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**A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence**

by

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# A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence

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**Abstract**

We study the asymptotic behaviour of (exact or approximate) minimizers of 3D nonlinear elasticity for plates of thickness $h$ in the limit $h \to 0$. We prove that 3D minimizers converge, after suitable rescaling, to minimizers of a hierarchy of plate models. What distinguishes the different limit models is the scaling of the elastic energy per unit volume $\sim h^\beta$. This is in turn governed by the strength of the applied force $\sim h^\alpha$. Membrane theory, derived earlier by Le Dret and Raoult, corresponds to $\alpha = \beta = 0$, nonlinear bending theory to $\alpha = \beta = 2$, Föppl von Kármán theory to $\alpha = 3$, $\beta = 4$ and linearized vK theory to $\alpha > 3$. Intermediate values of $\alpha$ lead to certain novel theories with constraints. A key ingredient in the proof is a generalization to higher derivatives of our rigidity result [31] that for maps $v : (0,1)^3 \to \mathbb{R}^3$, the $L^2$ distance of $\nabla v$ from a single rotation is bounded by a multiple of the $L^2$ distance from the set $SO(3)$ of all rotations.

# Contents

# 1 Introduction

A fundamental problem in nonlinear elasticity is to understand the relation between the three-dimensional theory and theories for lower-dimensional objects (plates, shells, rods, ...). There are many such theories (see e.g. [52, 3, 18]) and their derivation and validity has been a subject of many discussions. In particular the von Kármán plate equations, first formulated almost a hundred years ago [45], have been a subject of heated controversy in the continuum mechanics and applied mathematics communities. On the one hand these equations have been very widely used by engineers and nonlinear analysts alike; on the other hand their derivation has faced harsh criticism. Truesdell [76] writes about von Kármán's theory: "Analysts seem to love it, and it makes no sense to critical students of mechanics". His main criticisms (which he attributes to S.S. Antman) are: approximate geometry, assumptions one the way the stresses vary over the cross-section, commitment to some specific linear constitutive relation, neglect of some components of the strain and an apparent confusion of the referential and the spatial descriptions. These five criticisms are also quoted in [17] and we discuss them in more detail in Section 9. Villaggio [78] refers to the von Kármán theory as a typical example of a "bad theory" in the introduction of his text book on structural analysis and Ciarlet writes in his three volume treatise on nonlinear elasticity, plate and shell theories: "The two-dimensional von Kármán equations for nonlinearly elastic plates, originally proposed by T. von Kármán in 1910, play an almost mythical role in applied mathematics" [18, p. 367].

In this paper we show that the vK equations arise as a rigorous variational limit (or Γ-limit) of the equations of nonlinear three dimensional elasticity in the limit of vanishing thickness. In fact we derive a hierarchy of limiting theories which include the vK theory (see Table 1 in Section 2.6

for an overview and see Theorems 2 and 4 for precise statements). The different limiting theories are distinguished by different scaling exponents of the energy as a function of the thickness. The scaling of the energy in turn is controlled by the scaling of the applied forces.

Our approach begins with the elastic energy

$$\mathcal{E}^h(v) = \int_{\Omega_h} W(\nabla w(z))dz \tag{1}$$

of a deformation

$$w : \Omega_h = S \times (-\frac{h}{2}, \frac{h}{2}) \to \mathbb{R}^3.$$

It is natural to consider the energy per unit volume $\mathcal{E}^h/h$, see (2) below.

Heuristically one expects that deformations with $\mathcal{E}^h/h \sim 1$ correspond to a stretching of the midplane $S$ leading to a membrane theory, while $\mathcal{E}^h/h \sim h^2$ corresponds to a bending deformation (where $S$ remains unstretched) leading to nonlinear plate theory (first proposed by Kirchhoff [44]). If $\mathcal{E}^h/h \sim h^4$ one expects that the relevant rotations vary only by order $h$ and that one can linearize around a rigid motion. Scaling the in-plane and the out-of-plane deviation differently, one is formally lead to the von Kármán theory of plates.

Membrane theory was rigorously justified in [47, 48, 49] in the sense of Γ-convergence [24, 22] (for related work see also [1, 4] and for connections with the classical tension field theory in mechanics [80, 70, 71] see [65, 66]). The bending theory of plates and shells was recently obtained as a Γ-limit [30, 31, 33], see also [62, 63]. It is more delicate since the limit problem involves higher derivatives and hence the limit $h \to 0$ corresponds to a singular perturbation. (The earlier work [13] also uses Γ-convergence, but the authors need to impose additional constraints on the admissible three dimensional deformations to get enough compactness to complete the argument).

In this paper we study limiting theories corresponding to the scaling $\mathcal{E}^h/h \sim h^\beta$, $\beta > 2$, and we rigorously derive a hierarchy of theories by Γ-convergence. For $\beta = 4$ we obtain the vK theory, for $\beta > 4$ we obtain the usual linear theory (leading to the biharmonic equation for the out-of-plane component for isotropic energies) and for $2 < \beta < 4$ we obtain a theory with constraints. From the point of view of vK theory the constraint is that the vK stretching energy has to vanish, from the point of view of nonlinear bending theory the constraint can be seen as a geometrically linear version of the isometry constraint. The famous expression $(1/2)[\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v]$ for the membrane strain with its dependence on the in-plane and out-of-plane displacement, which leads to the nonlinearity in the vK equations, was used earlier in Föppl's work [27]. Consequently, in the physics literature one also finds frequently the term Föppl-von-Kármán theory. In contrast to von Kármán, however, Föppl considers only a membrane contribution to the energy and no bending contribution. In Section 2.5 we briefly discuss how (a relaxed version of) Föppl's theory arises if one assumes clamped boundary conditions. The different scalings and the corresponding limiting theories are summarized in Tables 1 and 2 in Section 2.6 below.

Various hierarchies of theories have been previously suggested in the literature based on formal asymptotic expansions or extra assumptions on the kinematics of the three-dimensional deformations; for recent contributions see, e.g., [29, 53]. However, the constrained theory we obtain for $2 < \beta < 4$ and which involves non-integer scaling exponents seems to be new among theories derived either rigorously or formally. One typical problem with formal expansions is that they can miss important effects if the class of ansatz functions is not rich enough. One example is the membrane theory considered in [29], which misses the fact the membranes have no resistance to compression (due to lack of bending energy they show "crumpling"). We do not discuss here the huge literature on the derivation of lower dimensional theories starting from geometrically linear three dimensional elasticity; rigorous convergence results go at least back to [54], see [18] for an extensive discussion of the literature.

That suitable bounds on the scaled displacements imply rigorous $\Gamma$-convergence of the energy to the vK functional has been shown by A. Raoult [69]. For a different rigorous approach to the von Kármán equations based on a clever use of the implicit function theorem see [56]. A justification of the von Kármán equations through formal asymptotics was given by Ciarlet [17]. Some of the results proved here have been announced in [32].

# 2 Main results

## 2.1 Setup

To state our results it is convenient to work in a fixed domain $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$, change variables $x = (z_1, z_2, \frac{z_3}{h})$ and rescale deformations according to $y(x) = w(z(x))$ so that $y : \Omega \to \mathbb{R}^3$. We abbreviate $x' = (x_1, x_2)$ and use the notation $\nabla' y = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ for the in-plane gradient so that

$$\nabla w = (\nabla' y, \frac{1}{h} y_{,3}) =: \nabla_h y$$

and

$$\frac{1}{h}\mathcal{E}(w) = I^h(y) := \int_\Omega W(\nabla_h y)\, dx. \tag{2}$$

We assume that the stored energy $W$ is Borel measurable with values in $[0, \infty]$ and satisfies

$$W(QF) = W(F) \qquad \forall Q \in SO(3), \tag{3}$$

$$W = 0 \qquad \text{on} \quad SO(3), \tag{4}$$

$$W(F) \geq c \operatorname{dist}^2(F, SO(3)), \qquad c > 0, \tag{5}$$

$$W \text{ is } C^2 \text{ in a neighborhood of } SO(3). \tag{6}$$

Since in many cases the relevant deformation gradients will be close to $SO(3)$ we also consider the quadratic form

$$Q_3(F) = \frac{\partial^2 W}{\partial F^2}(Id)(F, F), \tag{7}$$

which is twice the linearized energy, and $Q_2 : \mathbb{R}^{2\times 2} \to \mathbb{R}$,

$$Q_2(G) = \min_{a \in \mathbb{R}^3} \quad Q_3(G + a \otimes e_3 + e_3 \otimes a) \tag{8}$$

obtained by minimizing over stretches in the $x_3$ directions. In view of (4) and (5) both forms are positive semidefinite and hence convex. For the special case of isotropic elasticity we have

$$\begin{aligned} Q_3(F) &= 2\mu \left|\frac{F + F^T}{2}\right|^2 + \lambda (\operatorname{tr} F)^2, \\ Q_2(G) &= 2\mu \left|\frac{G + G^T}{2}\right|^2 + \frac{2\mu\lambda}{2\mu + \lambda} (\operatorname{tr} G)^2. \end{aligned} \tag{9}$$

In view of (4)-(6) $Q_2$ and $Q_3$ are positive-definite on symmetric matrices.

## 2.2 From membrane theory to nonlinear bending theory: $0 \leq \alpha \leq 2$

Here we quickly review the known (and some recent) results for forces with scaling exponent $\alpha$ between 0 and 2. In the case $\alpha = 0$, which leads to membrane theory, we need some additional assumptions and notation. We assume that the three-dimensional energy density $W$ satisfies

$$W(F) \leq C(1 + |F|^p), \quad W(F) \geq c|F|^p - C, \quad \text{for some } p \geq 2, c > 0. \quad (10)$$

Weaker growth hypotheses, which are compatible with the condition $W(F) \to \infty$ as $\det F \to 0$, are also possible, see [7]. For the corresponding membrane theory we consider the two-dimensional energy density $W_{2D} : \mathbb{R}^{3\times 2} \to [0, \infty]$, defined by minimizing out over stretches in the $x_3$ direction,

$$W_{2D}(F') := \min_{a\in\mathbb{R}^3} W(F' + a \otimes e_3) \quad (11)$$

and its quasiconvexification

$$\begin{aligned} W_{\text{membrane}}(F') &:= W_{2D}^{qc}(F') \\ &:= \inf\{\int_S W_{2D}(F' + \nabla'\eta) \ : \ \eta \in C_0^1(S, \mathbb{R}^3)\}. \end{aligned} \quad (12)$$

In this way the membrane energy $W_{\text{membrane}}$ takes into account the energy reducing effect of possible fine-scale oscillations. These do indeed arise in compression and $W_{\text{membrane}}(F')$ vanishes whenever $(F')^T F' \leq Id$. This effect is missed by theories based on formal asymptotic expansion.

The results about limiting theories can be stated in terms of convergence of minimizers or using the closely related notion of Γ-convergence. For the former we consider the functionals

$$J^h(y) = \int_\Omega W(\nabla_h y) - f^{(h)}(x') \cdot y \, dx, \quad (13)$$

where the applied forces $f^{(h)} : S \to \mathbb{R}^3$ satisfy

$$\frac{1}{h^\alpha} f^{(h)} \rightharpoonup f \quad \text{in } L^2(S; \mathbb{R}^3). \quad (14)$$

Here and in the following the half-arrow $\rightharpoonup$ denotes *weak convergence*. We assume that the total force and the total moment applied to the reference configuration is zero, i.e.

$$\int_\Omega f^{(h)} \, dx = 0, \quad \int_\Omega x \wedge f^{(h)} \, dx = 0. \quad (15)$$

Here, the former avoids the absence of a lower bound arising from the trivial invariance $y \to y + const.$. The latter can always be satisfied by rotating the forces $f^{(h)} \to Q^{(h)} f^{(h)}, Q^{(h)} \in SO(3)$, this being without loss of generality because of the rotational invariance of $W$ in (13). Since $f^{(h)}$ is independent of $x_3$ the conditions (15) are equivalent to

$$\int_S f^{(h)}\, dx' = 0, \quad \int_S x' f_3^{(h)}\, dx' = 0, \quad \int_S x_1 f_2^{(h)} - x_2 f_1^{(h)}\, dx' = 0. \tag{16}$$

For $\alpha > 2$ we also assume that the limiting force points in a single direction which we may choose to be the $x_3$ direction

$$f_1 = f_2 = 0. \tag{17}$$

If one imposes suitable boundary conditions which prevent a rigid motion of the plate one can also consider a combination of normal and tangential forces with different scalings. This will be discussed in detail in [35].

To cover cases where there may be nonattainment of the 3D energy, it is convenient to study not only convergence of exact minimizers but also of almost minimizers of $J^h$. Since the energy $J^h$ will typically scale like a power of $h$ we say that a sequence of deformations $y^{(h)}$ is a *$\beta$-minimizing sequence* if

$$\limsup_{h\to 0} \frac{1}{h^\beta}(J^h(y^{(h)}) - \inf J^h) = 0. \tag{18}$$

**Theorem 1 (Membrane to nonlinear bending theory)** *Suppose that the stored energy $W$ satisfies* (3)–(6) *and the forces satisfy* (14) *and* (16)*. The following assertions hold.*

*i) (membrane theory [47, 48, 49]) Suppose in addition* (10)*. Suppose that $\alpha = 0$ and set $\beta = 0$. Then $|\inf J^h| \le C$. If $y^{(h)}$ is a $\beta$-minimizing sequence then $y^{(h)} \rightharpoonup \bar{y}$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ (for a subsequence). The limit $\bar{y}$ is independent of $x_3$ and minimizes*

$$J_0^0(y) = \int_S W_{membrane}(\nabla' y) - f\cdot y\, dx' \tag{19}$$

*among all $y : S \to R^3$.*

*ii) (constrained membrane theory [19]) Suppose that $0 < \alpha < 1$ and set $\beta = \alpha$. Then $|\inf J^h| \le Ch^\beta$ and every $\beta$-minimizing sequence*

$y^{(h)}$ *has a subsequence with* $y^{(h)} \rightharpoonup \bar{y}$ *in* $W^{1,2}(\Omega;\mathbb{R}^3)$*. The limit* $\bar{y}$ *is independent of* $x_3$*, satisfies* $(\nabla'\bar{y})^T\nabla'\bar{y} \le Id$ *and minimizes*

$$J^0_\alpha(y) = \int_S -f\cdot y\,dx' \tag{20}$$

*among all* $y : S \to R^3$ *with* $(\nabla' y)^T\nabla' y \le Id$*.*

iii) *(nonlinear bending theory [30, 31]) Suppose that* $\alpha = 2$ *and set* $\beta = 2$*. Then* $|\inf J^h| \le Ch^2$ *and if* $y^{(h)}$ *is a* $\beta$*-minimizing sequence we have strong convergence* $y^{(h)} \to \bar{y}$ *in* $W^{1,2}(\Omega;\mathbb{R}^3)$ *(for a subsequence). The limit map is independent of* $x_3$*, is an isometric immersion, i.e.* $(\nabla'\bar{y})^T\nabla'\bar{y} = Id$*, and belongs to* $W^{2,2}(\Omega;\mathbb{R}^3)$*. Introducing the normal* $\bar{\nu} = \bar{y}_{,1}\wedge\bar{y}_{,2}$ *one can consider the second fundamental form* $\bar{A}_{ij} = -\bar{y}_{,ij}\cdot\bar{\nu}$*, where* $i,j \in \{1,2\}$*. Then* $\bar{y}$ *minimizes*

$$J^0_2(y) = \int_S \frac{1}{24}Q_2(A) - f\cdot y\,dx' \tag{21}$$

*among all isometries* $y : S \to \mathbb{R}^3$ *which belong to* $W^{2,2}(S;\mathbb{R}^3)$*. The nonlinear strain satisfies*

$$h^{-1}[(\nabla_h y^{(h)})^T\nabla_h y^{(h)} - Id] \to x_3(\bar{A}(x') + \operatorname{sym} a_{min}\otimes e_3),$$

*where* $2\operatorname{sym} G = G^T + G$ *and where* $a_{min}$ *is the vector which appears in the definition* (8) *of* $Q_2$*, i.e.* $Q_2(\bar{A}) = Q_3(\bar{A} + a_{min}\otimes e_3)$*.*

*In all cases we have convergence of energy, i.e.*

$$\lim_{h\to 0} h^{-\beta}J^h(y^{(h)}) = \lim_{h\to 0} h^{-\beta}\inf J^h = J^0_\alpha(\bar{y}) = \min J^0_\alpha. \tag{22}$$

Convergence of minimizers for the nonlinear bending theory was obtained independently by Pantz [62, 63] under the stronger assumption that $(\nabla_h y^{(h)})^T\nabla_h y^{(h)}$ is uniformly close to the identity.

The range $1 \le \alpha < 2$ is largely unexplored. In the context of delamination and blistering of thin films [37] one is lead to the study of compressive *Dirichlet* boundary conditions such as $y^{(h)}(x', x_3) = (\lambda x', hx_3)$ on $\partial S \times I$, with $0 \le \lambda < 1$ and one can show that $ch \le \inf I^h(y^{(h)}) \le Ch$, with $c > 0$, see [9] (as well as [40, 8] for related work), and [19] for the extension to anisotropic compression. The $\Gamma$-limit of $h^{-1}I^h$ is not known.

If instead of Dirichlet boundary conditions we only assume that $y^{(h)} \rightharpoonup (\lambda x', 0)$ in $W^{1,2}$ then much less is known. S. Venkataramani has constructed maps with periodic boundary conditions whose energy scales like $h^{5/3}$. His construction shows that for $\lambda = 0$ one can achieve an energy bound $Ch^{5/3}$. Conti and Maggi [20] have generalized this construction to a much large class of limit maps. They have also shown that every short map (i.e. every map satisfying $(\nabla' y)^T \nabla' y \leq Id$) can be approximated in $L^\infty$ (and weakly in $W^{1,2}$) by maps $y_h$ with energy bounded by $Ch^{5/3-\varepsilon}$. Thus part ii) of Theorem 1 can be extended to the regime $\alpha = \beta < 5/3$.

On the other hand no general lower bound is known, except for the trivial one $\liminf_{h\to 0} h^{-2} I^h(y^{(h)}) = \infty$, which follows from part iii) of the theorem. The scaling exponent $h^{5/3}$ has been suggested in the physics literature on crumpling as a natural exponent based on a formal scaling argument and an assumed equipartition of bending and stretching energy [51, 25] (for further discussion of crumpling see e.g. [6, 14]). Experimentally the structure of crumpled sheets is characterized by cone like singularities (smoothed at a scale $h$) which are connected by ridges whose width varies with the distance from the singularities. It is believed that the energy contribution of the cones is of order $h^2 \ln(1/h)$ while the ridges contribute $h^{5/3}$. For a single ridge with well-defined boundary conditions Venkataramani recently showed that the energy scales indeed like $h^{5/3}$ [77]. A related, but different, problem arises in the study of complex folding patterns at free boundaries after rupture and it has been suggested that similar patterns might be relevant in certain growth models in biology [73, 5].

### 2.3 von Kármán like theories: $\alpha > 2$

For $\alpha > 2$ we will show that the limit map $\bar{y}$ is not only an isometry (as in part iii) of Theorem 1 above) but is even close to a rigid motion. Then it is natural to study the deviation from the rigid motion and its scaling with $h$. To a map $y^{(h)} : \Omega \to \mathbb{R}^3$ we associate

$$\tilde{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)}, \quad \text{with constants } \bar{R}^{(h)} \in SO(3),\ c^{(h)} \in \mathbb{R}^3. \tag{23}$$

We set $I = (-1/2, 1/2)$ and consider the averaged in-plane and out-of-plane displacements

$$U^{(h)}(x') := \int_I \begin{pmatrix} \tilde{y}_1^{(h)} \\ \tilde{y}_2^{(h)} \end{pmatrix}(x', x_3) - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dx_3, \quad V^{(h)}(x') := \int_I \tilde{y}_3^{(h)}\, dx_3 \tag{24}$$

and their rescalings

$$u^{(h)} = \frac{1}{h^\gamma} U^{(h)}, \quad v^{(h)} = \frac{1}{h^\delta} V^{(h)} \tag{25}$$

defined by parameters $\gamma, \delta$.

For $u \in W^{1,2}(S, \mathbb{R}^2)$ and $v \in W^{2,2}(S)$ we introduce the generalized von Kármán functional

$$I_\alpha^{vK}(u,v) := \frac{\Lambda_\alpha}{2}\int_S Q_2(\frac{1}{2}[\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v])\,dx' + \frac{1}{24}\int_S Q_2((\nabla')^2 v)\,dx', \tag{26}$$

$$\Lambda_\alpha := \begin{cases} \infty & \text{if } 2 < \alpha < 3, \\ 1 & \text{if } \alpha = 3, \\ 0 & \text{if } \alpha > 3 \end{cases}$$

with the convention that $0 \cdot \infty = 0$. In other words for $\alpha = 3$ we have the usual von Kármán functional

$$I^{vK}(u,v) := \frac{1}{2}\int_S Q_2(\frac{1}{2}[\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v]) + \frac{1}{24} Q_2((\nabla')^2 v)\,dx', \tag{27}$$

for $\alpha > 3$ we have the "linearized" von Kármán functional

$$I_{\text{lin}}^{vK}(v) = \frac{1}{24}\int Q_2((\nabla')^2 v)\,dx', \tag{28}$$

and for $2 < \alpha < 3$ we also have $I_{\text{lin}}^{vK}$ but subject to the nonlinear constraint

$$\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v = 0. \tag{29}$$

A symmetrized gradient $e = \operatorname{sym} \nabla' u$ satisfies $e_{11,22} + e_{22,11} - 2e_{12,12} = 0$ (in the sense of distributions). Thus if (29) holds with $v \in W^{2,2}(S)$ we must have

$$\det((\nabla')^2 v) = 0. \tag{30}$$

Conversely (30) is sufficient for the existence of a map $u$ such that (29) holds (see Proposition 30 below).

Geometrically (30) is exactly the condition that the Gauss curvature of the graph of $v$ vanishes. Thus, at least for sufficiently smooth functions, (30) is equivalent to existence of an isometric map from the graph of $v$ to $\mathbb{R}^2$. See Theorem 25 for a precise statement.

**Theorem 2 (von Kármán like theories)** *Suppose that $W$ satisfies* (3)–(6) *and the applied forces satisfy* (14), (16) *and* (17). *Then the following assertions hold.*

*i) (linearized isometry constraint)* *Suppose $2 < \alpha < 3$ and set $\beta = 2\alpha - 2$, $\gamma = 2(\alpha - 2)$, $\delta = \alpha - 2$ (recall (25) for the definitions of $\gamma, \delta$). If $\alpha \in (2, 5/2)$ suppose in addition that $S$ is simply connected. Then $0 \geq \inf J^h \geq -Ch^\beta$. If $y^{(h)}$ is a $\beta$-minimizing sequence (in the sense of* (18)*) then there exists constants $\bar{R}^{(h)} \in SO(3)$ and $c^{(h)} \in \mathbb{R}^3$ such $\bar{R}^{(h)} \to \bar{R}$ and $\tilde{y}^{(h)}$ and the scaled in-plane and out-of-plane deformations given by* (23)–(25) *satisfy (for a subsequence)*

$$\nabla_h \tilde{y}^{(h)} \to Id \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times3}), \tag{31}$$

$$u^{(h)} \to \bar{u} \quad \text{in } W^{1,2}(S; \mathbb{R}^2), \qquad v^{(h)} \to \bar{v} \quad \text{in } W^{1,2}, \tag{32}$$

*equation* (29) *holds and $\bar{v} \in W^{2,2}$. Moreover the pair $(\bar{v}, \bar{R})$ minimizes the functional*

$$J^{vK}_{lin}(v, R) = \frac{1}{24} \int Q_2((\nabla')^2 v)\, dx' - R_{33} \int_S f_3 \cdot v\, dx', \tag{33}$$

*subject to*

$$\det(\nabla')^2 v = 0. \tag{34}$$

*ii) (vK theory)* *Suppose that $\alpha = 3$ and set $\beta = 4$, $\gamma = 2$, $\delta = 1$. Then $0 \geq J^h \geq -Ch^\beta$ and for a (subsequence of a) $\beta$-minimizing sequence* (31)-(32) *hold and the limit $(\bar{u}, \bar{v}, \bar{R})$ minimizes the usual von Kármán functional*

$$J^{vk}(u, v, R) = I^{vk}(u, v) - R_{33} \int_S f_3 \cdot v\, dx'. \tag{35}$$

*iii) (linearized vK theory)* *Suppose $\alpha > 3$ and set $\beta = 2\alpha - 2$, $\gamma = \alpha - 1$ and $\delta = \alpha - 2$. Then $0 \geq \inf J^h \geq -Ch^\beta$ and for a (subsequence of a) $\beta$-minimizing sequence* (32) *holds with $\bar{u} = 0$ and the pair $(\bar{v}, \bar{R})$ minimizes the linearized von Kármán functional*

$$J^{vK}_{lin}(v, R) = \frac{1}{24} \int_S Q_2((\nabla')^2 v)\, dx' - R_{33} \int_S f_3 \cdot v\, dx'. \tag{36}$$

*In all cases we have convergence of the scaled energy* $h^{-\beta}J^h(y^{(h)})$ *to the minimum of the limit functional* $I_\alpha^{vk}(u,v) - R_{33}\int_S f\cdot v$*. Moreover for* $f_3 \not\equiv 0$ *we have* $\bar{R}_{33} = 1$ *or* $\bar{R}_{33} = -1$*.*

**Remark 3** *If* $\bar{R}_{33} = 1$ *then* $\bar{R}$ *is an in-plane rotation and* $y^{(h)}$ *is close to* $\bar{R}\binom{x'}{0}$ *(up to translation). If* $\bar{R}_{33} = -1$ *then* $\bar{R}$ *is an in-plane rotation followed by a* $180^\circ$ *degree out-of-plane rotation* $R_0 = \mathrm{diag}(-1,1,-1)$*. Since* $J^0$ *is invariant under the transformation* $(u,v,R) \mapsto (u,-v,R_0R)$ *it suffices to consider the (conventional) situation* $R_{33} = 1$*.*

For convergence of the nonlinear strain

$$(\nabla_h y^{(h)})^T \nabla_h y^{(h)} = (\nabla_h \tilde{y}^{(h)})^T \nabla_h \tilde{y}^{(h)} \tag{37}$$

see (165) below. In particular the limiting strain is affine in $x_3$, see (110). In formal derivations of the vK equations such a form of the strain is often assumed *a priori*, whereas here it arises as a consequence of the scaling of the forces (and hence the energy).

## 2.4 Γ-convergence

Next we turn to the closely related description in terms of Γ-convergence. In this setting we consider the behavior of general sequences with bounded (scaled) energy rather than minimizing sequences. The Γ-limit $I$ of a sequence of functionals $I^h$ on a Banach space $X$ captures the lowest limiting value of $I^h(y^{(h)})$ among all sequences $y^{(h)}$ converging to $y$. More precisely Γ-convergence with respect to the weak (respectively strong) topology of $X$ requires that the following holds: (i) (Ansatz-free lower bound) For all sequences $y^{(h)}$ converging weakly (resp. strongly) to $y$, $\liminf_{h\to 0} I^{(h)}(y^{(h)}) \geq I^0(y)$, (ii) (Attainment of lower bound) For each $y \in X$ there exists a sequence $y^h$ converging weakly (resp. strongly) to $y$ such that

$$\lim_{h\to 0} I^{(h)}(y^{(h)}) = I^0(y). \tag{38}$$

See [22] for a comprehensive treatment of Γ-convergence.

We restrict attention to the von Kármán like theories; for membrane theory ($\alpha = 0$) see [47, 48, 49], for nonlinear bending theory of plates and shells ($\alpha = 2$) see [30, 31, 33, 62, 63] and for $0 < \alpha < 1$ see [19].

**Theorem 4 (Γ-convergence)** *Suppose that* $W$ *satisfies* (3)–(6) *and the applied forces satisfy* (14), (16) *and* (17)*. Let* $\alpha > 2$ *and let* $\beta, \gamma, \delta$ *be as in*

*Theorem 2 (see Table 1). If $\alpha \in (2, 5/2)$ suppose that $S$ is simply connected. Then the functionals $h^{-\beta} I^h$ are $\Gamma$-convergent to the generalized von Kármán functional $I^{vk}_\alpha$. More precisely we have*

*i) (Compactness and lower bound)   If*

$$\limsup_{h\to 0} \frac{1}{h^\beta} I^h(y^{(h)}) < \infty \tag{39}$$

*then there exists constants $\bar{R}^{(h)} \in SO(3)$ and $c^{(h)} \in \mathbb{R}^3$ such (for a subsequence) $\bar{R}^{(h)} \to \bar{R}$ and $\tilde{y}^{(h)}$ and the scaled in-plane and out-of-plane deformations given by (23)–(25) satisfy*

$$\nabla_h \tilde{y}^{(h)} \to Id \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}), \tag{40}$$

$$u^{(h)} \rightharpoonup u \quad \text{in } W^{1,2}(S; \mathbb{R}^2), \tag{41}$$

$$v^{(h)} \to v \quad \text{in } W^{1,2}(S), \quad v \in W^{2,2}(S). \tag{42}$$

*For $2 < \alpha < 3$ we have*

$$\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v = 0, \quad \det(\nabla')^2 v = 0 \tag{43}$$

$$\liminf_{h\to 0} \frac{1}{h^\beta} I^h(y^{(h)}) \geq \int_S \frac{1}{24} Q_2((\nabla')^2 v)\, dx' \tag{44}$$

*For $\alpha = 3$ we have*

$$\liminf_{h\to 0} \frac{1}{h^\beta} I^h(y^{(h)}) \geq I^{vK}(u, v), \tag{45}$$

*and for $\alpha > 3$ we have*

$$\liminf_{h\to 0} \frac{1}{h^\beta} I^h(y^{(h)}) \geq \int_S \frac{1}{24} Q_2((\nabla')^2 v)\, dx'. \tag{46}$$

*ii) (Optimality of lower bound)   If $2 < \alpha < 3$ and if $v \in W^{2,2}(S)$ with $\det(\nabla')^2 v = 0$ then there exist $u \in W^{1,2}(S; \mathbb{R}^2)$ such that (43) holds and there exists a sequence $\hat{y}^{(h)}$ such that (40)-(42) hold (with $\tilde{y}^{(h)}$ replaced with $\hat{y}^{(h)}$ and $\bar{R}^{(h)} = Id$, $c^{(h)} = 0$) and*

$$\lim_{h\to 0} \frac{1}{h^\beta} I^h(\hat{y}^{(h)}) = \int_S \frac{1}{24} Q_2((\nabla')^2 v)\, dx'. \tag{47}$$

*If $\alpha = 3$, $v \in W^{2,2}(S)$ and $u \in W^{1,2}(S;\mathbb{R}^2)$ then there exists $\hat{y}^{(h)}$ such that* (40)-(42) *hold and*

$$\lim_{h\to 0} \frac{1}{h^\beta} I^h(\hat{y}^{(h)}) = I^{vK}(u,v). \tag{48}$$

*If $\alpha > 3$ and $v \in W^{2,2}(S)$ then there exists $\hat{y}^{(h)}$ such that* (40) *and* (42) *hold*

$$\liminf_{h\to 0} \frac{1}{h^\beta} I^h(\hat{y}^{(h)}) = \int_S \frac{1}{24} Q_2((\nabla')^2 v)\, dx'. \tag{49}$$

**Remark 5** *In part i) one has in addition $u = 0$ provided that $\alpha > 3$. Further convergence results for $\nabla_h y^{(h)}$ are given in Lemmas 13 and 15 as well as in Corollary 14.*

### 2.5 Clamped boundary conditions and Föppl's theory

The scaling of the energy and the solutions can also depend strongly on the boundary conditions. The influence of boundary conditions will be discussed in more detail in [35]. Here we focus on an extreme case, the fully clamped plate. We thus assume that

$$y^{(h)}(x', x_3) = \begin{pmatrix} x' \\ hx_3 \end{pmatrix} \quad \text{on } \partial S \times I. \tag{50}$$

In terms of the averaged in-plane and out-of-plane displacements ((25) with $\bar{R}^{(h)} = Id,\ c^{(h)} = 0$) this implies in particular that

$$u = 0, \quad v = 0 \quad \text{on } \partial S, \tag{51}$$

where we have omitted the superscript $(h)$ here and below for simplicity. The first equation has an important consequence. It implies that

$$\int_S \operatorname{sym} \nabla' u\, dx' = 0.$$

Therefore control the membrane energy $(1/8)Q_2(2\operatorname{sym}\nabla' u + \nabla' v \otimes \nabla' v)$ alone provides an estimate for $v$. Indeed we have by Jensen's inequality and the obvious estimate $Q_2(A) \geq c|\operatorname{tr} A|^2$,

$$\begin{aligned} & \int_S Q_2(2\operatorname{sym}\nabla' u + \nabla' v \otimes \nabla' v)\, dx' \\ \geq\ & \frac{1}{|S|} Q_2\left( \int_S 2\operatorname{sym}\nabla' u + \nabla' v \otimes \nabla' v\, dx' \right) \\ \geq\ & c \left( \int_S |\nabla' v|^2\, dx' \right)^2. \end{aligned} \tag{53}$$

This implies that the clamped plate is much stiffer in response to applied normal loads than a plate with free boundaries (see the different exponents for $\delta$ in Tables 1 and 2). Note also that the above lower bound for the membrane energy scales like the fourth power of the displacement, while the bending energy $(1/24)\int Q_2((\nabla')^2 v)$ scales only quadratically, leading to a sublinear (in fact cubic root) behavior of the displacement in terms of the strength of the applied force, when the membrane term is dominant. This crossover from linear response for very weak forces ($\alpha > 3$) to sublinear behavior for stronger forces ($0 < \alpha < 3$) is exactly what Föppl [27] and von Kármán [45] wanted to capture with their extension of the linear, purely bending dominated theory.

A precise statement is contained in the following theorem. We define the relaxed membrane energy by

$$Q_2^{\rm rel}(A,b) = \min\{Q_2(A + \frac{b\otimes b}{2} + M) : \ M = M^T, M \geq 0\} \tag{54}$$

for all $A \in \mathbb{R}^{2\times 2}$, $b \in \mathbb{R}^2$. This relaxed energy is a geometrically linear version of the membrane energy of LeDret and Raoult: it vanishes if $\operatorname{sym} A + (1/2)b\otimes b$ is negative semidefinite (pure compression) and it agrees with $Q_2$ if the stress $\sigma = L(\operatorname{sym} A + (1/2)b\otimes b)$ is positive semidefinite (here $L$ is the self-adjoint operator associated to the quadratic form $Q_2$, i.e. $(LA, A) = Q_2(A)$). In fact $Q_2^{\rm rel}$ is both the quasiconvex and the rank-one convex envelope of $Q_2$ if the latter if viewed as a function on $3\times 2$ matrices $\binom{A}{b}$. We consider the limiting energy functional

$$J^{\rm Fö}_{\rm rel}(u,v) = \frac{1}{2}\int_S Q_2^{\rm rel}(\nabla u, \nabla v)\, dx' - \int_S f_3 v\, dx'. \tag{55}$$

**Theorem 6 ([21])** *Suppose that $W$ satisfies (3)–(6) and that $\Omega$ is strictly star-shaped with $C^2$ boundary. Suppose also that $0 < \alpha < 3$, that $f_1^{(h)} = f_2^{(h)} = 0$ and that $f_3^{(h)} : S \to \mathbb{R}$ satisfies*

$$\frac{1}{h^\alpha} f_3^{(h)} \rightharpoonup f_3 \quad \textit{in } L^2(S).$$

*Set*

$$\beta = \frac{4}{3}\alpha, \quad \gamma = \frac{2}{3}\alpha, \quad \delta = \frac{1}{3}\alpha.$$

*Then*

$$0 \geq \inf\{J^h(y) : \ y \textit{ satisfies } (50)\} \geq -Ch^\beta. \tag{58}$$

*If $y^{(h)}$ is a $\beta$-minimizing sequence (subject to (50)) then (for a subsequence)*

$$u^{(h)} \to u \quad \text{in } L^1(S;\mathbb{R}^2), \quad \operatorname{sym}\nabla' u^{(h)} \overset{*}{\rightharpoonup} \operatorname{sym}\nabla' u \quad \text{in } \mathcal{M}(S;\mathbb{R}^{2\times 2}), \tag{59}$$

$$v^{(h)} \rightharpoonup v \quad \text{in } W_0^{1,2}(S) \tag{60}$$

$$\liminf \frac{1}{h^\beta} J^h(y^{(h)}) = J_{\mathrm{rel}}^{\mathrm{Fö}}(u,v). \tag{61}$$

*and $(u,v)$ minimizes $J_{\mathrm{rel}}^{\mathrm{Fö}}$ subject to the boundary conditions $v = 0$ on $\partial S$ on $u^-(x') = \lambda(x')\nu(x')$, with $\lambda \geq 0$, on $\partial S$, where $\nu$ denotes the outer normal and $u^-$ is the inner trace of $u$ (which exists for functions whose symmetrized distributional gradient is a Radon measure).*

**Remark 7** *Föppl [27] considered the limiting functional with $Q_2$ instead of $Q_2^{\mathrm{rel}}$. This misses the degeneracy due to the possibility of crumpling, but Föppl's functional is correct if the plate is always in a state of stretch (and it may well be possible to prove that this is the case for certain reasonable loading conditions). Table 2 makes precise von Kármán's assertion that his theory lies in between the fully linear theory (which arises for $\alpha > 3$) and Föppl's theory.*

**Remark 8** *It might at first glance seem surprising that the limiting boundary condition for $u$ is not simply $u = 0$. The reason is that displacements satisfying the condition in the theorem can be approximated well in energy by deformations with zero boundary conditions. Indeed assume the inequality condition and consider an approximation $u_\delta$ with zero boundary conditions which agrees with $u$ except on a boundary layer of thickness $\delta$ in which $u_\delta$ is almost linear in the direction normal to the boundary. The $\nabla u_\delta$ is approximately $-u^- \otimes \nu$ in the boundary layer and therefore $\operatorname{sym}\nabla u_\delta$ is (almost) negative semidefinite since $\lambda \geq 0$, i.e. $u_\delta$ is (almost) compressive. Now the relaxed energy is zero on compressive deformations, hence there is almost no extra energy in the boundary layer.*

If the forces are scaled with exponent $\alpha \geq 3$ and we impose the clamped boundary conditions (50) then we obtain $\Gamma$-convergence to the same limit functionals as above, subject to the constraints

$$u = 0, \quad v = \nabla' v = 0. \tag{62}$$

The only slightly delicate point is to establish the new boundary condition $\nabla' v = 0$. For this one can use Corollary 14; see [35] for the details. The upper bound in the proof of $\Gamma$-convergence is easy since the ansatz functions essentially inherit the clamped boundary conditions from (62).

| $\alpha$ applied force | $\beta$ energy | $\gamma$ in-plane | $\delta$ out-of-plane | limit model |
|---|---|---|---|---|
| $\alpha = 0$ | 0 | 0 | 0 | Membrane |
| $0 < \alpha < 1$ | $\alpha$ | 0 | 0 | Constrained membrane |
| $\alpha = 2$ | $\alpha$ | 0 | 0 | Bending, isometric mid-plane |
| $2 < \alpha < 3$ | $2\alpha - 2$ | $2(\alpha - 2)$ | $\alpha - 2$ | Linearized isometry constraint |
| $\alpha = 3$ | $2\alpha - 2$ | $2(\alpha - 2)$ | $\alpha - 2$ | von Kármán |
| $\alpha > 3$ | $2\alpha - 2$ | $\alpha - 1$ | $\alpha - 2$ | Linearized vK |

Table 1: Relation between the scaling exponents $\alpha$ of the applied forces, $\beta$ of the energy $\gamma$ of the in-plane deformation and $\delta$ of the out-of-plane deformation. For $\alpha > 2$ we assume that the limit force is normal (see (17); cf., also Theorem 2)

### 2.6 Overview of scaling exponents and limit models

Given the scaling exponent of the applied force, the exponents describing the convergence of the energy and of the solution, together with the expression for the limiting theory, are determined by Theorems 1-6. Tables 1 and 2 give an overview of the different exponents, for unconstrained and clamped boundary conditions, respectively.

## 3 Outline of the proof

a) *Rigidity estimates.* As in the work on the nonlinear bending theory [31] the crucial ingredient is a quantitative estimate that bounds the squared $L^2$ distance of the deformation gradient $\nabla w$ from a rigid motion in terms of the energy $\int W(\nabla w)$. We recall this estimate in Theorem 9 at the beginning of the next section.

b) *Scaled rigidity estimates in thin domains.* In a thin domain $\Omega_h = S \times (-h/2, h/2)$ the constant in the rigidity estimate degenerates as $h \to 0$. We show that globally the constant degenerates like $h^{-2}$. Locally one can obtain a good approximation of $\nabla w$ by covering $\Omega_h$ by cubes of size $h$ and using a constant rotation in each cube. This leads to two important approximations: one by a piecewise constant maps $R^{(h)}$ with values in the rotations $SO(3)$ and another one $\tilde{R}^{(h)}$ (obtained by a difference quotient

| $\alpha$<br>applied force | $\beta$<br>energy | $\gamma$<br>in-plane | $\delta$<br>out-of-plane | limit model |
|---|---|---|---|---|
| $\alpha = 0$ | 0 | 0 | 0 | Membrane |
| $0 < \alpha < 3$ | $(4/3)\alpha$ | $(2/3)\alpha$ | $(1/3)\alpha$ | Relaxed Föppl<br>= lin. membrane |
| $\alpha = 3$ | $2\alpha - 2$ | $2(\alpha - 2)$ | $\alpha - 2$ | von Kármán |
| $\alpha > 3$ | $2\alpha - 2$ | $\alpha - 1$ | $\alpha - 2$ | Linearized vK |

Table 2: Relation between the scaling exponents for a *clamped* plate, assuming normal forces. Föppl's theory (or more precisely its relaxed version, can be seen as a geometrically linear version of membrane theory. von Kármán's theory which has both membrane and bending contributions lies in between Föppl's theory (capturing only membrane energy) and the linear theory (capturing only bending energy)

estimate and smoothing) which is differentiable, takes values close to $SO(n)$ (in an $L^2$ sense) and whose gradient can be bounded in terms of the energy. While this is straightforward in the interior some care has to be taken near the boundary. All these estimates are carried out in Theorem 10 in the next section. It is convenient to state them in the fixed domain $\Omega = S \times (-1/2, 1/2)$. Thus the gradient has to be replaced by the scaled gradient $\nabla_h$.

c) *Scaling of the in-plane and out-of-plane components.* From b) one sees that if the energy is small compared to $h^2$ then the deformation is close a rigid rotation even in a thin domain. Normalizing this rotation to the identity one easily derives the natural scaling exponents for the in-plane and out-of-plane components (see Table 1). This is done in Lemma 13 in Section 5.

d) *Identification of the limiting strain.* The estimates in b) show that the scaled approximate nonlinear strain $G^{(h)} = h^{2-\alpha}[(R^{(h)})^T \nabla_h w^{(h)} - Id]$ is bounded in $L^2$. Using a difference quotient argument we show that the limiting strain $G$ is affine in $x_3$, i.e. $G = G_0(x') + x_3 G_1(x')$ and we identify (the relevant submatrices) of the coefficients $G_0$ and $G_1$ in terms of the limiting in-plane and out-of-plane components $u$ and $v$. From this one obtains the lower bounds in Theorem 4 by a careful Taylor expansion. All this is done in Lemma 15 and Corollary 16.

e) $\Gamma$-*convergence* The lower bound follows directly from d). For the upper bound one has to identify good test functions. For $\alpha \geq 3$ one can use a more or less the standard ansatz (which is often assumed a priori in heuristic arguments in favour of the von-Kármán theory). For $\alpha < 5/2$ the situation is more delicate, since the standard ansatz only produces an approximate isometry of the mid-plane (the mid-plane is isometric only in the sense of geometrically linear elasticity) and the deviation from a true isometry leads to a too high elastic energy. To overcome this problem we need to study carefully the relation between geometrically linear isometries and full isometries. This is done in detail in Section 8. In addition we need to approximate $W^{2,2}$ isometries by $W^{2,\infty}$ maps which are isometries except on a very small set (such an approximation was already used in [31]).

f) *Convergence of minimizers* This follows from Γ-convergence by a suitable Poincaré inequality, see Section 7.1. To establish strong convergence of the in-plane components in $W^{1,2}$ (and not just weak convergence) we use an equiintegrable version of the rigidity estimates. This is carried out in Section 7.2.

Finally in Section 9 we discuss how our results address the criticisms raised against the vK theory. We also discuss possible extensions, open questions, and directions for future research.

For a first reading we recommend the reader to focus on Γ-convergence for the vK case, i.e. $\alpha = 3$, $\beta = 4$ and $E^h = h^4$. This case already contains the main ideas of our analysis and Sections 4, 5 and Subsection 6.1 are sufficient to obtain Γ-convergence to the vK functional. The main points in this case are the bounds on the scaled in-plane and out-of-plane displacements $u^{(h)}$ and $v^{(h)}$ (see Lemma 13) and the identification of the limiting strain in Lemma 15. This immediately yields the lower bound for Γ-convergence, and the upper bound follows by using the usual test function (123).

# 4 Geometric rigidity

**Theorem 9 (Quantitative rigidity estimates)** *Let $U$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. There exists a constant $C(U)$ with the following property. For each $v \in W^{1,2}(U, \mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that,*

$$\|\nabla v - R\|_{L^2(U)} \leq C(U) \, \| \operatorname{dist}(\nabla v, SO(n))\|_{L^2(U)}. \tag{63}$$

*The constant $C(U)$ can be chosen uniformly for a family of domains which are Bilipschitz equivalent with controlled Lipschitz constants. The constant $C(U)$ is invariant under dilations.*

For a proof see [31]. The estimate (63) was established by John [41, 42] under the stronger hypothesis that $v$ is locally Bilipschitz (for further developments along this line see [10]). The main difficulty with this assumption is that it does not follow from suitable bounds on the elastic energy alone. Reshetnyak [72] established a rigidity results for almost conformal maps (rather than almost isometries). He showed that if $\nabla v$ is close to the set $\mathbb{R}_0^+ SO(n)$ of all conformal matrices in $L^n$ then $\nabla v$ is close to the gradient of a single conformal map in $L^n$. His argument does, however, not give a quantitative estimate like (63).

In a thin domain $\Omega_h = S \times (-h/2, h/2)$ the constant $C(\Omega_h)$ degenerates like $h^{-2}$ (see (66) below). We can obtain a good approximation (at least in the interior) for $\nabla y$ by a piecewise constant map $R^{(h)}$ (with values in $SO(3)$) by covering $\Omega_h$ by cubes of size $h$. Application of Theorem 9 to two neighbouring cubes in addition yields a difference quotient estimate. Thus after mollification on a scale $h$ we can obtain another approximation $\tilde{R}^{(h)}$ (which in general no longer takes values exactly in $SO(3)$) whose gradient can be controlled in terms of the energy. This second approximation will prove useful to establish compactness and also higher regularity of the limits as $h \to 0$. The following result summarizes the estimates (up to the boundary) one can obtain in this way. As before we rescale to a fixed domain $\Omega$ and use the scaled gradient $\nabla_h = (\nabla', h^{-1}\partial_3)$.

**Theorem 10 (Approximation by rotations in thin domains)** *Suppose that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$. Let $y \in W^{1,2}(\Omega; \mathbb{R}^3)$ and*

$$E = \int_\Omega \operatorname{dist}^2(\nabla_h y, SO(3))dx,$$

*where $h \in (0, 1]$. Then there exist maps $R : S \to SO(3)$ and $\tilde{R} : S \to \mathbb{R}^{3\times 3}$, with $|\tilde{R}| \le C$, $\tilde{R} \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ such that*

$$||\nabla_h y - R||^2_{L^2(\Omega)} \le CE, \;\; ||R - \tilde{R}||^2_{L^2(S)} \le CE, \tag{64}$$

$$||\nabla \tilde{R}||^2_{L^2(S)} \le \frac{C}{h^2}E, \quad ||R - \tilde{R}||^2_{L^\infty(S)} \le \frac{C}{h^2}E. \tag{65}$$

*Moreover there exists a constant rotation $\bar{Q} \in SO(3)$ such that*

$$||\nabla_h y - \bar{Q}||^2_{L^2(\Omega)} \le \frac{C}{h^2}E. \tag{66}$$

*and*

$$||R - \bar{Q}||^2_{L^p}(S) \leq \frac{C_p}{h^2} E, \quad \forall p < \infty. \tag{67}$$

*Here all constants depend only on $S$ (and on $p$ where indicated).*

**Remark 11** *If $E \leq \delta_0 h^2$ for a sufficiently small value of $\delta_0$ then, in view of (80) we always have $\tilde{R}(x') \in \mathcal{U}$, where $\mathcal{U}$ is a tubular neighbourhood of $SO(3)$. Hence the map $R : S \to SO(3)$ obtained by projection to $SO(3)$ is in $W^{1,\infty}$ and $||\nabla R||^2_{L^2} \leq Ch^{-2}E$. Thus in this case the relevant estimates can be stated in terms of $R$ directly and $\tilde{R}$ appears only as an intermediate quantity.*

*Proof.* The result is implicit in [31]. We could follow the strategy used there and use Theorem 9 to first construct a map $R$ which is constant on squares of size $h$ and then mollify $R$ to obtain $\tilde{R}$ (using a suitable change of variables and tangential mollification near the boundary). For variety we follow a different approach, constructing $\tilde{R}$ first. One advantage of this approach is that the map $y \mapsto \tilde{R}$ is linear (as long as $E \leq Ch^2$, which is the main case of interest). To construct $\tilde{R}$ we use separate constructions in the interior and near the boundary and then glue them together by a partition of unity. We begin with the relevant local estimates.

*Step 1* (local estimates). Let $U$ be an open set in $\mathbb{R}^2$, and let $K \subset U$ be compact. We suppose that $3h < \mathrm{dist}_\infty(K, \partial U)$, where $\mathrm{dist}_\infty$ is the distance with respect to the norm $|(x_1, x_2)|_\infty = \max(|x_1|, |x_2|)$. Let $y \in W^{1,2}(U \times (-\frac{1}{2}, \frac{1}{2}); \mathbb{R}^3)$. To abbreviate the notation we write $F(x)$ for the scaled gradient and $\bar{F}(x')$ for its vertical average

$$F(x) = \nabla_h y(x), \quad \bar{F}(x') = \int_I F(x', x_3)\, dx_3, \quad I = (-1/2, 1/2).$$

For each point $x' \in K$ we consider the square

$$S_{x',h} = x' + (0, h)^2$$

with lower left corner $x'$. Let $\psi \in C_0^\infty((0, -1)^2)$ be a standard mollifier, i.e. $\psi \geq 0$, $\int \psi = 1$ and set $\psi_h(\cdot) = h^{-2}\psi(\cdot/h)$. On $K$ we define the smoothed rotation $R$ by

$$R = \psi_h * \bar{F}.$$

Here we write $R$ instead of $\tilde{R}$ or $\tilde{R}_h$ to simplify the notation. The $R$ defined above is not the map $R$ mentioned in the theorem. Explicitly we have

$$R(x') = \int_{S_{x',h}\times I} h^{-2}\psi\left(\frac{x'-z'}{h}\right) F(z)\, dz' dz_3.$$

We claim that

$$\int_{S_{x',h}\times I} |F(z) - R(z')|^2\, dz \le C \int_{S_{x',2h}\times I} \mathrm{dist}^2(F(z), SO(3))\, dz, \tag{68}$$

$$|\nabla R(x')|^2 \le \frac{C}{h^4}\int_{S_{x',h}\times I} \mathrm{dist}^2(F(z), SO(3))\, dz \tag{69}$$

$$\mathrm{dist}^2(R(x'), SO(3)) \le \frac{C}{h^2}\int_{S_{x',h}\times I} \mathrm{dist}^2(F(z), SO(3))\, dz \tag{70}$$

To prove (68) we use Theorem 9 applied to a cube of size $h$. Keeping in mind that $y(x_1, x_2, x_3) = \tilde{y}(x_1, x_2, hx_3)$ and $F(x) = \nabla\tilde{y}(x_1, x_2, hx_3)$ we see that there exists $R_{x',h} \in SO(3)$ such that

$$\int_{S_{x',h}\times I} |F(z) - R_{x',h}|^2\, dz \le C \int_{S_{x',h}\times I} \mathrm{dist}^2(F(z), SO(3))\, dz. \tag{71}$$

Since $\psi_h \ge 0$ and $\int \psi_h = 1$ Jensen's inequality yields

$$|R(x') - R_{x',h}|^2 \le \int_{S_{x',h}\times I} \psi_h(x'-z')|F(z) - R_{x',h}|^2 dz$$
$$\le \frac{C}{h^2}\int_{S_{x',h}\times I} |F(z) - R_{x',h}|^2\, dz \le \frac{C}{h^2}\int_{S_{x',h}\times I} \mathrm{dist}^2(F(z), SO(3))\, dz \tag{72}$$

This establishes (70). Using the fact that $\int \nabla\psi_h = 0$ we obtain similarly for $\tilde{x}' \in S_{x',h}$

$$|\nabla R(\tilde{x}')|^2 \le \frac{C}{h^4}\int_{S_{\tilde{x}',h}\times I} \mathrm{dist}^2(F(z), SO(3))\, dz \le \frac{C}{h^4}\int_{S_{x',2h}\times I} \mathrm{dist}^2(F(z), SO(3))\, dz. \tag{73}$$

and this proves in particular (69). We also conclude that for $\tilde{x}' \in S_{x',h}$

$$|R(\tilde{x}') - R(x')|^2 \le \frac{C}{h^2}\int_{S_{x',2h}\times I} \mathrm{dist}^2(F(z), SO(3))\, dz. \tag{74}$$

and combining this with (71), (72) and the elementary inequality

$$\int_{S_{x',h}\times I} |F(z)-R(z')|^2 dz \le 2\Big(\int_{S_{x',h}\times I} |F(z)-R_{x',h}|^2 dz + h^2 \sup_{(\tilde{x},x_3)\in S_{x',h}\times I} |R_{x',h}-R(\tilde{x}')|^2\Big)$$

we obtain (68).

Finally, for future reference consider a lattice of squares of size $h$ in $\mathbb{R}^2$, sum (68) over all squares which intersect $K$ and integrate (69) over $K\times I$. This yields

$$\int_{K\times I} |R-F|^2 + h^2|\nabla R|^2\, dz \le \int_{U\times I} \operatorname{dist}^2(F(z), SO(3))\, dz \tag{75}$$

*Step 2* (estimates near the boundary). We fix again an open set $U\subset\mathbb{R}^2$ and a compact subset $K\subset U$. We first consider the situation near a flat piece of the boundary of $S$. More precisely we suppose that $U\cap S = U\cap\mathbb{R}^2_+$, $U\cap\partial S = U\cap\partial\mathbb{R}^2_+$, where $R^2_+ = \{(x_1,x_2): x_2>0\}$ is the upper half plane. For $x'\in K\cap\mathbb{R}^2_+$ we define as before $R(x') = (\psi_h * \bar{F})(x')$. Since the support of $\psi$ is contained in the lower half plane $R$ is indeed well-defined when $h$ is small enough. Proceeding as in Step 1 (and using the standard lattice $(h\mathbb{Z})^2$ for the summation) we obtain (75) with $K$ and $U$ replaced by $K\cap S$ and $U\cap S$, respectively.

Now suppose that $S$ is locally the epigraph of a Lipschitz function, i.e. there exist a bounded open interval $J\subset\mathbb{R}$, a Lipschitz function $f: J\to\mathbb{R}$ and an orthonormal coordinate system (still denoted $(x_1,x_2)$) such that

$$U\cap S = \{x\in U: x_1\in J, x_2 > f(x_1)\}, \tag{76}$$

$$U\cap\partial S = \{x\in U: x_1\in J, x_2 = f(x_1)\}. \tag{77}$$

To flatten the boundary of $S$ consider the map $\Phi: U\cap\bar{S}\to\overline{\mathbb{R}^2_+}$ given by $\Phi(x) = (x_1, x_2 - f(x_1)$. Note that $\Phi$ is Bilipschitz and area preserving. Now set

$$(R\circ\Phi^{-1}) = \psi_h * (\bar{F}\circ\Phi^{-1}).$$

Then for $\xi'\in\Phi(K)$ and sufficiently small $h$ the value $(R\circ\Phi^{-1})(\xi')$ is well-defined. Application of Theorem 9 shows that there exists $R_{\xi',h}\in SO(3)$

such that

$$\begin{aligned}
&\int_{S_{\xi',h}\times I} |F(\Phi^{-1}(\zeta'), z_3) - R_{\xi',h}|^2 \, d\zeta' dz_3 \\
&\quad = \int_{\Phi^{-1}(S_{\xi',h})\times I} |F(z', z_3) - R_{\xi',h}|^2 \, dz' dz_3 \\
&\quad \le C \int_{\Phi^{-1}(S_{\xi',h})\times I} \operatorname{dist}^2(F(z', z_3), SO(3)) \, dz' dz_3 \\
&\quad = C \int_{S_{\xi',h}\times I} \operatorname{dist}^2(F, SO(3))(\Phi^{-1}(\zeta'), z_3)) \, d\zeta' dz_3.
\end{aligned}$$

The constant $C$ is independent of $\xi'$ since the estimate in Theorem 9 holds uniformly in domains which are Bilipschitzly equivalent. As before we deduce from the above estimate the following analogue of (72)

$$|R \circ \Phi^{-1}(\xi') - R_{\xi',h}|^2 \le \frac{C}{h^2} \int_{S_{\xi',h}\times I} \operatorname{dist}^2(F, SO(3))(\Phi^{-1}(\zeta'), z_3)) \, d\zeta' dz_3$$

as well as the pointwise estimates for $\nabla(R \circ \Phi^{-1})$ and $R \circ \Phi^{-1}$. This yields again an estimate for $||R \circ \Phi^{-1} - F \circ \Phi^{-1}||_{L^2(S_{\xi,h}\times I)}$ and after summation over the standard lattice we obtain the counterpart of (75), namely

$$\int_{(K\cap S)\times I} \left( |R - F|^2 + h^2 |\nabla R|^2 \right) dz \le C \int_{(U\cap S)\times I} \operatorname{dist}^2(F(z), SO(3)) \, dz. \tag{78}$$

*Step 3* (global estimates for $\tilde{R}$). Now it suffices to combine the estimates in Steps 1 and 2 via a partition of unity. Since $S$ is a Lipschitz domain its closure $\bar{S}$ can be covered by open sets $U_0, \dots, U_l$ where $\bar{U}_0 \subset S$ and where $U_1, \dots, U_l$ are of the form (76) and (77) (after a possible rotation of the coordinates). Consider a partition of unity subordinate to the cover $\{U_i\}$, i.e.

$$\eta_i \in C_0^\infty(U_i), \quad \eta_i \ge 0, \quad \sum \eta_i = 1 \text{ in } S$$

and set $K_i = \operatorname{supp} \eta_i$. Denote by $\tilde{R}_0, \dots \tilde{R}_l$ the maps on $K_0, \dots, K_l$ constructed in Steps 1 and 2 and set

$$\tilde{R} = \sum \eta_i \tilde{R}_i.$$

Using the fact that $\sum \nabla \eta_i = \nabla \sum \eta_i = 0$ in $S$ we find

$$\begin{aligned}\tilde{R} - F &= \sum \eta_i (\tilde{R}_i - F), \\ \nabla \tilde{R} &= \sum \eta_i \nabla \tilde{R}_i + \sum \nabla \eta_i (\tilde{R}_i - F).\end{aligned}$$

Applying (75) and (78) we deduce

$$\int_{S\times I} \left( |\tilde{R} - F|^2 + h^2 |\nabla \tilde{R}|^2 \right) dz \le CE. \tag{79}$$

This proves the first estimate in (65). Similarly we get

$$\sup_S \operatorname{dist}(\tilde{R}, SO(3)) \le \frac{C}{h^2} \sup_{x'\in S} \int_{(B_{x',C_0 h}\cap S)\times I} \operatorname{dist}^2(\nabla_h y, SO(3))\, dz \le \frac{C}{h^2} E. \tag{80}$$

Here $C_0$ depends only on $S$ (through the Lipschitz constants of the maps $\Phi_i$ used to flatten the boundary of $S$).

This essentially finishes the construction of $\tilde{R}$, but we need to address one more point. As defined, $\tilde{R}$ may not be bounded (unless $E \le Ch^2$ in which case we can use (80)). To remedy this it suffices to replace $\tilde{R}$ by its projection $\pi_\rho \tilde{R}$ onto a sufficiently large ball $B_\rho \subset \mathbb{R}^{3\times 3}$, which contains $SO(3)$. Indeed we have $|\nabla(\pi_\rho \circ \tilde{R})| \le |\nabla \tilde{R}|$ since $\pi_\rho$ is a contraction. Moreover

$$\begin{aligned}|\pi_\rho \circ \tilde{R} - F| &\le |\pi_\rho \circ \tilde{R} - \pi_\rho \circ F| + |\pi_\rho \circ F - F| \\ &\le |\tilde{R} - F| + \operatorname{dist}(F, SO(3)).\end{aligned}$$

Moreover, trivially $\operatorname{dist}(\pi_\rho \circ \tilde{R}, SO(3)) \le \operatorname{dist}(\tilde{R}, SO(3))$. Hence (79), (80) also hold with $\tilde{R}$ replaced by $\phi_\rho \circ \tilde{R}$, establishing the first estimate in (65).

*Step 4* (estimates for $R \in L^\infty(S, SO(3))$).

Since $SO(3)$ is a smooth manifold there exists a tubular neighbourhood $\mathcal{U}$ of $SO(3)$ such that the nearest-point projection $\pi : \mathcal{U} \to SO(3)$ is smooth. Let

$$R(x') = \begin{cases} \pi(\tilde{R}(x')) & \text{if } \tilde{R}(x') \in \mathcal{U} \\ \text{Id} & \text{else} \end{cases}.$$

Then $|R(x') - \tilde{R}(x')| = \operatorname{dist}(\tilde{R}(x'), SO(3))$ if $\operatorname{dist}(\tilde{R}(x'), SO(3)) < \delta$. Hence we always have $|R - \tilde{R}| \le C \operatorname{dist}(\tilde{R}, SO(3))$. Now (80) implies the second estimate in (65), and (79) proves (64).

*Step 5* (remaining estimates). The estimates in (66) and (67) follow from the Poincaré-Sobolev inequality. Indeed in view of the first estimate in (65) there exists a $\bar{Q}$ such that $||\tilde{R} - \bar{Q}||_{L^p}^2 \leq C_p h^{-2} E$ so $\bar{Q}$ satisfies (67). Taking $p = 2$ and using (64) we infer that $\bar{Q}$ satisfies (66). Finally, $\operatorname{dist}^2(\bar{Q}, SO(3)) = |\Omega|^{-1} \int_\Omega \operatorname{dist}^2(\bar{Q}, SO(3)) \leq CE$, since $\bar{Q}$ satisfies (67) and (66). Hence these two estimates continue to hold with $\bar{Q}$ replaced by any rotation with minimal distance from $\bar{Q}$. $\square$

For future reference we recall that Korn's inequality holds for Lipschitz domains (we will only need it for $S \subset \mathbb{R}^2$).

**Proposition 12 (Korn's inequality)** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and let $1 < p < \infty$. Consider the space*

$$E^p(\Omega) := \left\{ u \in L^p(\Omega; \mathbb{R}^n) : \operatorname{sym} \nabla u \in L^p(\Omega; \mathbb{R}^{n \times n}) \right\}$$

*Then*

$$E^p(\Omega) = W^{1,p}(\Omega; \mathbb{R}^n), \tag{81}$$

$$||u||_{1,p}^p := \int_\Omega |u|^p + |\nabla u|^p \, dx \leq C_p(\Omega) \int_\Omega |u|^p + |\operatorname{sym} \nabla u|^p \, dx, \tag{82}$$

$$\begin{aligned} &\min \left\{ ||u - Ax - b||_{1,p}^p : A + A^T = 0, A \in R^{n \times n}, b \in \mathbb{R}^n \right\} \\ &\quad \leq C_p(\Omega) \int_\Omega |\operatorname{sym} \nabla u|^p. \end{aligned} \tag{83}$$

*If $\Gamma \subset \partial\Omega$ has positive $\mathcal{H}^{n-1}$ measure then*

$$||u||_{1,p}^p \leq C_p(\Omega, \Gamma) \int_\Omega |\operatorname{sym} \nabla u|^p, \quad \textit{for all } u \textit{ with } u_{|\Gamma} = 0. \tag{84}$$

*Proof.* For (81) and (82) see [36], Theorem 1. To establish the assertion (83) one uses the compact embedding of $W^{1,p}$ into $L^p$ and the usual argument by contradiction starting with a sequence with $||u_k||_{1,p} = 1$, $\int u_k = 0$, $\int \nabla u_k - (\nabla u_k)^T = 0$ and $||\operatorname{sym} \nabla u_k||_p \to 0$. Any weak limit $u$ in $W^{1,p}$ satisfies $\operatorname{sym} \nabla u = 0$ and hence is an affine map with skew symmetric gradient and therefore zero by the normalization above. Together with (82) one obtains the desired contradiction. Similarly one obtains (84). $\square$

# 5 Scaling of in-plane and out-of-plane components and limiting strain

## 5.1 Scaling exponents

It follows from Theorem 10 that for energies $E^h$ small compared to $h^2$ the deformation $y^{(h)}$ is close to the trivial map $(x', x_3) \mapsto (x', hx_3)$, up to a rigid motion. The following lemma provides detailed estimates for the difference between $y^{(h)}$ and the trivial deformation (cf. Table 1). In view of future applications it is convenient to consider a general sequence $E^h$ and not to restrict attention to powers of $h$.

**Lemma 13 (Convergence of scaled out-of-plane and in-plane deformations)** *Suppose that*

$$I^h(y^{(h)}) \leq CE^h, \tag{85}$$

$$\lim_{h \to 0} h^{-2} E^h = 0. \tag{86}$$

*Then there exists a maps* $R^{(h)} : S \to SO(3)$ *and constants* $\bar{R}^{(h)} \in SO(3)$, $c^{(h)} \in \mathbb{R}^3$ *such that*

$$\tilde{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)}$$

*and the in-plane and out-of-plane displacements*

$$U^{(h)}(x') := \int_I \begin{pmatrix} \tilde{y}_1^{(h)} \\ \tilde{y}_2^{(h)} \end{pmatrix} (x', x_3) - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dx_3, \quad V^{(h)}(x') := \int_I \tilde{y}_3^{(h)} \, dx_3$$

*satisfy*

$$||\nabla_h \tilde{y}^{(h)} - R^{(h)}||_{L^2(\Omega)} \leq C\sqrt{E^h}, \tag{88}$$

$$||R^{(h)} - Id||_{L^q(S)} \leq C_q h^{-1} \sqrt{E^h} \quad \forall q < \infty, \quad ||\nabla' R^{(h)}||_{L^2(S)} \leq C h^{-1} \sqrt{E^h}. \tag{89}$$

*Moreover there exists a subsequence (not relabeled) such that*

$$v^{(h)} := \frac{h}{\sqrt{E^h}} V^{(h)} \to v \quad \text{in } W^{1,2}(S), \quad v \in W^{2,2}(S), \tag{90}$$

$$u^{(h)} := \min\left(\frac{h^2}{E^h}, \frac{1}{\sqrt{E^h}}\right) U^{(h)} \rightharpoonup u \quad \text{in } W^{1,2}(S; \mathbb{R}^2), \tag{91}$$

$$\frac{h}{\sqrt{E^h}}(R^{(h)} - Id) \to A \quad in\ L^q(\Omega;\mathbb{R}^{3\times 3}), \quad \forall q < \infty, \tag{92}$$

$$\frac{h}{\sqrt{E^h}}(\nabla_h \tilde{y}^{(h)} - Id)) \to A \quad in\ L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{93}$$

$$A_{,3} = 0, \quad A \in W^{1,2}(S;\mathbb{R}^{3\times 3}), \tag{94}$$

$$A = e_3 \otimes \nabla' v - \nabla' v \otimes e_3, \tag{95}$$

$$\frac{h^2}{E^h}\,\mathrm{sym}(R^{(h)} - Id) \to \frac{A^2}{2} \quad in\ L^q(S;\mathbb{R}^{3\times 3}), \quad \forall q < \infty \tag{96}$$

**Corollary 14** *In connection with boundary value problems it is also useful to study the convergence of the first moment*

$$\zeta^{(h)}(x') = \int_I x_3 \left[\tilde{y}^{(h)}(x', x_3) - \begin{pmatrix} x' \\ hx_3 \end{pmatrix}\right] dx_3 \tag{97}$$

*We have*

$$\frac{1}{\sqrt{E^h}}\zeta^{(h)} \rightharpoonup \frac{1}{12}Ae_3 = -\frac{1}{12}\begin{pmatrix} \nabla' v \\ 0 \end{pmatrix} \quad in W^{1,2}(S;\mathbb{R}^3). \tag{98}$$

*The analogous assertion holds if $E^h = h^2$. Then $\nabla_h y^{(h)} \to R$ in $L^2$ and $h^{-1}\zeta^{(h)} \rightharpoonup (1/12)(R - Id)e_3$ in $W^{1,2}$.*

Note that in the vK scaling $E^h \sim h^4$ the convergence results (90), (91) and (98) are consistent with the commonly used ansatz

$$y^{(h)}(x', x_3) \approx \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2 u(x') \\ hv(x') \end{pmatrix} - x_3 \begin{pmatrix} h^2 \nabla' v \\ 0 \end{pmatrix}. \tag{99}$$

Indeed it follows from (103) below that, for $\gamma \in \{1, 2\}$,

$$h^{-2}(\tilde{y}^{(h)}(x) - [x' + h^2 u(x') - x_3 h^2 \nabla' v(x')]) \to 0 \quad \text{in } L^2(\Omega;\mathbb{R}^2),$$

$$h^{-1}(\tilde{y}^{(h)}(x) - [hx_3 + hv(x')]) \to 0 \quad L^2(\Omega).$$

The right hand of (99) side by itself, however, does not lead an almost optimal approximation of the energy since it misses an extension or contraction of the vertical fibers in case of non-zero Poisson's ratio; see (123) below for an ansatz which includes this phenomenon and leads to an almost optimal approximation.

*Proof of the lemma. Step 1* (normalization) Estimates (88) and (89) follow immediately from Theorem 10 and Remark 11 since one can choose $\bar{R}^{(h)}$ so that (66) holds with $\bar{Q} = Id$. This implies that the average deformation gradient $\bar{F}^{(h)} = |\Omega|^{-1} \int_\Omega \nabla_h \tilde{y}^{(h)}$ satisfies $|\bar{F}^{(h)} - Id| \le Ch^{-1}\sqrt{E^h}$ and by applying an additional constant in-plane rotation to of order $h^{-1}\sqrt{E^h}$ to $y^{(h)}$ and $R^{(h)}$ we may assume that in addition to (88) and (89) we have

$$\int_\Omega (y^{(h)}_{1,2} - y^{(h)}_{2,1})\, dx = 0. \tag{100}$$

By choosing $c^{(h)}$ suitably we may also assume that

$$\int_\Omega y^{(h)} - \begin{pmatrix} x' \\ hx_3 \end{pmatrix} dx = 0. \tag{101}$$

*Step 2* (convergence of $A^{(h)} := (h/\sqrt{E^h})(R^{(h)} - Id)$). ¿From (89) we get for a subsequence

$$A^{(h)} \rightharpoonup A \quad \text{in } W^{1,2}(S; \mathbb{R}^{3\times3}).$$

Using the compact Sobolev embedding we deduce (92). Together with (88) we obtain (93). Since $R^{(h)}$ is independent of $x_3$ we also obtain (94).

*Step 3* (convergence of $(h^2/E^h)\,\mathrm{sym}(R^{(h)} - Id)$). Since $(R^{(h)})^T R^{(h)} = Id$ we have $A^{(h)} + (A^{(h)})^T = -(h/\sqrt{E^h})(A^{(h)})^T A^{(h)}$. Hence $A + A^T = 0$ and after multiplication by $h/\sqrt{E^h}$ we obtain (96) from the strong convergence of $A^{(h)}$.

*Step 4* (convergence of the scaled normal and tangential deviations). The convergence (90) of the scaled normal component immediately follows from (93). Moreover $v_{,i} = A_{3i}$ for $i = 1, 2$. Hence $v \in W^{2,2}$ as $A \in W^{1,2}$. ¿From (88) and (96) we see that $\mathrm{sym}\, \nabla' u$ is bounded in $L^2$. Using Korn's inequality (see Proposition 12) and the normalizations (100) and (101) we obtain (91).

*Step 5* (identification of $A$). By Steps 3 and 4 the matrix $A$ is skew-symmetric, $A_{31} = v_{,1}$ and $A_{32} = v_{,2}$. It only remains to identify $A_{12}$. Now (91) and (93) imply that (along the subsequence considered) $A_{12} = \lim_{h\to0} \max(\sqrt{E^h}/h, h) u^{(h)}_{1,2}$ and in view of the assumption (86) we see that $A_{12} = 0$. Thus (95) holds and the proof is finished. □

*Proof of the corollary.* Let

$$Y^{(h)} = \tilde{y}^{(h)} - \begin{pmatrix} x' \\ hx_3 \end{pmatrix}, \quad \bar{Y}^{(h)} = \int_I Y^{(h)}\, dx_3 = \begin{pmatrix} U^{(h)} \\ V^{(h)} \end{pmatrix}, \quad Z^{(h)} = Y^{(h)} - \bar{Y}^{(h)}. \tag{102}$$

Then

$$\frac{1}{h} Z^{(h)}_{,3} = \frac{1}{h} y^{(h)}_{,3} - e_3 = (\nabla_h y^{(h)} - R^{(h)}) e_3 + (R^{(h)} - Id) e_3,$$

and thus

$$||h^{-1} Z^{(h)}_{,3} - (R^{(h)} - Id) e_3||_{L^2(\Omega)} \leq C\sqrt{E^h}.$$

Since $\int_I Z^{(h)} = 0$ and $\int_I x_3 (R^{(h)} - Id) e_3\, dx_3 = 0$ this implies that

$$||h^{-1} Z^{(h)} - x_3 (R^{(h)} - Id) e_3||_{L^2(\Omega)} \leq C\sqrt{E^h}. \tag{103}$$

Now multiply the quantity inside the norm by $hx_3/\sqrt{E^h}$ and integrate in $x_3$. This yields

$$||\frac{1}{\sqrt{E^h}} \zeta^{(h)} - \frac{1}{12} \frac{h}{\sqrt{E^h}} (R^{(h)} - Id) e_3||_{L^2(S)} \leq Ch.$$

Together with (92) and (88) we obtain

$$\frac{1}{\sqrt{E^h}} \zeta^{(h)} \to \frac{1}{12} A e_3 \quad \text{in } L^2(S). \tag{104}$$

On the other hand we have for $\gamma \in \{1, 2\}$

$$\zeta^{(h)}_{,\gamma} = \int_I x_3 \tilde{y}^{(h)}_{,\gamma}\, dx_3 = \int_I x_3 (\nabla_h \tilde{y}^{(h)} - R^{(h)}) e_\gamma\, dx_3,$$

since $R^{(h)}$ is independent of $x_3$. In view of (88) this shows that $(1/\sqrt{E^h})\zeta^{(h)}_{,\gamma}$ is bounded in $L^2(S)$ and therefore the convergence in (104) is also weakly in $W^{1,2}(S)$. The above reasoning also applies for $E^h = h^2$, if one replaces (92) by the $L^2$ compactness of $R^{(h)}$. To obtain this compactness it suffices to note that by (64) and (65) in Theorem 10 there exist $\tilde{R}^{(h)}$ with $||\tilde{R}^{(h)}||_{W^{1,2}} \leq C$ and $||\tilde{R}^{(h)} - R^{(h)}||_{L^2} \leq Ch^2$. □

### 5.2 The limiting strain

We know that $\nabla_h y^{(h)}$ can be well approximated by rotations $R^{(h)}(x')$. Since $W$ is invariant under rotations the energy of $y^{(h)}$ is essentially controlled by the deviation of $(R^{(h)})^T \nabla_h y^{(h)}$ from the identity. In view of (88) the quantities $G^{(h)} := (1/\sqrt{E^h})[(R^{(h)})^T \nabla_h y^{(h)} - Id]$ converge weakly in $L^2$ (for a subsequence) to $G$. The following lemma shows that the relevant part of $G$ (i.e. the symmetric part of the in-plane components) can be identified in terms of $u$ and $v$, the limits of the scaled in-plane and out-of-plane displacements. In particular, we show that the relevant components of $G$ are affine in the thickness variable $x_3$, a fact which is often assumed *a priori*. The representation of $G$ immediately yields the lower bound in the definition of $\Gamma$-convergence (see the corollary immediately following the lemma).

We will later apply the lemma below to the sequences $\tilde{y}^{(h)}, R^{(h)}, u^{(h)}, v^{(h)}$ obtained in Lemma 13 above. In view of future applications we state the result in a slightly more general (and more self-contained) form, assuming only (105), (106) and (107) and not the other conclusions of Lemma 13.

**Lemma 15 (Identification of scaled limiting strain)** *Consider* $y^{(h)} : \Omega \to \mathbb{R}^3$ *and* $R^{(h)} : S \to SO(3)$ *and* $E^h > 0$ *and set*

$$u^{(h)} := \min\left(\frac{h^2}{E^h}, \frac{1}{\sqrt{E^h}}\right) \int_I \begin{pmatrix} y_1^{(h)} \\ y_2^{(h)} \end{pmatrix} (x', x_3) - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dx_3,$$

$$v^{(h)} := \frac{h}{\sqrt{E^h}} \int_I y_3^{(h)} \, dx_3.$$

*Suppose that*

$$\lim_{h \to 0} h^{-2} E^h = 0, \tag{105}$$

$$||\nabla_h y^{(h)} - R^{(h)}||_{L^2(\Omega)} \le C\sqrt{E^h}, \tag{106}$$

$$u^{(h)} \rightharpoonup u \quad \text{in } W^{1,2}(S;\mathbb{R}^2), \quad v^{(h)} \to v \quad \text{in } W^{1,2}(S), \quad v \in W^{2,2}(S). \tag{107}$$

*Then*

$$\frac{h}{\sqrt{E^h}}\left(R^{(h)} - Id\right) \to A = e_3 \otimes \nabla' v - \nabla' v \otimes e_3 \quad \text{in } L^2(S;\mathbb{R}^{3\times 3}). \tag{108}$$

*and*

$$G^{(h)} := \frac{(R^{(h)})^T \nabla_h \tilde{y}^{(h)} - Id}{\sqrt{E^h}} \rightharpoonup G \quad in\ L^2(\Omega; \mathbb{R}^{3\times 3}) \tag{109}$$

*and the* $2 \times 2$ *submatrix* $G''$ *given by* $G''_{\alpha\beta} = G_{\alpha\beta}$ *for* $1 \leq \alpha, \beta \leq 2$ *satisfies*

$$G''(x', x_3) = G_0(x') + x_3 G_1(x') \tag{110}$$

*where*

$$G_1 = -(\nabla')^2 v. \tag{111}$$

*Moreover*

$$\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v = 0, \quad if\ h^{-4} E^h \to \infty, \tag{112}$$

$$\operatorname{sym} G_0 = \tfrac{1}{2}(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v), \quad if\ h^{-4} E^h \to 1, \tag{113}$$

$$\operatorname{sym} G_0 = \tfrac{1}{2}(\nabla' u + (\nabla' u)^T), \quad if\ h^{-4} E^h \to 0. \tag{114}$$

**Corollary 16** *Let* $E^h$, $y^{(h)}$, $R^{(h)}$, $u^{(h)}$, $v^{(h)}$ *be as in the lemma above. Then we have the following lower semicontinuity results.*

*(i) If* $\lim_{h\to 0} h^{-4} E^h = 0$ *or* $\lim_{h \to 0} h^{-4} E^h = \infty$ *then*

$$\liminf_{h\to 0} \frac{1}{E^h} I^h(y^{(h)}) \geq \int_S \frac{1}{24} Q_2((\nabla')^2 v)\, dx'. \tag{115}$$

*(ii) If* $\lim_{h\to 0} h^{-4} E^h = 1$ *then*

$$\begin{aligned} \liminf_{h\to 0} \frac{1}{E^h} I^h(y^{(h)}) &\geq \int_S \frac{1}{2} Q_2(\tfrac{1}{2}[\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v]) \\ &+ \frac{1}{24} \int_S Q_2((\nabla')^2 v)\, dx'. \end{aligned} \tag{116}$$

**Remark 17** *If in the inequalities* (115) *or* (116) *we have equality then we obtain strong convergence of the nonlinear strain*

$$\frac{1}{\sqrt{E^h}} \Big( (\nabla_h y^{(h)})^T \nabla_h y^{(h)\,1/2} - Id \Big) \to \operatorname{sym} G \quad in\ L^2(S; \mathbb{R}^{3\times 3}),$$

*see* (165) *below.*

*Proof of Lemma 15. Step 1.* We first assume (108) (for the special sequence coming from Lemma 13 we know this anyhow) and establish the main assertion, namely the representation formula for $G$.

Using the identity $\mathrm{sym}(Q - Id) = -(Q - Id)^T(Q - Id)$ which holds for all $Q \in SO(3)$ we immediately deduce from (108) that

$$\frac{h^2}{E^h}\,\mathrm{sym}\left(R^{(h)} - Id\right) \to A^2 = -\nabla' v \otimes \nabla' v - |\nabla' v|^2 e_3 \otimes e_3. \tag{117}$$

By assumption $G^{(h)}$ is bounded in $L^2$, thus a subsequence converges weakly.

To show that the limit matrix $G''$ is affine in $x_3$ we consider the difference quotients

$$H^{(h)}(x', x_3) = s^{-1}[G^{(h)}(x', x_3 + s) - G^{(h)}(x', x_3)].$$

Multiply the definition of $G^{(h)}$ by $R^{(h)}$, take the difference quotient and express the difference quotient acting on $y$ by an integral over $y_{,3}$. This yields for $\alpha, \beta \in \{1, 2\}$

$$\left(\frac{h}{\sqrt{E^h}}\,\frac{1}{s}\int_0^s \frac{1}{h}\tilde{y}^{(h)}_{\alpha,3}(x', x_3 + \sigma)d\sigma\right)_{,\beta} = (R^{(h)}H^{(h)})_{\alpha\beta}(x', x_3).$$

In view of (108) and (106) the left hand side converges in $W^{-1,2}(S \times (-1, 1-s))$ to $A_{\alpha 3,\beta}(x') = -v_{,\alpha\beta}(x')$. We have $R^{(h)} \to Id$ boundedly a.e. and $H^{(h)} \rightharpoonup H$ in $L^2$ and we thus obtain $H_{\alpha\beta} = -v_{,\alpha\beta}$. Since $v$ is independent of $x_3$ and since $s > 0$ was arbitrary we conclude that $G''$ is affine in $x_3$ and $G_1$ has the form given in the lemma. In order to prove the formula for $G_0$ it suffices to study

$$G_0^{(h)}(x') = \int_{-\frac{1}{2}}^{\frac{1}{2}} G^{(h)}(x', x_3)dx_3.$$

We have for $\alpha, \beta \in \{1, 2\}$

$$(G^{(h)})_{\alpha\beta}(x') = \frac{(\nabla'\tilde{y}^{(h)} - Id)_{\alpha\beta}}{\sqrt{E^h}} - \frac{(R^{(h)} - Id)_{\alpha\beta}}{\sqrt{E^h}} + \left[(R^{(h)} - Id)^T\frac{\nabla_h\tilde{y}^{(h)} - R^{(h)}}{\sqrt{E^h}}\right]_{\alpha\beta} \tag{118}$$

First suppose $h^{-4}E^h \to 1$, i.e. $E^h \approx h^4$. Using the convergence of $u^{(h)}$, (117), (108) and the hypothesis (106) we see that

$$(\mathrm{sym}\,G_0^{(h)})_{\alpha\beta} \to \left[\mathrm{sym}\,\nabla' u - \frac{A^2}{2}\right]_{\alpha\beta} \quad \text{in } L^1(S) \tag{119}$$

and using again (117) we obtain (113). Similarly we obtain (114).

To derive (112) multiply (118) by $h^2/\sqrt{E^h}$ and use again the weak convergence of $u^{(h)}$ and (117), as well the hypotheses (106) and (108).

*Step 2.* We now prove (108). Since $R^{(h)}$ is independent of $x_3$ we have for $\alpha, \beta \in \{1, 2\}$

$$(R^{(h)} - Id)_{\alpha\beta} = \int_I (R^{(h)} - \nabla_h y^{(h)})_{\alpha\beta}\, dx_3 + \max\left(\sqrt{E^h}, \frac{E^h}{h^2}\right) u^{(h)}_{\alpha,\beta},$$

$$(R^{(h)} - Id)_{3\beta} = \int_I (R^{(h)} - \nabla_h y^{(h)})_{3\beta}\, dx_3 + \frac{\sqrt{E^h}}{h} v^{(h)}_{,\beta}.$$

Thus

$$\frac{h}{\sqrt{E^h}}\left(R^{(h)} - Id\right)_{\alpha\beta} \to 0, \quad \frac{h}{\sqrt{E^h}}\left(R^{(h)} - Id\right)_{3\beta} \to v_{,\beta} \quad \text{in } L^2(S). \quad (120)$$

Using the fact that $R^{(h)}$ takes values in $SO(3)$ we deduce that $||R^{(h)}_{\beta 3}||_{L^2} \leq C\sqrt{E^h}/h$. To get control on $R^{(h)}_{33}$ we use that fact that for $Q \in SO(3)$ we have

$$|1 - Q_{33}| = |\det Q - Q_{33}| \leq C \sum_{\alpha,\beta=1}^{2} |(Q - Id)_{\alpha\beta}| + C(|Q_{13}Q_{31}| + |Q_{23}Q_{32}|).$$

Substituting $Q = R^{(h)}$ and using (120) and the generalized dominated convergence theorem (with $L^1$ convergent majorant rather than constant majorant) we easily deduce that $(h/\sqrt{E^h})(R^{(h)}_{33} - 1) \to 0$ in $L^2$. To control $R^{(h)}_{13}$ we thus the fact that the first and third row of $R^{(h)}$ are orthogonal. This yields

$$|R^{(h)}_{13} + R^{(h)}_{31}| \leq C(|R^{(h)}_{11} - 1| + |R^{(h)}_{33} - 1| + |R^{(h)}_{12}|)$$

and together with (120) and the convergence of $R^{(h)}_{33}$ this gives the desired convergence for $R^{(h)}_{13}$. The same argument applies to $R^{(h)}_{23}$ and this finishes the proof. □

*Proof of Corollary 16.* To show (115) and (116) we use a careful Taylor expansion. Let $\omega : [0, \infty) \to [0, \infty)$ denote a modulus of continuity of $D^2W$ near the identity and consider the good set $\Omega_h := \{x \in \Omega : |G^{(h)}(x)| < h^{-1}\}$. Its characteristic function $\chi_h$ is bounded and satisfies $\chi_h \to 1$ in $L^1(\Omega)$. Thus we have $\chi_h G_h \rightharpoonup G$ in $L^2(\Omega)$. By Taylor expansion

$$\frac{1}{E^h}\chi_h W(Id + \sqrt{E^h}G^{(h)}) \geq \frac{1}{2}Q_3(\chi_h G^{(h)}) - \omega(h^{-1}\sqrt{E^h})|G^{(h)}|^2.$$

Thus using (105)

$$
\begin{aligned}
& \liminf_{h\to 0} \frac{1}{E^h} I^h(y^{(h)}) \\
= \; & \liminf_{h\to 0} \frac{1}{E^h} \int_\Omega W((R^{(h)})^T \nabla_h y^{(h)})\, dx \\
\geq \; & \liminf_{h\to 0} \left[ \frac{1}{2} \int_\Omega Q_3(\chi_h G^{(h)})\, dx + \frac{1}{E^h} \int_\Omega (1-\chi_h) W(\nabla_h y^{(h)})\, dx \right] \\
\geq \; & \frac{1}{2} \int_\Omega Q_3(G)\, dx \geq \frac{1}{2} \int_\Omega Q_2(G'')\, dx. \qquad (121)
\end{aligned}
$$

Here we used the fact that $Q_3$ is a positive semidefinite quadratic form and therefore the functional $v \mapsto \int_\Omega Q_3(v)$ is weakly lower semicontinuous in $L^2$. Now by (110)

$$
\int_{-1/2}^{1/2} Q_2(G'')(x', x_3)\, dx_3 = Q_2(G_0(x')) + \frac{1}{12} Q_2(G_1(x')). \qquad (122)
$$

Together with the representations (111), (113) and (114) this implies (115) and (116) and the proof of Corollary 16 is finished. □

# 6 von Kármán like theories: Γ-convergence

*Proof of Theorem 4.* We now return to the situation were the energy scaling is given by powers of $h$ rather than more general functions. The situation of Theorem 4 corresponds to the choice

$$
E^h = h^{2\alpha-2},
$$

and the borderline case $E^h = h^4$ corresponds to the exponent $\alpha = 3$.

With these choices part i) of Theorem 4 follows immediately from Lemma 13 and Lemma 15.

To prove part ii) of the theorem we consider the cases $\alpha = 3$, $\alpha < 3$ and $\alpha > 3$ separately.

## 6.1 Upper bound, $\alpha = 3$

We assume first that $u$ and $v$ are smooth and we make the ansatz

$$
\hat{y}^{(h)}(x', x_3) = \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2 u \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} v_{,1} \\ v_{,2} \\ 0 \end{pmatrix} + h^3 x_3 d^{(0)} + \frac{h^3}{2} x_3^2 d^{(1)}, \qquad (123)
$$

Then

$$\nabla_h \hat{y}^{(h)} = Id + \left(\begin{array}{c|c} h^2\nabla' u & -h(\nabla' v)^T \\ \hline h\nabla' v & 0 \end{array}\right) - h^2 x_3 \left(\begin{array}{c|c} (\nabla')^2 v & 0 \\ \hline 0 & 0 \end{array}\right)$$
$$+ h^2 d^{(0)} \otimes e_3 + h^2 x_3 d^{(1)} \otimes e_3 + \mathcal{O}(h^3) \qquad (124)$$

Using the identities $(I+A)^T(I+A) = I + 2\,\mathrm{sym}\,A + A^T A$ and $(e_3 \otimes a' - a' \otimes e_3)^T(e_3 \otimes a' - a' \otimes e_3) = a' \otimes a' + |a'|^2 e_3 \otimes e_3$ for $a' \in \mathbb{R}^2$ we obtain for the nonlinear strain

$$(\nabla_h \hat{y}^{(h)})^T \nabla_h \hat{y}^{(h)}$$
$$= Id + 2h^2(\mathrm{sym}\,\nabla' u - x_3(\nabla')^2 v) + h^2(\nabla' v \otimes \nabla' v + |\nabla' v|^2 e_3 \otimes e_3)$$
$$+2h^2\,\mathrm{sym}[(d^{(0)} + x_3 d^{(1)}) \otimes e_3] + \mathcal{O}(h^3). \qquad (125)$$

Taking the square root and using the frame indifference (3) of $W$ and Taylor expansion we get

$$h^{-4}W(\nabla_h \hat{y}^{(h)}) = h^{-4}W([(\nabla_h \hat{y}^{(h)})^T \nabla_h \hat{y}^{(h)}]^{1/2}) \to \frac{1}{2}Q_3(A + x_3 B),$$

where

$$A = \mathrm{sym}\,\nabla' u + \tfrac{1}{2}\nabla' v \otimes \nabla' v + \tfrac{1}{2}|\nabla' v|^2 e_3 \otimes e_3 + \mathrm{sym}\,d^{(0)} \otimes e_3,$$
$$B = -(\nabla')^2 v + \mathrm{sym}\,d^{(1)} \otimes e_3.$$

For a symmetric $2 \times 2$ matrix $A''$ let $c = \mathcal{L}A'' \in \mathbb{R}^3$ denote the vector which realizes the minimum in the definition of $Q_2$, i.e.

$$Q_2(A'') = Q_3(A'' + c \otimes e_3 + e_3 \otimes c).$$

Since $Q_3$ is positive definite on symmetric matrices, $c$ is uniquely determined and the map $\mathcal{L}$ is linear. We now take

$$d^{(0)} = -\tfrac{1}{2}|\nabla' v|^2 e_3 + \mathcal{L}(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v) \qquad (126)$$
$$d^{(1)} = -2\mathcal{L}((\nabla')^2 v). \qquad (127)$$

This finishes the proof of Theorem 4 ii) for $\alpha = 3$ and smooth $u, v$. For general $u$,$v$ it suffices to consider suitable smooth approximations $u^{(h)}$, $v^{(h)}$, $d^{(0,h)}$, $d^{(1,h)}$.

### 6.2 Upper bound, $\alpha > 3$

This is simpler. We take $u = 0$ and

$$\hat{y}^{(h)}(x', x_3) = \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + \begin{pmatrix} 0 \\ h^{\alpha-2} v \end{pmatrix} - h^{\alpha-1} x_3 \begin{pmatrix} v_{,1} \\ v_{,2} \\ 0 \end{pmatrix} + \frac{h^\alpha}{2} x_3^2 d^{(1)}. \tag{128}$$

In this case the term in the nonlinear strain involving $\nabla' v$ becomes of higher order and we obtain $h^{-2+2\alpha} W(\nabla_h \hat{y}^{(h)}) \to Q_3(x_3 B)$, where $B$ is as above, and we conclude easily.

### 6.3 Upper bound, $2 < \alpha < 3$

In analogy with the case $\alpha = 3$ we could make the ansatz

$$\begin{aligned} \hat{y}^{(h)}(x', x_3) \quad &= \quad \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^{2(\alpha-2)} u \\ h^{\alpha-2} v \end{pmatrix} - h^{\alpha-1} x_3 \begin{pmatrix} v_{,1} \\ v_{,2} \\ 0 \end{pmatrix} \\ &+ \quad \frac{h^{2\alpha-3}}{2} x_3 d^{(0)} + \frac{h^\alpha}{2} x_3^2 d^{(1)}. \end{aligned} \tag{129}$$

Proceeding as above we obtain the desired conclusion at least for $\alpha > 5/2$. This ansatz can, however, not work for $\alpha$ close to 2. Indeed we obtain for the strain in the midplane

$$[(\nabla_h \hat{y}^{(h)})^T \nabla_h \hat{y}^{(h)}]_{ij}(x', 0) = h^{4(\alpha-2)} (\nabla u)^T \nabla u \quad \text{for } i, j \in \{1, 2\}$$

and this leads to an energy contribution of order $h^{8(\alpha-2)}$ which is larger than the desired estimate $h^{2\alpha-2}$ if $\alpha < 7/3$.

Thus instead of the ansatz (129) which only leads to an approximate isometry of the midplane we will first construct an exact isometry

$$\bar{y}_\varepsilon : S \to \mathbb{R}^3, \quad \bar{y}_\varepsilon(x') = \begin{pmatrix} x' + \varepsilon^2 u_\varepsilon(x') \\ \varepsilon v(x') \end{pmatrix}. \tag{130}$$

We then consider the normal $\nu_\varepsilon := \bar{y}_{\varepsilon,1} \wedge \bar{y}_{\varepsilon,2}$ and as for the nonlinear bending theory we make the ansatz

$$\hat{y}^{(h)}(x', x_3) = \bar{y}_\varepsilon(x') + \varepsilon h x_3 \nu_\varepsilon(x') + \varepsilon \frac{h^2}{2} x_3^2 d(x'), \quad \text{where } \varepsilon = h^{\alpha-2}. \tag{131}$$

Assume for the moment that $v$ belongs to $W^{2,\infty}$ (then also $u \in W^{2,\infty}(S; \mathbb{R}^2)$ in view of (199) below). Then the existence of $\hat{y}_\varepsilon^{(h)}$ and uniform $W^{2,\infty}$

bounds on $u_\varepsilon$ are established in Section 8 (see Theorem 25 and the explicit expressions (187), (190) and (194) below). Assume in addition that $d$ is Lipschitz. Keeping in mind that $(\nu_\varepsilon)_{,j} \cdot \nu_\varepsilon = 0$ we find

$$\nabla \hat{y}^{(h)} = Q_\varepsilon \left( Id + hx_3 \left[ (\nabla' y_\varepsilon)^T \nabla' \nu_\varepsilon + Q_\varepsilon^T \varepsilon d \otimes e_3 \right] \right) + \mathcal{O}(h^2 \varepsilon), \qquad (132)$$

where

$$Q_\varepsilon(x') = (\nabla' \bar{y}_\varepsilon, \nu_\varepsilon) \in SO(3). \qquad (133)$$

Now

$$\nabla' \nu_\varepsilon = -\varepsilon \left( \begin{array}{c|c} (\nabla')^2 v & 0 \\ \hline 0 & 0 \end{array} \right) + \mathcal{O}(\varepsilon^2), \quad Q_\varepsilon = Id + \mathcal{O}(\varepsilon) \qquad (134)$$

and thus, using frame indifference, we get

$$\begin{aligned} h^{2-2\alpha} W(\nabla \hat{y}^{(h)}) &= \varepsilon^{-2} h^{-2} W(Q_\varepsilon^T \nabla \hat{y}^{(h)}) \\ &\to \frac{1}{2} x_3^2 Q_3 \left( \left( \begin{array}{c|c} -(\nabla')^2 v & 0 \\ \hline 0 & 0 \end{array} \right) + d \otimes e_3 \right) \\ &= \frac{1}{2} x_3^2 Q_2((\nabla')^2 v), \end{aligned}$$

where in the last equality we used the choice

$$d = -2\mathcal{L}((\nabla')^2 v). \qquad (135)$$

This finishes the proof of the upper bound for $2 < \alpha < 3$ for $v \in W^{2,\infty}$. The general case is treated in the following subsection. □

### 6.4 Approximation of $W^{2,2}$ data for $2 < \alpha < 3$

In general we only have $v \in W^{2,2}(S)$ and standard mollification arguments would destroy the crucial constraint $\det(\nabla')^2 v = 0$. Pakzad [61] showed (using earlier work of Kirchheim [43]) that for convex domains $S$ there nonetheless exist approximations $v_k \in C^2$ which satisfy $\det(\nabla')^2 v_k = 0$ and converge to $v$ in $W^{2,2}$. Since the limit functional is continuous with respect to this convergence a standard argument in Γ-convergence shows that it suffices to construct the upper bound for $v \in C^2$ and this we have already achieved.

For general domains $S$ we face two difficulties. First, the construction of the isometry $\bar{y}_\varepsilon$ requires that $|\varepsilon \nabla v| < 1$ (see Theorem 25) but for Lipschitz domains we do not always have a global Lipschitz bound for $v$. Second, we do not have a bound for the term $h \nabla' \nu_\varepsilon$ in the supremum norm and hence

Taylor expansion may not be justified. The second difficulty will be handled by a truncation argument for Sobolev functions as in [31].

To overcome the first difficulty we use Theorem 37 and Proposition 30. Thus for each admissible pair $(u, v)$ there exist $v_k \in W^{2,2} \cap W^{1,\infty}$ such that $\det(\nabla')^2 v_k = 0$ and $v_k \to v$ in $W^{2,2}$ and an admissible pair $(u_k, v_k)$ with $u_k \to u$ in $W^{2,q}$ for all $q < 2$ (cf. (199)). In particular $I^{vK}(u_k, v_k) \to I^{vK}(u, v)$. Thus by a standard density and diagonalization argument in $\Gamma$-convergence we may suppose that $v \in W^{1,\infty}(S)$.

Applying Theorem 25 to $V = \varepsilon v_\varepsilon$ we find a $W^{2,2}$ map $\Phi_\varepsilon : S \to \mathbb{R}^2$ such that

$$\bar{y}_\varepsilon = \begin{pmatrix} \Phi_\varepsilon \\ \varepsilon v_\varepsilon \end{pmatrix} \tag{136}$$

is an isometric immersion. Moreover $\Phi_\varepsilon = Id + \varepsilon^2 u_\varepsilon$ and

$$||u_\varepsilon||_{W^{2,2}(S)} \leq C \tag{137}$$

Next we replace $\bar{y}_\varepsilon$ by a $W^{2,\infty}$ map which agrees with $\bar{y}_\varepsilon$ except on a very small set. We use the following truncation result, which is a special case of results by Liu [50] and Ziemer [81].

**Proposition 18 (Approximation by $W^{k,\infty}$ maps)** *Let $S$ be a bounded Lipschitz domain in $\mathbb{R}^n$, let $1 < p < \infty$, $k \in \mathbb{N}$ and $\lambda > 0$. Suppose that $u \in W^{k,p}$ and let*

$$|u|_k(x) := \sum_{|\alpha| \leq k} |\nabla^\alpha u|(x).$$

*Then there exists $u^\lambda \in W^{k,\infty}$ such that*

$$\begin{aligned} ||u^\lambda||_{W^{k,\infty}} &\leq C(p,k,S)\,\lambda, \\ |\{x \in S : u^\lambda(x) \neq u(x)\}| &\leq \frac{C(p,k)}{\lambda^p} \int_{|u|_k \geq \lambda/2} |u|_k^p \qquad (138) \\ ||u^\lambda||_{W^{k,p}} &\leq C(p,k,S) ||u||_{W^{k,p}} \qquad (139) \end{aligned}$$

*In particular,*

$$\lim_{\lambda \to \infty} \lambda^p |\{x \in S : u^\lambda(x) \neq u(x)\}| = 0 \tag{140}$$

*and*

$$\lim_{\lambda \to \infty} ||u^\lambda - u||_{W^{k,p}} = 0. \tag{141}$$

**Remark 19** *One can also include boundary conditions as follows (see [31], Proposition A.2 for the details). Let* $\Gamma$ *be a closed subset of* $\partial S$ *which satisfies* $\mathcal{H}^{n-1}(B(x,r)\cap\Gamma)\geq cr^{n-1}$ *for all* $x\in\Gamma$ *and all* $r\in(0,r_0)$*. If* $u\in W^{2,p}(S)$ *and* $u=\nabla u=0$ *on* $\Gamma$ *(in the sense of trace) then the approximation* $u^\lambda$ *can be chosen such that* $u^\lambda=\nabla u^\lambda=0$ *on* $\Gamma$*.*

We now make the specific choice

$$\varepsilon = h^{\alpha-2}, \quad \lambda = \frac{\varepsilon}{h} = h^{\alpha-3}. \tag{142}$$

We apply Proposition 18 to each component of $Y_\varepsilon = \bar{y}_\varepsilon - id$ and set $\bar{y}_\varepsilon^\lambda = Y_\varepsilon^\lambda + id$ and $\nu_{\varepsilon,\lambda} = \bar{y}_{\varepsilon,1}^\lambda \wedge \bar{y}_{\varepsilon,2}^\lambda$. As before we set $d = -2\mathcal{L}((\nabla')^2 v)$ and we choose Lipschitz approximations $d_h$ satisfying

$$d_h \to d \text{ in } L^2, \quad hd_h \to 0 \text{ in } W^{2,\infty}. \tag{143}$$

Finally we denote the set of bad points by

$$E^\lambda = \left\{x\in S : \bar{y}_\varepsilon^\lambda \neq \bar{y}_\varepsilon\right\} = \bigcup_{i=1}^{3}\left\{x\in S : (Y_\varepsilon^\lambda)_i \neq (Y_\varepsilon)_i\right\}. \tag{144}$$

As before we consider the ansatz

$$\hat{y}^{(h)}(x', x_3) = \bar{y}_\varepsilon^\lambda(x') + hx_3\nu_{\varepsilon,\lambda}(x') + \varepsilon h^2 \frac{x_3^2}{2} d_h(x'), \tag{145}$$

We have

$$\nabla_h \hat{y}^{(h)} = Q^{(h)} + \varepsilon h x_3 a^{(h)} + \varepsilon h x_3 b^{(h)},$$

where

$$Q^{(h)} = (\nabla'\bar{y}_\varepsilon^\lambda, \nu_\varepsilon^\lambda), \quad a^{(h)} = (\varepsilon^{-1}\nabla'\nu_\varepsilon^\lambda, 0), \quad |b^{(h)} - d_h\otimes e_3| \leq h|\nabla' d_h|.$$

We claim that

$$Q^{(h)} \to Id \quad \text{uniformly}, \tag{146}$$

$$a^{(h)} \to \left(\begin{array}{c|c} (\nabla')^2 v & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{in } L^2, \quad \varepsilon h a^{(h)} \to 0 \quad \text{in } L^\infty, \tag{147}$$

$$b^{(h)} \to d\otimes e_3 \quad \text{in } L^2, \quad \varepsilon h b^{(h)} \to 0 \quad \text{in } L^\infty. \tag{148}$$

In $S\backslash E^\lambda$ we have $Q^{(h)} \in SO(3)$ and thus $W(\nabla_h \hat{y}^{(h)}) = W(Q^{(h)^T}\nabla_h \hat{y}^{(h)})$. One can then use (146)–(148) in connection with the dominated convergence theorem to conclude that (cf. Proposition 20 below)

$$\begin{aligned}
&\limsup_{h\to 0} \frac{1}{h^{2(\alpha-2)}} \int_{(S\backslash E^\lambda)\times I} W(\nabla_h \hat{y}^{(h)}) \\
&\leq \limsup_{h\to 0} \int_{(S\backslash E^\lambda)\times I} \frac{1}{(\varepsilon h)^2} W(Q^{(h)^T}\nabla_h \hat{y}^{(h)}) \\
&= \int_\Omega \frac{1}{2} Q_3 \left( x_3 \left( \begin{array}{c|c} (\nabla')^2 v & 0 \\ \hline 0 & 0 \end{array} \right) + x_3 d \otimes e_3 \right) = \frac{1}{24} \int_S Q_2((\nabla')^2 v). \qquad (149)
\end{aligned}$$

We now first prove (146)–(148) and then bound the energy contribution on $E^\lambda$. We first derive $L^\infty$ bounds for $\nabla' u_\varepsilon^\lambda$ and $v^\lambda$. By the definition of $\bar{y}_\varepsilon^\lambda$ we have

$$||\varepsilon^2 u_\varepsilon^\lambda||_{W^{2,\infty}} \leq C\lambda, \quad ||\varepsilon v^\lambda||_{W^{2,\infty}} \leq C\lambda. \qquad (150)$$

Proposition 18 yields

$$||u_\varepsilon^\lambda||_{W^{2,2}} \leq ||u_\varepsilon||_{W^{2,2}} \leq C, \quad ||v^\lambda||_{W^{2,2}} \leq ||v||_{W^{2,2}} \leq C.$$

Hence by the Brezis-Wainger inequality

$$||\nabla' u_\varepsilon^\lambda||_{L^\infty} \leq C \ln \frac{\lambda}{\varepsilon^2} \leq C \ln \frac{1}{h^{\alpha-1}} \leq C\varepsilon^{-1/2}$$

and similarly

$$||\nabla' v^\lambda||_{L^\infty} \leq C\varepsilon^{-1/2}.$$

The normal $\nu_\varepsilon^\lambda$ can be expanded as

$$\nu_\varepsilon^\lambda = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon \begin{pmatrix} \nabla' v^\lambda \\ 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 \\ \mathrm{div} u_\varepsilon^\lambda \end{pmatrix} + \varepsilon^3 B_1(\nabla' v^\lambda, \nabla' u_\varepsilon^\lambda) + \varepsilon^4 B_2(\nabla' u_\varepsilon^\lambda, \nabla' u_\varepsilon^\lambda),$$

where $B_1$ and $B_2$ are bilinear forms whose precise expression does not matter. Taking into account the $L^\infty$ bounds for $\nabla' u_\varepsilon^\lambda$ and $\nabla' v^\lambda$ we see that $|Q^{(h)} - Id| \leq C\varepsilon^{1/2}$, which proves (146). Differentiating the expression for $\nu_\varepsilon^\lambda$ we see using (150) that

$$||\nabla' \nu_\varepsilon^\lambda||_{L^\infty} \leq C(\varepsilon \frac{\lambda}{\varepsilon} + \varepsilon^2 \frac{\lambda}{\varepsilon^2} + \varepsilon^3 \varepsilon^{-1/2} \frac{\lambda}{\varepsilon^2}) \leq C\lambda. \qquad (151)$$

and

$$\left\| \nabla' \nu_\varepsilon^\lambda - \varepsilon \left( \begin{array}{c|c} (\nabla')^2 v^\lambda & 0 \\ \hline 0 & 0 \end{array} \right) \right\|_{L^2} \leq C(\varepsilon^2 + \varepsilon^3 \varepsilon^{-1/2} + \varepsilon^4 \varepsilon^{-1/2}) \leq C\varepsilon^2.$$

Since $h\lambda = \varepsilon$ goes to zero as $h$ goes to zero this implies (147). Finally (148) follows immediately from the properties of $d_h$. This finishes the proof of (149).

It only remains to estimate the contribution from $E^\lambda$. We claim that

$$\frac{1}{h^2}|E^\lambda| = \frac{\lambda^2}{\varepsilon^2}|E^\lambda| \to 0, \quad \text{as } h \to 0, \tag{152}$$

$$|\operatorname{dist}(Q^{(h)}, SO(3))| \leq C\lambda |E^\lambda|^{1/2} \leq C\varepsilon. \tag{153}$$

To prove the first inequality we use (138) for each component of $u_\varepsilon$ and for $v$ separately. With the notation $|v|_2 = |v| + |\nabla' v| + |(\nabla')^2 v|$ this yields

$$\begin{aligned} \frac{\lambda^2}{\varepsilon^2}|E^\lambda| &\leq C\frac{1}{\varepsilon^2}\int_{|\varepsilon v|_2 \geq \lambda/2} |\varepsilon v|_2^2 \, dx' + \frac{1}{\varepsilon^2}\int_S |\varepsilon^2 u_\varepsilon|_2^2 \, dx' \\ &\leq C\int_{|v|_2 \geq 1/(2h)} |v|_2^2 \, dx' + C\varepsilon^2, \end{aligned}$$

and (152) follows. To prove (153) we first recall that $|(\nabla')^2 \bar{y}_\varepsilon^\lambda| \leq C\lambda$. Together with (151) this yields $|\nabla' \operatorname{dist}(\nabla_h \hat{y}^{(h)}, SO(3))| \leq C\lambda$. Moreover $\operatorname{dist}(\nabla_h \hat{y}^{(h)}, SO(3)) = 0$ on $(S \setminus E^\lambda) \times I$. Now every point in $\Omega$ has distance at most $C|E^\lambda|^{1/2}$ from the set $(S \setminus E^\lambda) \times I$ and this yields (153).

Using (153) and the $L^\infty$ bounds for $h\varepsilon a^{(h)}$ and $h\varepsilon b^{(h)}$ and the behaviour of $W$ in a neighbourhood of $SO(3)$ we see that

$$W(\nabla_h \hat{y}^{(h)}) \leq C\varepsilon^2 + C(\varepsilon h)^2(|a^{(h)}|^2 + |b^{(h)}|^2).$$

Together with the $L^2$ convergence of $a^{(h)}$ and $b^{(h)}$ and the fact that $|E^\lambda| \to 0$ we obtain from (152)

$$\limsup_{h \to 0} \frac{1}{(h\varepsilon)^2}\int_{E^\lambda \times I} W(\nabla_h \hat{y}^{(h)}) \, dx \leq \limsup_{h \to 0} \frac{1}{h^2}|E^\lambda| = 0.$$

Combining this with (149) we obtain the desired upper bound. □

In the estimate (149) above we have made use of the following version of the dominated convergence theorem.

**Proposition 20** *Suppose that for $\delta \to 0$ we have*

$$G^\delta \to G \quad in\, L^2(\Omega), \qquad \delta G^\delta \to 0 \quad in\, L^\infty. \tag{154}$$

*Then*

$$\delta^{-2} W((Id + \delta G^\delta) \to \frac{1}{2} Q_3(G) \quad in\, L^1(\Omega). \tag{155}$$

*Proof.* For a subsequence we have $G^\delta \to G$ a.e. Hence, for this subsequence,

$$\delta^{-2} W((Id + \delta G^\delta) \to \frac{1}{2} Q_3(G) \quad \text{a.e.} \tag{156}$$

In view of the $L^\infty$ convergence we also have

$$\delta^{-2} W((Id + \delta G^\delta)) \leq C \delta^{-2} |\delta G^\delta|^2 \leq |G^\delta|^2. \tag{157}$$

Since the right hand side converges in $L^1(\Omega)$ the generalized dominated convergence theorem implies that (155) holds along the subsequence considered. Since the limit is unique we have convergence of the full sequence. □

# 7 von Kármán like theories: convergence of minimizers

The convergence of minimizers follows from the Γ-convergence result and a Poincaré like inequality related to the rigidity estimates.

## 7.1 A priori estimates and application of Γ-convergence

*Proof of Theorem 2.* By (66) there exist $\bar{R}^{(h)} \in SO(3)$ and $c^{(h)} \in \mathbb{R}^3$ such that

$$Y^{(h)}(x) := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)} - \begin{pmatrix} x' \\ hx_3 \end{pmatrix} \tag{158}$$

satisfies

$$||Y^{(h)}||^2_{L^2(\Omega)} + ||\nabla_h Y^{(h)}||^2_{L^2(\Omega)} \leq C h^{-2} I^{(h)}(y^{(h)}). \tag{159}$$

Using the test function $x \mapsto (x', hx_3)$ and the conditions (16) and (17) on the total force and total moment of $f^{(h)}$ we obtain the trivial bound

$$\inf J^{(h)} \leq 0. \tag{160}$$

Using once more the conditions on $f^{(h)}$ and the fact the $y^{(h)}$ is a $\beta$-minimizing sequence (see (18) we deduce that

$$\begin{aligned} I^{(h)}(y^{(h)}) &= J^{(h)}(y^{(h)}) + \int_\Omega (R^{(h)})^T f^{(h)} \cdot (R^{(h)})^T y^{(h)}\, dx \\ &= J^{(h)}(y^{(h)}) + \int_\Omega (R^{(h)})^T f^{(h)} \cdot Y^{(h)}\, dx \\ &\leq Ch^\beta + Ch^{\alpha-1}\left(I^{(h)}(y^{(h)})\right)^{1/2}. \end{aligned}$$

Since $\beta = 2\alpha - 2$ this immediately yields

$$I^{(h)}(y^{(h)}) \leq Ch^\beta. \tag{161}$$

Now all the assertions of Theorem 2 follow from Theorem 4 and Lemma 13 except for the strong convergence of $u^{(h)}$ in (32). This will be addressed in the following subsection.

## 7.2 Strong convergence of the in-plane components

To establish strong convergence of the in-plane components we first show that

$$h^{1-\alpha}\,\mathrm{dist}(\nabla_h \tilde{y}^{(h)}, SO(3)) \to |\,\mathrm{sym}\, G| \quad \text{in } L^2(\Omega). \tag{162}$$

Indeed, since $y^{(h)}$ is a $\beta$-minimizing sequence in the sense of (18) we must have equality in (121) and all the limits inferior can be replaced by limits. Thus

$$\lim_{h\to 0} \int_\Omega Q_3(\chi_h G^{(h)})\, dx = \int Q_3(G)\, dx,$$

$$\lim_{h\to 0} h^{2-2\alpha} \int_\Omega (1-\chi_h) W(\nabla_h \tilde{y}^{(h)})\, dx = 0, \tag{163}$$

where $\chi_h$ is the characteristic function of the set $\{|G^{(h)}| < h^{-1}\}$. Since $Q_3$ is positive definite on symmetric matrices the first identity yields

$$\chi_h\, \mathrm{sym}\, G^{(h)} \to \mathrm{sym}\, G \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}). \tag{164}$$

Thus by the definition (109) of $G^{(h)}$ (recall that $E^h = h^{2\alpha-2}$)

$$\begin{aligned} &h^{1-\alpha}\chi_h\, \mathrm{dist}(\nabla_h \tilde{y}^{(h)}, SO(3)) = h^{1-\alpha}\chi_h\, \mathrm{dist}(Id + h^{\alpha-1}G^{(h)}, SO(3)) \\ &= \chi_h |\,\mathrm{sym}\, G^{(h)}| + \chi_h \mathcal{O}(h^{\alpha-1}|G^{(h)}|^2) \to |\,\mathrm{sym}\, G| \quad \text{in } L^2(\Omega), \end{aligned}$$

since $\sup \chi_h h^{\alpha-1}|G^{(h)}| \le h^{\alpha-2} \to 0$. Together with (163) and the coercivity condition (5) on $W$ we deduce (162). We remark in passing that using the pointwise estimate $|(F^T F)^{1/2} - Id)| \le C \operatorname{dist}(F, SO(3))$ and (164) we can deduce in the same way the convergence of the nonlinear strain, i.e.,

$$h^{1-\alpha} \left( [(\nabla_h \tilde{y}^{(h)})^T \nabla_h \tilde{y}^{(h)}]^{1/2} - Id \right) \to \operatorname{sym} G \quad \text{in } L^2(S; \mathbb{R}^{3\times 3}). \qquad (165)$$

From (162) we deduce in particular that $h^{2-2\alpha} \operatorname{dist}^2(\nabla_h \tilde{y}^{(h)}, SO(3))$ is equiintegrable. Using a refined version of Theorems 9 and 10 (see Propositions 21 and 22 below) we deduce that

$$|G^{(h)}|^2 = h^{2-2\alpha} |\nabla_h \tilde{y}^{(h)} - R^{(h)}|^2 \quad \text{is equiintegrable.} \qquad (166)$$

In connection with (164) this implies that

$$\operatorname{sym} G^{(h)} \to \operatorname{sym} G \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}).$$

Since $R^{(h)} \to Id$ in $L^2$ and $|R^{(h)}| = \sqrt{n}$ we also deduce (using e.g. Egoroff's theorem) from (166) that

$$(R^{(h)} - Id) G^{(h)} \to 0 \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}).$$

Thus by the definition (109) of $G^{(h)}$ and the convergence of $\operatorname{sym} G^{(h)}$

$$h^{1-\alpha} \operatorname{sym}(\nabla_h \tilde{y}^{(h)} - R^{(h)}) = h^{1-\alpha} \operatorname{sym}(R^{(h)} G^{(h)}) \to \operatorname{sym} G \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}). \qquad (167)$$

Now by (96)

$$h^{4-2\alpha} \operatorname{sym}(R^{(h)} - Id) \to \frac{A^2}{2} \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}).$$

Hence $h^{4-2\alpha} \operatorname{sym}(\nabla_h y^{(h)} - Id)$ converges in $L^2$ if $\alpha \le 3$, while for $\alpha > 3$ the expression $h^{1-\alpha} \operatorname{sym}(\nabla_h y^{(h)} - Id)$ converges in $L^2$. Recalling the definition of $u^{(h)}$ (see (91) and Table 1) we see that $\operatorname{sym} \nabla' u^{(h)}$ converges strongly in $L^2(S; \mathbb{R}^{2\times 2})$, for all $\alpha > 2$. Finally Korn's inequality and weak convergence of $\nabla' u^{(h)}$ in $L^2$ imply strong convergence of $\nabla' u^{(h)}$. This finishes the proof of Theorem 2 □

In the proof of strong convergence we used the following equiintegrable version of the rigidity estimate in thin sets.

**Proposition 21** *Suppose that* $\alpha > 2$ *and*

$$\operatorname{dist}(\nabla_h y^{(h)}, SO(3)) \le h^{\alpha-1}(M + f_2)$$

*with* $M \in \mathbb{R}$, $M \ge 0$ *and* $f_2 \in L^2(\Omega)$. *Let* $R^{(h)}$ *be the map constructed in the proof of Theorem 10. Then*

$$|\nabla_h y^{(h)} - R^{(h)}| \le h^{\alpha-1}(G_1 + G_2) \tag{168}$$

*with*

$$||G_1||_{L^p(\Omega)} \le CM, \quad \textit{for some } p > 2, \quad ||G_2||_{L^2(\Omega)} \le C||f_2||_{L^2(\Omega)}. \tag{169}$$

*In particular, if*

$$h^{2-2\alpha} \operatorname{dist}^2(\nabla_h y^{(h)}, SO(3)) \quad \textit{is equiintegrable}$$

*then*

$$h^{2-2\alpha} |\nabla_h y^{(h)} - R^{(h)}|^2 \quad \textit{is equiintegrable.}$$

To prove this we use a refined version of Theorem 9 which can be proved in essentially the same way as the original result, see [34].

**Proposition 22** *Let* $U$ *be a bounded Lipschitz domain in* $\mathbb{R}^n$, $n \ge 2$, *and let* $1 < p_1 < p_2 < \infty$. *Then there exist constants* $C(p_1, p_2, U)$ *with the following properties. If* $v \in W^{1,1}(U; \mathbb{R}^n)$ *and*

$$\operatorname{dist}(\nabla v, SO(n)) \le f_1 + f_2, \quad \textit{with } f_i \in L^{p_i}(U) \tag{170}$$

*then there exist* $g_i \in L^{p_i}(U)$ *and* $R \in SO(n)$ *such that*

$$|\nabla v - R| \le g_1 + g_2, \quad ||g_i||_{L^{p_i}(U)} \le C(p_1, p_2, U)||f_i||_{L^{p_i}(U)}. \tag{171}$$

*The constants* $C(p_1, p_2, U)$ *are invariant under dilations of* $U$.

*Proof of Proposition 21.* We only show the interior estimate. The estimates near $\partial S$ are obtained in a similar way by first flattening a sufficiently small piece of $\partial S$ as in the proof of Theorem 10. Thus let $K \subset S$ be compact and suppose that $\operatorname{dist}(K, \partial S) \ge Ch$. Let $\mathcal{L}'$ denote the set of points $x'$ in the lattice $(h\mathbb{Z})^2$ for which the lattice cell

$$S_{x',h} = x' + (0, h)^2$$

intersects $K$, i.e.

$$\mathcal{L}' = \{x' \in (h\mathbb{Z})^2 : S_{x',h} \cap K \neq \emptyset\}.$$

Let $f_1 \equiv M$, $f = M + f_2$, fix $x' \in \mathcal{L}'$ and write $F^{(h)} = \nabla_h y^{(h)}$. By Proposition 22 applied to $f_i \chi_{S_{x',h}\times I}$ there exist $g_{1,x'}$ and $g_{2,x'}$ and a rotation $R_{x',h}$ such that

$$|F^{(h)}(z) - R_{x',h}| \leq h^{\alpha-1}(g_{1,x'}(z) + g_{2,x'}(z)), \quad \text{for } z \in S_{x',h} \times I, \tag{172}$$

$$\int_{S_{x',h}\times I} |g_{1,x'}|^p \, dz \leq CM^p h^2, \quad \int_{S_{x',h}\times I} |g_{2,x'}|^2 \, dz \leq C \int_{S_{x',h}\times I} |f_2|^2 \, dz. \tag{173}$$

Strictly speaking we apply Proposition 22 to $S_{x',h} \times hI$, rescale the functions $f_i$ in $x_3$ accordingly and then unscale again.

Using the definition of $R = R^{(h)}$ in the proof of Theorem 10 and arguing as done there we see that this implies

$$\begin{aligned} |R^{(h)}(x') - R_{x',h}|^2 &\leq Ch^{2\alpha-2} \frac{1}{h^2} \int_{S_{x',h}\times I} |F^{(h)} - R_{x',h}|^2 \, dz \\ &\leq Ch^{2\alpha-2} \left( M^2 + \frac{1}{h^2} \int_{S_{x',h}\times I} |f_2|^2 \, dz \right). \end{aligned}$$

By (74) we have for $\tilde{x}' \in S_{x',h}$

$$\begin{aligned} |R^{(h)}(\tilde{x}') - R^{(h)}(x')|^2 &\leq Ch^{2\alpha-2} \frac{1}{h^2} \int_{S_{x',2h}\times I} |f|^2 \, dz \\ &\leq Ch^{2\alpha-2} \left( M^2 + \frac{1}{h^2} \int_{S_{x',2h}\times I} |f_2|^2 \, dz \right). \end{aligned}$$

Thus

$$\frac{|F^{(h)}(z) - R^{(h)}(z')|^2}{h^{2\alpha-2}} \leq C \left( M^2 + g_{1,x'}^2 + g_{2,x'}^2 + \frac{1}{h^2} \int_{S_{x',2h}\times I} |f_2|^2 \, dz \right), \tag{174}$$

for $z \in S_{x',h} \times I$. Hence

$$|F^{(h)} - R^{(h)}| \leq h^{\alpha-1}(G_1 + G_2) \quad \text{in } K,$$

where

$$G_1 = C\Big(M + \sum_{x' \in \mathcal{L}'} g_{1,x'} \chi_{S_{x',h}}\Big),$$

$$G_2 = C\Big(\sum_{x' \in \mathcal{L}'} g_{2,x'} \chi_{S_{x',h}} + \sum_{x' \in \mathcal{L}'} \Big(\frac{1}{h^2} \int_{Sx',2h \times I} |f_2|^2 \, dz\Big)^{1/2} \chi_{S_{x',h}}\Big).$$

From this one easily deduces (169). The assertion about equiintegrability is an immediate consequence. □

# 8 Construction of isometries from infinitesimal isometries

## 8.1 Construction of isometries

In this section we always deal with maps or functions defined on a bounded Lipschitz domain $S \subset \mathbb{R}^2$. To simplify the notation we write $\nabla$ instead of $\nabla'$ for the two-dimensional gradient. Given a map $V \in W^{2,2}(S)$ we seek to construct an isometric immersion

$$y : S \to R^3, \quad \text{of the form } y = \begin{pmatrix} \Phi \\ V \end{pmatrix}.$$

We thus need to solve the equation

$$(\nabla y)^T \nabla y = (\nabla \Phi)^T \nabla \Phi + \nabla V \otimes \nabla V = Id.$$

The main result of this section is that (for simply connected domains) the condition $\det \nabla^2 V = 0$ is necessary and sufficient for this, see Theorem 25 below. The same condition is sufficient and necessary to obtain a linearized isometric immersion, i.e. a solution of

$$\nabla W + (\nabla W)^T + \nabla V \otimes \nabla V = 0, \tag{175}$$

where $W : S \to \mathbb{R}^2$, see Proposition 30 below.

To put these results in perspective we first review some general properties of isometric immersions for the convenience of the reader. These properties are classical for smooth maps, but we will need them for $W^{2,2}$ maps. For a general $W^{2,2}$ map $y : S \to \mathbb{R}^3$ we define the induced metric by $g_{ij} = y_{,i} \cdot y_{,j}$ and we set $n = y_{,1} \wedge y_{,2}$ and

$$A_{ij} = -y_{,ij} \cdot n. \tag{176}$$

If $y$ is an isometric immersion, i.e. if $g_{ij} = \delta_{ij}$, then $n$ is a unit normal to the image of $y$ and $A$ is the second fundamental form.

**Proposition 23** *Suppose that $S \in \mathbb{R}^2$ is a bounded Lipschitz domain and $y \in W^{2,2}(S;\mathbb{R}^3)$ is an isometric immersion. Then*

$$y_{,ij} = A_{ij} n, \tag{177}$$

$$A_{i1,2} = A_{i2,1}, \quad \textit{for} \ \ i = 1, 2, \tag{178}$$

*in the sense of distributions. Moreover*

$$\det A = 0. \tag{179}$$

*Proof.* Since $g_{ij} = \delta_{ij}$ we have $|n| = 1$. Differentiation of $g_{ij}$ yields $y_{,ij} \cdot y_{,k} = 0$. Thus $y_{,ij}$ is parallel to $n$ and this proves (177). To establish (178) first note that for smooth $y$ we have the identity

$$A_{i1,2} - A_{i2,1} = y_{,i1} \cdot n_{,2} - y_{,i2} \cdot n_{,1} \tag{180}$$

By approximation this identity holds in the sense of distributions if $y \in W^{2,2}$. By (177) the vector $y_{,ij}$ is parallel to $n$ (a.e.), but $n_{,k}$ is perpendicular to $n$, since $|n| = 1$. This proves (178).

Finally to establish (179) we start from the identity

$$g_{11,22} + g_{22,11} - 2g_{12,12} = 2y_{,12} \cdot y_{,12} - 2y_{,11} \cdot y_{,22}. \tag{181}$$

This holds pointwise for smooth $y$ and by approximation it holds in the sense of distribution for $y \in W^{2,2}$. For an isometric immersion the left hand side vanishes and together with (177) this proves (179). □

**Remark 24** *If $y$ is smooth then one can deduce from* (179) *that locally $\nabla y$ is either a constant or is constant on a smooth curve. In the latter case one can further conclude that the smooth curve is a line defined by the kernel of $A$. It turns out that the latter conclusion is still true for $y \in W^{2,2}$ (see Theorem 35 below). The proof, however, requires a much finer analysis [43, 61]) (for even weaker conditions see Pogorelov's work [67, 68]). The following arguments do not require this geometric property, except for the fine regularity estimates in Subsection 8.3.*

Now we come the the announced result on the construction of isometric immersions from linearized isometric immersions.

**Theorem 25** *Let $S \in \mathbb{R}^2$ be a bounded, simply connected domain with Lipschitz boundary. Suppose that $V \in W^{2,2}(S)$ and $||\nabla V||_{L^\infty} < 1$. Then there exists $\Phi \in W^{1,2}(S)$ with $\det \Phi > 0$ and*

$$(\nabla \Phi)^T \nabla \Phi = Id - \nabla V \otimes \nabla V \tag{182}$$

*if and only of*

$$\det \nabla^2 V = 0. \tag{183}$$

*Moreover* $\Phi$ *is unique up to a rigid motion. If* (183) *holds and* $||\nabla V||_{L^\infty} \le 1/2$ *then* $\Phi$ *can be chosen such that* $U := \Phi - id$ *satisfies*

$$||\nabla^2 U||_{L^2} \le C||\nabla V||_{L^\infty}||\nabla^2 V||_{L^2}, \tag{184}$$

$$||U||_{W^{2,2}} \le C||\nabla V||_{L^\infty}||\nabla^2 V||_{L^2} + C||\nabla V||^2_{L^2}. \tag{185}$$

**Remark 26** *If* $S$ *is not simply connected then the condition* $\det \nabla^2 V = 0$ *is not sufficient. Consider e.g. the annulus* $S = \{x : 1/2 < |x| < 1\}$ *and the map* $V = \varepsilon|x|$*. Let* $r = |x|$, $\Theta = x/r$*. Then, using the notation below in (189), (190) one easily computes that* $\nabla\theta = h_F = (1/r)((\sqrt{1-\varepsilon^2} - 1)/(\sqrt{1-\varepsilon^2}))\Theta^\perp$*. Hence in polar coordinates* $(r, \alpha)$ *we get that* $\theta(r,\alpha) - \theta(r, 0) = ((1/\sqrt{1-\varepsilon^2}) - 1)\alpha$*. We see that* $\theta$ *is well defined only if* $1/\sqrt{1-\varepsilon^2} \in \mathbb{Z}$*.*

We will see in Proposition 31 that for $V \in W^{2,2}$ the condition (183) actually implies that $V \in C^1(S)$. If $S$ is of class $C^{1,\alpha}$ then $\nabla V$ is continuous up to the boundary, see Theorem 33 below.

*Proof of Theorem 25.* Let $g = Id - \nabla V \otimes \nabla V$ and let $F = g^{1/2}$, i.e.

$$F = Id - \hat{\lambda}(|\nabla V|^2)\nabla V \otimes \nabla V, \quad \text{where } \hat{\lambda}(s) = \frac{1 - \sqrt{1-s}}{s}. \tag{186}$$

Then we need to show that there exists a rotation

$$e^{i\theta} := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \text{with } \nabla\Phi = e^{i\theta}F. \tag{187}$$

The following result gives a necessary and sufficient condition for this. We associate to $F$ a vector field

$$h_F := \frac{1}{\det F} F^T \operatorname{curl} F = (\operatorname{cof} F^{-1}) \operatorname{curl} F, \tag{188}$$

where

$$(\operatorname{curl} F)_p = F_{p2,1} - F_{p1,2}.$$

**Proposition 27 ([26], Prop. 3.1)** *Let $S$ be a bounded, simply connected domain with Lipschitz boundary. Suppose that $F \in W^{1,1}(S;\mathbb{R}^{2\times 2})$, $\det F > 0$ and $|F^{-1}| \leq C$. Then $F$ can be written in the form $F = e^{-i\theta}\nabla\Phi$ with $\Phi \in W^{2,1}(S;\mathbb{R}^{2\times 2})$ and $\theta \in W^{1,1}(S;\mathbb{R}^{2\times 2})$ if and only if*

$$\operatorname{curl} h_F = 0 \tag{189}$$

*in the sense of distributions. Moreover in this case*

$$\nabla\theta = h_F. \tag{190}$$

From (190) we easily read off the estimate (184) for $\nabla^2\Phi = \nabla^2 U$. To estimate the lower derivatives of $U$ we use the fact that $\theta$ is only defined up to a constant. Hence we may suppose $\int \theta = 0$ and thus $||\theta||_{L^2} \leq ||\nabla\theta||_{L^2}$. Therefore (187) yields

$$||\nabla U||_{L^2} = ||\nabla\Phi - Id||_{L^2} \leq C||\theta||_{L^2} + C||F - Id||_{L^2} \leq C||\nabla\theta||_{L^2} + C|||\nabla V|^2||_{L^2}. \tag{191}$$

To control the last term we use the estimate

$$||f^2||_{L^2} \leq C||f^2||_{L^1} + C||\nabla f^2||_{L^1} \leq C||f||_{L^2}^2 + C||f||_{L^2}||\nabla f||_{L^2} \tag{192}$$

for $f = |\nabla V|^2$. Together with the previous estimate for $\nabla\theta$ this yields

$$||\nabla U||_{L^2} \leq C||\nabla V||_{L^\infty}||\nabla^2 V||_{L^2} + C||\nabla V||_{L^2}^2. \tag{193}$$

Finally using the freedom to add a constant to $U$ we obtain (185).

To prove Theorem 25 it only remains to show that the condition $\operatorname{curl} F = 0$ is equivalent to $\det \nabla^2 V = 0$. To check this we write

$$a = \nabla V, \quad \lambda = \hat\lambda(|a|^2), \quad \hat\lambda(s) = \frac{1-\sqrt{1-s}}{s}.$$

Then $F = Id - \lambda a \otimes a$ and using $a_{1,2} = a_{2,1}$ we get

$$(\operatorname{curl} F)_p = -\operatorname{curl}(\lambda a_p a) = -a_p \nabla\lambda \wedge a - \lambda\nabla a_p \wedge a,$$

where we used the notation $\alpha \wedge \beta = \alpha_1\beta_2 - \alpha_2\beta_1$. Thus

$$\begin{aligned}
&(F^T \operatorname{curl} F)_j \\
&\quad = -a_j\nabla\lambda \wedge a - \lambda\nabla a_j \wedge a + \lambda a_j|a|^2\nabla\lambda \wedge a + \lambda^2 a_j a_p \nabla a_p \wedge a \\
&\quad = a_j g \wedge a - \lambda\nabla a_j \wedge a,
\end{aligned}$$

where

$$g = -\nabla\lambda + |a|^2\lambda\nabla\lambda + \lambda^2\tfrac{1}{2}\nabla|a|^2 = \nabla(-\lambda + \tfrac{1}{2}\lambda^2|a|^2).$$

A short calculation shows that $g = 0$. Indeed writing $s = |a|^2$ we have

$$\begin{aligned}\lambda^2|a|^2 &= (\hat{\lambda})^2(s)\, s \\ &= s\frac{(1-\sqrt{1-s})^2}{s^2} = \frac{2-s-2\sqrt{1-s}}{s} = 2\lambda - 1.\end{aligned}$$

Since $\det F = 1 - \lambda|a|^2$ and $\nabla a_j = a_{,j}$ we get

$$(h_F)_j = -\frac{\lambda}{1-\lambda|a|^2} a_{,j} \wedge a \tag{194}$$

and the following proposition shows that $\operatorname{curl} h_F = 0$ if and only if $\det \nabla^2 V = 0$. □

**Proposition 28** *Let $S$ be a domain in $\mathbb{R}^2$ and suppose that $V \in W^{2,2}(S)$ and $\phi \in C^1(\mathbb{R})$ with $\phi$ and $\phi'$ bounded. Define $h : S \to \mathbb{R}^2$ by*

$$h_j := \phi(|\nabla V|^2))\, \nabla V_{,j} \wedge \nabla V.$$

*Then*

$$\operatorname{curl} h = \psi(|\nabla V|^2) \det \nabla^2 V, \tag{195}$$

*where*

$$\psi(s) = -4s\phi'(s) - 2\phi(s) = -4\sqrt{s}(\sqrt{s}\phi)'. \tag{196}$$

**Remark 29** *If in addition $|\nabla V|^2 \leq M$ a.e. then it suffices that $\phi$ be $C^1$ on $[0, M]$ since such $\phi$ can be extended to $\mathbb{R}$ in such a way that $\phi$ and $\phi'$ remain globally bounded.*

*Proof.* If suffices to show the result for $V \in C^3$ since the general case follows by approximation. As before let $a = \nabla V$ and note that $\alpha \wedge \beta = -\beta^\perp \cdot \alpha$. Since $a_{,12} = a_{,21}$ we have

$$\begin{aligned}&\operatorname{curl} h \\ &\quad = \Big[\phi(|a|^2)(a^\perp \cdot a_{,1})\Big]_{,2} - \Big[\phi(|a|^2)(a^\perp \cdot a_{,2})\Big]_{,1} \\ &\quad = \phi(|a|^2)(a^\perp_{,2} \cdot a_{,1} - a^\perp_{,1} \cdot a_{,2}) \\ &\quad +2\phi'(|a|^2)\Big[(a \cdot a_{,2})(a^\perp \cdot a_{,1}) - (a \cdot a_{,1})(a^\perp \cdot a_{,2})\Big].\end{aligned}$$

Now

$$a_{,1}\cdot a_{,2}^{\perp} = -a_{,2}\cdot a_{,1}^{\perp} = -\det\nabla a = -\det\nabla^2 V,$$

and using the linearity of the determinant in rows we compute

$$\det\begin{pmatrix} a^{\perp}\cdot a_{,1} & a^{\perp}\cdot a_{,2} \\ a\cdot a_{,1} & a\cdot a_{,2}\end{pmatrix} = \sum_{i,j=1}^{2} a_i^{\perp} a_j \det\begin{pmatrix} e_i\cdot a_{,1} & e_i\cdot a_{,2} \\ e_j\cdot a_{,1} & e_j\cdot a_{,2}\end{pmatrix}$$
$$= (a_1^{\perp}a_2 - a_2^{\perp}a_1)\det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2}\end{pmatrix} = -2|a|^2\det\nabla^2 V.$$

This finishes the proof, once it is noted that $\psi(s)$ has no zeroes on $[0,1)$ for $\phi(s) = -(\lambda/(1-s\lambda)$. □

**Proposition 30** *Suppose that $S$ is a simply connected, bounded Lipschitz domain. Let $V\in W^{2,2}(S)$. Then the equation*

$$\nabla W + (\nabla W)^T + \nabla V\otimes\nabla V = 0 \tag{197}$$

*admits a solution $W\in W^{1,2}(S;\mathbb{R}^2)$ if and only if*

$$\det\nabla^2 V = 0. \tag{198}$$

*If* (198) *holds then $W\in W^{2,2}(S;\mathbb{R}^2)$ and*

$$W_{i,jk} = -V_{,i}V_{,jk}. \tag{199}$$

*In particular* $\det\nabla^2 W_i = 0$*, for $i=1,2$. Moreover $W$ is uniquely determined up to an affine map with skew-symmetric gradient. In particular one can choose $W$ such that*

$$||\nabla^2 W||_{L^2} \le C||\nabla V||_{L^\infty}||\nabla^2 V||_{L^2}, \tag{200}$$

$$||W||_{W^{2,2}} \le C||\nabla V||_{L^\infty}||\nabla^2 V||_{L^2} + C||\nabla V||_{L^2}^2. \tag{201}$$

*Proof.* Let

$$e = -\frac{1}{2}\nabla V\otimes\nabla V. \tag{202}$$

For a smooth $V$ we have

$$e_{11,22} + e_{22,11} - 2e_{12,12} = \det\nabla^2 V, \tag{203}$$

and by approximation this identity holds in the sense of distributions if $V \in W^{2,2}$. Now an $L^2$ map $e : S \to \mathbb{R}^{2\times 2}_{\rm sym}$ is the symmetrized gradient of a $W^{1,2}$ function $W$, i.e.

$$2e = (\nabla W)^T + \nabla W, \tag{204}$$

if and only if

$$e_{11,22} + e_{22,11} - 2e_{12,12} = 0, \tag{205}$$

see e.g. [52], Chapter 1, §17, equation (25) (note that in Love's notation $e_{xx} = e_{11}$, $e_{yy} = e_{22}$, but $e_{xy} = 2e_{12}$, see his equation (24) or [17], p. 372). Hence the condition $\det \nabla^2 V = 0$ is sufficient and necessary for the existence of $W$. Now (204) implies that

$$W_{i,jk} = e_{ij,k} + e_{ik,j} - e_{jk,i} \tag{206}$$

and after a short calculation this yields (199), which in turn implies (200). Using the equation (197) for $W$ and the estimate (192) with $f = |\nabla V|^2$ we see that

$$||\,\mathrm{sym}\,\nabla W||_{L^2} \leq C||\nabla V||^2_{L^2} + C||\nabla V||_{L^2}||\nabla^2 V||_{L^2}. \tag{207}$$

Now (201) follows from Korn's inequality. $\square$

## 8.2 Simple regularity estimates

In general functions in $W^{2,2}(S)$ just fail to be in $C^1$. The situation is, however, better for isometric immersions. We begin with a scalar result.

**Proposition 31** *Suppose that $V \in W^{2,2}(S)$ and $\det \nabla^2 V = 0$. Then $V \in C^1(S)$. If $B_\rho(x) \subset B_R(x) \subset S$ we have more precisely*

$$\mathrm{osc}_{B_\rho} \nabla V \leq C(\ln \frac{R}{\rho})^{-1/2} ||\nabla^2 V||_{L^2(B_R)}, \tag{208}$$

*where* $\mathrm{osc}_{B_\rho} f := \mathrm{diam}\, f(B_\rho)$.

*Proof.* Following Kirchheim [43] we consider the map $f^\delta(x_1, x_2) = (\nabla V)(x_1, x_2) + \delta(-x_2, x_1)$ and compute

$$\det \nabla f^\delta = \det \begin{pmatrix} V_{,11} & V_{,12} - \delta \\ V_{,12} + \delta & V_{,22} \end{pmatrix} = \delta^2 > 0.$$

Since the map $f^\delta$ in addition belongs $W^{1,2}$ it it is monotone and hence continuous by a result of Vodopyanov and Goldstein [79] ( see also [75], [39] and

[28], Theorem 5.14). In fact $f^\delta$ is monotone in the sense of Lebesgue (i.e. $\partial f^\delta(B(x,r)) = f^\delta(\partial B(x,r))$ and in particular $\operatorname{osc}_{B(x,\rho)} f^\delta \le \operatorname{osc}_{B(x,r)} f^\delta \le \operatorname{osc}_{\partial B(x,r)} f^\delta$, for $r \in (\rho, R)$. Now we can apply the Sobolev embedding theorem $W^{1,2}(\partial B(x,r)) \hookrightarrow C^{0,1/2}(\partial B(x,r))$ for a.e. $r \in (\rho, R)$, take squares, divide by $r$ and integrate from $\rho$ to $R$ to obtain the desired logarithmic modulus of continuity for $f$ (see e.g. [55], (4.3.17), p. 110). Since the constants involved are independent of $\delta$ we obtain the assertion for $f$ by taking the limit $\delta \to 0$. □

Now each component of an isometric immersion satisfies $\det \nabla^2 y_i = 0$ (see Proposition 23). Hence we obtain the following corollary.

**Corollary 32** *Let $S$, $V$, $\Phi$ and $U$ be as in Theorem 25. Then $V$, $\Phi$ und $U$ are $C^1$ in $S$. Moreover, for any compactly contained subset $S'$ we have*

$$||\nabla U||_{L^\infty(S')} \le C(S')||\nabla V||_{L^\infty(S)}||\nabla^2 V||_{L^2(S)}. \tag{209}$$

## 8.3 Refined regularity estimates

For sufficiently smooth domains the continuity estimates in Proposition 31 hold up to the boundary.

**Theorem 33** *Suppose that $S$ is a $C^{1,\alpha}$ domain (for some $\alpha > 0$) and that $V \in W^{2,2}(S)$ with $\det \nabla^2 V = 0$. Then $V \in C^1(\bar{S})$ and for sufficiently small $\rho, R$ with $0 < \rho < R$ we have*

$$\operatorname{osc}_{B_\rho \cap S} \nabla V \le C(\ln \frac{R}{\rho})^{-1/2} ||\nabla^2 V||_{L^2(B_R \cap S)}, \tag{210}$$

*In particular*

$$||\nabla V||_{L^\infty(S)} \le \frac{1}{|S|} \left| \int_S \nabla V \, dx \right| + C||\nabla^2 V||_{L^2(S)}. \tag{211}$$

**Remark 34** *The result does not hold for Lipschitz domains. Consider for example the truncated cone $\{(x_1, x_2) : x_1 \in (0, 1/2), |x_2| < x_1\}$ and $V(x) = v(x_1)$ with $v'(0) = \infty$ and $\int_0^1 t|v''(t)|^2 < \infty$. One may take e.g. $v'(t) = |\ln t|^\alpha$, $0 < \alpha < 1/2$. A slight modification shows that even $C^1$ domains are not sufficient. One needs a certain logarithmic modulus of continuity of the normal.*

*Proof.* See [60].

In the setting of Theorem 25 we thus obtain for $C^{1,\alpha}$ domains the estimates

$$\|\nabla V\|_{L^\infty(S)} \leq C\|V\|_{W^{2,2}(S)} \tag{212}$$

$$\|\nabla U\|_{L^\infty(S)} + \|\nabla^2 U\|_{L^2(S)} \leq C\|V\|^2_{W^{2,2}(S)}. \tag{213}$$

The proof of Theorem 33 uses the fact that the gradient of an isometric immersion is either locally constant or constant along a line segment which touches $\partial S$ at both ends. This is classical for smooth maps. For $C^2$ maps it follows as a special case of more general results of Hartman and Nirenberg [38]. Pogorelov [67, 68] has established the result under very weak hypotheses. He only requires that the surface is $C^1$ and that the image of the Gauss map (which maps each point on then surface to its normal) has measure zero on $S^2$. Pakzad recently gave a shorter proof (using results of Kirchheim) for $W^{2,2}$ isometric immersions. For later use we state both the scalar version (for functions with $\det \nabla^2 V = 0$) of this result and the version for isometric immersions.

**Theorem 35** *[43, 61] Let $S$ be a bounded Lipschitz domain. Suppose that $V \in W^{2,2}(S)$ with $\det \nabla^2 V = 0$. Consider the open set*

$$S_1 = \{x \in S : \nabla V \text{ is constant in a neighbourhood of } x\}. \tag{214}$$

*Then through every point $x \in S \setminus S_1$ there exists a line segment which intersects $\partial S$ at both ends and on which $\nabla V$ is constant.*

*The same characterization holds for an isometric immersion in $W^{2,2}(S; \mathbb{R}^3)$.*

**Remark 36** *The statement for isometric immersions follows from that for scalar functions as follows. By Proposition 23 the second fundamental form $A$ is curl-free and thus can be locally written as $A = \nabla f$. Since $A$ is symmetric we also have locally $f = \nabla V$. Hence $\det \nabla^2 V = \det A = 0$. Thus if $f$ is not locally constant, it is constant on a line segment. Now* (177) *and Lemma 39 imply that for each component $y_i$ the gradient $\nabla y_i$ is constant on that segment.*

The above characterization can also be used to approximate $W^{2,2}$ functions which satisfy $\det \nabla^2 V = 0$ by functions in $W^{2,2} \cap W^{1,\infty}$ which satisfy the same constraint. The idea is that each component of the set $\{|\nabla V| < k\}$ is bounded by line segments on which $\nabla V$ is constant and by pieces of $\partial S$. If $k$ is sufficiently big then by local regularity there is one large component $U$ of $\{|\nabla V| < k\}$ and we obtain a good approximation by replacing $V$ by a constant in the regions between $\partial U$ and $\partial S$. The precise statement is as follows.

**Theorem 37 ([60])** *Suppose that $S$ is a bounded Lipschitz domain. Let $V \in W^{2,2}(S)$ with $\det \nabla^2 V = 0$. Then there exist an increasing sequence of open sets $S_k \subset S$ and $V_k \in W^{2,2}(S)$ such that*

$$V_k = V \quad in\, S_k, \qquad \nabla^2 V_k = 0 \quad a.e.\ in\, S \setminus S_k, \tag{215}$$

$$|\nabla V_k| \leq k \quad in\, S, \tag{216}$$

$$\bigcup_{k=1}^{\infty} S_k = S \tag{217}$$

*In particular* $\det \nabla^2 V_k = 0$, $||\nabla^2 V_k||_{L^2(S)} \leq ||\nabla^2 V||_{L^2(S)}$ *and* $V_k \to V$ *in* $W^{2,2}(S)$.

**Remark 38 ([60])** *If $\Gamma \subset \partial S$ is a finite union of intervals and $\nabla V = 0$ on $\Gamma$ (in the sense of trace) then we can achieve that $V_k = V$ and $\nabla V_k = \nabla V = 0$ on $\Gamma$. In fact there exists an subset of $S$ whose boundary includes $\Gamma$ on which $V_k = V$.*

In Remark 36 we have used the fact that if $\nabla u$ and $\nabla v$ are parallel in an $L^2$ sense in $S$ and if $v$ is constant on a line so is $u$. The following lemma gives a precise statement.

**Lemma 39** *Let $\Gamma = \{(x_1, x_2) : x_2 = h(x_1), x_1 \in (0, a)\}$ be a Lipschitz graph and let*

$$U = \{(x_1, x_2) : h(x_1) < x_2 < h(x_1) + b, x_1 \in (0, a)\} \tag{218}$$

*be a strip above $\Gamma$. Suppose that $u \in W^{1,1}(U)$, $b_k, v_k \in W^{1,2}(U)$ and*

$$\nabla u = \sum_k b_k \nabla v_k. \tag{219}$$

*If the functions $v_k$ are constant on $\Gamma$ (in the sense of trace) then $u$ is constant on $\Gamma$.*

*Proof.* We may assume that $h = 0$, since otherwise we can consider the functions $\tilde{u}, \tilde{b_k}, \tilde{v_k}$ given by $\tilde{u}(x_1, x_2) = u(x_1, h(x_1), x_2)$ etc. We may also suppose that the $v_k$ vanish on $\Gamma$ since otherwise we can first subtract suitable constants. So suppose $h = 0$ and let $\bar{u}(x_1) = u(x_1, 0)$ denote the trace of $u$ on $\Gamma$. We claim that

$$\int_\Gamma \bar{u}\bar{\varphi}'\, dx_1 = 0, \quad \forall \bar{\varphi} \in C_0^\infty(0, a). \tag{220}$$

This immediately applies the assertion. To prove (220) fix $\bar{\varphi}$ and consider $\psi \in C_0^\infty([0,1))$ with $\psi = 1$ on $[0,1/2]$. Set $\varphi^\delta(x) = \varphi(\bar{x}_1)\psi(x_2/\delta)$. Then for sufficiently small $\delta > 0$ the function $\varphi^\delta$ vanishes on $\partial U \setminus \Gamma$. Thus

$$\int_U \nabla u \wedge \nabla \varphi^\delta \, dx = \int_U (u\varphi^\delta_{,2})_{,1} - (u\varphi^\delta_{,1})_{,2} \, dx = \int_\Gamma \bar{u}\bar{\varphi}' \, dx_1. \tag{221}$$

On the other hand we have

$$\begin{aligned} \int_U \nabla u \wedge \nabla \varphi^\delta \, dx &= \int_U \nabla v_k \wedge b_k \nabla \varphi^\delta \, dx \\ = \int_U \nabla v_k \wedge \nabla(b_k \varphi^\delta) \, dx &- \int_U \nabla v_k \wedge \varphi^\delta \nabla b_k \, dx. \end{aligned}$$

Now the first term vanishes. To see this, approximate the $b_k$ by smooth functions and use that $v_k$ vanishes on $\Gamma$ and the calculation in (221). The second term goes to zero as $\delta \to 0$. Hence (220) holds. □

# 9 Conclusions

We have shown that variational methods and rigidity estimates yield rigorous convergence proofs of 3D energy-minimizing solutions to solutions of a hierarchy of plate theories as the plate thickness tends to zero. Unlike in heuristic derivations of such plate theories, no a priori assumptions whatsoever on the structure of the 3D solutions are made. The different theories in the hierarchy are distinguished by the relation between the strength of the applied force and the thickness.

## 9.1 vK theory revisited

It is instructive to see how our results address the criticisms raised by Truesdell and Antman against the usual derivations of von Kármán's plate theory. This were [76]:

(i) approximate geometry

(ii) assumptions on the way the stresses vary over the cross-section

(iii) commitment to some specific linear constitutive relation

(iv) neglect of some components of the strain

(v) an apparent confusion of the referential and the spatial descriptions

The first and last criticisms in particular refer to the fact that nonlinear elasticity is invariant under the full group $SO(n)$ of rotations while von Kármán's theory is based on geometrically linear elasticity which is only invariant under the tangent space of skew-symmetric matrices. Since the three-dimensional elastic energy is invariant under rotations, large rotations could in principal occur and these would not be compatible with the use of geometrically linear theory. This point is addressed by our rigidity estimate, Theorem 10. In the energy scaling regime in which von Kármán theory is valid (i.e. $E^h \sim h^4$) the three-dimensional deformation gradient is very well approximated by a *constant* rotation (see (66)), which we may assume to be the identity. This is the reason why geometrically linear theory works. The crucial expression $\operatorname{sym} \nabla' u + (1/2)\nabla' v \otimes \nabla' v$ which represents the membrane strain in vK theory is derived rigorously in Lemma 15, see (113). At the heart of the matter is a second order estimate for the deviation of the local deformation gradient from a constant rotation (see (96) and (88), where the constant rotation is taken as the identity and where $E^h = h^4$).

Point (iii) is again essentially addressed by (66). This allows one to use a Taylor expansion of the energy and therefore only the linearization of the full elastic energy matters and we get a linear constitutive relation in the limiting model. One subtle point is that from smallness of the energy one can only prove that the gradient is close to the identity in $L^2$, while Taylor expansion requires an $L^\infty$ estimate. Here lies the strength of the variational character of $\Gamma$-convergence. For the lower bound we can ignore the very small set were the gradient may not be uniformly close the identity (see (121)). For the upper we only need to construct a test function and this we can choose so that uniform convergence holds.

Point (ii) is also taken care of by Lemma 15. It shows that the relevant components of the limiting strain are affine in $x_3$. Since the limiting stress-strain relation is linear, the same holds for the stress.

Finally (iv) emerges naturally in our analysis. Again Lemma 15 shows that certain components of the limiting strain can be predicted from the limiting in-plane and out-of-plane deformations. Minimization over the remaining components leads a lower bound for the energy and the construction of test functions for the upper bound shows that this lower bound is already sharp. Hence for (almost) minimizing sequences in the sense of (18) the remaining components of the strain are essentially slaved to the ones which enter directly into the theory.

## 9.2 Extensions

The approach discussed here can be extended to many other settings and a lot of work is currently in progress. Let us just mention shells [49, 33], rods [57, 58, 64], other boundary conditions and stability [35], heterogeneous films and multilayers [74] and multiwell problems [11, 74, 15, 16, 46, 59]. We also believe that the estimates developed here should be useful for the numerical analysis of thin elastic bodies.

## 9.3 Theories which involve membrane and bending energy

A wide open problem is the question whether one can in a rigorous way justify theories which are two-dimensional but still involve the small thickness parameter $h$. There are many cases of interest with boundary conditions that do not fall into any of the categories. A typical case involves boundary conditions that cause part of the shell to stretch, but another part to bend with no stretching. This apparently can also happen near a singularity.

The theories derived in the limit $h \to 0$ (see Table 1) also often show certain degeneracies. In such cases membrane theory exhibits no resistance to compression, leading to undesirable instabilities. While membrane theory is thus too soft, Kirchhoff's bending theory is often too stiff. It captures the bending energy correctly but only isometric immersions have finite energy. Practitioners often use theories which involve both membrane and bending contributions to the energy (and thus retain the small parameter $h$), i.e., geometrically nonlinear versions of the von Kármán theory. The question whether any one of these theories has a rigorous asymptotic status is unclear. There have been some attempts to extend the concept of Γ-convergence to Γ-expansion in order to capture not only the limit, but also higher order terms, but so far this approach has been mostly successful for linear problems [4]. It seems that these Γ-expansions tend to separate the regimes more than is desirable; each successive term in the Γ-expansion can only make an arbitrarily small perturbation to the preceeding theory. In ongoing work Braides and Truskinovsky [12] are studying a number of nonlinear examples where such a Γ-expansion would be very desirable. The issue of simplified theories which still contain the small parameter is of particular interest in the force range $1 \leq \alpha < 2$, as we already discussed in Section 2.2.

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