A Hierarchical Structure Approach to Finite-Time Filter Design for Fuzzy Markov Switching Systems With Deception Attacks

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***Abstract*—This work is concerned with the issue of finite-time filter design for a type of Takagi–Sugeno (T–S) fuzzy Markov switching system (MSSs) with deception attacks (DAs). In view of communication network security, the randomly occurring DAs are considered in the measurement output (MO), in which the malicious unknown but bounded signals are launched by the adversary. Notably, to characterize the fallibility of the communication links between the MO and the filter, the packet dropouts, DAs, and quantization effects are taken into account simultaneously, which signifies that the resulting system is much more applicable than the existing results. Meanwhile, to deal with the phenomenon of asynchronous switching, a hierarchical structure approach is adopted, which involves the existing nonsynchronous/synchronous strategy as special cases. By means of a fuzzy-basis-dependent Lyapunov strategy, sufficient criteria are formulated such that the resulting system is stochastic finite-time boundedness under randomly occurring DAs. Finally, a double-inverted pendulum model and a numerical example are provided to validate the feasibility of the attained method.**

***Index Terms*—Deception attacks (DAs), finite-time, fuzzy Markov switching systems (MSSs), hierarchical structure (HS) approach.**

I. Introduction

OVER THE past decades, due to the extensive utilization in physical applications subject to abrupt structure changes, including sudden environmental alters and component fiascos, Markov switching systems (MSSs) have attained increasing attention [1]–[3]. Note that MSSs consist of a set of subsystems, where the information exchange among subsystems is ruled by the Markov chain. Following this trend, a large quantity of theoretical results is reported, including stability analysis, control, filter, etc., [4]–[10]. However, it is identified that many presented results are mainly concerned with linear MSSs instead of nonlinear MSSs. In reality, it has been well recognized that nonlinear MSSs are more general. In reality, many efficient tools have been addressed to tackle the nonlinear systems. By incorporating nonlinear dynamics into optimization, the nonlinear systems have been studied in [11] and [12]. Another efficient way to tackle the nonlinear systems is the Takagi–Sugeno (T–S) fuzzy model. The T–S fuzzy model was proposed by Takagi and Sugeno [13], in which the complex nonlinear systems can be divided into various linear submodels subject to IF-THEN rules. Since then, the model has proven to be an efficient tool to approximate nonlinear systems. Aided by the T–S fuzzy model technique, nonlinear MSSs are recognized as fuzzy MSSs (FMSSs), which have been widely investigated in many practical systems [14]–[20]. Especially, all the aforementioned FMSSs are concerned with an infinite time interval (ITI). It is well identified that the ITI cannot be applied in many practical scenarios subject to the fixed time interval. To fill this gap, the concept of finite-time stability (FTS) is proffered and has been treated as a hot research issue [21]–[25]. Nevertheless, not enough attention has been devoted to the issue of FTS for FMSSs, and their dynamic behavior still remains an open challenge, which is the first motivation of this article.

In all the aforementioned FMSSs, the designed filters are classified into two categories: 1) mode-independent filters (MIFs) and 2) mode-dependent filters (MDFs). By neglecting the useful mode information of the operational plant, MIFs are easy to design, which may lead to the conservativeness of attained results. As for MDFs, the plant-mode information is always presupposed to be accessible to filters; whereas the aforementioned hypothesis is hard to be contented and difficult to apply. In practice, because of the network-induced communication delays, signal losses, and other factors, it is a tough task to attain the plant-mode information when designing MDFs. Therefore, the investigation of asynchronous filters (AFs) for FMSSs becomes natural. In [26] and [27], by applying a hidden Markov model (HMM) technique, the quantized

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controls for nonlinear MSSs have been addressed. By utilizing the improved HMM strategy, the robust control and state estimation for MSSs have been inspected in [28] and [29], respectively. In [30], by employing a nonhomogenous asynchronous approach, the asynchronous $l_2 - l_\infty$ filter design for MSSs has been analyzed. In [31], by resorting to the hierarchical structure (HS) method, the quantized nonstationary filtering for MSSs has been explored. As stated in [31], the HS strategy is most remarkable, which characterizes the plant, quantizer, and filter in a more reasonable way. Up to now, few efforts have been dedicated to the AF design for FMSSs subject to the HS strategy, which is the second motivation behind this article.

Furthermore, in the networked control systems (NCSs), the signals among sensors, filters/controller, and actuator are imperfect since they are transmitted via communication networks. The unreliability of the network may cause unexpected phenomena, for instance, packet dropouts (PDs) [32], [33]; quantization effect [34], [35]; network-induced delay [36]; and so on. These unpredictable factors may decay the target system's performance. In addition, with regard to the security of NCSs, such as autonomous vehicles, unmanned aerial systems, and so on, the deception attacks (DAs) are not preventable. In reality, the DAs are launched by the adversaries in the cyber layer to measurement outputs (MOs). Note that DAs are major sources of dangerous attacks, which have been cogitated in [37]–[39]. Due to the significance of the network-induced phenomenon and the security of the network, how to tackle with the coexistence of PDs, quantization effects, and DAs in FMSSs is another motivation of this article.

Summarizing the aforementioned deliberation, we focus on the issue of finite-time filter design for FMSSs with DAs in this article. The three major contributions to this study are summarized as follows.

1) With the perspective of network security, the randomly occurring DAs are extended in MO, where the malicious unknown but bounded signals are launched by the adversary. With the consideration of the network-induced phenomenon, a novel MO model is provided for the coexistence of DAs, quantization, and PDs, simultaneously.
2) In light of the HS approach, the phenomenon of the asynchronous switching among plant, quantizer, and filter is described in a more reasonable way.
3) By resorting to a fuzzy-basis-dependent Lyapunov strategy, the association of the finite-time boundedness and the unknown but bounded DAs are formulated to quantify the level of damage.

Notations: The notations utilized in this work are listed in Table I.

II. Preliminaries and System Description

A. Fuzzy Markov Switching Systems

Fixing a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and considering the nonlinear MSSs described by a fuzzy model as follows.

Plant Rule p: IF ϑ_{1k} is $M_{p1}, \ldots, \vartheta_{fk}$ is M_{pf},

TABLE I
NOTATIONS

Notations	Denotes
$\mathscr{E}\{\cdot\}$	The mathematical expectation operator
$\mathrm{sym}\{\mathcal{K}\}$	$\mathcal{K}+\mathcal{K}^\top$
$\lambda_{\min}(\mathcal{K})$	The minimum eigenvalues of $\mathcal{K}$
$\lambda_{\max}(\mathcal{K})$	The maximum eigenvalues of $\mathcal{K}$
$\mathbb{R}^{n_x}$	n_x-dimensional Euclidean space
$*$	Symmetric matrix
$\mathcal{K}^{-1}$	Matrix inverse
$\mathcal{K}^\top$	Matrix transpose

THEN

$$\begin{cases} x(k+1) = A(p,\varphi_k)x(k) + B(p,\varphi_k)\omega(k) \\ \quad\; y(k) = C(p,\varphi_k)x(k) + D(p,\varphi_k)\omega(k) \end{cases} \tag{1}$$

where $\vartheta_{qk} \in \{\vartheta_{1k}, \vartheta_{2k}, \ldots, \vartheta_{fk}\}$ stands for the premise variable of the original system and M_{pq} indicates the fuzzy membership functions (FMFs). Specifically, $p \in \{1, 2, \ldots, r\}$ and r is the number of the plant IF-THEN rules. $x(k) \in \mathbb{R}^{n_x}$ symbolizes the system state vector, $y(k) \in \mathbb{R}^{n_y}$ expresses the measured output (MO), and $\omega(k) \in \mathbb{R}^{n_w}$ stands for the exogenous disturbance that satisfies $\omega^\top(k)\omega(k) \le \kappa$ $(\forall k \in [0, N])$.

The stochastic variable (SV) φ_k is recognized as a discrete-time Markov chain (DMC), and $\varphi_k \in \mathcal{A} = \{1, 2, \ldots, n_A\}$. The switching transition probability (STP) of the original plant obeys generator $\Gamma_1 = [\pi_{ab}]$ as follows:

$$\Pr\{\varphi_{k+1} = b | \varphi_k = a\} = \pi_{ab}$$

where $a, b \in \mathcal{A}$, $\pi_{ab} \in [0, 1]$, and $\forall a \in \mathcal{A}$, $\sum_{b\in\mathcal{A}} \pi_{ab} = 1$. For $\varphi_k = a$ $(a \in \mathcal{A})$, denote $\mathbf{A}(p, \varphi_k) = \mathbf{A}_{pa}$, where $\mathbf{A}(p, \varphi_k) = \{A(p, \varphi_k), B(p, \varphi_k), C(p, \varphi_k), D(p, \varphi_k)\}$ are known matrices subject to proper dimensions.

Adopting the T–S fuzzy inference technique, the FMSS (1) subject to the T–S fuzzy model can be reestablished as follows:

$$\begin{cases} x(k+1) = A_{ha}x(k) + B_{ha}\omega(k) \\ \quad\; y(k) = C_{ha}x(k) + D_{ha}\omega(k) \end{cases} \tag{2}$$

where

$$A_{ha} = \sum_{p=1}^{r} h_p(\vartheta_k)A_{pa}, \quad B_{ha} = \sum_{p=1}^{r} h_p(\vartheta_k)B_{pa}$$

$$C_{ha} = \sum_{p=1}^{r} h_p(\vartheta_k)C_{pa}, \quad D_{ha} = \sum_{p=1}^{r} h_p(\vartheta_k)D_{pa}$$

and $M_{pq}(\vartheta_{qk})$ symbolizes the grade of membership of ϑ_{qk} in M_{pq}, and $g_p(\vartheta_k) = \Pi_{q=1}^{r} M_{pq}(\vartheta_{qk})$, normalized fuzzy basis functions $h_p(\vartheta_k) = g_p(\vartheta_k)/\sum_{p=1}^{r} g_p(\vartheta_k)$, $h_p(\vartheta_k) \ge 0$, and $\sum_{p=1}^{r} h_p(\vartheta_k) = 1$.

B. Deception Attacks

In the NCSs, the DAs scenario may be experienced during the signal transmission via a communication network. As a source of deteriorating plant performance, DAs are injected by adversaries in a random way. In FMSS (2), a bounded attacker is injected to randomly eliminate the MO $y(k)$

$$\bar{y}(k) = y(k) + \beta_{\varphi_k}(k)\hbar(k) \tag{3}$$

where $\hbar(k) = -y(k) + \xi(k)$ stands for the deception signal. $\beta_{\varphi_k}(k) = 1$ and $\beta_{\varphi_k}(k) = 0$ refer to whether attack launched is successful or not. $\|\xi(k)\| \leq \zeta$ represents the embedded information launched by an adversary and confined to a bound signal ζ. The SV $\beta_{\varphi_k}(k)$ is determined by the Bernoulli distribution as follows:

$$\Pr\{\beta_{\varphi_k}(k) = 1\} = \beta_{\varphi_k}, \ \Pr\{\beta_{\varphi_k}(k) = 0\} = 1 - \beta_{\varphi_k}$$

where $\beta_{\varphi_k} \in [0, 1]$.

Remark 1: In (3), $\hbar(k)$ contains both the accurate information of MO $y(k)$ and the embedded DAs $\xi(k)$. Here, $\xi(k)$ is unknown and hard to distinguish, is which recognized as a bounded signal. Meanwhile, the mode-dependent $\beta_{\varphi_k}(k)$ is adopted to illustrate the probability occurring DAs, and abided by a Bernoulli distribution. If $\beta_{\varphi_k}(k) = 1$, this implies the attacks occurred and MO is replaced by DAs; if $\beta_{\varphi_k}(k) = 0$, it indicates no attacks occur.

C. Packet Dropout

In reality, it is unacceptable that signal communication between MO is perfect, which means $\bar{y}(k) = y_f(k)$. In networked FMSSs, the PDs cannot be neglected since MO drops intermittently, that is, $\bar{y}(k) \neq y_f(k)$. In this case, a Bernoulli method is borrowed to depict the PD phenomenon, and the relationship of $\bar{y}(k)$ and $y_f(k)$ is revealed as follows:

$$y_f(k) = \alpha_{\varphi_k}(k)\bar{y}(k) \tag{4}$$

where $\alpha_{\varphi_k}(k)$ is a mode-dependent SV. Similar to $\beta_{\varphi_k}(k)$, $\alpha_{\varphi_k}(k)$ abides by the Bernoulli distribution law

$$\Pr\{\alpha_{\varphi_k}(k) = 1\} = \alpha_{\varphi_k}, \ \Pr\{\alpha_{\varphi_k}(k) = 0\} = 1 - \alpha_{\varphi_k}$$

where $\alpha_{\varphi_k} \in [0, 1]$. Obviously, the PD occurs when $\alpha_{\varphi_k} = 0$ and no PD occurs when $\alpha_{\varphi_k} = 1$.

For the SVs $\alpha_{\varphi_k}(k)$ and $\beta_{\varphi_k}(k)$, it is easy to infer that

$$\mathscr{E}\{(\Im(k) - \Im)\} = 0, \ \mathscr{E}\left\{(\Im(k) - \Im)^2\right\} = \Im(1 - \Im) \tag{5}$$

where $\Im(k) = \{\alpha_{\varphi_k}(k), \beta_{\varphi_k}(k)\}$ and $\Im = \{\alpha_{\varphi_k}, \beta_{\varphi_k}\}$.

Remark 2: Note that the PD rate (PDR) has been studied in [43], where the PDR is assumed to be uncertain. Different from [43], the mode-dependent SVs $\alpha_{\varphi_k}(k)$ and $\beta_{\varphi_k}(k)$ are adopted to depict the random occurrence of DAs and PDs, respectively. Meanwhile, the above two SVs are presupposed to be mutually independent. If $\alpha_{\varphi_k}(k) = 1$ and $\beta_{\varphi_k}(k) = 0$, only communication link of MO occurs; if $\alpha_{\varphi_k}(k) = 1$ and $\beta_{\varphi_k}(k) = 1$, only the communication link of DAs occurs; otherwise, the MO losses occur.

D. Quantized Measured Output

Aiming to tackle the limited communication bandwidth between the MO and filter, the signals are required to be quantized before being transmitted. Benefitting from its advantage, an improved mode-dependent logarithmic quantizer (MDLQ) is adopted

$$\begin{aligned}\mathfrak{Q}_{\phi_k}(y_f(k)) = \big[&\mathfrak{Q}_{\phi_k 1}(y_{1f}(k)), \mathfrak{Q}_{\phi_k 2}(y_{2f}(k)) \\ &\ldots, \mathfrak{Q}_{\phi_k t}(y_{tf}(k))\big]^\top\end{aligned} \tag{6}$$

where $\mathfrak{Q}_{j\phi_k}(y_{ja}(k))$, $j \in \{1, 2, \ldots, t\}$ symbolizes the jth component of $\mathfrak{Q}(\theta_k, y_a(k))$. Furthermore, $-\mathfrak{Q}_j(\theta_k, y_{ja}(k)) = -\mathfrak{Q}_j(\theta_k, -y_{ja}(k))$.

The MDLQ can be sketched by a battery of quantization levels as follows:

$$\begin{aligned}\mathfrak{R}_{j,\phi_k} = \Big\{&\pm v_j^{(i)}(\phi_k) : v_j^{(i)}(\phi_k) = \rho_j^i(\phi_k)v_{j0} \\ &i = \pm 1, \pm 2, \ldots\Big\} \cup \{0\}\end{aligned}$$

where $v_{j0} > 0$ and $\rho_j(\phi_k) \in (0, 1)$. The quantizer $\mathfrak{Q}_j(\phi_k, y_{jf}(k))$ is portrayed by

$$\mathfrak{Q}_{j\phi_k}(y_{jf}(k)) = \begin{cases} v_j^{(i)}, \ \frac{v_j^{(i)}(\phi_k)}{1+\sigma_j(\phi_k)} < y_{jf}(k) \leq \frac{v_j^{(i)}(\phi_k)}{1-\sigma_j(\phi_k)} \\ 0, \ y_{jf}(k) = 0 \\ -\mathfrak{Q}_{j\phi_k}(-y_{jf}(k)), \ y_{jf}(k) < 0 \end{cases}$$

where $\sigma_j(\phi_k) = (1 - \rho_j(\phi_k))(1 + \rho_j(\phi_k))$ $\forall j \in \{1, 2, \ldots, t\}$. It is clear that

$$\mathfrak{Q}_{j\phi_k}(y_{jf}(k)) = (I + \Delta_{j\phi_k}(k))y_{jf}(k)$$

where $|\Delta_{j\phi_k}(k)| \leq \chi_{j\phi_k}$.

Defining $\Delta_{\phi_k}(k) = \mathrm{diag}\{\Delta_{j\phi_k}(k), \ldots, \Delta_{t\phi_k}(k)\}$, aided by the aforementioned equation, $\mathfrak{Q}_{\phi_k}(y_f(k))$ can be reformulated as

$$\mathfrak{Q}_{\phi_k}(y_f(k)) = (I + \Delta_{\phi_k}(k))y_f(k). \tag{7}$$

In (7), the SV ϕ_k is identified as another DMC and taking the value in a set $\mathcal{C} = \{1, 2, \ldots, n_c\}$. The STP of MDLQ following generator $\Gamma_2^{\varphi_{k+1}} = [\eta_{cd}^{\varphi_{k+1}}]$ is

$$\Pr\{\phi_{k+1} = d | \phi_k = c\} = \eta_{cd}^{\varphi_{k+1}}$$

where $c, d \in \mathcal{C}$, $\eta_{cd}^{\varphi_{k+1}} \in [0, 1]$, and $\sum_{d\in\mathcal{C}} \eta_{cd}^{\varphi_{k+1}} = 1$.

For all $a, b \in \mathcal{A}$, $c, d \in \mathcal{C}$ and $\varphi_k = a$, $\phi_k = c$, collaborating with (3), (4), and (7), $\mathfrak{Q}_c(y_f(k))$ can be reformulated as follows:

$$\begin{aligned}\mathfrak{Q}_c(y_f(k)) &= \alpha_a(k)(1 - \beta_a(k))(I + \Delta_c(k))y(k) \\ &\quad + \alpha_a(k)\beta_a(k)(I + \Delta_c(k))\xi(k).\end{aligned} \tag{8}$$

Remark 3: For σ-algebra produced by $\mathfrak{T}_{k-1} = \sigma\{\varphi_1, \varphi_2, \ldots, \varphi_{k-1}\}$, the DMC ϕ_k is presupposed to be $\mathfrak{T}_{k-1}$-independent. As stated in [31], even though the MDLQ mode ϕ_k is different from the plant-mode φ_k, the values of STP of ϕ_k are affected by φ_k and ϕ_{k-1}, simultaneously.

E. Asynchronous Fuzzy-Based Filter

In this section, the asynchronous fuzzy-based full-order filter for FMSS (2) is inferred as

$$\widehat{x}(k+1) = A_{fh\psi_k}\widehat{x}(k) + B_{fh\psi_k}\mathfrak{Q}_{\phi_k}(y_f(k)) \tag{9}$$

where $\widehat{x}(k)$ represents the filter state and

$$A_{fh\psi_k} = \sum_{p=1}^{r} h_p(\vartheta_k)A_{fq\psi_k}, \quad B_{fh\psi_k} = \sum_{p=1}^{r} h_p(\vartheta_k)B_{fq\psi_k}$$

in which $A_{fq\psi_k}$ and $B_{fq\psi_k}$ are matrices to resolve.

Recalling (9), the SV ψ_k is regarded as another DMC and taking value over a set $\mathcal{M} = \{1, 2, \ldots, n_M\}$. The STP of the filter-keeping generator $\Gamma_3^{\varphi_k\phi_k} = [\tau_{\mu\nu}^{\varphi_k\phi_k}]$ is

$$\Pr\{\psi_{k+1}=\nu|\psi_k=\mu\}=\tau_{\mu\nu}^{\varphi_k\phi_k}$$

$\forall\mu,\nu\in\mathcal{M}$, $\tau_{\mu\nu}^{\varphi_k\phi_k}\in[0,1]$, and $\sum_{\nu\in\mathcal{M}}\tau_{\mu\nu}^{\varphi_k\phi_k}=1$.

Remark 4: It is remarkable that many asynchronous issues for FMSSs have been reported. For example, the nonsynchronized state estimation for FMSSs has been explored in [44], where the estimator mode is different from the operating region mode. Differently, to model the mismatch of modes between filter and plant, this article focuses on the AF. In (9), the σ-algebra formed by $\mathfrak{W}_{k-1}=\sigma\{\phi_1,\phi_2,\ldots,\phi_{k-1}\}$, and the DMC ψ_k is surmised to be $(\mathfrak{T}_{k-1},\mathfrak{W}_{k-1})$-independent. Similarly, the values of STP of ψ_k are influenced by φ_k, ϕ_k, and φ_{k-1}, concurrently.

Remark 5: In light of (9), when $\varphi_k=\phi_k=\psi_k$, the designed AF is reduced to a mode-dependent one; when $\varphi_k=\phi_k$ or $\varphi_k=\psi_k$, the AF degrades to a partly mode-dependent one; when $\mathcal{C}=\mathcal{M}=1$, the AF decreases to a mode-independent one; and when $\mathcal{A}=\mathcal{C}=\mathcal{M}=1$, the AF demotes to a conventional one. Therefore, the designed AF covers aforementioned four special cases.

Defining $\delta(k)=[x^\top(k)\ \widehat{x}^\top(k)]^\top$, $\mathcal{I}_1=[0\ I]^\top$, $\mathcal{I}_2=[I\ 0]$, and $\forall a,b\in\mathcal{A}, c,d\in\mathcal{C}, \mu,\nu\in\mathcal{M}$, merging with (2) and (9), the following filtering FMSS is elicited:

$$\begin{aligned}\delta(k+1)=&\ \mathscr{A}_{hac\mu}\delta(k)+\mathscr{C}_{ha\mu}\xi(k)+\mathscr{B}_{hac\mu}\omega(k)\\&+\{\alpha_a(k)(1-\beta_a(k))-\alpha_a(1-\beta_a)\}\\&\times\big[\mathcal{I}_1B_{fha}(I+\Delta_c(k))C_{ha}\mathcal{I}_2\delta(k)\\&+\mathcal{I}_1B_{fha}(I+\Delta_c(k))D_{ha}\omega(k)\big]\\&+(\alpha_a(k)\beta_a(k)-\alpha_a\beta_a)\mathcal{I}_1B_{fha}\\&\times(I+\Delta_c(k))\xi(k)\end{aligned}\tag{10}$$

where

$$\begin{aligned}\mathscr{A}_{hac\mu}&=\sum_{p=1}^{r}\sum_{q=1}^{r}h_p(\vartheta_k)h_q(\vartheta_k)\mathscr{A}_{pqac\mu}\\\mathscr{B}_{hac\mu}&=\sum_{p=1}^{r}\sum_{q=1}^{r}h_p(\vartheta_k)h_q(\vartheta_k)\mathscr{B}_{pqac\mu}\\\mathscr{C}_{hac\mu}&=\sum_{q=1}^{r}h_q(\vartheta_k)\mathscr{C}_{qac\mu}\\\mathscr{D}_{ha\mu}&=\sum_{p=1}^{r}\sum_{q=1}^{r}h_p(\vartheta_k)h_q(\vartheta_k)\mathscr{C}_{pqa\mu}\\\mathscr{A}_{pqac\mu}&=\begin{bmatrix}A_{pa}&0\\\alpha_a(1-\beta_a)B_{fq\mu}(I+\Delta_c(k))C_{pa}&A_{fq\mu}\end{bmatrix}\\\mathscr{B}_{pqac\mu}&=\begin{bmatrix}B_{pa}\\\alpha_a(1-\beta_a)B_{fq\mu}(I+\Delta_c(k))D_{pa}\end{bmatrix}\\\mathscr{C}_{qa\mu}&=\begin{bmatrix}0\\\alpha_a\beta_aB_{fq\mu}(I+\Delta_c(k))\end{bmatrix}.\end{aligned}$$

To proceed further, a definition is embraced as below.

Definition 1 [24]: Given a time interval $T>0$, scalars $c_2>c_1>0$, and a matrix $R>0$, the resulting MSSs (10) with $\omega^\top(k)\omega(k)\le\kappa$ are said to be stochastically finite-time boundedness (SFTB) with respect to $(c_1,c_2,N,R,\kappa,\zeta)$ $\forall k\in\{1,2,\ldots,T\}$ such that

$$\delta^\top(0)R\delta(0)\le c_1\Rightarrow\mathfrak{E}\Big\{\delta^\top(k)R\delta(k)\Big\}<c_2.$$

III. Main Results

In this section, sufficient conditions of asynchronous filtering for FMSS (10) will be exhibited.

Theorem 1: Given scalars $\epsilon>0$, $\upsilon>0$, the FMSS (10) is SFTB with respect to $(c_1,c_2,N,R,\kappa,\zeta)$, if there exist matrices $P_{hac\mu}>0$, $S_c>0$ $\forall a,b\in\mathcal{A}, c,d\in\mathcal{C}, \mu,\nu\in\mathcal{M}$, such that

$$\Upsilon_{gppac\mu}<0\tag{11}$$

$$\Upsilon_{gpqac\mu}+\Upsilon_{gqpac\mu}<0,\ p<q\tag{12}$$

$$\upsilon^Nc_1\sigma_{\max}(P_R)+\frac{\upsilon^N-1}{\upsilon-1}\Big(\upsilon d+\epsilon\zeta^2\Big)<\sigma_{\min}(P_R)c_2\tag{13}$$

where

$$\begin{aligned}\Upsilon_{gpqac\mu}&=\begin{bmatrix}\Xi_{gpqac\mu}&\Sigma_{pqac\mu}\\ *&\Psi_c\end{bmatrix}\\\Xi_{gpqac\mu}&=\begin{bmatrix}\Xi^{(1)}_{pqac\mu}&\Xi^{(2)}_{pqac\mu}\\ *&\Xi^{(3)}_{gac\mu}\end{bmatrix}\\\Sigma_{pqac\mu}&=\Big[\mathscr{G}^{(1)}_{pa}\Omega_cS_c\ \ \mathscr{G}^{(2)}_{pa}\Omega_cS_c\ \ \mathscr{G}^{(3)}_{pa}\Omega_cS_c\\&\qquad\mathscr{H}^{(1)\top}_{q\mu}\ \ \mathscr{H}^{(2)\top}_{q\mu}\ \ \mathscr{H}^{(3)\top}_{q\mu}\Big]\\\Psi_c&=\{-S_c,-S_c,-S_c,-S_c,-S_c,-S_c\}\\\Xi^{(1)}_{pqac\mu}&=\mathrm{diag}\{-\upsilon P_{pac\mu},-\epsilon I,-\upsilon I\}\\\Xi^{(2)}_{pqac\mu}&=\Big[\Xi^{(2)\top}_{1pqac\mu}\ \ \Xi^{(2)\top}_{2pqac\mu}\ \ \Xi^{(2)\top}_{3pqac\mu}\Big]\\\Xi^{(3)}_{gac\mu}&=\mathrm{diag}\Big\{-\mathscr{P}^{-1}_{gac\mu},-\mathscr{P}^{-1}_{gac\mu}-\mathscr{P}^{-1}_{gac\mu}\Big\}\\\Xi^{(2)}_{1pqac\mu}&=\Big[\mathscr{A}^{\ell}_{pac\mu}\ \ \mathscr{C}^{\ell}_{pac\mu}\ \ \mathscr{B}^{\ell}_{pac\mu}\Big]\\\Xi^{(2)}_{2pqac\mu}&=\big[\theta_1\mathcal{I}_1B_{fq\mu}C_{ha}\mathcal{I}_2\ \ 0\ \ \theta_1\mathcal{I}_1B_{fq\mu}D_{ha}\big]\\\Xi^{(2)}_{3pqac\mu}&=\big[0\ \ 0\ \ \theta_2\mathcal{I}_1B_{fh\mu}\big]\\\mathscr{P}_{gac\mu}&=\sum_{a\in\mathcal{A}}\sum_{c\in\mathcal{C}}\sum_{\nu\in\mathcal{M}}\pi_{ab}\tau^{b}_{cd}\eta^{bd}_{\mu\nu}P_{gbd\nu}\\\mathscr{A}^{\ell}_{pqac\mu}&=\begin{bmatrix}A_{pa}&0\\\alpha_a(1-\beta_a)B_{fq\mu}C_{pa}&A_{fq\mu}\end{bmatrix}\\\mathscr{B}^{\ell}_{pqac\mu}&=\begin{bmatrix}B_{pa}\\\alpha_a(1-\beta_a)B_{fq\mu}D_{pa}\end{bmatrix}\\\mathscr{C}^{\ell}_{qa\mu}&=\begin{bmatrix}0\\\alpha_a\beta_aB_{fq\mu}\end{bmatrix}\\\mathscr{G}^{(1)}_{pa}&=\big[\alpha_a(1-\beta_a)C_{pa}\mathcal{I}_2\ \ \alpha_a(1-\beta_a)D_{pa}\ \ 0\ 0\ 0\ 0\big]^\top\\\mathscr{G}^{(2)}_{pa}&=\big[\theta_1C_{pa}\mathcal{I}_2\ \ 0\ \ \theta_1D_{pa}\ \ 0\ 0\ 0\big]^\top\\\mathscr{G}^{(3)}_{pa}&=[0\ 0\ I\ 0\ 0\ 0]^\top\\\mathscr{H}^{(1)}_{q\mu}&=\Big[0\ \ 0\ \ 0\ \ B^\top_{fq\mu}\mathcal{I}_1^\top\ \ 0\ \ 0\Big]\\\mathscr{H}^{(2)}_{q\mu}&=\Big[0\ \ 0\ \ 0\ \ 0\ \ B^\top_{fq\mu}\mathcal{I}_1^\top\ \ 0\Big]\\\mathscr{H}^{(3)}_{q\mu}&=\Big[0\ \ 0\ \ 0\ \ \alpha_a\beta_aB^\top_{fq\mu}\mathcal{I}_1^\top\ \ 0\ \ \theta_2B^\top_{fq\mu}\mathcal{I}_1^\top\Big]\end{aligned}$$

$$\begin{aligned}\theta_1 &= \sqrt{\alpha_a(1-\beta_a)(1-\alpha_a(1-\beta_a))}\\ \theta_2 &= \sqrt{\alpha_a\beta_a(1-\alpha_a\beta_a)}\\ \Omega_c &= \mathrm{diag}\{\chi_{1c}, \chi_{2c}, \dots, \chi_{tc}\}\\ \sigma_{\min}(P_R) &= \min_{a\in\mathcal{A},c\in\mathcal{C},\mu\in\mathcal{M}}\left\{\lambda_{\min}\left(R^{-1/2}P_{ac\mu}R^{-1/2}\right)\right\}\\ \sigma_{\max}(P_R) &= \max_{a\in\mathcal{A},c\in\mathcal{C},\mu\in\mathcal{M}}\left\{\lambda_{\max}\left(R^{-1/2}P_{ac\mu}R^{-1/2}\right)\right\}.\end{aligned}$$

Proof: First, let us define the following:

$$\begin{aligned}&h_+\ \Delta q\ (h_1(\vartheta_{k+1}), h_2(\vartheta_{k+1}), \dots, h_r(\vartheta_{k+1}))\\ &P_{h_+bd\nu}\ =\ \sum_{g=1}^{r} h_g(\vartheta_{k+1})P_{gbd\nu}.\end{aligned}$$

Combining (11) and (12) gives

$$\begin{aligned}\Upsilon_{hac\mu} &= \sum_{g=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{r} h_g(\vartheta_k)h_p(\vartheta_k)h_q(\vartheta_k)\Upsilon_{gpqac\mu}\\ &= \sum_{g=1}^{r} h_g(\vartheta_k)\left(\sum_{p=1}^{r} h_p^2(\vartheta_k)\Upsilon_{ppac\mu} + \sum_{p=1}^{r-1}\sum_{q=p+1}^{r}\right.\\ &\qquad \left.\times\, h_p(\vartheta_k)h_q(\vartheta_k)\left(\Upsilon_{gpqac\mu} + \Upsilon_{gqpac\mu}\right)\right) < 0 \qquad (14)\end{aligned}$$

where

$$\begin{aligned}\Upsilon_{hac\mu} &= \begin{bmatrix}\Xi_{hac\mu} & \Sigma_{hac\mu}\\ * & \Psi_c\end{bmatrix}\\ \Xi_{hac\mu} &= \begin{bmatrix}\Xi^{(1)}_{hac\mu} & \Xi^{(2)}_{hac\mu}\\ * & \Xi^{(3)}_{hac\mu}\end{bmatrix}\\ \Sigma_{hac\mu} &= \Big[\mathscr{G}^{(1)}_{ha}\Omega_c S_c \ \ \mathscr{G}^{(2)}_{ha}\Omega_c S_c \ \ \mathscr{G}^{(3)}_{ha}\Omega_c S_c\\ &\qquad \mathscr{H}^{(1)\top}_{h\mu} \ \ \mathscr{H}^{(2)\top}_{h\mu} \ \ \mathscr{H}^{(3)\top}_{h\mu}\Big]\\ \Xi^{(1)}_{hac\mu} &= \mathrm{diag}\{-\upsilon P_{hac\mu}, -\epsilon I, -\upsilon I\}\\ \Xi^{(3)}_{hac\mu} &= \mathrm{diag}\left\{-\mathscr{P}^{-1}_{h_+ac\mu}, -\mathscr{P}^{-1}_{h_+ac\mu} - \mathscr{P}^{-1}_{h_+ac\mu}\right\}\\ \Xi^{(2)}_{hac\mu} &= \left[\Xi^{(2)\top}_{1hac\mu} \ \ \Xi^{(2)\top}_{2hac\mu} \ \ \Xi^{(2)\top}_{3hac\mu}\right]\\ \Xi^{(2)}_{1hac\mu} &= \left[\mathscr{A}^{\ell}_{hac\mu} \ \ \mathscr{C}^{\ell}_{hac\mu} \ \ \mathscr{B}^{\ell}_{hac\mu}\right]\\ \Xi^{(2)}_{2hac\mu} &= \left[\theta_1\mathcal{I}_1 B_{fh\mu}C_{ha}\mathcal{I}_2 \ \ 0 \ \ \theta_1\mathcal{I}_1 B_{fh\mu}D_{ha}\right]\\ \Xi^{(2)}_{3hac\mu} &= \left[0 \ \ 0 \ \ \theta_2^2\mathcal{I}_1 B_{fh\mu}\right].\end{aligned}$$

Applying the Schur complement (SC) to (14), one can get that

$$\begin{aligned}\Xi_{hac\mu} &+ \sum_{s=1}^{3}\mathscr{G}^{(s)}_{ha}\Omega_c S_c\Omega_c\mathscr{G}^{(s)\top}_{ha}\\ &+ \sum_{s=1}^{3}\mathscr{H}^{(s)\top}_{h\mu}S_c^{-1}\mathscr{H}^{(s)}_{h\mu} < 0 \qquad (15)\end{aligned}$$

which is equivalent to

$$\begin{aligned}\Xi_{hac\mu} &+ \sum_{s=1}^{3}\mathscr{G}^{(s)}_{ha}\Delta_c(k)S_c\Delta_c(k)\mathscr{G}^{(s)\top}_{ha}\\ &+ \sum_{s=1}^{3}\mathscr{H}^{(s)\top}_{h\mu}S_c^{-1}\mathscr{H}^{(s)}_{h\mu} < 0. \qquad (16)\end{aligned}$$

By Theorem 2.1, (16) can be reformulated as

$$\Xi_{hac\mu} + \mathrm{sym}\left\{\sum_{s=1}^{3}\mathscr{G}^{(s)}_{ha}\Delta_c^{\top}(k)\mathscr{H}^{(s)}_{h\mu}\right\} < 0 \qquad (17)$$

where

$$\begin{aligned}\mathscr{G}^{(1)}_{ha} &= [\alpha_a(1-\beta_a)C_{ha}\mathcal{I}_2 \ \ \alpha_a(1-\beta_a)D_{ha} 0\ 0\ 0\ 0]^{\top}\\ \mathscr{G}^{(2)}_{ha} &= [\theta_1 C_{ha}\mathcal{I}_2\ 0\ \theta_1 D_{ha}\ 0\ 0\ 0]^{\top}\\ \mathscr{G}^{(3)}_{ha} &= [0\ 0\ I\ 0\ 0\ 0]^{\top}\\ \mathscr{H}^{(1)}_{h\mu} &= \left[0\ \ 0\ \ 0\ \ B^{\top}_{fh\mu}\mathcal{I}_1^{\top}\ \ 0\ \ 0\right]\\ \mathscr{H}^{(2)}_{h\mu} &= \left[0\ \ 0\ \ 0\ \ 0\ \ B^{\top}_{fh\mu}\mathcal{I}_1^{\top}\ \ 0\right]\\ \mathscr{H}^{(3)}_{h\mu} &= \left[0\ \ 0\ \ 0\ \ \alpha_a\beta_a B^{\top}_{fh\mu}\mathcal{I}_1^{\top}\ \ 0\ \ \theta_2 B^{\top}_{fh\mu}\mathcal{I}_1^{\top}\right].\end{aligned}$$

Carrying out SC to (17), one can get that

$$\overline{\Xi}_{hac\mu} = \begin{bmatrix}\Xi^{(1)}_{hac\mu} & \overline{\Xi}^{(2)}_{hac\mu}\\ * & \Xi^{(3)}_{hac\mu}\end{bmatrix} < 0 \qquad (18)$$

where

$$\begin{aligned}\overline{\Xi}^{(2)}_{hac\mu} &= \left[\overline{\Xi}^{(2)}_{1hac\mu}\ \ \overline{\Xi}^{(2)}_{2hac\mu}\ \ \overline{\Xi}^{(2)}_{3hac\mu}\right]\\ \overline{\Xi}^{(2)}_{1hac\mu} &= \left[\mathscr{A}_{hac\mu}\ \ \mathscr{C}_{hac\mu}\ \ \mathscr{B}_{hac\mu}\right]\\ \overline{\Xi}^{(2)}_{2hac\mu} &= \big[\theta_1\mathcal{I}_1 B_{fh\mu}(I+\Delta_c(k))C_{ha}\mathcal{I}_2\ \ 0\\ &\qquad \theta_1\mathcal{I}_1 B_{fh\mu}(I+\Delta_c(k))D_{ha}\big]\\ \overline{\Xi}^{(2)}_{3hac\mu} &= \left[0\ \ 0\ \ \theta_2^2\mathcal{I}_1 B_{fh\mu}(I+\Delta_c(k))\right].\end{aligned}$$

Next, we establish a Lyapunov function as follows:

$$V(k, \delta_k, h, \varphi_k, \phi_k, \psi_k) = \delta^{\top}(k)P_{hac\mu}\delta(k) \qquad (19)$$

where $P_{hac\mu} = \sum_{p=1}^{r} h_p(\vartheta_k)P_{pac\mu}$.

For $\varphi_k = a$, $\phi_k = c$, and $\psi_k = \mu$, similar to [31], it leads to

$$\begin{aligned}\Pr\ \{\varphi_{k+1} = b, \phi_{k+1} = d, \psi_{k+1} = \nu | \varphi_k = a\\ \phi_k = c, \psi_k = \mu\} = \pi_{ab}\tau^{b}_{cd}\eta^{bd}_{\mu\nu}. \qquad (20)\end{aligned}$$

Recollecting (10), we derive that

$$\begin{aligned}&\mathscr{E}\{V(k+1, \delta_{k+1}, h_+, \varphi_{k+1} = b, \phi_{k+1} = d, \psi_{k+1} = \nu\\ &\qquad |k, \delta_k, h, \varphi_k = a, \phi_k = c, \psi_k = \mu)\}\\ &= \mathscr{E}\left\{\delta^{\top}(k+1)\sum_{a\in\mathcal{A}}\sum_{c\in\mathcal{C}}\sum_{\nu\in\mathcal{M}}\pi_{ab}\tau^{b}_{cd}\eta^{bd}_{\mu\nu}\right.\\ &\qquad \left.\times\, P_{h_+bd\nu}\delta(k+1)\right\}\end{aligned}$$

$$
\begin{aligned}
&= \mathscr{E}\Big\{\big[\mathscr{A}_{hac\mu}\delta(k) + \mathscr{C}_{hac\mu}\xi(k) + \mathscr{B}_{hac\mu}\omega(k)\big]^\top \\
&\quad \times \mathscr{P}_{h_+ac\mu}\big[\mathscr{A}_{hac\mu}\delta(k) + \mathscr{C}_{hac\mu}\xi(k) \\
&\quad + \mathscr{B}_{hac\mu}\omega(k)\big] \\
&\quad + \theta_1^2\big[\mathcal{I}_1 B_{fh\mu}(I + \Delta_c(k))C_{ha}\mathcal{I}_2\delta(k) \\
&\quad + \mathcal{I}_1 B_{fh\mu}(I + \Delta_c(k))D_{ha}\omega(k)\big]^\top \\
&\quad \times \mathscr{P}_{h_+ac\mu}\big[\mathcal{I}_1 B_{fh\mu}(I + \Delta_c(k))C_{ha}\mathcal{I}_2\delta(k) \\
&\quad + \mathcal{I}_1 B_{fh\mu}(I + \Delta_c(k))D_{ha}\omega(k)\big] \\
&\quad + \xi^\top(k)\theta_2^2\big(\mathcal{I}_1 B_{fh\mu}(I + \Delta_c(k))\big)^\top \\
&\quad \times \mathscr{P}_{h_+ac\mu}\big(\mathcal{I}_1 B_{fh\mu}(I + \Delta_c(k))\big)\xi(k)\Big\}
\end{aligned} \tag{21}
$$

where

$$
\mathscr{P}_{h_+ac\mu} = \sum_{a\in\mathcal{A}}\sum_{c\in\mathcal{C}}\sum_{\nu\in\mathcal{M}} \pi_{ab}\tau_{cd}^b\eta_{\mu\nu}^{bd}P_{h_+bd\nu}.
$$

In light of the condition $\|\xi(k)\| \le \zeta$, it is concluded that

$$
\begin{aligned}
&\mathscr{E}\{V(k+1, \delta_{k+1}, h_+, \varphi_{k+1}, \phi_{k+1}, \psi_{k+1}\} \\
&\quad \le \varpi^\top(k)\sum_{s=2}^{4}\overline{\Xi}_{hac\mu}^{(s)\top}\mathscr{P}_{h_+ac\mu}\overline{\Xi}_{hac\mu}^{(s)}\varpi(k) \\
&\quad\quad + \epsilon\Big(\zeta^2 - \xi^\top(k)\xi(k)\Big)
\end{aligned} \tag{22}
$$

where $\varpi(k) = [\delta^\top(k)\ \ \xi^\top(k)\ \ \omega^\top(k)]^\top$.

For $\upsilon > 1$, by letting

$$
\begin{aligned}
\mathscr{J}(k) &= \mathscr{E}\{V(k+1, \delta_{k+1}, h_+, \varphi_{k+1}, \phi_{k+1}, \psi_{k+1}\} \\
&\quad - \upsilon V(k, \delta_k, h, \varphi_k, \phi_k, \psi_k) - \upsilon\omega^\top(k)\omega(k)
\end{aligned} \tag{23}
$$

and substituting (22) into (23), we have

$$
\mathscr{J}(k) \le \varpi^\top(k)\widehat{\Xi}_{hac\mu}\varpi(k) + \epsilon\zeta^2 \tag{24}
$$

where

$$
\widehat{\Xi}_{hac\mu} = \Xi_{hac\mu}^{(1)} + \sum_{s=2}^{4}\overline{\Xi}_{hac\mu}^{(s)\top}\mathscr{P}_{h_+ac\mu}\overline{\Xi}_{hac\mu}^{(s)}.
$$

Using SC to (18), it is clear that $\widehat{\Xi}_{hac\mu} < 0$. Therefore, $\mathscr{J}(k)$ in (24) is bounded as

$$
\mathscr{J}(k) \le \epsilon\zeta^2. \tag{25}
$$

Recalling (23) and (25) implies

$$
\begin{aligned}
&\mathscr{E}\{V(k+1, \delta_{k+1}, h_+, \varphi_{k+1}, \phi_{k+1}, \psi_{k+1})\} \\
&\quad < \upsilon V(k, \delta_k, h, \varphi_k, \phi_k, \psi_k) + \upsilon\omega^\top(k)\omega(k) + \epsilon\zeta^2.
\end{aligned} \tag{26}
$$

Summing up (26) on both sides from 0 to $k+1$ for $k \le N$, it leads to

$$
\begin{aligned}
&\mathscr{E}\{V(k, \delta_k, h, \varphi_k, \phi_k, \psi_k)\} \\
&\quad \le \upsilon^k\mathscr{E}\{V(0, \delta_0, h, \varphi_0, \phi_0, \psi_0)\} \\
&\quad\quad + \sum_{m=0}^{k-1}\upsilon^{k-m}\omega^\top(m)\omega(m) + \sum_{m=0}^{k-1}\upsilon^{k-1-m}\epsilon\zeta^2 \\
&\quad \le \upsilon^N\mathscr{E}\{V(0, \delta_0, h, \varphi_0, \phi_0, \psi_0)\} + \frac{\upsilon^N - 1}{\upsilon - 1}\Big(\upsilon\kappa + \epsilon\zeta^2\Big).
\end{aligned} \tag{27}
$$

Identifying (19), it is easy to derive that

$$
V(0, \delta_0, h, \varphi_0, \phi_0, \psi_0) \le \sigma_{\max}(P_R)\delta^\top(0)R\delta(0) \tag{28}
$$

and

$$
V(k, \delta_k, h, \varphi_k, \phi_k, \psi_k) \ge \sigma_{\min}(P_R)\delta^\top(k)R\delta(k). \tag{29}
$$

Together, with (27)–(29), we get

$$
\begin{aligned}
\mathscr{E}\Big\{\delta^\top(k)R\delta(k)\Big\} &\le \upsilon^N c_1\sigma_{\min}^{-1}(P_R)\sigma_{\max}(P_R) \\
&\quad + \sigma_{\min}^{-1}(P_R)\frac{\upsilon^N - 1}{\upsilon - 1}\Big(\upsilon\kappa + \epsilon\zeta^2\Big).
\end{aligned} \tag{30}
$$

By (13), it is concluded that $\mathscr{E}\{\delta^\top(k)R\delta(k)\} \le c_2\ \forall k \in \{1, 2, \ldots, N\}$. From Definition 1, the FMSS (10) is SFTB. The proof is completed. ■

In what follows, the fuzzy-based filter gains will be awarded in Theorem 2.

Theorem 2: Given scalars $\epsilon > 0$ and $\upsilon > 0$, the FMSS (10) is SFTB with respect to $(c_1, c_2, N, R, \kappa, \zeta)$, if there exist matrices $S_c > 0$, Z_μ, $Z_{ac\mu}^{(1)}$, $Z_{ac\mu}^{(2)}$, $\overline{A}_{q\mu}$, $\overline{B}_{q\mu}$, and $P_{pac\mu} = \begin{bmatrix} P_{pac\mu}^{(1)} & P_{pac\mu}^{(2)} \\ * & P_{pac\mu}^{(3)} \end{bmatrix}$ $\forall a, b \in \mathcal{A}, c, d \in \mathcal{C}, \mu, \nu \in \mathcal{M}$, such that

$$
\widetilde{\Upsilon}_{gppac\mu} < 0 \tag{31}
$$

$$
\widetilde{\Upsilon}_{gpqac\mu} + \widetilde{\Upsilon}_{gqpac\mu} < 0, \quad p < q \tag{32}
$$

$$
\lambda_1 R \le P_{pac\mu} \le \lambda_2 R \tag{33}
$$

$$
\upsilon^N c_1\lambda_2 + \frac{\upsilon^N - 1}{\upsilon - 1}\Big(\upsilon\kappa + \epsilon\zeta^2\Big) < \lambda_1 c_2 \tag{34}
$$

where

$$
\begin{aligned}
\widetilde{\Upsilon}_{gpqac\mu} &= \begin{bmatrix} \widetilde{\Xi}_{gpqac\mu} & \widetilde{\Sigma}_{pqac\mu} \\ * & \Psi_c \end{bmatrix} \\
\widetilde{\Xi}_{gpqac\mu} &= \begin{bmatrix} \widetilde{\Xi}_{pqac\mu}^{(1)} & \widetilde{\Xi}_{pqac\mu}^{(2)} \\ * & \widetilde{\Xi}_{gac\mu}^{(3)} \end{bmatrix}, \\
\widetilde{\Sigma}_{pqac\mu} &= \Big[\mathscr{G}_{pa}^{(1)}\Omega_c S_c\ \ \mathscr{G}_{pa}^{(2)}\Omega_c S_c\ \ \mathscr{G}_{pa}^{(3)}\Omega_c S_c \\
&\quad\quad \widetilde{\mathscr{H}}_{q\mu}^{(1)\top}\ \ \widetilde{\mathscr{H}}_{q\mu}^{(2)\top}\ \ \widetilde{\mathscr{H}}_{q\mu}^{(3)\top}\Big] \\
\widetilde{\Xi}_{pqac\mu}^{(1)} &= \begin{bmatrix} -\upsilon P_{pac\mu}^{(1)} & -\upsilon P_{pac\mu}^{(2)} & 0 & 0 \\ * & -\upsilon P_{pac\mu}^{(3)} & 0 & 0 \\ * & * & -\epsilon I & 0 \\ * & * & * & -\upsilon I \end{bmatrix} \\
\widetilde{\Xi}_{pqac\mu}^{(2)} &= \Big[\widetilde{\Xi}_{1pqac\mu}^{(2)\top}\ \ \widetilde{\Xi}_{2pqac\mu}^{(2)\top}\ \ \widetilde{\Xi}_{3pqac\mu}^{(2)\top}\Big] \\
\widetilde{\Xi}_{gac\mu}^{(3)} &= \mathrm{diag}\big\{\mathfrak{P}_{gac\mu}, \mathfrak{P}_{gac\mu}, \mathfrak{P}_{gac\mu}\big\} \\
\widetilde{\Xi}_{1pqac\mu}^{(2)} &= \Big[\mathfrak{A}_{pqac\mu}^\top\ \ \mathfrak{C}_{pqac\mu}^\top\ \ \mathfrak{B}_{pqac\mu}^\top\Big] \\
\widetilde{\Xi}_{2pqac\mu}^{(2)} &= \Big[\mathfrak{M}_{pqac\mu}^\top\ \ 0\ \ \mathfrak{N}_{pqac\mu}^\top\Big] \\
\widetilde{\Xi}_{3pqac\mu}^{(2)} &= \Big[0\ 0\ \mathfrak{L}_{pqac\mu}^\top\Big] \\
\mathfrak{A}_{pqac\mu} &= \begin{bmatrix} A_{pa}^\top Z_{ac\mu}^{(1)\top} + \alpha_a(1-\beta_a)C_{pa}^\top\overline{B}_{q\mu}^\top & A_{pa}^\top Z_{ac\mu}^{(2)\top} + \alpha_a(1-\beta_a)C_{pa}^\top\overline{B}_{q\mu}^\top \\ \overline{A}_{q\mu}^\top & \overline{A}_{q\mu}^\top \end{bmatrix}
\end{aligned}
$$

$$\begin{aligned}
\mathfrak{B}_{pqac\mu} &= \Big[B_{pa}^{\top}Z_{ac\mu}^{(1)\top} + \alpha_a(1-\beta_a)D_{pa}^{\top}\overline{B}_{q\mu}^{\top} \\
&\qquad B_{pa}^{\top}Z_{ac\mu}^{(2)\top} + \alpha_a(1-\beta_a)D_{pa}^{\top}\overline{B}_{q\mu}^{\top}\Big] \\
\mathfrak{C}_{pqac\mu} &= \Big[\alpha_a\beta_a\overline{B}_{q\mu}^{\top} \quad \alpha_a\beta_a\overline{B}_{q\mu}^{\top}\Big] \\
\mathfrak{M}_{pqac\mu} &= \begin{bmatrix} \theta_1 C_{pa}^{\top}\overline{B}_{q\mu}^{\top} & \theta_1 C_{pa}^{\top}\overline{B}_{q\mu}^{\top} \\ 0 & 0 \end{bmatrix} \\
\mathfrak{N}_{pqac\mu} &= \Big[\theta_1 D_{pa}^{\top}\overline{B}_{q\mu}^{\top} \quad \theta_1 D_{pa}^{\top}\overline{B}_{q\mu}^{\top}\Big] \\
\mathfrak{L}_{pqac\mu} &= \Big[\theta_2\overline{B}_{q\mu}^{\top} \quad \theta_2\overline{B}_{q\mu}^{\top}\Big] \\
\mathfrak{P}_{gac\mu} &= \begin{bmatrix} \mathscr{P}_{gac\mu}^{(1)} - \mathrm{sym}\Big(Z_{ac\mu}^{(1)}\Big) & \mathscr{P}_{gac\mu}^{(2)} - Z_{ac\mu}^{(2)} - Z_{\mu} \\ * & \mathscr{P}_{gac\mu}^{(3)} - \mathrm{sym}\big(Z_{\mu}\big) \end{bmatrix} \\
\mathscr{P}_{gac\mu}^{(l)} &= \sum_{a\in\mathcal{A}}\sum_{c\in\mathcal{C}}\sum_{v\in\mathcal{M}} \pi_{ab}\tau_{cd}^{b}\eta_{\mu v}^{bd}P_{gbdv}^{(l)} \ \ (l=1,2,3) \\
\widetilde{\mathscr{H}}_{q\mu}^{(1)} &= \Big[0\ 0\ 0\ 0\ \overline{B}_{q\mu}^{\top}\ \overline{B}_{q\mu}^{\top}\ 0\ 0\ 0\ 0\Big] \\
\widetilde{\mathscr{H}}_{q\mu}^{(2)} &= \Big[0\ 0\ 0\ 0\ 0\ 0\ \overline{B}_{q\mu}^{\top}\ \overline{B}_{q\mu}^{\top}\ 0\ 0\Big] \\
\widetilde{\mathscr{H}}_{q\mu}^{(3)} &= \Big[0\ 0\ 0\ 0\ \alpha_a\beta_a\overline{B}_{q\mu}^{\top}\ \alpha_a\beta_a\overline{B}_{q\mu}^{\top}\ 0\ 0\ \theta_2\overline{B}_{q\mu}^{\top}\ \theta_2\overline{B}_{q\mu}^{\top}\Big].
\end{aligned}$$

In addition, if the inequalities (31)–(34) are solvable, the filter gains can be expressed by

$$A_{fq\mu} = Z_{\mu}^{-1}\overline{A}_{q\mu}, B_{fq\mu} = Z_{\mu}^{-1}\overline{B}_{q\mu}. \tag{35}$$

Proof: Aiming to tackle the nonlinearity $\mathscr{P}_{ac\mu}^{-1}$, an invertible matrix $\mathscr{Z}_{ac\mu}$ is developed as

$$\mathscr{Z}_{ac\mu} = \begin{bmatrix} Z_{ac\mu}^{(1)} & Z_{\mu} \\ Z_{ac\mu}^{(2)} & Z_{\mu} \end{bmatrix}. \tag{36}$$

In light of (11), premultiplying and postmultiplying (11) by $\mathrm{diag}\{I, I, I, \mathscr{Z}_{gac\mu}, \mathscr{Z}_{gac\mu}, \mathscr{Z}_{gac\mu}, \underbrace{I,\ldots,I}_{6\times 6}\}$ and its transpose, where the matrix I shares the compatible dimensions with the corresponding block in (11). Thus, (11) is reformulated as

$$\Upsilon_{gppac\mu}' = \begin{bmatrix} \Xi_{gppac\mu}' & \Sigma_{ppac\mu}' \\ * & \Psi_c \end{bmatrix} < 0 \tag{37}$$

where

$$\begin{aligned}
\Xi_{gppac\mu}' &= \begin{bmatrix} \Xi_{ppac\mu}^{(1)} & \Xi_{ppac\mu}^{(2)\prime} \\ * & \Xi_{gac\mu}^{(3)\prime} \end{bmatrix} \\
\Sigma_{ppac\mu}' &= \Big[\mathscr{G}_{pa}^{(1)}\Omega_c S_c \quad \mathscr{G}_{pa}^{(2)}\Omega_c S_c \quad \mathscr{G}_{pa}^{(3)}\Omega_c S_c \\
&\qquad \mathscr{Z}_{gac\mu}^{\top}\mathscr{H}_{p\mu}^{(1)\top} \quad \mathscr{Z}_{gac\mu}^{\top}\mathscr{H}_{p\mu}^{(2)\top} \quad \mathscr{Z}_{gac\mu}^{\top}\mathscr{H}_{p\mu}^{(3)\top}\Big] \\
\Xi_{ppac\mu}^{(2)\prime} &= \Big[\Xi_{1ppac\mu}^{(2)\top}\mathscr{Z}_{gac\mu}^{\top} \quad \Xi_{2ppac\mu}^{(2)\top}\mathscr{Z}_{gac\mu}^{\top} \quad \Xi_{3ppac\mu}^{(2)\top}\mathscr{Z}_{gac\mu}^{\top}\Big] \\
\Xi_{gac\mu}^{(3)\prime} &= \mathrm{diag}\Big\{-\mathscr{Z}_{gac\mu}\mathscr{P}_{gac\mu}^{-1}\mathscr{Z}_{gac\mu}^{\top} \\
&\qquad -\mathscr{Z}_{gac\mu}\mathscr{P}_{gac\mu}^{-1}\mathscr{Z}_{gac\mu}^{\top}, -\mathscr{Z}_{gac\mu}\mathscr{P}_{gac\mu}^{-1}\mathscr{Z}_{gac\mu}^{\top}\Big\}.
\end{aligned}$$

In addition, for $\mathscr{P}_{gac\mu}^{-1} > 0$, it is well recognized that

$$\big(\mathscr{P}_{gac\mu} - \mathscr{Z}_{gac\mu}\big)\mathscr{P}_{gac\mu}^{-1}\big(\mathscr{P}_{gac\mu} - \mathscr{Z}_{gac\mu}\big)^{\top} \geq 0 \tag{38}$$

which can be rewritten as

$$-\mathscr{Z}_{gac\mu}\mathscr{P}_{gac\mu}^{-1}\mathscr{Z}_{gac\mu}^{\top} \leq \mathscr{P}_{gac\mu} - \mathrm{sym}\big\{\mathscr{Z}_{gac\mu}\big\}. \tag{39}$$

Aided by (39), $\Upsilon_{gppac\mu}'$ is inferred as

$$\Upsilon_{gppac\mu}'' = \begin{bmatrix} \Xi_{gppac\mu}'' & \Sigma_{ppac\mu}' \\ * & \Psi_c \end{bmatrix} < 0 \tag{40}$$

where

$$\begin{aligned}
\Xi_{gppac\mu}'' &= \begin{bmatrix} \Xi_{ppac\mu}^{(1)} & \Xi_{ppac\mu}^{(2)\prime} \\ * & \Xi_{gac\mu}^{(3)\prime\prime} \end{bmatrix} \\
\Xi_{gac\mu}^{(3)\prime\prime} &= \mathrm{diag}\Big\{-\mathscr{Z}_{gac\mu}\mathscr{P}_{gac\mu}^{-1}\mathscr{Z}_{gac\mu}^{\top} \\
&\qquad -\mathscr{Z}_{gac\mu}\mathscr{P}_{gac\mu}^{-1}\mathscr{Z}_{gac\mu}^{\top}, -\mathscr{Z}_{gac\mu}\mathscr{P}_{gac\mu}^{-1}\mathscr{Z}_{gac\mu}^{\top}\Big\}.
\end{aligned}$$

From (35), it is clear that

$$\overline{A}_{p\mu} = Z_{\mu}A_{fp\mu}, \overline{B}_{p\mu} = Z_{\mu}B_{fp\mu}. \tag{41}$$

By the specific form of $\mathscr{A}_{ppac\mu}^{\ell}$, $\mathscr{B}_{ppac\mu}^{\ell}$, and $\mathscr{C}_{pac\mu}^{\ell}$, it is easy to conclude that (31) is derived by (40). Similarly, the inequalities in (32) can be guaranteed. This completes the proof. ■

IV. Numerical Example

To verify that the achieved method is valid, two simulation examples are delivered in the following section.

A. Example 1

Considering the MSS (1) with two plant modes and two fuzzy rules, the system parameters are expressed as follows [40]:

$$\left[\begin{array}{c|c} A_{11} & B_{11} \\ \hline C_{11} & D_{11} \end{array}\right] = \left[\begin{array}{cc|c} 0.58 & 0.39 & 0.11 \\ -0.12 & 0.45 & 0.19 \\ \hline 1 & 0 & 1.1 \end{array}\right]$$

$$\left[\begin{array}{c|c} A_{12} & B_{12} \\ \hline C_{12} & D_{12} \end{array}\right] = \left[\begin{array}{cc|c} 0.65 & 0.48 & 0.30 \\ -0.16 & 0.52 & 0.20 \\ \hline 1 & 0 & 1.05 \end{array}\right]$$

$$\left[\begin{array}{c|c} A_{21} & B_{21} \\ \hline C_{21} & D_{21} \end{array}\right] = \left[\begin{array}{cc|c} 0.82 & 0.12 & 0.21 \\ -0.20 & 0.60 & 0.29 \\ \hline 1 & 0 & 0.9 \end{array}\right]$$

$$\left[\begin{array}{c|c} A_{22} & B_{22} \\ \hline C_{22} & D_{22} \end{array}\right] = \left[\begin{array}{cc|c} 0.88 & 0.13 & 0.10 \\ -0.13 & 0.70 & 0.20 \\ \hline 1 & 0 & 0.95 \end{array}\right].$$

The STP matrix of the target MSS (1) is chosen by

$$\Gamma_1 = \begin{bmatrix} 0.2 & 0.8 \\ 0.55 & 0.45 \end{bmatrix}.$$

Besides, the asynchronous quantizer and AF are obeying the following STP matrices:

$$\Gamma_2^1 = \begin{bmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{bmatrix}, \ \Gamma_2^2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.25 & 0.75 \end{bmatrix}$$

$$\Gamma_3^{11} = \begin{bmatrix} 0.85 & 0.15 \\ 0.25 & 0.75 \end{bmatrix}, \ \Gamma_3^{12} = \begin{bmatrix} 0.55 & 0.45 \\ 0.35 & 0.65 \end{bmatrix}$$

$$\Gamma_3^{21} = \begin{bmatrix} 0.05 & 0.95 \\ 0.15 & 0.85 \end{bmatrix}, \ \Gamma_3^{22} = \begin{bmatrix} 0.7 & 0.3 \\ 0.85 & 0.15 \end{bmatrix}.$$

TABLE II
UPPER BOUNDS OF ϵ FOR DIFFERENT $\overline{\alpha}$, $\overline{\beta}$, AND ζ

	$\zeta=2$			$\zeta=5$		
	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=0.1$	$\alpha=0.5$	$\alpha=0.9$
$\beta=0.1$	1.9581	1.9584	1.9586	2.3910	2.3919	2.3928
$\beta=0.5$	1.9561	1.9572	1.9584	2.3872	2.3885	2.3919
$\beta=0.9$	1.9541	1.9561	1.9581	2.3902	2.3872	2.3910

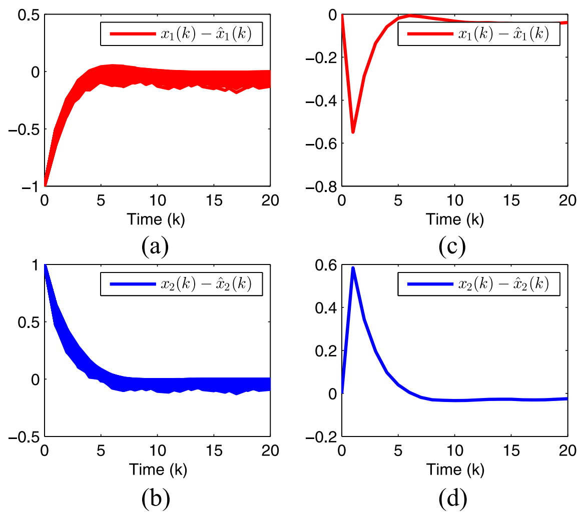

Fig. 1. (a) SR $x_1(k)-\widehat{x}_1(k)$ (100 repetitions). (b) SR $x_2(k)-\widehat{x}_2(k)$ (100 repetitions). (c) MV of $x_1(k)-\widehat{x}_1(k)$. (d) MV of $x_2(k)-\widehat{x}_2(k)$.

The FMFs are given by

$$h_1(1)=0.5\Big(\sin^2(x_1(k))+\sin^2(x_2(k))\Big),\ \ h_1(2)=1-h_1(1)$$
$$h_2(1)=0.6\Big(\sin^2(x_1(k))+\sin^2(x_2(k))\Big),\ \ h_2(2)=1-h_2(1).$$

Besides, letting $\alpha_1=0.95$, $\alpha_2=0.9$, $\beta_1=0.55$, $\beta_2=0.60$, $\rho_1=0.85$, $\rho_2=0.8$, $\upsilon=0.1$, $\epsilon=20$, $c_1=1.2$, $c_2=5$, $N=10$, and $\zeta=5$, by Theorem 2, the filter gains are achieved as follows:

$$\begin{bmatrix}A_{f11} | & B_{f11}\end{bmatrix}=\begin{bmatrix}0.0396 & 0.0416 & -0.0237\\ 0.0268 & 0.0339 & -0.0225\end{bmatrix}$$
$$\begin{bmatrix}A_{f12} | & B_{f12}\end{bmatrix}=\begin{bmatrix}0.0371 & 0.0394 & -0.0236\\ 0.0258 & 0.0309 & -0.0224\end{bmatrix}$$
$$\begin{bmatrix}A_{f21} | & B_{f21}\end{bmatrix}=\begin{bmatrix}0.0342 & 0.0299 & -0.0242\\ 0.0235 & 0.0290 & -0.0262\end{bmatrix}$$
$$\begin{bmatrix}A_{f22} | & B_{f22}\end{bmatrix}=\begin{bmatrix}0.0366 & 0.0310 & -0.0241\\ 0.0244 & 0.0313 & -0.0260\end{bmatrix}.$$

Letting $\overline{\alpha}=\alpha_l$ and $\overline{\beta}=\beta_l$ $(l=1,2)$, according to the derived method in Theorem 2, the upper bound of ϵ with various of $\overline{\alpha}$, $\overline{\beta}$, and ζ is exhibited in Table II. It can be detected from the table that the values of ϵ are increasing when ζ increases.

For given initial condition $x(0)=[1\ \ -1]^\top$ and $\widehat{x}(0)=[0\ \ 0]^\top$, and external disturbance $\omega(k)=1/(1+k^2)$, by the achieved filter gains, Fig. 1(a) and (c) depicts the state responses (SRs) of $x_1(k)-\widehat{x}_1(k)$ and $x_2(k)-\widehat{x}_2(k)$ without DAs, while Fig. 1(b) and (d) shows the mean values (MVs) of $x_1(k)-\widehat{x}_1(k)$ and $x_2(k)-\widehat{x}_2(k)$ without DAs. Similarly, the SRs of $x_1(k)-\widehat{x}_1(k)$ and $x_2(k)-\widehat{x}_2(k)$ with DAs are displayed in Fig. 2(a) and (c), respectively. The MVs of $x_1(k)-\widehat{x}_1(k)$ and $x_2(k)-\widehat{x}_2(k)$ with DAs are shown in Fig. 2(b) and (d), respectively. It can be observed from Figs. 1 and 2 that the MSS (10) is SFTB whether the attacker exists or not.

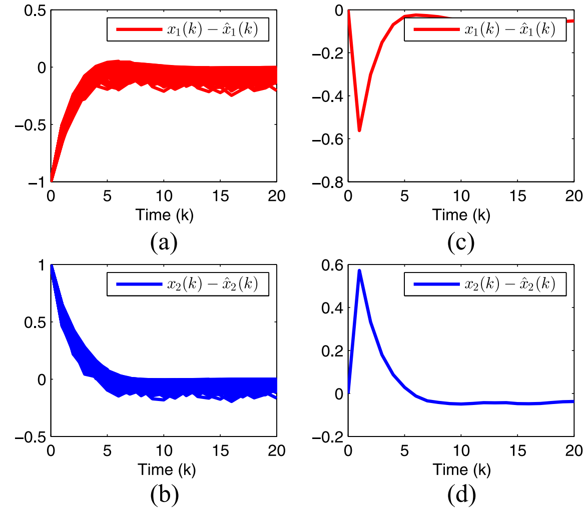

Fig. 2. (a) SR $x_1(k)-\widehat{x}_1(k)$ with DAs (100 repetitions). (b) SR $x_2(k)-\widehat{x}_2(k)$ with DAs (100 repetitions). (c) MV of $x_1(k)-\widehat{x}_1(k)$ with DAs. (d) MV of $x_2(k)-\widehat{x}_2(k)$ with DAs.

B. Example 2

A double-inverted pendulum (DIP) is borrowed to verify the applicability of developed results [41], [42]. As stated in [41] and [42], the equation of DIP is described by

$$\begin{aligned}\dot{x}_{i1}&=x_{i2}\\ \dot{x}_{i2}&=\frac{1}{100J_i}u_i-\frac{kr^2}{4J_i}x_{i1}+\left[\frac{m_igr}{J_i}-\frac{kr^2}{4J_i}x_{i2}\right]\sin(x_{i1})\\ &\quad+\frac{1}{J_i}x_{i2}+\sum_{k=1,k\neq i}^{2}\frac{3kr^2}{4J_k}x_{i1}\end{aligned}$$

where

x_{i1}	angle of the pendulum;
x_{i2}	angular velocity;
J_i	moment of inertia;
m_i	mass of the pendulum;
k	constant of connecting torsional spring;
r	length of the pendulum;
g	gravity constant.

The values of the parameters Ji, m_i, k, r, and g are, respectively, set as $J_1=2$ kg, $J_2=2.5$ kg, $m_1=2$ kg, $m_2=2.5$ kg, $k=8$ N $\cdot$ n/rad, $r=1$, and $g=9.8$ m/s^2.

Similar to [41], by discretizing the DIP with sampling period $T=0.01$ s, the corresponding DIP is reestablished as the FMSSs: Plant Rule 1: If $x_1(k)$ is M^i, then

$$\begin{aligned}x(k+1)&=A_{pa}x(k)+B_{pa}\omega_k\\ y(k)&=C_{pa}x(k)+D_{pa}\omega(k)\end{aligned}$$

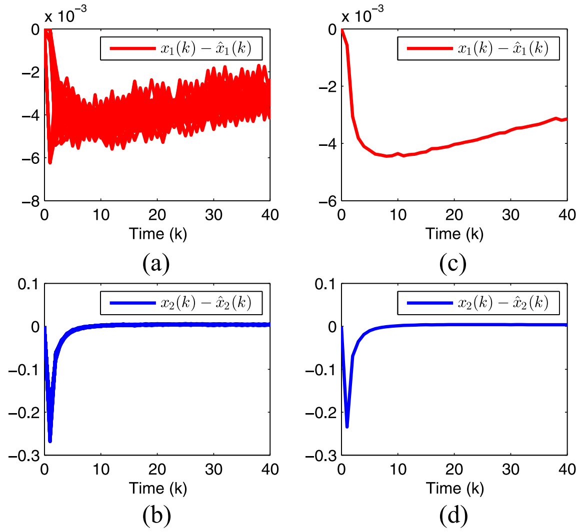

Fig. 3. (a) SR $x_1(k)-\widehat{x}_1(k)$ (100 repetitions). (b) SR $x_2(k)-\widehat{x}_2(k)$ (100 repetitions). (c) MV of $x_1(k)-\widehat{x}_1(k)$. (d) MV of $x_2(k)-\widehat{x}_2(k)$.

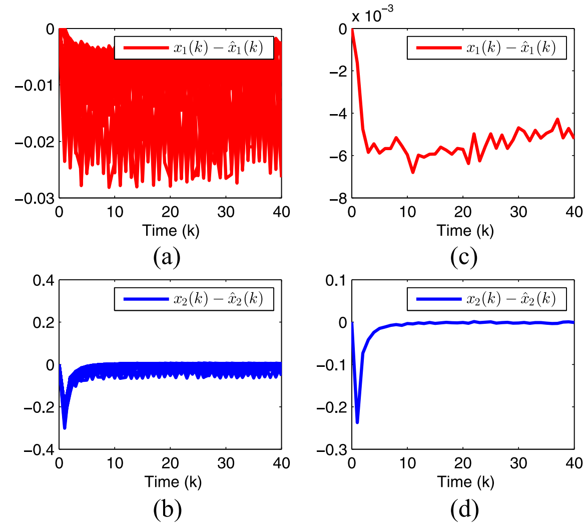

Fig. 4. (a) SR $x_1(k)-\widehat{x}_1(k)$ with DAs (100 repetitions). (b) SR $x_2(k)-\widehat{x}_2(k)$ with DAs (100 repetitions). (c) MV of $x_1(k)-\widehat{x}_1(k)$ with DAs. (d) MV of $x_2(k)-\widehat{x}_2(k)$ with DAs.

where

$$\left[\begin{array}{c|c} A_{11} & B_{11} \\ \hline C_{11} & D_{11}\end{array}\right] = \left[\begin{array}{cc|c} 1 & 0.0120 & 0 \\ -1.3200 & -0.1540 & 0.5 \\ \hline 1 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{12} & B_{12} \\ \hline C_{12} & D_{12}\end{array}\right] = \left[\begin{array}{cc|c} 1 & 0.0120 & 0 \\ -1.3760 & -0.0352 & 0.5 \\ \hline 1 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{21} & B_{21} \\ \hline C_{21} & D_{21}\end{array}\right] = \left[\begin{array}{cc|c} 1 & 0.0120 & 0 \\ -1.1818 & -0.1658 & 0.5 \\ \hline 1 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{22} & B_{22} \\ \hline C_{22} & D_{22}\end{array}\right] = \left[\begin{array}{cc|c} 1 & 0.0120 & 0 \\ -1.2378 & -0.0447 & 0.4 \\ \hline 1 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{31} & B_{31} \\ \hline C_{31} & D_{31}\end{array}\right] = \left[\begin{array}{cc|c} 1 & 0.0120 & 0 \\ -1.3760 & -0.0352 & 0.5 \\ \hline 1 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{32} & B_{32} \\ \hline C_{32} & D_{32}\end{array}\right] = \left[\begin{array}{cc|c} 1 & 0.0120 & 0 \\ -1.2485 & -0.0448 & 0.4 \\ \hline 1 & 0 & 1\end{array}\right].$$

In addition, the FMFs are the same as [42], $\alpha_l = 0.9$, $\beta_l = 0.8$ $(l = 1, 2)$, $\zeta = 2$, $\kappa = 0.6$, $c_1 = 0.2$, $c_2 = 10$, $\epsilon = 8$, and other parameters are expressed as the same as Example 1. By solving inequalities in Theorem 2, the filter gains are acquired as follows:

$$\left[\begin{array}{c|c} A_{f11} & B_{f11}\end{array}\right] = \left[\begin{array}{cc|c} 0.0496 & -0.0144 & -0.0009 \\ -0.0698 & 0.0084 & -0.0348\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{f12} & B_{f12}\end{array}\right] = \left[\begin{array}{cc|c} 0.0568 & -0.0153 & -0.0011 \\ -0.0880 & 0.0122 & -0.0346\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{f21} & B_{f21}\end{array}\right] = \left[\begin{array}{cc|c} 0.0469 & -0.0154 & -0.0016 \\ -0.0576 & 0.0048 & -0.0340\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{f22} & B_{f22}\end{array}\right] = \left[\begin{array}{cc|c} 0.0496 & -0.0159 & -0.0017 \\ -0.0671 & 0.0084 & -0.0339\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{f31} & B_{f31}\end{array}\right] = \left[\begin{array}{cc|c} 0.0495 & -0.0146 & -0.0010 \\ -0.0683 & 0.0146 & -0.0346\end{array}\right]$$

$$\left[\begin{array}{c|c} A_{f32} & B_{f32}\end{array}\right] = \left[\begin{array}{cc|c} 0.0572 & -0.0159 & -0.0012 \\ -0.0843 & 0.0169 & -0.0344\end{array}\right].$$

For the initial condition $x(0) = [0\ 0]^\top$ and $\widehat{x}(0) = [0\ 0]^\top$, and external disturbance $\omega(k) = 1/(1+k^2)$, by the attained filter gains, the SRs of $x_1(k)-\widehat{x}_1(k)$ and $x_2(k)-\widehat{x}_2(k)$ without DAs are revealed in Fig. 3(a) and (c); the MVs of $x_1(k)-\widehat{x}_1(k)$ and $x_2(k)-\widehat{x}_2(k)$ without DAs are exhibited in Fig. 3(b) and (d); the SRs of $x_1(k)-\widehat{x}_1(k)$ and $x_2(k)-\widehat{x}_2(k)$ with DAs are presented in Fig. 4(a) and (c); and the MVs of $x_1(k)-\widehat{x}_1(k)$ and $x_2(k)-\widehat{x}_2(k)$ with DAs are unveiled in Fig. 4(b) and (d), respectively. It can be witnessed from Figs. 3 and 4 that the MSS (10) is SFTB whether the attacker exists or not.

V. Conclusion

In this work, the issue of finite-time filter design for FMSSs with DAs has been discussed by resorting to an HS approach, where the randomly occurring DAs, mode-dependent quantization, and PDs are involved. Especially, the quantizer and the filter ruin mutual asynchronous with the plant. By means of a fuzzy-basis-dependent Lyapunov method, the SFTB of the resulting plant subject to randomly occurred DAs has been achieved. Finally, two examples have been included to evaluate the validity of promoted strategies.

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