-isomorphic $\mathscr{L}_\infty$-spaces. It has also been noted [1] that if we take Y to be a Hilbert space and X to be $X_{a,b}$ with, for instance, $0 < b < \frac{1}{2} < a < 1$ and $a^4 + b^4 = 1$, then all operators from X to Y and all operators from Y to X are compact. The constructions in the present paper allow us to exhibit a continuum of spaces X_α, $\alpha \in \mathfrak{c}$, such that $\mathcal{L}(X_\alpha, X_\beta) = \mathcal{K}(X_\alpha, X_\beta)$ for all $\alpha \neq \beta$.

We start by taking an almost-disjoint family $(L_\alpha)_{\alpha \in \mathfrak{c}}$ of infinite subsets of $\mathbb{N}$. For each α we enumerate L_α in increasing order as l_j^α and define

$$m_j^\alpha = m_{l_j^\alpha} \quad \text{and} \quad n_j^\alpha = n_{l_j^\alpha},$$

where $(m_j, n_j) = (2^{2^j}, 2^{2^{j^2+1}})$ is the sequence mentioned in §2.4.

Now we may take X_α to be either $\mathfrak{B}_{\mathrm{mT}}[(\mathscr{A}_{n_j^\alpha}, 1/m_j^\alpha)_{j \in \mathbb{N}}]$ or $\mathfrak{X}_{\mathrm{K}}[(\mathscr{A}_{n_j^\alpha}, 1/m_j^\alpha)_{j \in \mathbb{N}}]$.

LEMMA 10.3. *Assume that* $\alpha\neq\beta$ *and let* $T: X_\alpha\to X_\beta$ *be a bounded linear operator. For any* RIS $(x_i)_{i\in\mathbb{N}}$ *in* X_α, *we have* $\|T(x_i)\|\to 0$ *as* $i\to\infty$.

Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a C-RIS in X_α and suppose, if possible, that $\|Tx_i\|>\delta>0$ for all i. Since $(Tx_i)_{i\in\mathbb{N}}$ is weakly null, we may, by taking a subsequence, assume that $(Tx_i)_{i\in\mathbb{N}}$ is a small perturbation of a skipped-block sequence in X_β. Thus, if $l=l^\beta_{2j}\in L_\beta$, we may apply Proposition 4.8 to conclude that

$$\left\| n_l^{-1}\sum_{i=1}^{n_l} Tx_i \right\|_{X_\beta} \geqslant \frac{1}{4} m_l^{-1} n_l^{-1} \sum_{r=1}^{n_l} \|Tx_i\| \geqslant \frac{\delta m_l^{-1}}{4}.$$

On the other hand, Corollary 5.5 tells us that

$$\left\| n_l^{-1}\sum_{i=1}^{n_l} x_i \right\|_{X_\alpha} \leqslant 10C \left\| n_l^{-1}\sum_{i=1}^{n_l} e_i \right\|,$$

where the norm on the right-hand side is calculated in $T[(\mathscr{A}_{3n_j}, m_j^{-1})_{j\in L_\alpha}]$. If l is not in L_α, then this norm is at most m_l^{-2} by Lemma 2.5, so that

$$\left\| n_l^{-1}\sum_{i=1}^{n_l} x_i \right\|_{X_\alpha} \leqslant 10C m_l^{-2}.$$

By the assumed almost-disjointness of L_β and L_α, we can certainly choose j such that $l^\beta_{2j}\notin L_\alpha$ and $m_l>40C\|T\|\delta^{-1}$, yielding a contradiction. □

THEOREM 10.4. *If* $\alpha\neq\beta$, *every bounded linear operator from* X_α *to* X_β *is compact.*

Proof. This is immediate from the preceding lemma and Proposition 5.11. □

Remark. The topologies $\sigma(\ell_1, X_\alpha)$ provide a continuum of very incomparable weak* topologies on ℓ_1: indeed, any linear mapping on ℓ_1 which is continuous from $\sigma(\ell_1, X_\alpha)$ to $\sigma(\ell_1, X_\beta)$, with $\alpha\neq\beta$, is necessarily compact.

10.3. The space of operators $\mathcal{L}(\mathfrak{X}_{\mathrm{K}})$

Of course, the spaces $\mathcal{L}(X)$ and $\mathcal{K}(X)$ of bounded (resp. compact) linear operators on an infinite-dimensional Banach space X are always decomposable. (Indeed, for finite-dimensional subspaces $E\subset X$ and $F\subset X^*$, the subspaces $X^*\otimes E$ and $F\otimes X$ are complemented.) So we must not hope for too much exotic structure in these spaces of operators. In this section we shall look briefly at subspaces of $\mathcal{L}(\mathfrak{X}_{\mathrm{K}})$. Certainly, $\mathcal{L}(\mathfrak{X}_{\mathrm{K}})=\mathcal{K}(\mathfrak{X}_{\mathrm{K}})\oplus\mathbb{R}I$ has HI subspaces, such as those isomorphic to $\mathfrak{X}_{\mathrm{K}}$, and subspaces isomorphic to $\mathfrak{X}_{\mathrm{K}}^*=\ell_1$. It has no subspace isomorphic to c_0, by a result of Emmanuele [16]. (The main result of [16] shows that c_0 does not embed into $\mathcal{K}(X_{a,b})$ and the same proof works for $\mathfrak{X}_{\mathrm{K}}$.) We shall now see that $\mathcal{K}(\mathfrak{X}_{\mathrm{K}})$ does have other subspaces with unconditional basis. It is a general fact that if $(x_n)_{n\in\mathbb{N}}$ is a basic sequence in a Banach space X then the injective tensor product $\ell_1\widehat{\otimes}_\varepsilon X$ contains a sequence equivalent to the "unconditionalization" of the basic sequence $(x_n)_{n\in\mathbb{N}}$. This follows immediately from the following exact formula for the norm of a finite sum of elementary tensors in $\ell_1\widehat{\otimes}_\varepsilon X$:

$$\left\|\sum_{j=1}^n e_j^*\otimes x_j\right\|_\varepsilon = \sup\left\|\sum_{j=1}^n \pm x_j\right\|,$$

where the supremum is over all choices of signs.

In the case of $\mathfrak{X}_{\mathrm{K}}$, the space of compact operators $\mathcal{K}(\mathfrak{X}_{\mathrm{K}})$ is isomorphic to $\ell_1\widehat{\otimes}_\varepsilon\mathfrak{X}_{\mathrm{K}}$, and so contains the unconditionalization of any basic sequence in $\mathfrak{X}_{\mathrm{K}}$. An interesting special case is that of the basis $(d_\gamma)_{\gamma\in\Gamma}$; we have chosen to prove the following proposition in a way that does not depend on the general theory of tensor products.

Proposition 10.5. *The family* $(e_\gamma^*\otimes d_\gamma)_{\gamma\in\Gamma}$ *is an unconditional basis of a reflexive subspace of* $\mathcal{K}(\mathfrak{X}_{\mathrm{K}})$.

Proof. Let us write $U_\gamma=e_\gamma^*\otimes d_\gamma$ considered as the rank-1 operator

$$\begin{aligned} U_\gamma\colon \mathfrak{X}_{\mathrm{K}} &\longrightarrow \mathfrak{X}_{\mathrm{K}},\\ x &\longmapsto x(\gamma)d_\gamma. \end{aligned}$$

For a finite linear combination $W=\sum_{\gamma\in\Gamma_n} w(\gamma)U_\gamma$ and any $x\in\operatorname{ball}\mathfrak{X}_{\mathrm{K}}$ we have

$$\|W(x)\| = \left\|\sum_{\gamma\in\Gamma_n} w(\gamma)x(\gamma)d_\gamma\right\| \leqslant \max_{\pm}\left\|\sum_{\gamma\in\Gamma_n} \pm w(\gamma)d_\gamma\right\|.$$

We shall write $|||W|||$ for the last expression on the line above. We have thus shown that $\|W\|\leqslant|||W|||$.

On the other hand, if we choose $u(\gamma)=\pm1$ for $\gamma\in\Gamma_n$ in such a way as to achieve the maximum in the definition of $|||W|||$ and then set $y=i_n(u)$, we have

$$|||W||| = \left\|\sum_{\gamma\in\Gamma_n} w(\gamma)u(\gamma)d_\gamma\right\| = \|W(y)\| \leqslant \|W\|\,\|i_n\| \leqslant 2\|W\|.$$

Thus the operator norm $\|\cdot\|$ and the unconditionalized norm $|||\cdot|||$ are equivalent on $[U_\gamma:\gamma\in\Gamma]$. It will be convenient to work with the latter norm.

Given a linear combination $V=\sum_{\gamma\in\Gamma_n} v(\gamma)U_\gamma$, any vector $\sum_{\gamma\in\Gamma_n} \pm v(\gamma)d_\gamma$ in $\mathfrak{X}_{\mathrm{K}}$ (whether or not the signs achieve the supremum in the definition of the unconditionalized norm) will be called a *realization* of W.

If the subspace $[U_\gamma : \gamma \in \Gamma]$ is not reflexive, then by unconditionality there is a skipped block sequence equivalent to the unit vector basis of either c_0 or ℓ_1. We shall treat the case of ℓ_1, leaving the (very easy) other case to the reader.

We consider a normalized skipped block sequence with

$$V_i = \sum_{\gamma \in \Gamma_{p_i - 1} \setminus \Gamma_{p_{i-1}}} v(\gamma) U_\gamma$$

and suppose, if possible, that $(V_i)_{i\in\mathbb{N}}$ is C-equivalent to the usual ℓ_1-basis for the norm $|||\cdot|||$. More precisely, let us suppose that $|||V_i||| \leqslant C$ for all i, and that

$$\left|\left|\left| \sum_{i=1}^{\infty} a(i) V_i \right|\right|\right| \geqslant \sum_{i=1}^{\infty} |a(i)|$$

for all scalars a_i. Let us note that if W is a linear combination of the form

$$W = n^{-1} \sum_{i=l+1}^{l+n} V_i,$$

then any realization $\widehat{W}$ of W is a C-ℓ_1-average (as in Definition 8.1). Indeed $\widehat{W}$ is expressible as

$$n^{-1} \sum_{i=l+1}^{l+n} \widehat{V}_i,$$

where the $\widehat{V}_i$ are realizations of V_i, and so satisfy $\|\widehat{V}_i\| \leqslant |||V_i||| \leqslant C$ for all i.

We now look at Lemma 8.4. It should be clear that, by choosing sequences $(j_k)_{k\in\mathbb{N}}$ and $(l_k)_{k\in\mathbb{N}}$ growing sufficiently fast, we may define

$$W_k = n_{j_k}^{-1} \sum_{i=l_j+1}^{l_j+n_{j_k}} V_i,$$

in such a way that any realizations $\widehat{W}_k$ form a $2C$-RIS in $\mathfrak{X}_{\mathrm{K}}$. In particular,

$$\left|\left|\left| n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} W_k \right|\right|\right| = \left\| n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} \hat{W}_k \right\|$$

for suitable realizations $\widehat{W}_k$, yielding

$$\left|\left|\left| n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} W_k \right|\right|\right| \leqslant 12 C m_{j_0}^{-1},$$

by Proposition 5.6. On the other hand,

$$\left|\left|\left| n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} W_k \right|\right|\right| = \left|\left|\left| n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} n_{j_k}^{-1} \sum_{i=l_k+1}^{l_k+n_{n_k}} V_i \right|\right|\right|,$$

which is at least 1, by our assumption on $(V_i)_{i\in\mathbb{N}}$.

So we have a contradiction for suitably large values of j_0. □

10.4. $\mathcal{L}(\mathfrak{X}_{\mathrm{K}})$ as a Banach algebra

In his 1972 memoir [30], B. E. Johnson set up the theory of cohomology of Banach algebras, and introduced the notion of an *amenable* Banach algebra. He posed the question of whether the algebra $\mathcal{L}(X)$ can ever be amenable for an infinite-dimensional Banach space X. Whether $\mathcal{L}(X)$ is amenable remains an important open problem for a number of concrete Banach spaces, including ℓ_p $(p\neq 1,2)$ [15]. For more about amenability of Banach algebras, including the definition, the reader is referred to [14]. Without going into such details we can note that $\mathfrak{X}_{\mathrm{K}}$ gives an example that answers Johnson's question. We are grateful to H. G. Dales for bringing this question, and the relevant references, to our attention.

PROPOSITION 10.6. *The Banach algebra $\mathcal{L}(\mathfrak{X}_{\mathrm{K}})$ is amenable.*

Proof. It is shown in [24] that $\mathcal{K}(X)$ is amenable if X is an $\mathscr{L}_p$-space, $1\leqslant p\leqslant\infty$; thus $\mathcal{K}(\mathfrak{X}_{\mathrm{K}})$ is amenable. It is shown in [14, Proposition 2.8.58 (i)] that the algebra obtained by adjoining an identity to a non-unital amenable Banach algebra is again amenable. In our case this gives us the amenability of $\mathcal{L}(\mathfrak{X}_{\mathrm{K}})=\mathcal{K}(\mathfrak{X}_{\mathrm{K}})\oplus\mathbb{R}I$. □

10.5. Open problems

Our constructions give no clue as to whether there exists a reflexive Banach space on which all operators are scalar-plus-compact. The construction of such a space, if one exists, will need new ideas. We thus have no example of a reflexive space on which all operators have non-trivial proper invariant subspaces. It is piquant to observe that, at the other end of the spectrum, the construction of a reflexive space on which some operator has no non-trivial proper invariant subspace has also proved to be very resistant to attack. We refer the reader to the papers of Enflo [17], [18] and Read [37], [38] for more about the "invariant subspace problem", noting the more recent paper [39] of Read, in which a strictly singular operator is constructed which has no non-trivial proper invariant subspace.

As we remarked in the introduction, we do not know whether an isomorphic predual of ℓ_1 which has the "few-operators" property in the scalar–plus–strictly-singular sense necessarily also has this property in the scalar-plus-compact sense. We therefore pose the following problem.

Problem 10.7. Let X be an $\mathscr{L}_\infty$-space on which every bounded linear operator is a strictly singular perturbation of a scalar multiple of the identity. Is every strictly singular operator on X necessarily compact?

In this context, we are grateful to Yolanda Moreno Salguero for pointing out to us that, in general, a strictly singular operator on an $\mathscr{L}_\infty$-space does not need to be weakly compact. Indeed, it is easy to see that for any separable, infinite-dimensional $\mathscr{L}_\infty$-space X, there is a quotient operator from X onto c_0. If X has no subspace isomorphic to c_0, such an operator is strictly singular.

Working with T. Raikoftsalis, the present authors have recently constructed another counterexample to the scalar-plus-compact problem. Like the space presented here, it is an $\mathscr{L}_\infty$-space constructed by the Bourgain–Delbaen method. However, the new space has non-separable dual and has a subspace isomorphic to ℓ_1. We believe it to be the first example of an indecomposable space containing ℓ_1. The only obvious obstruction to embeddability of a given Banach space X into an indecomposable space is the existence in X of a subspace isomorphic to c_0. We therefore are led to pose another problem.

Problem 10.8. Let X be a separable Banach space with no subspace isomorphic to c_0. Does X necessarily embed in an indecomposable space?

It is tempting to push this conjecture one step further by asking if a separable Banach space not containing c_0 can be embedded in a space with the scalar-plus-compact property. However, it is easy to see that this is too much to hope for.

PROPOSITION 10.9. *The space $\ell_1 \oplus \ell_2$ does not embed in a separable Banach with the scalar-plus-compact property.*

Proof. Let Y be a separable space that contains subspaces Y_1 and Y_2, isomorphic to ℓ_1 and ℓ_2, respectively. By a theorem of Pełczyński [36], the existence of a subspace isomorphic to ℓ_1 implies that there is a quotient operator Q from Y onto $\mathcal{C}[0,1]$. As is well known, there exists a quotient operator $R: \mathcal{C}[0,1] \to Y_2$. The composition $RQ: Y \to Y$ is weakly compact and non-compact, and hence not of the form $\lambda I + K$. $\square$

Finally, we would like to draw the reader's attention to the problems posed by Bourgain [11, p. 46] about the spaces $X_{a,b}$ and $\mathscr{L}_\infty$-spaces in general. Problems 1, 2 and 3 remain open. Hoping that his or her appetite has been whetted by the present paper, we leave it to the reader to find out what these problems are. Concerning Problem 4, we now know [21] that there is an infinite-dimensional Banach space with separable dual, no reflexive subspace and no subspace isomorphic to c_0. The present paper yields an example of an $\mathscr{L}_\infty$-space with no unconditional basis sequence. But we still do not have an example of a space X with X^* isomorphic to ℓ_1 and not containing c_0 nor a reflexive subspace. Only Problem 5 has been completely settled: each $X_{a,b}$ is saturated with ℓ_p for some p [27].

Note added in proof. Problem 10.7 has been answered in negative by M. Tarbard [41]. Related to Problem 10.8 the first author and Th. Raikoftsalis [7] have shown that every separable reflexive space is embedded into an indecomposable reflexive. Moreover, recently, the authors in collaboration with D. Freeman, E. Odell, Th. Raikoftsalis, Th. Schlumprecht and D. Zisimopoulou have proved that every Banach space X with separable dual not containing a complemented subspace isomorphic to ℓ_1 is embedded into an $\mathscr{L}_\infty$-space with the scalar-plus-compact property.

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