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A High Stability, Smooth Walking Pattern for a Biped Robot

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Abstract

Biped robots have better mobility than conventional wheeled robots, but they tip over easily. In order to walk stably in various environments such as rough terrain, up and down slopes, or regions containing obstacles, it is desirable to adapt to such ground conditions with a suitable foot motion, and maintain the stability of the robot by a smooth hip motion. In this paper, we propose a method to plan a walking pattern consisting of a foot trajectory and a hip trajectory. First, we formulate the constraints of a foot trajectory, and generate the foot trajectory by 3rd order spline interpolation. By setting the values of constraint parameters, it is easy to produce different types of foot motion. Then, we formulate a hip trajectory using a 3rd order periodic spline function, and derive the hip trajectory with high stability. Finally, the effectiveness of the proposed method is illustrated by simulation examples.

1 Introduction

Biped robots have better mobility than conventional wheeled robots, especially for moving on rough terrain, steep stairs and obstacle environments. The study of biped robots has aroused the interest of a number of researchers [1-16].

One main topic of interest is walking pattern synthesis. Zerrugh, et al. [1] have investigated the walking pattern for a biped robot by recording human kinematic data. McGeer [2] presented a natural walking pattern generated by the passive interaction of gravity and inertia on a downhill slope. In order to extend the energy-optimal walking method to level ground and uphill slopes, Channon, et al. [3], Rostami, et al. [4], and Roussel, et al. [5] have proposed methods of gait generation by minimizing the energy consumption cost function.

However, these investigations rarely consider the stability of a biped robot. Since a biped robot easily tips over, it is necessary to take stability into account when determining a walking pattern. Zheng, et al. [6] have proposed a method of gait synthesis in the case of static stability. Chevallereau, et al. [7] have discussed dynamic stability by analyzing the reaction force between the sole and the ground when specifying a low energy reference trajectory by designing a ballistic motion in single-support phase. Unfortunately, this low energy reference trajectory does not ensure that the stability constraint is satisfied.

In order to ensure dynamic stability for a biped robot, Takanishi et al. [8], Shin et al. [9], and Hirose et al. [10] have proposed methods of walking pattern synthesis based on the ZMP (Zero Moment Point) [11]. Basically, these investigations first design a desired ZMP trajectory, then derive the torso motion to realize the desired ZMP trajectory. However, since the change of the ZMP due to the torso motion is limited, not all desired ZMP trajectories will be realizable [17]. Furthermore, to realize a desired ZMP trajectory, the torso motion may vary radically. In this case, since the torso is relatively massive, energy consumption will become large, and the control for task execution of the upper limbs will become difficult. Therefore, it is necessary to consider both the stability of the biped robot and the smoothness of the torso motion.

In addition, in order for a biped robot to adapt to walking conditions such as level ground, rough terrain and obstacle-filled environments, the robot must have various types of foot motion. For example, it is desirable that the biped robot can lift its feet high enough to negotiate obstacles, or absorb the impact force by the heel landing the ground first. However, previous investigations have not sufficiently discussed such a complete foot trajectory.

In this paper, we focus on two problems: the first one is how to generate a complete foot trajectory; the second one is how to derive a smooth torso motion with high stability. The paper is organized as follows. We introduce the ZMP criterion in section 2, and describe the walking cycle in section 3. In section 4, we formulate the constraints of the foot trajectory and generate the foot trajectory by 3rd spline interpolation. In section 5, we formulate a hip trajectory by a 3rd periodic spline function, and determine the hip trajectory with high stability by iterative computation. Finally, we provide simulation results in section 6 and a conclusion in section 7.

2 ZMP Criterion

We consider an anthropomorphic biped robot with a trunk. Each leg consists of a thigh, a shin and a foot, and has 6 degrees of freedom: 3 degrees of freedom in the hip joint, 1 in the knee joint, and 2 in the ankle joint.

For a robot with four or more legs, it is possible to consider static stability using the center of gravity. But for a biped robot, it is necessary to take into account dynamic stability. To evaluate dynamic stability, we use the concept

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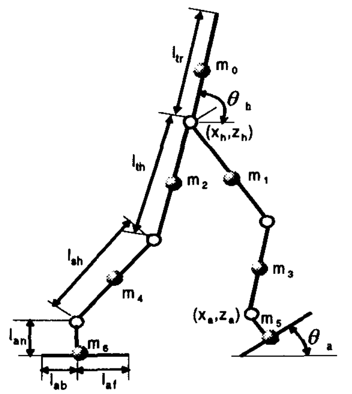


Fig. 1 Model of the biped robot

of the ZMP first introduced by Vukobratovic [11].

The ZMP is defined as the point on the ground about which the sum of all the moments of active forces is equal to zero. If the ZMP is inside the contact polygon between the foot and the ground, the biped robot is stable. In the following, the contact polygon is called the stable region. The ZMP can be computed by the following equations:

$$x_{zmp} = \frac{\sum_{i=0}^{6} m_i(\ddot{z}_i + g)x_i - \sum_{i=0}^{6} m_i \ddot{x}_i z_i - \sum_{i=0}^{6} I_{iy}\dot{\Omega}_{iy}}{\sum_{i=0}^{6} m_i(\ddot{z}_i + g)} \qquad (1)$$

$$y_{zmp} = \frac{\sum_{i=0}^{6} m_i(\ddot{z}_i + g)y_i - \sum_{i=0}^{6} m_i \ddot{y}_i z_i + \sum_{i=0}^{6} I_{iy}\dot{\Omega}_{iy}}{\sum_{i=0}^{6} m_i(\ddot{z}_i + g)} \qquad (2)$$

where m_i is the mass of link i (Fig. 1), $(I_{ix}, I_{iy})^T$ is the inertial vector of link i, $(\Omega_{ix}, \Omega_{iy})^T$ is the angular velocity vector of link i, g is the gravitational acceleration, $(x_{zmp}, y_{zmp}, 0)$ is the coordinate of the ZMP, and (x_i, y_i, z_i) is the coordinate of the mass center of link i on the absolute Cartesian coordinate system.

If the ZMP is near the center of the stable region, that is, the minimum distance between the ZMP and the boundary of the stable region is large, the biped robot will have high stability. In the following, the minimum distance between the ZMP and the boundary of the stable region is called the stability margin (Fig. 2). The stability margin can be regarded as a measure of the quality of stability.

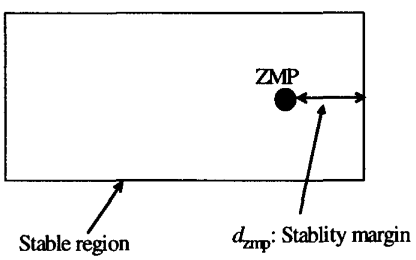


Fig. 2 Stable region and stability margin

3 Walking Cycle

A complete walking cycle may be divided into a double-support phase and a single-support phase. In the double-support phase, both feet are in contact with the ground. This phase begins with the heel of the forward foot touching the ground, and ends with the toe of the rear foot taking off the ground. In the single-support phase, one foot is stationary on the ground and the other foot swings from the rear to the front.

Many studies on gait planning [2-5] have assumed that the double-support phase is instantaneous. But in that case, the hip has to move too fast during the double-support phase. This is because, in order to maintain the stability of the biped robot, the center of gravity in the case of static stability or the ZMP in the case of dynamic stability must be moved to the front foot from the rear foot during the short double-support phase. On the other hand, if the time interval of the double-support phase is too large, it is difficult for the biped robot to walk at a high speed. To get a suitable value, we can specify the time interval of the double-support phase to be about 20% of the period of one step walking according to human locomotion.

If both foot trajectories and the hip trajectory are known, all joint trajectories of the biped robot will be determined by the kinematic constraints. The walking pattern can therefore be denoted uniquely by both foot trajectories and the hip trajectory. When the robot moves straight forward, the positions of both feet in the lateral direction are constant. The lateral hip motion can be derived similarly as the hip motion in the sagittal direction. In the following sections, we only discuss the trajectories in the sagittal plane.

For a sagittal plane, each foot trajectory can be denoted by a vector $\mathbf{X}_a = [x_a(t), z_a(t), \theta_a(t)]^T$, where $(x_a(t), z_a(t))$ is the coordinate of the ankle position, and $\theta_a(t)$ denotes the slope of the foot. The hip trajectory can be denoted by a vector $\mathbf{X}_h = [x_h(t), z_h(t), \theta_h(t)]^T$, where $(x_h(t), z_h(t))$ denote the coordinate of the hip position, and $\theta_h(t)$ denotes the slope of the hip (Fig. 1).

In order to adapt to ground conditions, it is necessary to first specify both foot trajectories, then determine the hip trajectory. In the following, we first plan foot trajectories, then discuss the hip trajectory (Fig. 3).

4 Foot Trajectories

Supposing that the period necessary for one walking step is T_c, the time of the kth step walking is from kT_c to $(k+1)T_c$, $k = 1, 2, \cdots$. To simplify the analysis, we define that

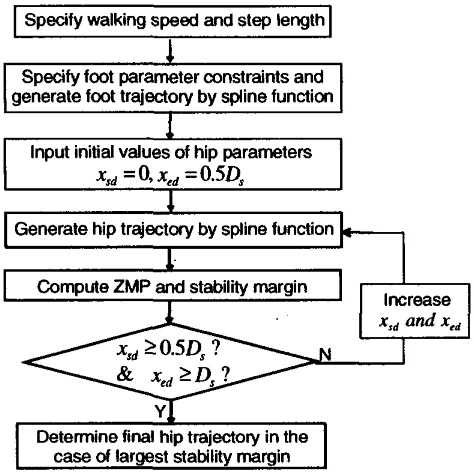


Fig. 3 Algorithm for planning walking patterns

the kth walking step begins with the heel of the left foot leaving the ground at $t = kT_c$, and ends with the heel of the left foot touching the ground at $t=(k+1)T_c$. In the following, we discuss only the generation of the left foot trajectory. The right foot trajectory is same as the left foot trajectory except for a T_c delay.

Most previous studies have defined foot trajectories such that the feet are always level to the ground, that is, the foot slope $\theta_a(t)$ is always zero. In that case, since the whole sole of the front foot will suddenly land on the ground at the beginning of the double-support phase, the center of gravity will move to the central part of the front foot in a very short period, particularly for high speed walking. The impact force between the sole of the front foot and the ground may become very large, and the biped robot easily becomes unstable. On the other hand, if the foot slope $\theta_a(t)$ is not zero at the beginning of the double-support phase, the heel can land first before the whole sole comes into contact with the ground. In this case, it is possible for the center of gravity to move smoothly from the heel to the toe, and the impact force can become small. Similarly, it is desirable that the rear foot leaves the ground with a suitable foot slope to push the body forward, particularly for high speed walking. Additionally, from the viewpoint of human natural locomotion and aesthetics, it is undesirable that the foot slope always be level.

Let q_b and q_f be the designated slope angles of the left foot as it leaves and lands on the ground (Fig. 4), respectively. We then get the following constraints:

$$\theta_a(kT_c + T_d) = q_b \tag{3}$$

$$\theta_a(kT_c + T_c) = q_f \tag{4}$$

where T_d denotes the time interval of the double-support phase.

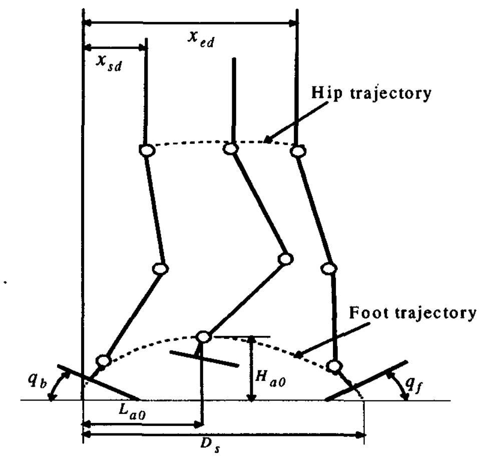


Fig. 4 Parameters of walking pattern

From the viewpoint of minimum energy, it is desired that the biped robot can walk without lifting its swing foot. However, in the case of rough terrain or environments in the presence of obstacles, it is necessary to lift the swing foot to at least a certain height. Let (L_{ao}, H_{ao}) be the coordinate of the highest point of the swing foot (Fig. 4), the following constraints must be satisfied.

$$x_a(kT_c + t_o) = kD_s + L_{ao} \tag{5}$$

$$z_a(kT_c + t_o) = H_{ao} \tag{6}$$

where, $kT_c + t_o$ denotes the time when the left foot is at its highest point, and D_s denotes the length of one step.

From equations (3) and (4), and considering that the whole sole of the left foot is in contact with the ground at $t=kT_c$ and $t=(k+1)T_c+T_d$, we get the following constraints:

$$\theta_a(t) = \begin{cases} q_s & t = kT_c \\ q_b & t = kT_c + T_d \\ q_f & t = kT_c + T_c \\ q_e & t = (k+1)T_c + T_d \end{cases} \tag{7}$$

where q_s and q_e are the slopes of the ground surface at the contact points (in particular $q_s = q_e = 0$ in the case of level ground).

From equations (5), (6), (7) and the kinematic constraints, the following constraints must be satisfied.

$$x_a(t) = \begin{cases} kD_s & t=kT_c \\ kD_s + l_{af}(1-\cos q_b) + l_{an}\sin q_b & t=kT_c+T_d \\ kD_s + L_{a0} & t=kT_c+T_o \\ (k+1)D_s - l_{ab}(1-\cos q_f) - l_{an}\sin q_f & t=kT_c+T_c \\ (k+1)D_s & t=(k+1)T_c+T_d \end{cases} \tag{8}$$

$$z_a(t)=\begin{cases} l_{an} & t=kT_c \\ l_{an}\cos q_b + l_{af}\sin q_b & t=kT_c+T_d \\ H_{a0} & t=kT_c+T_o \\ l_{an}\cos q_f + l_{ab}\sin q_f & t=kT_c+T_c \\ l_{an} & t=(k+1)T_c+T_d \end{cases} \tag{9}$$

where l_{an} denotes the height of the foot, l_{af} the length from the ankle joint to the toe, and l_{ab} the length from the ankle joint to the heel (Fig. 1).

Since the whole sole of the left foot is in contact with the ground at $t=kT_c$ and $t=(k+1)T_c+T_d$, the following derivative constraints must be satisfied.

$$\begin{cases} \dot{\theta}_a(kT_c)=0 \\ \dot{\theta}_a((k+1)T_c+T_d)=0 \end{cases} \tag{10}$$

$$\begin{cases} \dot{x}_a(kT_c)=0 \\ \dot{x}_a((k+1)T_c+T_d)=0 \end{cases} \tag{11}$$

$$\begin{cases} \dot{z}_a(kT_c)=0 \\ \dot{z}_a((k+1)T_c+T_d)=0 \end{cases} \tag{12}$$

In order to generate a smooth trajectory, it is necessary that second derivatives (accelerations) $\ddot{\theta}_a(t)$, $\ddot{x}_a(t)$ and $\ddot{z}_a(t)$ be continuous at all t including all breakpoints $t=kT_c$, kT_c+T_d, kT_c+T_0, $(k+1)T_c$, $(k+1)T_c+T_d$.

It is possible to solve for the foot trajectory which satisfies constraint equations (7), (8), (9), (10), (11), (12) and the second derivative continuity conditions using polynomial interpolation. But in this case, the order of the polynomial is too high, the computation of polynomial is difficult, and the trajectory may oscillate. Therefore, we obtain the foot trajectory by 3rd order spline interpolation (see appendix). In that case, $x_a(t)$, $z_a(t)$ and $\theta_a(t)$ are characterized by 3rd order polynomial expressions, and the second derivatives $\ddot{\theta}_a(t)$, $\ddot{x}_a(t)$ and $\ddot{z}_a(t)$ are always continuous.

5 Hip Trajectory

Some researchers have discussed the generation of the hip (torso) trajectory based on the ZMP [7,8,9]. Basically, these investigations first design a desired ZMP trajectory (including time and path), then derive the torso motion to realize the desired ZMP trajectory. The advantage of this method is that the stability margin can be large if the desired ZMP is designed near the center of the stable region. However, since the change of the ZMP due to torso motion is limited, not all desired ZMP trajectories will be realizable [17]. Furthermore, even if the desired ZMP trajectory is realized, the torso motion may be too large. Therefore, it is necessary to propose a new method to derive a hip motion with high stability.

We can specify $\theta_h(t)$ to be constant; in particular, $\theta_h(t)=90^0$ in the case of a level ground. Hip motion along the z-axis hardly affects the position of the ZMP. In order to reduce the impact force between the sole and the ground, we can specify $z_h(t)$ to be constant, or vary within a small range. In order to get a smooth hip motion along the x-axis with high stability, we take the following steps:

(1) Generate a series of smooth $x_h(t)$.

(2) Determine the final $x_h(t)$ with a large stability margin

A complete walking process may be divided into three phases: a starting phase that the speed is from zero to a desired constant velocity, a steady phase with a desired constant velocity, and a ending phase that the speed is from a desired constant velocity to zero.

It is reasonable that $x_h(t)$ during one step cycle can be divided into one function of the double-support phase and one function of the single-support phase. Let x_{sd} and x_{ed} denote distances along the x-axis from the hip to the ankle of the stance foot at the beginning and the end of the double-support phase, respectively (Fig. 4), we get the following equation:

$$x_h(t)=\begin{cases} kD_s+x_{sd} & t=kT_c \\ kD_s+x_{ed} & t=kT_c+T_d \\ (k+1)D_s+x_{sd} & t=(k+1)T_c \end{cases} \tag{13}$$

Since the initial and the final constraints of $\dot{x}_h(t_0)=0$, $\dot{x}_h(t_e)=0$ must be satisfied, $x_h(t)$ of the starting phase and the ending phase can be obtained by a 3rd order spline interpolation. But, in order to obtain a smooth periodic $x_h(t)$ of the steady phase, the following derivative constraints must be satisfied:

$$\begin{cases} \dot{x}_h(kT_c)=\dot{x}_h(kT_c+T_c) \\ \ddot{x}_h(kT_c)=\ddot{x}_h(kT_c+T_c) \end{cases} \tag{14}$$

By using 3rd order periodic spline interpolation [18], we obtain $x_h(t)$ satisfying constraints (13), (14) as follows:

$$x_h(t)=\begin{cases} kD_s+\frac{(x_{ed}-x_{sd})}{T_d^2(T_c-T_d)} \\ [(T_d+kT_c-t)^3-(T_d+kT_c-t)T_d^2- \\ (t-kT_c)^3+T_d^2(t-kT_c)]+ \\ \frac{x_{sd}}{T_d}(T_d+kT_c-t)+\frac{x_{ed}}{T_d}(t-kT_c) & t\in(kT_c,kT_c+T_d) \\ kD_s+\frac{(x_{sd}-x_{ed})}{T_d(T_c-T_d)^2} \\ [(T_c+kT_c-t)^3-(T_c+kT_c-t)(T_c-T_d)^2- \\ (t-kT_c-T_d)^3+(T_c-T_d)^2(t-kT_c-T_d)]+ \\ \frac{x_{ed}}{T_c-T_d}(T_c+kT_c-t)+\frac{x_{sd}}{T_c-T_d}(t-kT_c-T_d) & t\in(kT_c+T_d,kT_c+T_c) \end{cases} \tag{15}$$

By defining different values of x_{sd} and x_{ed}, we get a series of smooth $x_h(t)$ according to equation (15). We specify x_{sd} and x_{ed} to vary within a fixed range. In particular,

$$\begin{cases} 0 < x_{sd} < 0.5D_s \\ 0.5D_s < x_{ed} < D_s \end{cases} \tag{16}$$

Based on equations (1), (15) and (16), the problem of finding the smooth trajectory $x_h(t)$ with the largest stability margin can be formulated as follows:

$$\operatorname*{Max.}_{x_{sd}\in[0,0.5D_s],x_{ed}\in[0.5D_s,D_s]} \; d_{zmp}(x_{sd}, x_{ed}) \tag{17}$$

where $d_{zmp}(x_{sd}, x_{ed})$ denotes the stability margin (Fig. 2). We obtain the solutions of equation (17) by iterative computation (Fig. 3).

6 Simulation

We have constructed a simulator of an anthropomorphic biped robot (Fig. 5) on an SGI workstation by using the dynamic software package called DADS. This allows us to analyze the necessary joint torque, the contact force between the feet and the ground, etc. Parameters of the biped robot (Fig. 1) are set according to Table 1. We suppose that the step length is 0.5 [m], and the step period is 0.9 [s], and the ground is level, that is, $q_s = q_e = 0$.

Simulation results are shown in Fig. 6 and Fig. 7 in the case of $q_b = -0.4$ [rad], $q_f = 0.2$ [rad], $L_{ao} = 0.25$ [m],

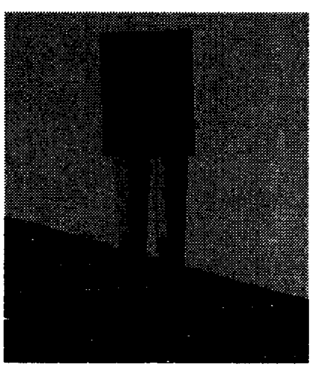
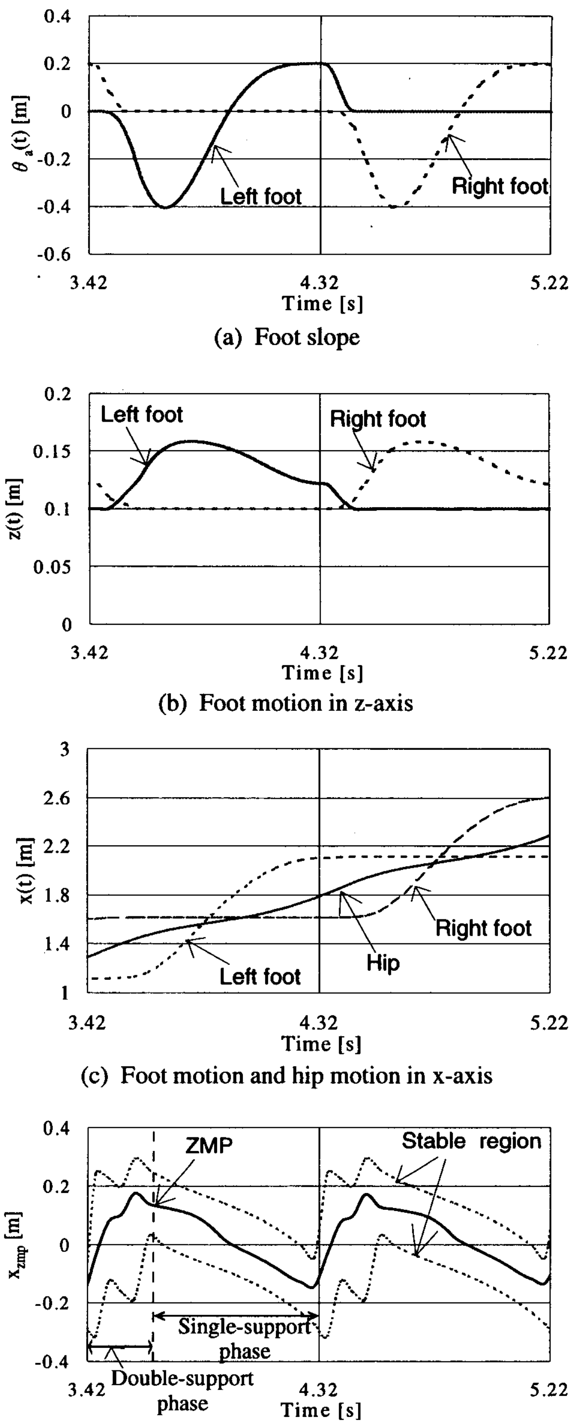


Fig. 5 Biped robot in simulator

Table 1 Parameters of the biped robot

Length (cm)	l_{tr}	l_{th}	l_{sh}	l_{an}	l_{ab}	l_{af}	
	60	40	40	10	10	13	
Weight (kg)	m_0	m_1	m_2	m_3	m_4	m_5	m_6
	36	7.0	7.0	3.7	3.7	1.3	1.3
Inertia (kgm²)	I_{0y}	I_{1y}	I_{2y}	I_{3y}	I_{4y}	I_{5y}	I_{6y}
	0.69	0.16	0.16	0.08	0.08	0.01	0.01



(d) ZMP trajectory in the hip coordinate system

Fig. 6 Trajectories of foot, hip and ZMP
$q_b = -0.4\,[rad]$, $q_f = 0.2\,[rad]$
$L_{a0} = 0.25\,[m]$, $H_{a0} = 0.08\,[m]$

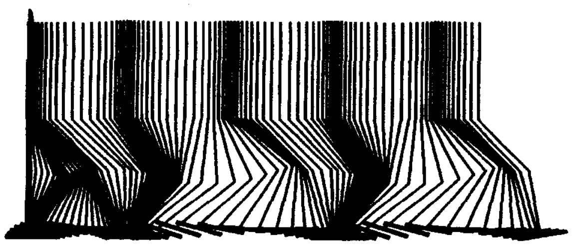


Fig. 7 Walking pattern in the case of
$q_b = -0.4\,[rad]$, $q_f = 0.2\,[rad]$
$L_{a0} = 0.25\,[m]$, $H_{a0} = 0.08[m]$

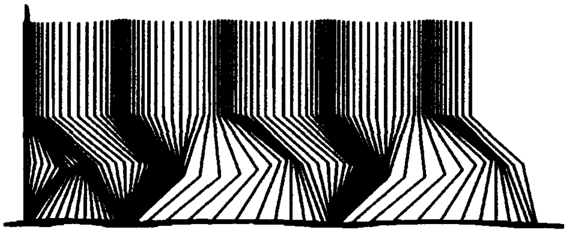
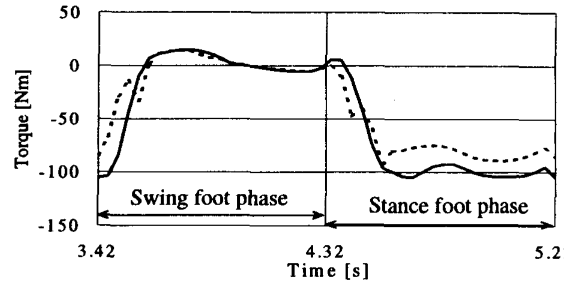


Fig. 8 Walking pattern in the case of
$q_b = 0.0\,[rad]$, $q_f = 0.0\,[rad]$
$L_{a0} = 0.25\,[m]$, $H_{a0} = 0.02[m]$

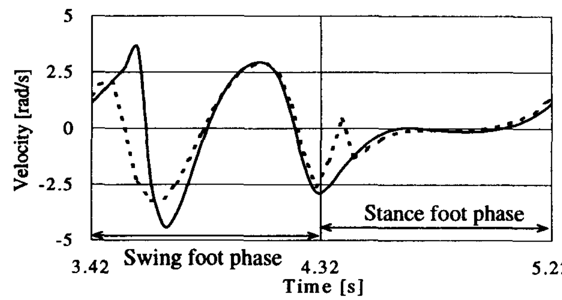
$H_{ao} = 0.16$ [m]. From the foot slope (Fig. 6 (a) and the walking pattern (Fig. 7), we can see the two feet leave and land on the ground with the desired slopes. It is known that both foot trajectories (Fig. 6 (a) (b) (c)) and the hip trajectory (Fig. 6 (c)) are smooth, and the ZMP trajectory (Fig. 6 (d)) is always near the center of the stable region, that is, the robot has a large stability margin.

The change of the stable region boundary during the double-support phase (Fig. 6(d)) can be explained as follows. The heel of the forward foot touches the ground at the beginning of the double-support phase, then the heel of the rear foot takes off the ground and the whole sole of the forward foot comes into contact with the ground. Finally the toe of the rear foot leaves the ground at the end of the double-support phase. Thus, the boundary of the stable region during the double-support phase changes from the length between the heel of the rear foot and the heel of the forward foot, to the length between the toe of the rear foot and the heel of the forward foot, and finally to the length of the forward foot.

By specifying different values of q_b, q_f, H_{ao} and L_{ao} we can get different walking patterns. The walking pattern can be chosen according to ground constraints such as obstacles and the roughness of the terrain. Based on satisfaction of ground constraints, it is also possible to select a walking pattern with small specifications of joint actuators. Fig. 8 shows the walking pattern in the case of q_b =0.0 [rad], q_f =0.0 [rad], L_{ao} =0.25 [m], H_{ao} =0.02 [m]. This walking pattern is not generally suitable, because it requires a relatively large knee torque and velocity (Fig. 9).



(a) Knee torque



(b) Knee velocity

Fig. 9 Specifications of different walking patterns
- - - : $q_b = -0.4\,[rad]$, $q_f = 0.2\,[rad]$
$L_{a0} = 0.25\,[m]$, $H_{a0} = 0.08[m]$
—— : $q_b = -0.0\,[rad]$, $q_f = 0.0\,[rad]$
$L_{a0} = 0.25\,[m]$, $H_{a0} = 0.02[m]$

7 Conclusion

Biped robots have better mobility than wheeled robots but tip over easily. In order to walk stably in various environmental conditions such as rough terrain, up and down slopes, or regions with obstacles, it is desirable to adapt to such ground constraints by using a suitable foot motion, and to maintain the stability of the robot with a smooth hip motion. In this paper, we propose a method to plan a walking pattern consisting of a foot trajectory and a hip trajectory. First, we formulate the constraints of a complete foot trajectory, and generate the foot trajectory by 3rd order spline interpolation. Then, we formulate the hip trajectory by a 3rd order periodic spline function, and determine the final hip trajectory with a large stability margin by iterative computation. Finally, the effectiveness of the proposed method is illustrated by simulation examples.

By setting the values of foot constraint parameters, we can produce walking patterns to adapt to ground conditions involving obstacles or rough terrain. Also, it is possible to select a walking pattern with small specifications of joint actuators. This is also suggested by simulation.

Appendix

For n breakpoints $t_1 < t_2 < \cdots < t_n$, $S(t_j) = f_j$, $j = 1, 2, \cdots, n$, the 3rd order spline function $S(t)$ is a 3rd order polynomial for each (t_j, t_{j+1}), and the first derivative $S'(t)$ and second derivative $S''(t)$ are continuous on (t_1, t_n).

Let $I_j = (t_j, t_{j+1})$, $h_j = t_{j+1} - t_j$, then $S(t)$ is denoted by the following equation:

$$S(t) = \frac{M_j}{6h_j}(t_{j+1} - t)^3 + \frac{M_{j+1}}{6h_j}(t - t_j)^3 + (f_j - \frac{M_j h_j^2}{6})\frac{t_{j+1} - t}{h_j} + (f_{j+1} - \frac{M_{j+1} h_j^2}{6})\frac{t - t_j}{h_j} \tag{18}$$

M_j is the solution of the following equations:

$$\begin{cases} 2M_1 + b_1 M_2 = d_1 \\ \frac{h_{j-1}}{6} M_{j-1} + \frac{h_j + h_{j-1}}{3} M_j + \frac{h_j}{6} M_{j+1} \\ \qquad = \frac{f_{j+1} - f_j}{h_j} - \frac{f_j - f_{j-1}}{h_{j-1}} \quad (j = 2, 3, \cdots, n-1) \\ a_n M_{n-1} + 2M_n = d_n \end{cases} \tag{19}$$

where,

$$\begin{cases} a_j = \frac{h_{j-1}}{h_j + h_{j-1}} \quad b_j = 1 - a_j \quad (j = 2, 3, \cdots n-1) \\ c_j = \frac{f_{j+1} - f_j}{h_j} \quad (j = 1, 2, \cdots n-1) \\ d_j = \frac{6(c_j - c_{j-1})}{h_j + h_{j-1}} \quad (j = 2, 3, \cdots n-1) \end{cases} \tag{20}$$

In the case of initial constraint $S'(t_1) = 0$ and end constraint $S'(t_n) = 0$, the following equations are obtained:

$$\begin{cases} b_1 = a_n = 1 \\ d_1 = \frac{6(f_2 - f_1)}{h_1} \\ d_n = -\frac{6(f_n - f_{n-1})}{h_{n-1}} \end{cases} \tag{21}$$

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