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# A Hesitant Fuzzy Linguistic TODIM Method Based on a Score Function

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**Abstract**

Hesitant fuzzy linguistic term sets (HFLTSs) are very useful for dealing with the situations in which the decision makers hesitate among several linguistic terms to assess an alternative. Some multi-criteria decision-making (MCDM) methods have been developed to deal with HFLTSs. These methods are derived under the assumption that the decision maker is completely rational and do not consider the decision maker's psychological behavior. But some studies about behavioral experiments have shown that the decision maker is bounded rational in decision processes and the behavior of the decision maker plays an important role in decision analysis. In this paper, we extend the classical TODIM (an acronym in Portuguese of interactive and multi-criteria decision-making) method to solve MCDM problems dealing with HFLTSs and considering the decision maker's psychological behavior. A novel score function to compare HFLTSs more effectively is defined. This function is also used in the proposed TODIM method. Finally, a decision-making problem that concerns the evaluation and ranking of several telecommunications service providers is used to illustrate the validity and applicability of the proposed method.

*Keywords:* Multi-criteria decision-making; Hesitant fuzzy linguistic term set; TODIM method; Distance measure; Score function; Comparison operator

## 1. Introduction

In multi-criteria decision-making (MCDM) problems, many criteria are of qualitative nature, so it is more suitable to evaluate them by using linguistic information [39]. For example, when we evaluate the "comfort" or "design" of a car, linguistic terms such as, "excellent", "good", "poor" etc. are preferred. Fuzzy linguistic approach [39] has obtained successful results dealing with linguistic information in decision making [18,23,27,38]. Many linguistic models have been presented to extend and improve the fuzzy linguistic approach in information modeling and computing processes [5,11,35]. These linguistic models use

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a single linguistic term to assess a linguistic variable. However, due to the lack of information about the decision problem or the lack of decision makers' knowledge on it, decision makers might hesitate among several linguistic terms to express their evaluations being necessary more flexible and complex linguistic expressions than single linguistic terms. In order to model this type of uncertainty provoked by hesitancy, Rodríguez et al. [29,30] introduced the notion of hesitant fuzzy linguistic term sets (HFLTSs) that facilitates the elicitation of linguistic expressions close to the human beings' cognitive model by means of context-free grammars. The study on the theory and applications of HFLTSs has been quickly spread [2,17,19,20,21,31,34,37,41] because of its usefulness in different applications.

It is necessary to point out that these approaches are derived under the assumption that the decision maker is completely rational. But some studies about behavioral experiments [3,15,33] have shown that the decision maker is bounded rational in decision processes and his/her behavior plays an important role in decision analysis. For example, when selecting an investment project, the decision maker usually has psychological expectations for some criteria such as profit, cost and risk, i.e., reference points. If a criterion value is over the reference point, the decision maker will be satisfied and regard the excess part as the "gain". Conversely, if a criterion value is under the reference point, the decision maker will be unsatisfied and regard the lacking part as the "loss" [15,33]. In addition, the decision maker is more sensitive to losses than to gains [1]. Therefore, it seems necessary to introduce the decision maker's psychological behavior to solve decision making problems.

The TODIM (an acronym in Portuguese of interactive and multiple attribute decision making) method early developed by Gomes and Lima in 7,8 is a tool that considers the decision maker's behavior to solve MCDM problems. In the classical TODIM method, according to Prospect Theory [15], the prospect value function is built to measure the dominance degree of each alternative over the remaining ones, which reflects the decision maker's behavioral characteristic such as reference dependence and loss aversion, and then the overall value of each alternative is calculated and whereby the ranking of alternatives can be obtained. Up to now, the TODIM method has been extensively applied in various fields of decision-making, such as selection of the destination of natural gas [10], evaluations of residential properties [9] and oil spill response [26]. The classical TODIM method uses numerical information to assess the criteria. But in many situations, numerical values are inadequate or insufficient to model real-life decision problems and the fuzzy sets and their extensions are more appropriate to model human judgments. Thus, different extensions of the classical TODIM method have been developed for dealing with different types of information such as, fuzzy numbers [16], intuitionistic fuzzy sets [22], hesitant fuzzy sets [40], etc. However, in qualitative contexts, when decision makers hesitate about their evaluations and the use of only one linguistic term is not sufficient to reflect their hesitation, it is necessary to use another type of information representation able to model this type of hesitation, such as HFLTS. Therefore, the aim of this paper is to develop an extended TODIM method to solve MCDM problems able to manage the hesitation of the decision makers by using HFLTSs. To do so, first it is proposed a novel score function to compare HFLTSs which takes into account both the average linguistic term of an HFLTS and its hesitant degree reflected by the number of the possible linguistic terms that compound the HFLTS. This function and a distance measure for HFLTSs are used to build a prospect value function, which can measure the dominance degree of one alternative over the remaining ones concerning each criterion. Therefore, the overall dominance degree of each alternative can be obtained by aggregating the dominance degrees of each alternative over the remaining under all the criteria, and the alternatives can be ranked according to their overall dominance degree. Finally, a decision-making problem about several telecommunications service providers is used to illustrate the validity and applicability of the proposed method.

The paper is organized as follows. Section 2 reviews some concepts about HFLTSs and the classical TODIM approach. Section 3 proposes a novel score function to compare HFLTSs. Section 4 presents an extension of the TODIM method which deals with HFLTSs. Section 5 presents an example to illustrate the use of the proposed method, and finally, some conclusions are pointed out in Section 6.

## 2. Preliminaries

This section revises some basic concepts, operations and distance measures of HFLTSs, and introduces in short the classical TODIM approach.

### 2.1. Hesitant fuzzy linguistic term sets

Due to the complexity of the real world decision making problems, it is often that decision makers hesitate among several linguistic terms to express their knowledge and they would like to use more than one linguistic term or more complex linguistic expressions that can reflect their knowledge in a proper way. In order to deal with these hesitant situations Rodríguez et al. introduced the concept of HFLTS [29,30] which is based on hesitant fuzzy sets [32].

**Definition 1**[29,30]. Let $S=\{s_0,\ldots,s_\tau\}$ be a linguistic term set. An HFLTS, $H_S$, is an ordered finite subset of consecutive linguistic terms of $S$,

$$H_S=\{s_i,s_{i+1},\ldots,s_j\} \ \ such\,that \ \ s_k\in S,\ k\in\{i,\ldots,j\}. \tag{1}$$

**Example 1**. Let $S=\{none, very\ low, low, medium, high, very\ high, perfect\}$ be a linguistic term set and $\vartheta$ be a linguistic variable, then $H_S^1(\vartheta)=\{medium, high, very\ \ high, perfect\}$ and $H_S^2(\vartheta)=\{low, medium, high\}$ are two HFLTSs on $S$.

Rodríguez et al. [30,31] proposed the use of context-free grammars to generate simple but rich linguistic expressions which can be easily represented by means of HFLTSs. A context-free grammar $G_H$, was defined in 30 and extended in 31 to generate suitable expressions for decision making. Such expressions can be transformed into HFLTSs by using the transformation function $E_{G_H}$.

**Definition 2**[30]. Let $S=\{s_0,\ldots,s_\tau\}$ be a linguistic term set, and $E_{G_H}$ be a function that transforms linguistic expressions $ll\in S_{ll}$, obtained by using the context-free grammar $G_H$, into HFLTSs, $H_S$.

$$E_{G_H}: S_{ll}\longrightarrow H_S \tag{2}$$

being $S$ the linguistic term set used by $G_H$ and $S_{ll}$ the expression domain generated by $G_H$.

The transformation of the linguistic expressions into HFLTSs will depend on the linguistic expressions generated by the context-free grammar $G_H$.

The use of linguistic information implies to carry out computing with words processes [14,24,25]. In order to facilitate such computations with HFLTSs, Rodríguez et al. introduced the envelope of an HFLTS.

**Definition 3**[30]. The envelope of an HFLTS $H_S$, $env(H_S)$, is a linguistic interval whose limits are obtained through its upper and lower bounds.

$$env(H_S)=[H_{S^-},H_{S^+}],\ \ H_{S^-}\leqslant H_{S^+},$$

where $H_{S^-}=min\{s_i\mid s_i\in H_S\}$ and $H_{S^+}=max\{s_i\mid s_i\in H_S\}$.

### 2.2. Distance measure for HFLTSs

Distance measures are very important in many scientific fields, such as decision making, machine learning, pattern recognition etc. These measures are the basis of some well-known multicriteria decision making methods, such as TOPSIS, VIKOR and ELECTRE and have been applied to manage different types of information. Liao et al. [19] introduced the axiomatic definition of the distance measure and some distance formulas for HFLTSs.

Let $S=\{s_0,\ldots,s_\tau\}$ be a linguistic term set, $H_S^1=\{s_{\delta_l^1}|l=1,2,\ldots,\#H_S^1\}$ and $H_S^2=\{s_{\delta_l^2}|l=1,2,\ldots,\#H_S^2\}$ be two HFLTSs, where $\#H_S$ is the number of linguistic terms in an HFLTS $H_S$. Generally $\#H_S^1\neq\#H_S^2$. Therefore, in order to operate correctly, the shorter one should be extended until the length of both is the same. The best way to extend the shorter one is to add the same linguistic term several times in it until the changed linguistic term set has the same length as the longer one. The added linguistic term can be obtained by the following method.

Suppose that $H_S^2$ is the shortest, $H_S^{2^+}=max\{s_i\mid s_i\in H_S^2\}$, $H_S^{2^-}=min\{s_i\mid s_i\in H_S^2\}$

and $\xi(0 \leqslant \xi \leqslant 1)$ is an optimized parameter. The added linguistic term $s$ in $H_S^2$ can be obtained by $s = C^2(\xi, H_S^{2^+}, 1-\xi, H_S^{2^-}) = \xi \odot H_S^{2^+} \oplus (1-\xi) \odot H_S^{2^-} = s_k$, where $C^2(\xi, H_S^{2^+}, 1-\xi, H_S^{2^-})$ is the convex combination of two linguistic terms [4], $k = min\{\tau, round(\xi Ind(H_S^{2^+}) + (1-\xi) Ind(H_S^{2^-}))\}$, "*round*" is the usual *round* operation, and $Ind(\cdot)$ is the subscript of a linguistic term.

Following the Example 1, $H_S^1 = \{s_3, s_4, s_5, s_6\}$ and $H_S^2 = \{s_2, s_3, s_4\}$. We can see that $\#H_S^1 > \#H_S^2$, hence it should be extended $H_S^2$ by adding a linguistic term several times until having the same length than $H_S^1$ and then to calculate the distance between $H_S^1$ and $H_S^2$. The selection of this linguistic term mainly relies on the decision makers' risk attitudes, which determine the optimized parameter $\xi$. From the optimistic point of view $\xi = 1$, thus $H_S^2$ is extended as $H_S^2 = \{s_2, s_3, s_4, s_4\}$, and from the pessimistic point of view $\xi = 0$, $H_S^2$ is extended as $H_S^2 = \{s_2, s_2, s_3, s_4\}$. If the decision makers are neutral, then $\xi = 0.5$. So the added linguistic term $s$ is $s_3$ and $H_S^2$ is extended as $H_S^2 = \{s_2, s_3, s_3, s_4\}$. Although different operations may obtain different results, it is reasonable because the decision makers' risk attitudes have a direct influence on the final decision. Without loss of generality, in this paper, we assume $\xi = \frac{1}{2}$.

Let $S = \{s_0, \ldots, s_\tau\}$ be a linguistic term set, $H_S^1 = \{s_{\delta_l^1} | l = 1, 2, \ldots, \#H_S^1\}$ and $H_S^2 = \{s_{\delta_l^2} | l = 1, 2, \ldots, \#H_S^2\}$ be two HFLTSs on $S$ with the same length $L = \#H_S^1 = \#H_S^2$, where $\delta_l^i (i = 1, 2)$ are the subscripts of the linguistic terms $s_{\delta_l^i}$ and $\tau$ the granularity of the linguistic term set $S$. Then the distance between $H_S^1$ and $H_S^2$ can be calculated by the following generalized distance measure proposed by Liao et al. [19]:

$$d_{gd}\left(H_S^1, H_S^2\right) = \left(\frac{1}{L}\sum_{l=1}^{L}\left(\frac{|\delta_l^1 - \delta_l^2|}{\tau+1}\right)^{\lambda}\right)^{1/\lambda}. \quad (3)$$

When $\lambda = 1$, $d_{hd}\left(H_S^1, H_S^2\right)$ is the Hamming distance of $H_S^1$ and $H_S^2$:

$$d_{hd}\left(H_S^1, H_S^2\right) = \frac{1}{L}\sum_{l=1}^{L}\frac{|\delta_l^1 - \delta_l^2|}{\tau+1}, \quad (4)$$

and when $\lambda = 2$, $d_{ed}\left(H_S^1, H_S^2\right)$ is the Euclidean distance of $H_S^1$ and $H_S^2$:

$$d_{ed}\left(H_S^1, H_S^2\right) = \left(\frac{1}{L}\sum_{l=1}^{L}\left(\frac{|\delta_l^1 - \delta_l^2|}{\tau+1}\right)^{2}\right)^{1/2}. \quad (5)$$

By using the Example 1, if $\xi = \frac{1}{2}$, $H_S^2$ is extended to $\{s_2, s_3, s_3, s_4\}$. The generalized distance between $H_S^1$ and $H_S^2$ is computed as follows: $d_{gd}\left(H_S^1, H_S^2\right) = \left(\frac{1}{4}\left(\left(\frac{|3-2|}{7}\right)^{\lambda} + \left(\frac{|4-3|}{7}\right)^{\lambda} + \left(\frac{|5-3|}{7}\right)^{\lambda} + \left(\frac{|6-4|}{7}\right)^{\lambda}\right)\right)^{1/\lambda}$.
If $\lambda = 1$, then the Hamming distance between $H_S^1$ and $H_S^2$ is $d_{hd}(H_S^1, H_S^2) = \frac{1}{4}\left(\frac{|3-2|}{7} + \frac{|4-3|}{7} + \frac{|5-3|}{7} + \frac{|6-4|}{7}\right) \approx 0.2143$;
if $\lambda = 2$, then the Euclidean distance between $H_S^1$ and $H_S^2$ is $d_{ed}(H_S^1, H_S^2) = \left(\frac{1}{4}\left(\left(\frac{|3-2|}{7}\right)^{2} + \left(\frac{|4-3|}{7}\right)^{2} + \left(\frac{|5-3|}{7}\right)^{2} + \left(\frac{|6-4|}{7}\right)^{2}\right)\right)^{1/2} \approx 0.2259$.

### 2.3. *Classical TODIM method*

The basic idea of the classical TODIM method proposed in 8,9 is to measure the dominance degree of each alternative over the remaining ones by establishing a prospect value function based on Prospect Theory [15]. Based on the obtained dominance degrees, the ranking of the alternatives can be determined. The main advantage of the TODIM method is its capability of capturing the decision maker's behavior. The classical TODIM method is suitable to handle MCDM problems in which decision makers use numerical values to express their assessments. An algorithm for the TODIM method is summarized as follows [6].

Let $P = \{p_1, p_2, \ldots, p_m\}$ be a set of alternatives, $C = \{c_1, c_2, \ldots, c_n\}$ be a set of criteria and $w = \{w_1, w_2, \ldots, w_n\}$ be a weighting vector of criteria, where $w_j$ denotes the weight or the importance degree of criterion $c_j$. Let $X = (x_{ij})_{m\times n}$ be a decision matrix, where $x_{ij}$ represents the assessment provided by the decision maker for the alternative $p_i \in P$ with respect to the criterion $c_j \in C$.

***Step 1.*** To normalize the decision matrix $X = (x_{ij})_{m\times n}$ into $Y = (y_{ij})_{m\times n}$ using the normalization method.

***Step 2.*** To calculate the relative weights $w_{jr}$ of criteria $c_j(j=1,2,\ldots,n)$ to the reference criterion $c_r$, i.e.,

$$w_{jr}=w_j/w_r, \quad (6)$$

where $w_r=max\{w_j|j=1,2,\ldots,n\}$.

***Step 3.*** To calculate the dominance degree for each alternative $p_i(i=1,2,\ldots,m)$ over the remaining alternatives $p_k(k=1,2,\ldots,m)$ concerning criteria $c_j(j=1,2,\ldots,n)$, i.e.,

$$\Phi_j(p_i,p_k) = \begin{cases} \sqrt{(y_{ij}-y_{kj})w_{jr}/(\sum_{j=1}^n w_{jr})}, & y_{ij}-y_{kj}>0; \\ 0, & y_{ij}-y_{kj}=0; \\ -\frac{1}{\theta}\sqrt{(y_{kj}-y_{ij})\left(\sum_{j=1}^n w_{jr}\right)/w_{jr}}, & y_{ij}-y_{kj}<0, \end{cases} \quad (7)$$

where $\theta$ is the attenuation factor of the losses, $y_{ij}-y_{kj}$ denotes the gain of the alternative $p_i$ over the alternative $p_k$ concerning the criterion $c_j$ if $y_{ij}-y_{kj}>0$, and the loss if $y_{ij}-y_{kj}<0$.

***Step 4.*** To calculate the dominance degree for each alternative $p_i(i=1,2,\ldots,m)$ over the remaining alternatives $p_k(k=1,2,\ldots,m)$ as follows:

$$\delta(p_i,p_k)=\sum_{j=1}^n \Phi_j(p_i,p_k). \quad (8)$$

***Step 5.*** To compute the overall dominance degree for each alternative $p_i(i=1,2,\ldots m)$,

$$\xi(p_i)=\frac{\sum_{k=1}^m \delta(p_i,p_k)-\min_i\{\sum_{k=1}^m \delta(p_i,p_k)\}}{\max_i\{\sum_{k=1}^m \delta(p_i,p_k)\}-\min_i\{\sum_{k=1}^m \delta(p_i,p_k)\}}. \quad (9)$$

***Step 6.*** To rank the alternatives and select the most desirable one(s) according to the overall dominance degrees of the alternatives. The greater $\xi(p_i)$ is, the better alternative $p_i$ will be.

## 3. A new score function to compare HFLTSs

This section revises and analyzes two different methods to compare HFLTSs, and shows by means of an example that sometimes such methods cannot distinct between two HFLTSs. Therefore, this section presents a new score function to compare HFLTSs in a better way than the other two methods, because it is able to compare two HFLTSs when the other methods cannot do it.

### *3.1. Comparison methods for HFLTSs*

The comparison operation is used in many decision making models to obtain a ranking of alternatives. Thus, it is necessary to define a comparison method for HFLTSs which can be used in decision making. In spite of the novelty of the concept HFLTS, two methods to compare HFLTSs have been already proposed [30,37].

The first comparison method for HFLTSs was proposed by Rodríguez et al. [30]. This method uses the envelope of an HFLTS, that is a linguistic interval, for the comparison by adapting the Wang et al.'s approach [36] for linguistic intervals.

**Definition 4**[36]. Let $a=[a^L,a^U]$ and $b=[b^L,b^U]$ be two numerical intervals, the preference degree of $a$ over $b$ is defined by

$$\rho(a>b)=\frac{max(0,a^U-b^L)-max(0,a^L-b^U)}{(a^U-a^L)+(b^U-b^L)}. \quad (10)$$

**Definition 5**. Let $H_S^1$ and $H_S^2$ be two HFLTSs and $env(H_S^1)=[H_{S^-}^1,H_{S^+}^1]$, $env(H_S^2)=[H_{S^-}^2,H_{S^+}^2]$ the envelope of $H_S^1$ and $H_S^2$ respectively. The comparison between $H_S^1$ and $H_S^2$ is as follows:

$H_S^1>H_S^2$ iff $env(H_S^1)>env(H_S^2)$, $H_S^1\sim H_S^2$ iff $env(H_S^1)\sim env(H_S^2)$,
that means, $H_S^1>H_S^2$ iff $\rho([Ind(H_{S^-}^1),Ind(H_{S^+}^1)]>[Ind(H_{S^-}^2),Ind(H_{S^+}^2)])>0.5$,
$H_S^1\sim H_S^2$ iff $\rho([Ind(H_{S^-}^1),Ind(H_{S^+}^1)]>[Ind(H_{S^-}^2),Ind(H_{S^+}^2)])=0.5$, where $Ind(s_i)=i$ (it is the subscript of the linguistic term), $s_i\in S=\{s_0,\ldots,s_\tau\}$.

Wei et al. also proposed a comparison method to compare HFLTSs [37] based on the probability theory. Due to the complexity of this method, an example is introduced to explain it easily.

Let $S=\{s_0:\text{nothing},s_1:\text{very low},s_2:\text{low},s_3:\text{medium},s_4:\text{high},s_5:\text{very high},s_6:\text{perfect}\}$ be a linguistic term set, and $H_S^1=\{s_3,s_4,s_5,s_6\}$, $H_S^2=\{s_2,s_3,s_4\}$ be two different HFLTSs. Clearly, $H_S^1$

and $H_S^2$ have the common linguistic terms $s_3$ and $s_4$. The HFLTSs are written as follows:

$H_S^1$ : $s_3, s_4, s_5, s_6,$
$H_S^2$ : $s_2, s_3, s_4.$

In order to compare $H_S^1$ and $H_S^2$, two linguistic term sets denoted by $H_1^*$ and $H_2^*$ are built by adding the linguistic term $\bar{s}_2$ in $H_S^1$ and two linguistic terms $\bar{s}_5$ and $\bar{s}_6$ in $H_S^2$.

$H_1^*$ : $\bar{s}_2, s_3, s_4, s_5, s_6,$
$H_2^*$ : $s_2, s_3, s_4, \bar{s}_5, \bar{s}_6.$

where $\bar{s}_2$ can be any linguistic term in $H_S^1$, and $\bar{s}_5$, $\bar{s}_6$ can be any linguistic term in $H_S^2$.

Therefore, in order to compare $H_S^1$ and $H_S^2$, we compare the linguistic terms in the corresponding place in $H_1^*$ and $H_2^*$ by computing a possibility degree.

**Definition 6**[37]. Let $H_S^1$ and $H_S^2$ be two HFLTSs, the possibility degree of $H_S^1$ being not less than $H_S^2$, denoted by $\overline{\rho}(H_S^1 \geqslant H_S^2)$ is computed as follows:

$$\overline{\rho}(H_S^1 \geqslant H_S^2) = \frac{0.5|H^*_{S(1,2)}| + |H_{H_1^* > H_2^*}|}{|H_1^*|}, \qquad (11)$$

being $H^*_{S(1,2)} = \{s_i \mid s_i \in H_S^1 \text{ and } s_i \in H_S^2\}$ the set of the common linguistic terms in $H_S^1$ and $H_S^2$, $H_{H_1^* > H_2^*} = \{s_i^1 \mid s_i^1 \in H_1^*, s_i^2 \in H_2^*, s_i^1 > s_i^2\}$ the set of all linguistic terms in $H_1^*$ larger than the corresponding terms in $H_2^*$, and $|X|$ the cardinal number of a set $X$.

**Definition 7**[37]. Let $H_S^1$ and $H_S^2$ be two HFLTSs. If $\overline{\rho}(H_S^1 \geqslant H_S^2) > 0.5$, then we say that $H_S^1$ is superior to $H_S^2$ with the degree of $\overline{\rho}(H_S^1 \geqslant H_S^2)$, denoted by $H_S^1 \succ^{\overline{\rho}(H_S^1 \geqslant H_S^2)} H_S^2$. Especially, if $\overline{\rho}(H_S^1 \geqslant H_S^2) = 1$, then we call that $H_S^1$ is absolutely superior to $H_S^2$. If $\overline{\rho}(H_S^1 \geqslant H_S^2) = 0.5$, then we say that $H_S^1$ is indifferent with $H_S^2$, denoted by $H_S^1 \sim H_S^2$.

Once revised the comparison methods for HFLTSs, they are analyzed by using the following example.

**Example 2**. Let $S = \{s_0, \ldots, s_6\}$ be a linguistic term set, $H_S^1 = \{s_3\}$, $H_S^2 = \{s_3, s_4\}$ and $H_S^3 = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ be three HFLTSs defined in $S$.

In order to apply the comparison method proposed by Rodríguez et al. firstly the envelopes of these three HFLTSs are obtained by using the Def. 3.

$env(H_S^1) = [s_3, s_3]$, $env(H_S^2) = [s_3, s_4]$ and $env(H_S^3) = [s_1, s_6]$.

Afterwards, the preference degrees $\rho(env(H_S^2) > env(H_S^1))$, $\rho(env(H_S^2) > env(H_S^3))$ and $\rho(env(H_S^3) > env(H_S^1))$ are computed by means of the Def. 4.

$\rho(env(H_S^2) > env(H_S^1)) = \frac{max(0, Ind(s_4) - Ind(s_3)) - max(0, Ind(s_3) - Ind(s_3))}{(Ind(s_4) - Ind(s_3)) + (Ind(s_3) - Ind(s_3))}$ $= 1$, hence $H_S^2 > H_S^1$;
$\rho(env(H_S^2) > env(H_S^3)) = \frac{max(0, Ind(s_4) - Ind(s_1)) - max(0, Ind(s_3) - Ind(s_6))}{(Ind(s_4) - Ind(s_3)) + (Ind(s_6) - Ind(s_1))}$ $= \frac{1}{2}$, hence $H_S^2 \sim H_S^3$;
$\rho(env(H_S^3) > env(H_S^1)) = \frac{max(0, Ind(s_6) - Ind(s_3)) - max(0, Ind(s_1) - Ind(s_3))}{(Ind(s_6) - Ind(s_1)) + (Ind(s_3) - Ind(s_3))}$ $= \frac{3}{5}$, hence $H_S^3 > H_S^1$.

Note that $s_3$ is a linguistic term which appears both in $H_S^1$ and in $H_S^2$, so we should not say that $H_S^2$ is absolutely superior to $H_S^1$. Thus, Wei et al. [37] pointed out that sometimes it is not suitable to use this method to compare HFLTSs, and defined a new comparison method.

By using Def. 6 and Def. 7, the three HFLTSs are compared as follows.

$$\overline{\rho}(H_S^2 \geqslant H_S^1) = \frac{0.5|H^*_{S(2,1)}| + |H_{H_2^* > H_1^*}|}{|H_2^*|} = \frac{0.5*1+1}{2} = 0.75,$$

where $H_1{}^* = \{s_3, s_3\}$, $H_2{}^* = \{s_3, s_4\}$, $H^*_{S(2,1)} = \{s_i | s_i \in H_2{}^*, s_i \in H_1{}^*\} = \{s_3\}$, $H_{H_2^* > H_1^*} = \{s_i^2 | s_i^2 \in H_2{}^*, s_i^1 \in H_1{}^*, s_i^2 > s_i^1\} = \{s_4\}$. Thus, we get that $H_S^2$ is superior to $H_S^1$ with degree 0.75.

$$\overline{\rho}(H_S^3 \geqslant H_S^2) = \frac{0.5|H^*_{S(3,2)}| + |H_{H_3^* > H_2^*}|}{|H_3^*|} = \frac{0.5*2+2}{6} = \frac{1}{2},$$

$$\overline{\rho}(H_S^3 \geqslant H_S^1) = \frac{0.5|H^*_{S(3,1)}| + |H_{H_3^* > H_1^*}|}{|H_3^*|} = \frac{0.5*1+3}{6} = \approx 0.583.$$

Thus, we obtain that $H_S^2$ is indifferent to $H_S^3$, and $H_S^3$ is superior to $H_S^1$ with the degree 0.583.

Both methods obtain 0.5 to compare the HFLTSs $H_S^2$ and $H_S^3$, thus it is not possible to distinguish them. Consequently, the following conclusion is obtained.

Let $H_S^1 = \{s_{\delta_l^1} | l = 1, \ldots, \#H_S^1\}$ and $H_S^2 = \{s_{\delta_l^2} | l = 1, \ldots, \#H_S^2\}$ be two HFLTSs, and $\overline{\delta^1} = \frac{1}{\#H_S^1} \sum_{l=1}^{\#H_S^1} \delta_l^1$ and $\overline{\delta^2} = \frac{1}{\#H_S^2} \sum_{l=1}^{\#H_S^2} \delta_l^2$ be the average linguistic terms of $H_S^1$ and $H_S^2$ respectively. If $\overline{\delta^1} = \overline{\delta^2}$, that is, $H_S^1$ and $H_S^2$ have the same average linguistic term, then $H_S^1$ is indifferent to $H_S^2$ by using the above two methods. But we note that in the Example 2, $H_S^3$ contains more possible terms than $H_S^2$ and has bigger hesitant degree than $H_S^2$, so $H_S^2$ should be more reliable and should be greater than $H_S^3$.

Therefore, after analyzing this example, it seems that it is necessary to define a new comparison method which is able to compare HFLTSs in a better way.

### 3.2. A score function for comparing HFLTSs

According to the previous analysis, we take into account two aspects to define the new score function: i) the average linguistic term, and ii) the hesitant degree.

The greater the average linguistic term of an HFLTS, the greater the HFLTS should be. This means that the result of the score function should increase when the average linguistic term of the HFLTS increases.

On the other hand, an HFLTS has bigger hesitant degree if it contains more possible terms, and the result of the score function should decrease when its hesitant degree increases. In order to measure the hesitant degree it is computed the normalized variance of the subscripts of the linguistic terms that compound the HFLTS.

**Definition 8**. Let $S = \{s_0, \ldots, s_\tau\}$ be a linguistic term set, and $H_S = \{s_{\delta_l} | l = 1, \ldots, \#H_S\}$ be an HFLTS on $S$. A score function $\mathbb{F}(H_S)$ of $H_S$ is defined as follows,

$$\mathbb{F}(H_S) = \overline{\delta} - \frac{\frac{1}{\#H_S}\sum_{l=1}^{\#H_S}(\delta_l - \overline{\delta})^2}{var(\tau)}, \qquad (12)$$

where $\overline{\delta} = \frac{1}{\#H_S}\sum_{l=1}^{\#H_S}\delta_l$ and $var(\tau) = \frac{(0-\tau/2)^2+\ldots+(\tau-\tau/2)^2}{\tau+1}$.

**Definition 9**. The definition of the comparison between two HFLTSs is based on the score function of the HFLTSs, $\mathbb{F}(H_S)$. Hence, the comparison between $H_S^1$ and $H_S^2$ is defined as follows:

$H_S^1 > H_S^2$ iff $\mathbb{F}(H_S^1) > \mathbb{F}(H_S^2)$;
$H_S^1 = H_S^2$ iff $\mathbb{F}(H_S^1) = \mathbb{F}(H_S^2)$.

By using the Example 2, we compare the three HFLTSs applying the Def. 8 and Def. 9.

$\mathbb{F}(H_S^1) = 3 - \frac{0}{4} = 3$,
$\mathbb{F}(H_S^2) = 3.5 - \frac{(3-3.5)^2+(4-3.5)^2}{2\times 4} = 3.4375$ and
$\mathbb{F}(H_S^3) = 3.5 - \frac{(1-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(6-3.5)^2}{6\times 4}$
$= 2.7708$.

Therefore, the ranking of the HFLTSs is $H_S^2 > H_S^1 > H_S^3$.

We can see that the proposed score function allows comparing HFLTSs more effectively, since it considers not only the average linguistic term of an HFLTS, but also its hesitant degree. This score function will be used in the proposed TODIM method.

## 4. An hesitant fuzzy linguistic TODIM method

This section proposes an extension of the TODIM approach to handle MCDM problems with HFLTSs and introduces an algorithm for the proposed method.

### 4.1. Description of a MCDM problem under hesitant fuzzy linguistic information

Generally, a MCDM problem consists of identifying a desirable compromise solution from the feasible alternatives which are defined by a set of conflicting criteria [12,13,28]. Let $P = \{p_1, \ldots, p_m\}$ be a set of alternatives, $C = \{c_1, \ldots, c_n\}$ be a set of criteria, and $w = \{w_1, w_2, \ldots, w_n\}$ be a weighting vector of criteria satisfying $\sum_{j=1}^n w_j = 1$ and $0 \leqslant w_j \leqslant 1$. In this decision making problem, we suppose a linguistic term set $S = \{s_0, \ldots, s_\tau\}$, and a context-free grammar $G_H$, as the one defined in 30 which generates comparative linguistic expressions to assess criteria and alternatives. The linguistic expressions provided by the decision maker are transformed into HFLTSs by using the transformation function $E_{G_H}$, introduced in Def. 2 to construct a hesitant fuzzy linguistic decision matrix $R = (r_{ij})_{m\times n}$, where $r_{ij}$ is an HFLTS on $S$ and represents the linguistic assessment provided by the decision maker for the alternative $p_i$ with respect to the criterion $c_j$.

The criteria may be of different types, cost and benefit. Since, the criteria of cost are transformed into criteria of benefit by normalizing the hesitant fuzzy linguistic decision matrix $R=(r_{ij})_{m\times n}$ to yield a corresponding normalized hesitant fuzzy linguistic decision matrix $G=(g_{ij})_{m\times n}$, where

$$g_{ij}=\begin{cases} r_{ij}, & \text{for benefit criterion } c_j,\\ Neg(r_{ij}), & \text{for cost criterion } c_j,\end{cases} \quad (13)$$

being $Neg(r_{ij})=\{Neg(s_{\delta_l})|s_{\delta_l}\in r_{ij}, l=1,\ldots,\#r_{ij}\}$.

**Definition 10**. [11] Let $S=\{s_0,\ldots,s_\tau\}$ be a linguistic term set, the negation of a linguistic term $s_i\in S$, is defined as follows:

$$Neg(s_i)=s_{\tau-i}. \quad (14)$$

### 4.2. *A TODIM approach with HFLTS*

Similarly to the classical TODIM method introduced in section 2.3, the first step is to normalize the original decision matrix using Eq. (13). Afterwards, it is computed the dominance degree for each alternative by using a prospect value function based on Prospect Theory [15]. To do so, it is necessary to identify a reference criterion and calculate the relative weight of each criterion to the reference criterion. Usually, the criterion with the highest weight can be regarded as the reference criterion and then the relative weight $w_{jr}$ of the criterion $c_j$ to the reference criterion $c_r$ can be obtained by Eq. (6). By using the Def. 8 and Def. 9 the assessments provided over the alternatives and criteria which are represented by HFLTSs are compared. Analogously to the Eq. (7), the dominance degree of the alternative $p_i$ over the alternative $p_k$ concerning the criterion $c_j$ is calculated using the following function:

$$\Phi_j(p_i,p_k)=\begin{cases} \sqrt{w_{jr}d_{gd}(g_{ij},g_{kj})/\sum_{j=1}^n w_{jr}}, & \text{if } \mathbb{F}(g_{ij})-\mathbb{F}(g_{kj})>0;\\ 0, & \text{if } \mathbb{F}(g_{ij})-\mathbb{F}(g_{kj})=0;\\ -\frac{1}{\theta}\sqrt{\left(\sum_{j=1}^n w_{jr}\right)d_{gd}(g_{kj},g_{ij})/w_{jr}}, & \text{if } \mathbb{F}(g_{ij})-\mathbb{F}(g_{kj})<0,\end{cases} \quad (15)$$

where the distance $d_{gd}(g_{ij},g_{kj})$ defined by Eq. (3) denotes the gain of the alternative $p_i$ over the alternative $p_k$ concerning the criterion $c_j$ if $\mathbb{F}(g_{ij})-\mathbb{F}(g_{kj})>0$, and the loss if $\mathbb{F}(g_{ij})-\mathbb{F}(g_{kj})<0$. The parameter $\theta>0$ represents the attenuation factor of the losses. Thus, the greater $\theta$ is, the lower the degree of loss aversion is.

The dominance degree $\delta(p_i,p_k)$ of the alternative $p_i$ over the alternative $p_k$ can be obtained by aggregating $\Phi_j(p_i,p_k)$ under each criterion $c_j$ applying Eq. (8). And the overall dominance degree $\xi(p_i)$ of the alternative $p_i$ can be computed by Eq. (9).

Obviously, $0\leqslant\xi(p_i)\leqslant 1$, and the greater $\xi(p_i)$ is, the better the alternative $p_i$ will be. Finally, the ranking of the alternatives $p_i(i=1,\ldots,m)$ is obtained according to their overall dominance degrees.

An algorithm for the proposed TODIM approach with HFLTSs is defined as follows.

***Step 1.*** To transform the comparative linguistic expressions provided by the decision maker into HFLTSs applying the transformation function introduced in Def. 2.

***Step 2.*** To construct the hesitant fuzzy linguistic decision matrix $R=(r_{ij})_{m\times n}$ using the HFLTSs obtained in the previous step.

***Step 3.*** To normalize the decision matrix $R=(r_{ij})_{m\times n}$ into $G=(g_{ij})_{m\times n}$ by Eq. (13).

***Step 4.*** To determine the reference criterion $c_r$, and calculate the relative weights $w_{jr}(j=1,\ldots,n)$ of the criteria $c_j(j=1,\ldots,n)$ to the reference criterion $c_r$ using Eq. (6).

***Step 5.*** To calculate the dominance degrees $\Phi_j(p_i,p_k)$ of the alternatives $p_i(i=1,\ldots,m)$ over the alternatives $p_k(k=1,\ldots,m)$ concerning each criterion $c_j$ using Eq. (15).

***Step 6.*** To calculate the dominance degrees $\delta(p_i,p_k)$ of the alternatives $p_i(i=1,\ldots,m)$ over the alternatives $p_k(k=1,\ldots,m)$ using Eq. (8).

***Step 7.*** To calculate the overall dominance degrees $\xi(p_i)$ of the alternatives $p_i(i=1,\ldots,m)$ using Eq. (9).

***Step 8.*** To rank the alternatives according to the overall dominance degrees $\xi(p_i)(i=1,\ldots,m)$.

## 5. Illustrative example

In this section a MCDM problem is solved following the algorithm defined for the proposed TODIM approach.

### 5.1. Problem description

Nowadays, the competition among telecommunications services is increasing and it is much more difficult for SMEs (Small and Medium-sized Enterprises) to choose a suitable telecommunications service to improve their business operations, since ample resources can be a big obstacle. Let suppose that a SME has to select the best telecommunications service provider to improve its benefits. There are four possible alternatives: provider 1 ($p_1$), provider 2 ($p_2$), provider 3 ($p_3$) and provider 4 ($p_4$). Based on the society research, we consider four major criteria to evaluate these four telecommunications service providers. These criteria are: The Satisfaction of Price ($c_1$), Quality ($c_2$), Service ($c_3$), and Safeguard ($c_4$). A detailed description of such criteria is given in Table 1.

Table 1: Criteria to evaluate a telecommunications service

| Criterion | Description of criterion |
|---|---|
| Price $c_1$ | How the company is satisfied with the price, which will be paid for the telecommunications service |
| Quality $c_2$ | What level the telecommunications service can reach |
| Service $c_3$ | The maintenance and repair |
| Safeguard $c_4$ | The reliability of information protection |

In this decision problem, it is used the context-free grammar $G_H$, defined in [30], that generates comparative linguistic expressions suitable for this decision making problem. The linguistic term set used is $S = \{none(n), very\ low(vl), low(l), medium(m), high(h), very\ high(vh), perfect(p)\}$. In this problem the criteria have different importance being the weighting vector $w = (0.2, 0.15, 0.15, 0.5)^T$.

The assessments provided are shown in Table 2.

Table 2: Assessments over the alternatives and criteria.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $p_1$ | bt l and h | vh | bt vl and m | bt vl and m |
| $p_2$ | lower than l | bt l and m | bt h and vh | bt h and vh |
| $p_3$ | lower than m | bt l and vh | vh | p |
| $p_4$ | m | bt l and h | bt vl and l | greater than h |

The symbol "bt" in Table 2 stands for the word "between".

### 5.2. Solving procedure

In order to solve the problem, we follow the steps described in the algorithm defined for the proposed hesitant fuzzy linguistic TODIM model.

***Step 1.*** The comparative linguistic expressions provided by the decision maker are transformed into HFLTSs as is shown in Table 3.

Table 3: Assessments transformed into HFLTSs.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $p_1$ | {l,m,h} | {vh} | {vl,l,m} | {vl,l,m} |
| $p_2$ | {n,vl} | {l,m} | {h,vh} | {h,vh} |
| $p_3$ | {n,vl,l} | {l,m,h,vh} | {vh} | {p} |
| $p_4$ | {m} | {l,m,h} | {vl,l} | {vh,p} |

***Step 2.*** To construct the hesitant fuzzy linguistic decision matrix $R = (r_{ij})_{m\times n}$,

$$\begin{pmatrix} \{l,m,h\} & \{vh\} & \{vl,l,m\} & \{vl,l,m\} \\ \{n,vl\} & \{l,m\} & \{h,vh\} & \{h,vh\} \\ \{n,vl,l\} & \{l,m,h,vh\} & \{vh\} & \{p\} \\ \{m\} & \{l,m,h\} & \{vl,l\} & \{vh,p\} \end{pmatrix}.$$

***Step 3.*** The decision matrix is already normalized, so it is not necessary to normalize it.

***Step 4.*** To take the criterion *Safeguard* ($c_4$) as the reference criterion, because it is considered the most important factor. Thus the weight of the reference criterion $w_r = 0.5$. We take $\theta = 1$, which means that the losses will contribute with their real value to the global value.

***Step 5.*** To calculate the dominance degrees $\Phi_j(p_i, p_k)$ of the alternatives $p_i(i = 1,2,3,4)$ over the alternatives $p_k(k = 1,2,3,4)$ concerning each criterion $c_j$. In the distance measure, $\lambda = 1.5$. The results are shown in Table 4.

***Step 6.*** To calculate the dominance degrees of the alternatives over the others (see Table 5).

***Step 7.*** To obtain the overall dominance degrees for each alternative:

$\xi(p_1) = 0.3373$, $\xi(P_2) = 0$, $\xi(P_3) = 1$, $\xi(P_4) = 0.3899$.

***Step 8.*** To obtain the ranking for the four telecommunication service providers,

$$p_3 \succ p_4 \succ p_1 \succ p_2$$

Finally, $p_3$ is the most desirable telecommunication service provider.

Table 4: Dominance degrees of each alternative over the others concerning each criterion.

| $c_1$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
|---|---|---|---|---|
| $p_1$ | 0 | 0.2682 | 0.239 | -0.7383 |
| $p_2$ | -1.3408 | 0 | -0.6482 | -1.343 |
| $p_3$ | -1.1952 | 0.1296 | 0 | -1.22 |
| $p_4$ | 0.1477 | 0.2686 | 0.244 | 0 |
| $c_2$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
| $p_1$ | 0 | 0.2326 | 0.192 | 0.2113 |
| $p_2$ | -1.5507 | 0 | -1.0526 | -0.7485 |
| $p_3$ | -1.28 | 0.1579 | 0 | 0.1162 |
| $p_4$ | -1.4088 | 0.1123 | -0.7746 | 0 |
| $c_3$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
| $p_1$ | 0 | -1.5482 | -1.7059 | 0.1123 |
| $P_2$ | 0.2322 | 0 | -0.7746 | 0.2535 |
| $p_3$ | 0.2559 | 0.1162 | 0 | 0.2746 |
| $p_4$ | -0.7485 | -1.6903 | -1.8304 | 0 |
| $c_4$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
| $p_1$ | 0 | -0.848 | -1.0746 | -1.0017 |
| $p_2$ | 0.424 | 0 | -0.6637 | -0.5345 |
| $p_3$ | 0.5373 | 0.3318 | 0 | 0.2121 |
| $p_4$ | 0.5008 | 0.2673 | -0.4243 | 0 |

Table 5: Overall dominance degrees of each alternative over the others.

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
|---|---|---|---|---|
| $p_1$ | 0 | -1.8954 | -2.3495 | -1.4164 |
| $p_2$ | -2.2353 | 0 | -3.1391 | -2.3724 |
| $p_3$ | -1.682 | 0.7356 | 0 | -0.6172 |
| $p_4$ | -1.5088 | -1.0422 | -2.7852 | 0 |

In the literature has been proposed some MCDM approaches that deals with HFLTSs [17,19,20,30,34,37]. Nevertheless, they do not consider the psychological behavior of the decision makers. The proposed hesitant fuzzy linguistic TODIM approach can consider the psychological behavior by calculating the dominance degrees of the alternatives.

## 6. Conclusions

The classical TODIM is a valuable tool to solve M-CDM problems with crisp values and consider the decision makers' psychological behavior, but it is not able to manage hesitant fuzzy linguistic term sets (HFLTS). HFLTS is an effective tool to express human beings' hesitancy by means of linguistic evaluations and has wide applications in MCDM. In this paper, we have extended the TODIM method to solve MCDM problems with HFLTS. The most important advantage of the proposed approach is that it can handle decision-making problems in which the assessments are represented by HFLTSs, and it can take into account the decision makers' psychological behavior. In addition, we have also introduced a novel score function to compare HFLTSs and have used an example to show that the proposed comparison method can compare HFLTSs when other methods cannot do it.

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