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J. N. Reddy and C. F. Liu

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A HIGHER-ORDER THEORY FOR  
GEOMETRICALLY NONLINEAR ANALYSIS OF  
COMPOSITE LAMINATES

by

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ABSTRACT

A refined, third-order laminate theory that accounts for the transverse shear strains is developed, the Navier solutions are derived, and its finite element models are developed. The theory allows parabolic description of the transverse shear stresses, and therefore the shear correction factors of the usual shear deformation theory are not required in the present theory. The theory also accounts for small strains but moderately large displacements (i.e., von Karman strains). Closed-form solutions of the linear theory for certain cross-ply and angle-ply plates and cross-ply shells are derived. The finite-element model is based on independent approximations of the displacements and bending moments (i.e., mixed formulation), and therefore only  $C^0$ -approximations are required. Further, the mixed variational formulations developed herein suggest that the bending moments can be interpolated using discontinuous approximations (across interelement boundaries). The finite element is used to analyze cross-ply and angle-ply laminated plates and shells for bending and natural vibration. Many of the numerical results presented here for laminated shells should serve as references for future investigations.

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## 1. INTRODUCTION

### 1.1 Background

The analyses of composite laminates in the past have been based on one of the following two classes of theories:

- (i) Three-dimensional elasticity theory
- (ii) Laminated plate theories

In three-dimensional elasticity theory, each layer is treated as an elastic continuum with possibly distinct material properties in adjacent layers. Thus the number of governing differential equations will be  $3N$ , where  $N$  is the number of layers in the laminate. At the interface of two layers, the continuity of displacements and stresses give additional relations. Solution of the equations becomes intractable as the number of layers becomes large.

In a 'laminated plate theory', the laminate is assumed to be in a state of plane stress, the individual laminae are assumed to be elastic, and perfect bonding between layers is assumed. The laminate properties (i.e. stiffnesses) are obtained by integrating the lamina properties through the thickness. Thus, laminate plate theories are equivalent single-layer theories. In the 'classical laminate plate theory' (CLPT), which is an extension of the classical plate theory (CPT) to laminated plates, the transverse stress components are ignored. The classical laminate plate theory is adequate for many engineering problems. However, laminated plates made of advanced filamentary composite materials, whose elastic to shear modulus ratios are very large, are susceptible to thickness effects because their effective transverse shear moduli are significantly smaller than the effective elastic moduli

along fiber directions. These high ratios of elastic to shear moduli render the classical laminated plate theory inadequate for the analysis of thick composite plates.

The first, stress-based, shear deformation plate theory is due to Reissner [1-3]. The theory is based on a linear distribution of the inplane normal and shear stresses through the thickness,

$$\sigma_1 = \frac{M_1}{(h^2/6)} \frac{z}{(h/2)}, \quad \sigma_2 = \frac{M_2}{(h^2/6)} \frac{z}{(h/2)}, \quad \sigma_6 = \frac{M_6}{(h^2/6)} \frac{z}{(h/2)} \quad (1.1)$$

where  $(\sigma_1, \sigma_2)$  and  $\sigma_6$  are the normal and shear stresses,  $(M_1, M_2)$  and  $M_6$  are the associated bending moments (which are functions of the inplane coordinates  $x$  and  $y$ ),  $z$  is the thickness coordinate and  $h$  is the total thickness of the plate. The distribution of the transverse normal and shear stresses ( $\sigma_3$ ,  $\sigma_4$  and  $\sigma_5$ ) is determined from the equilibrium equations of the three-dimensional elasticity theory. The differential equations and the boundary conditions of the theory were obtained using Castigliano's theorem of least work.

The origin of displacement-based theories is attributed to Basset [4]. Basset assumed that the displacement components in a shell can be expanded in a series of powers of the thickness coordinate  $\zeta$ . For example, the displacement component  $u_1$  along the  $\xi_1$  coordinate in the surface of the shell can be written in the form

$$u_1(\xi_1, \xi_2, \zeta) = u_1^0(\xi_1, \xi_2) + \sum \zeta^n u_1^{(n)}(\xi_1, \xi_2) \quad (1.2a)$$

where  $\xi_1$  and  $\xi_2$  are the curvilinear coordinates in the middle surface of the shell, and  $u_1^{(n)}$  have the meaning

$$u_1^{(n)}(\xi_1, \xi_2) = \left. \frac{d^n u_1}{d\zeta^n} \right|_{\zeta=0}, \quad n = 0, 1, 2, \dots \quad (1.2b)$$

Basset's work did not receive as much attention as it deserves. In a 1949 NACA technical note, Hildebrand, Reissner and Thomas [5] presented a displacement-based shear deformation theory for shells (also see Hencky [6]). They assumed the following displacement field,

$$\begin{aligned} u_1(\xi_1, \xi_2, \zeta) &= u(\xi_1, \xi_2) + \zeta \phi_x(\xi_1, \xi_2) \\ u_2(\xi_1, \xi_2, \zeta) &= v(\xi_1, \xi_2) + \zeta \phi_y(\xi_1, \xi_2) \\ u_3(\xi_1, \xi_2, \zeta) &= w(\xi_1, \xi_2) \end{aligned} \quad (1.3)$$

The differential equations of the theory are then derived using the principle of minimum total potential energy. This approach results in five differential equations in the five displacement functions,  $u$ ,  $v$ ,  $w$ ,  $\phi_x$ , and  $\phi_y$ .

The shear deformation theory based on the displacement field in Eq. (1.3) for plates is often referred to as the Mindlin plate theory. Mindlin [7] presented a complete dynamic theory of isotropic plates based on the displacement field (1.3) taken from Hencky [6]. We shall refer to the shear deformation theory based on the displacement field (1.3) as the first-order shear deformation theory.

A generalization of the first-order shear deformation plate theory for homogeneous isotropic plates to arbitrarily laminated anisotropic plates is due to Yang, Norris, and Stavsky [8] and Whitney and Pagano [9]. Extensions of the classical von Karman plate theory [10] to shear deformation theories can be found in the works of Reissner [11,12] Medwadowski [13], Schmidt [14] and Reddy [15].

Higher-order, displacement-based, shear deformation theories have been investigated by Librescu [16] and Lo, Christensen and Wu [17]. In

these higher-order theories, with each additional power of the thickness coordinate an additional dependent unknown is introduced into the theory. Levinson [18] and Murthy [19] presented third-order theories that assume transverse inextensibility. The nine displacement functions were reduced to five by requiring that the transverse shear stresses vanish on the bounding planes of the plate. However, both authors used the equilibrium equations of the first-order theory in their analysis. As a result, the higher-order terms of the displacement field are accounted for in the calculation of the strains but not in the governing differential equations or in the boundary conditions. These theories can be shown (see Librescu and Reddy [20]) to be the same as those described by Reissner [1-3]. Recently, Reddy [21-23] developed a new third-order plate theory, which is extended in the present study to laminated shells.

## 1.2 Review of Literature

Shell structures are abundant on the earth and in space. Use of shell structures dates back to ancient Rome, where the roofs of the Pantheon can be classified today as thick shells. Shell structures for a long time have been built by experience and intuition. No logical and scientific study had been conducted on the design of shells until the eighteenth century.

The earliest need for design criterion for shell structures probably came with the development of the steam engine and its attendant accessories. However, it was not until 1888, by Love [24], that the first general theory was presented. Subsequent theoretical efforts have



been directed towards improvements of Love's formulation and the solutions of the associated differential equations.

An ideal theory of shells require a realistic modeling of the actual geometry and material properties, and an appropriate description of the deformation. Even if we can take care of these requirements and derive the governing equations, analytical solutions to most shell problems are nevertheless limited in scope, and in general do not apply to arbitrary shapes, load distributions, and boundary conditions. Consequently, numerical approximation methods must be used to predict the actual behavior.

Many of the classical theories were developed originally for thin elastic shells, and are based on the Love-Kirchhoff assumptions (or the first approximation theory): (1) plane sections normal to the undeformed middle surface remain plane and normal to the deformed middle surface, (2) the normal stresses perpendicular to the middle surface can be neglected in the stress-strain relations, and (3) the transverse displacement is independent of the thickness coordinate. The first assumption leads to the neglect of the transverse shear strains.

Surveys of various classical shell theories can be found in the works of Naghdi [25] and Bert [26]. These theories, known as the Love's first approximation theories (see Love [24]) are expected to yield sufficiently accurate results when (i) the lateral dimension-to-thickness ratio ( $a/h$ ) is large; (ii) the dynamic excitations are within the low-frequency range; (iii) the material anisotropy is not severe. However, application of such theories to layered anisotropic composite

shells could lead to as much as 30% or more errors in deflections, stresses, and frequencies.

As noted by Koiter [27], refinements to Love's first approximation theory of thin elastic shells are meaningless, unless the effects of transverse shear and normal stresses are taken into account in the refined theory. The transverse normal stress is, in general, of order  $h/R$  (thickness to radius ratio) times the bending stresses, whereas the transverse shear stresses, obtained from equilibrium conditions, are of order  $h/a$  (thickness to the length of long side of the panel) times the bending stresses. Therefore, for  $a/R < 10$ , the transverse normal stress is negligible compared to the transverse shear stresses.

Ambartsumyan [28,29] was considered to be the first to analyze laminates that incorporated the bending-stretching coupling due to material anisotropy. The laminates that Ambartsumyan analyzed are now known as laminated orthotropic shells because the individual orthotropic layers were oriented such that the principal axes of material symmetry coincided with the principal coordinates of the shell reference surface. In 1962, Dong, Pister and Taylor [30] formulated a theory of thin shells laminated of anisotropic shells. Cheng and Ho [31] presented an analysis of laminated anisotropic cylindrical shells using Flugge's shell theory [32]. A first approximation theory for the unsymmetric deformation of nonhomogeneous, anisotropic, elastic cylindrical shells was derived by Widera and his colleagues [33,34] by means of asymptotic integration of the elasticity equation. For a homogeneous, isotropic material, the theory reduced to the Donnell's

equation. An exposition of various shell theories can be found in the article by Bert [26] and monograph by Librescu [35].

The effect of transverse shear deformation and transverse isotropy as well as thermal expansion through the shell thickness were considered by Gulati and Essenberg [36] and Zukas and Vinson [37]. Dong and Tso [38] presented a theory applicable to layered, orthotropic cylindrical shells. Whitney and Sun [39] developed a higher-order shear deformation theory. This theory is based on a displacement field in which the displacements in the surface of the shell are expanded as linear functions of the thickness coordinate and the transverse displacement is expanded as a quadratic function of the thickness coordinate. Recently, Reddy [40] presented a shear deformation version of the Sanders shell theory for laminated composite shells. Such theories account for constant transverse shear stresses through thickness, and therefore require a correction to the transverse shear stiffness.

As far as the finite element analysis of shells is concerned, the early works can be attributed to those of Dong [41], Dong and Selna [42], Wilson and Parsons [43], and Schmit and Monforton [44]. The studies in [41-44] are confined to the analysis of orthotropic shells of revolution. Other finite element analyses of laminated anisotropic composite shells include the works of Panda and Natarajan [45], Shivakumar and Krishna Murty [46], Rao [47], Seide and Chang [48], Venkatesh and Rao [49], Reddy and his colleagues [15,50-51], and Noor and his colleagues [52-54].

### 1.3 Present Study

While the three-dimensional theories [55-60] give more accurate results than the lamination (classical or shear deformation) theories, they are intractable. For example, the 'local' theory of Pagano [60] results in a mathematical model consisting of  $23N$  partial differential equations in the laminate's midplane coordinates and  $7N$  edge boundary conditions, where  $N$  is the number of layers in the laminate. The computational costs, especially for geometrically nonlinear problems or transient analysis using the finite element method, preclude the use of such a theory. As demonstrated by Reddy [21-23] and his colleagues [61-63], the refined plate theory provides improved global response estimates for deflections, vibration frequencies and buckling loads for laminated composite plates. The present study, motivated by the above findings, deals with the extension of the third-order plate theory of Reddy [21-23] to laminated composite shells. The theory also accounts for the von Karman strains. The resulting theory contains, as special cases, the classical and first-order theories of plates and shells. Mixed variational formulations and associated finite-element models are developed in this study. The significant and novel contributions of the research conducted (in addition to those reported in the first year's report [23]) are:

1. The formulation of a new third-order theory of laminated shells that accounts for a parabolic distribution of the transverse shear stresses and the von Karman strains.
2. The derivation of the exact solutions of the new theory for certain simply supported laminated composite shells.

3. The development of a mixed variational principle for the new theory of shells, which yields as special cases those of the classical (e.g., Love-Kirchhoff) and the first-order theory.
4. The development of a mixed,  $C^0$ -finite-element and its application to the bending and vibration analysis of laminated composite shells.

## 2. FORMULATION OF THE NEW THEORY

### 2.1 Kinematics

Let  $(\xi_1, \xi_2, \zeta)$  denote the orthogonal curvilinear coordinates (or shell coordinates) such that the  $\xi_1$ - and  $\xi_2$ -curves are lines of curvature on the midsurface  $\zeta=0$ , and  $\zeta$ -curves are straight lines perpendicular to the surface  $\zeta=0$ . For cylindrical and spherical shells the lines of principal curvature coincide with the coordinate lines. The values of the principal radii of curvature of the middle surface are denoted by  $R_1$  and  $R_2$ .

The position vector of a point on the middle surface is denoted by  $\underline{r}$ , and the position of a point at distance  $\zeta$  from the middle surface is denoted by  $\underline{R}$ . The distance  $ds$  between points  $(\xi_1, \xi_2, 0)$  and  $(\xi_1+d\xi_1, \xi_2+d\xi_2, 0)$  is determined by (see Fig. 2.1)

$$\begin{aligned} (ds)^2 &= d\underline{r} \cdot d\underline{r} \\ &= \alpha_1^2 (d\xi_1)^2 + \alpha_2^2 (d\xi_2)^2 \end{aligned} \quad (2.1)$$

where  $d\underline{r} = \underline{g}_1 d\xi_1 + \underline{g}_2 d\xi_2$ , the vectors  $\underline{g}_1$  and  $\underline{g}_2$  ( $\underline{g}_i = \frac{\partial \underline{r}}{\partial \xi_i}$ ) are tangent to the  $\xi_1$  and  $\xi_2$  coordinate lines, and  $\alpha_1$  and  $\alpha_2$  are the surface metrics

$$\alpha_1^2 = \underline{g}_1 \cdot \underline{g}_1, \quad \alpha_2^2 = \underline{g}_2 \cdot \underline{g}_2. \quad (2.2)$$

The distance  $dS$  between points  $(\xi_1, \xi_2, \zeta)$  and  $(\xi_1+d\xi_1, \xi_2+d\xi_2, \zeta+d\zeta)$  is given by

$$\begin{aligned} (dS)^2 &= d\underline{R} \cdot d\underline{R} \\ &= L_1^2 (d\xi_1)^2 + L_2^2 (d\xi_2)^2 + L_3^2 (d\zeta)^2 \end{aligned} \quad (2.3)$$

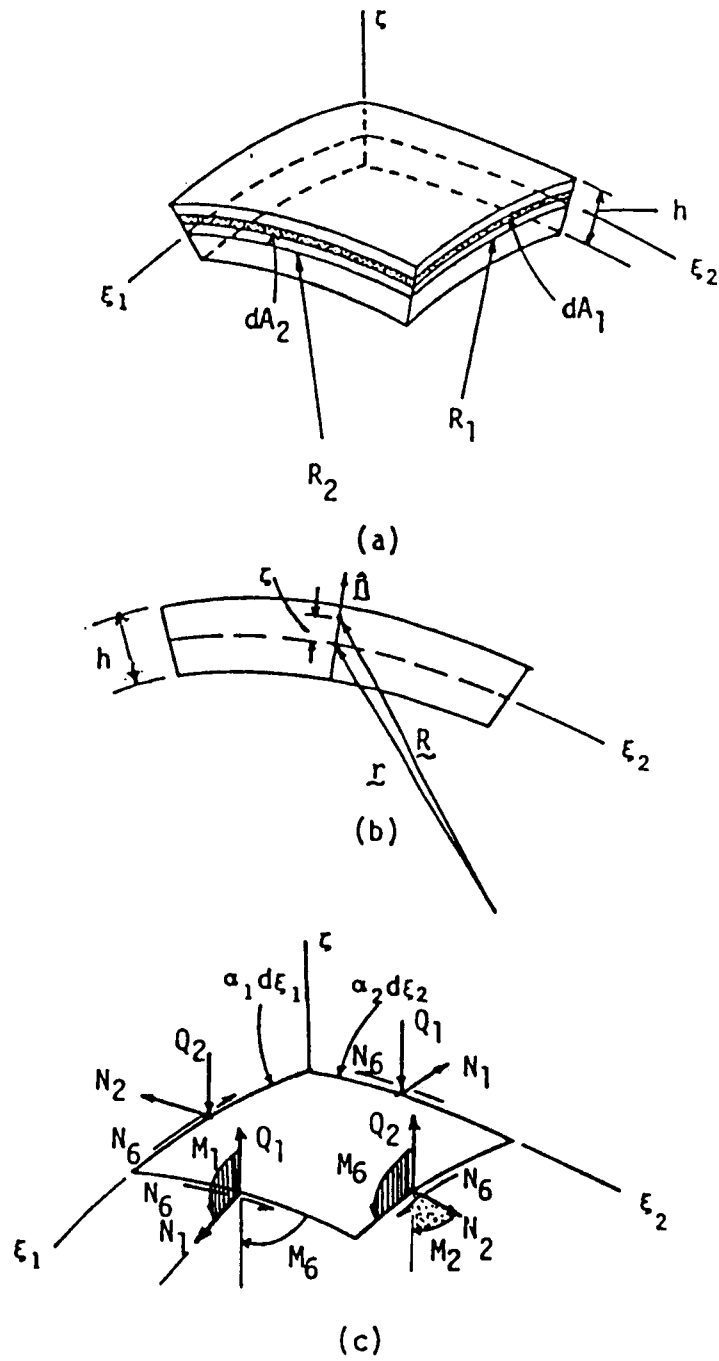


Figure 2.1 Geometry and stress resultants of a shell

where  $d\tilde{R} = \frac{\partial \tilde{R}}{\partial \xi_1} d\xi_1 + \frac{\partial \tilde{R}}{\partial \xi_2} d\xi_2 + \frac{\partial \tilde{R}}{\partial \zeta} d\zeta$ , and  $L_1$ ,  $L_2$  and  $L_3$  are the Lame' coefficients

$$L_1 = \alpha_1 \left(1 + \frac{\xi}{R_1}\right), \quad L_2 = \alpha_2 \left(1 + \frac{\xi}{R_2}\right), \quad L_3 = 1 \quad (2.4)$$

It should be noted that the vectors  $\tilde{G}_1 \equiv \frac{\partial \tilde{R}}{\partial \xi_1}$  and  $\tilde{G}_2 \equiv \frac{\partial \tilde{R}}{\partial \xi_2}$  are parallel to the vectors  $\tilde{g}_1$  and  $\tilde{g}_2$ , respectively.

## 2.2 Displacement Field

The shell under consideration is composed of a finite number of orthotropic layers of uniform thickness (see Fig. 2.2). Let  $N$  denote the number of layers in the shell, and  $\zeta_k$  and  $\zeta_{k-1}$  be the top and bottom  $\zeta$ -coordinates of the  $k$ -th layer. Before we proceed, a set of simplifying assumptions that provides a reasonable description of the behaviors are as follows:

1. thickness to radius and other dimensions of shell are small.
2. transverse normal stress is negligible
3. strains are small, yet displacements can be moderately large compared to thickness

Following the procedure similar to that presented in [21] for flat plates, we begin with the following displacement field:

$$\bar{u}(\xi_1, \xi_2, \zeta, t) = \left(1 + \frac{\xi}{R_1}\right)u + \zeta\phi_1 + \zeta^2\psi_1 + \zeta^3\theta_1$$

$$\bar{v}(\xi_1, \xi_2, \zeta, t) = \left(1 + \frac{\xi}{R_2}\right)v + \zeta\phi_2 + \zeta^2\psi_2 + \zeta^3\theta_2$$

$$\bar{w}(\xi_1, \xi_2, \zeta, t) = w \quad (2.5)$$

where  $t$  is time,  $(\bar{u}, \bar{v}, \bar{w})$  are the displacements along the  $(\xi_1, \xi_2, \zeta)$



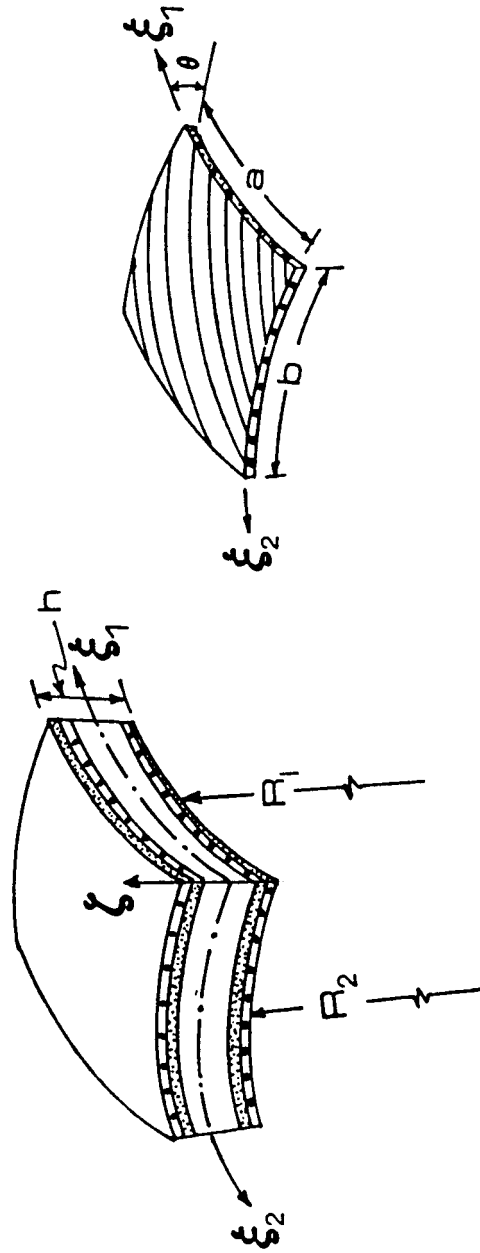


Figure 2.2 Geometry of a laminated doubly-curved shell

coordinates,  $(u, v, w)$  are the displacements of a point on the middle surface and  $\phi_1$  and  $\phi_2$  are the rotations at  $z = 0$  of normals to the midsurface with respect to the  $\xi_2$ - and  $\xi_1$ -axes, respectively. All of  $(u, v, w, \phi_1, \phi_2, \psi_1, \psi_2, \theta_1, \theta_2)$  are functions of  $\xi_1$  and  $\xi_2$  only. The particular choice of the displacement field in Eq. (2.5) is dictated by the desire to represent the transverse shear strains by quadratic functions of the thickness coordinate,  $z$ , and by the requirement that the transverse normal strain be zero. The kinematics of deformation of a transverse normal in various theories is shown in Fig. 2.3.

The functions  $\psi_i$  and  $\theta_i$  will be determined using the condition that the transverse shear stresses,  $\sigma_{13} = \sigma_5$  and  $\sigma_{23} = \sigma_4$  vanish on the top and bottom surfaces of the shell:

$$\sigma_5(\xi_1, \xi_2, \pm \frac{h}{2}, t) = 0 \quad , \quad \sigma_4(\xi_1, \xi_2, \pm \frac{h}{2}, t) = 0 \quad (2.6)$$

These conditions are equivalent to, for shells laminated of orthotropic layers, the requirement that the corresponding strains be zero on these surfaces. The transverse shear strains of a shell with two principal radii of curvature are given by

$$\begin{aligned} \epsilon_5 &= \frac{\partial \bar{u}}{\partial z} + \frac{1}{\alpha_1} \frac{\partial \bar{w}}{\partial \xi_1} - \frac{u}{R_1} \\ &= \frac{u}{R_1} + \phi_1 + 2z\psi_1 + 3z^2\theta_1 + \frac{1}{\alpha_1} \frac{\partial w}{\partial \xi_1} - \frac{u}{R_1} \\ \epsilon_4 &= \frac{\partial \bar{v}}{\partial z} + \frac{1}{\alpha_2} \frac{\partial \bar{w}}{\partial \xi_2} - \frac{v}{R_2} \\ &= \frac{v}{R_2} + \phi_2 + 2z\psi_2 + 3z^2\theta_2 + \frac{1}{\alpha_2} \frac{\partial w}{\partial \xi_2} - \frac{v}{R_2} \end{aligned} \quad (2.7)$$

Setting  $\epsilon_5(\xi_1, \xi_2, \pm \frac{h}{2}, t)$  and  $\epsilon_4(\xi_1, \xi_2, \pm \frac{h}{2}, t)$  to zero, we obtain

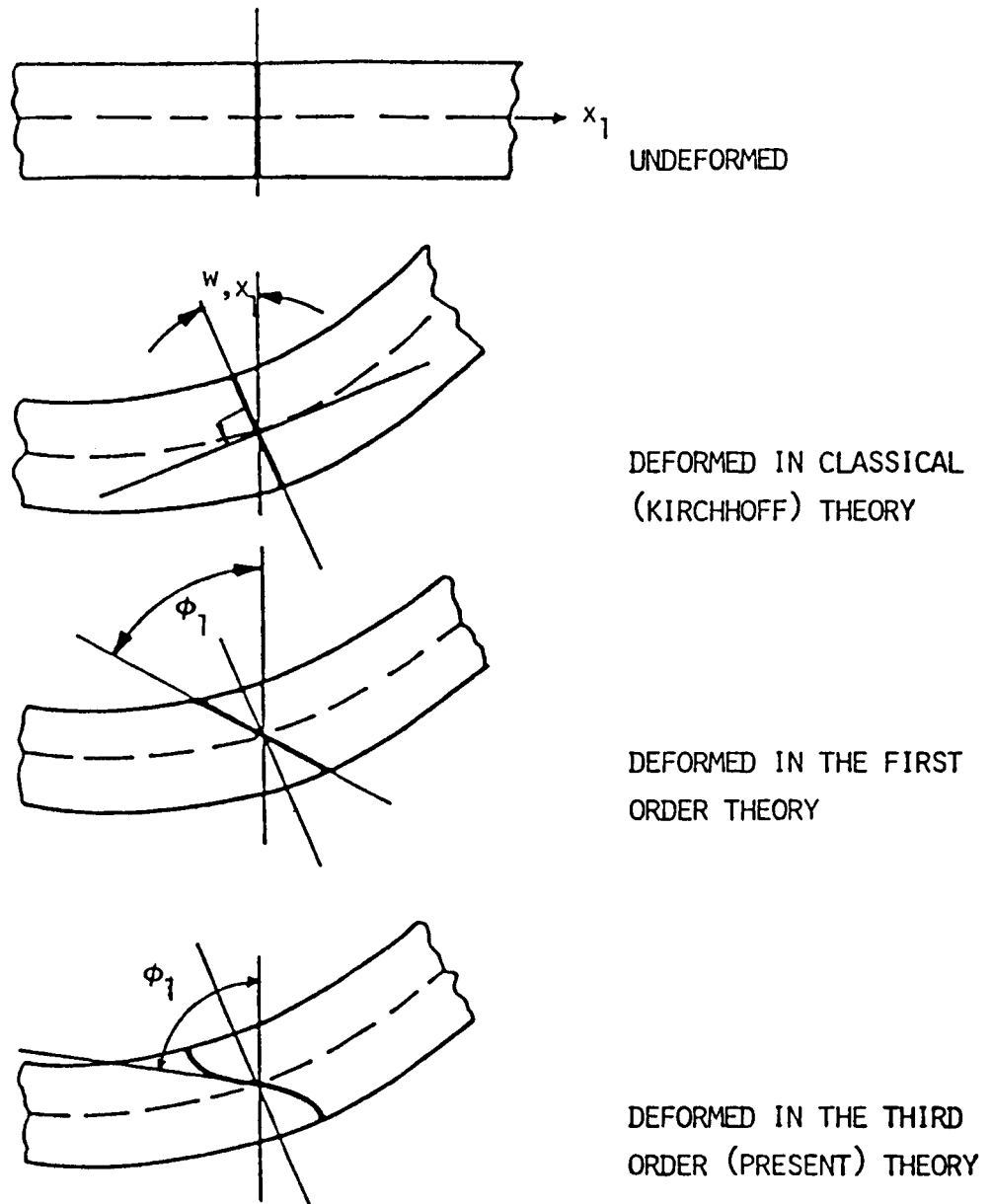


Figure 2.3 Assumed deformation patterns of the transverse normals in various displacement-based theories.

$$\begin{aligned}\psi_1 &= \psi_2 = 0 \\ \theta_1 &= -\frac{4}{3h^2} \left( \phi_1 + \frac{1}{\alpha_1} \frac{\partial w}{\partial \xi_1} \right) \\ \theta_2 &= -\frac{4}{3h^2} \left( \phi_2 + \frac{1}{\alpha_2} \frac{\partial w}{\partial \xi_2} \right)\end{aligned}\quad (2.8)$$

Substituting Eq. (2.8) into Eq. (2.5), we obtain

$$\begin{aligned}\bar{u} &= \left(1 + \frac{\zeta}{R_1}\right)u + \zeta\phi_1 + \zeta^3 \frac{4}{3h^2} \left[-\phi_1 - \frac{1}{\alpha_1} \frac{\partial w}{\partial \xi_1}\right] \\ \bar{v} &= \left(1 + \frac{\zeta}{R_2}\right)v + \zeta\phi_2 + \zeta^3 \frac{4}{3h^2} \left[-\phi_2 - \frac{1}{\alpha_2} \frac{\partial w}{\partial \xi_2}\right]\end{aligned}\quad (2.9)$$

This displacement field is used to compute the strains and stresses, and then the equations of motion are obtained using the dynamic analog of the principle of virtual work.

### 2.3 Strain-Displacement Relations

Substituting Eq. (2.9) into the strain-displacement relations referred to an orthogonal curvilinear coordinate system, we obtain

$$\begin{aligned}\epsilon_1 &= \epsilon_1^0 + \zeta(\kappa_1^0 + \zeta^2 \kappa_1^2) \\ \epsilon_2 &= \epsilon_2^0 + \zeta(\kappa_2^0 + \zeta^2 \kappa_2^2) \\ \epsilon_4 &= \epsilon_4^0 + \zeta^2 \kappa_4^1 \\ \epsilon_5 &= \epsilon_5^0 + \zeta^2 \kappa_5^1 \\ \epsilon_6 &= \epsilon_6^0 + \zeta(\kappa_6^0 + \zeta^2 \kappa_6^2)\end{aligned}\quad (2.10)$$

where

$$\begin{aligned}
\varepsilon_1^0 &= \frac{\partial u}{\partial x_1} + \frac{w}{R_1} + \frac{1}{2} \left( \frac{\partial w}{\partial x_1} \right)^2, \quad \kappa_1^0 = \frac{\partial \phi_1}{\partial x_1}, \quad \kappa_1^2 = -\frac{4}{3h^2} \left( \frac{\partial \phi_1}{\partial x_1} + \frac{\partial^2 w}{\partial x_1^2} \right) \\
\varepsilon_2^0 &= \frac{\partial v}{\partial x_2} + \frac{w}{R_2} + \frac{1}{2} \left( \frac{\partial w}{\partial x_2} \right)^2, \quad \kappa_2^0 = \frac{\partial \phi_2}{\partial x_2}, \quad \kappa_2^2 = -\frac{4}{3h^2} \left( \frac{\partial \phi_2}{\partial x_2} + \frac{\partial^2 w}{\partial x_2^2} \right) \\
\varepsilon_4^0 &= \phi_2 + \frac{\partial w}{\partial x_2} \quad \kappa_4^1 = -\frac{4}{h^2} \left( \phi_2 + \frac{\partial w}{\partial x_2} \right) \\
\varepsilon_5^0 &= \phi_1 + \frac{\partial w}{\partial x_1} \quad \kappa_5^1 = -\frac{4}{h^2} \left( \phi_1 + \frac{\partial w}{\partial x_1} \right) \\
\varepsilon_6^0 &= \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2}, \quad \kappa_6^0 = \frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2} \\
\kappa_6^2 &= -\frac{4}{3h^2} \left( \frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2} + 2 \frac{\partial^2 w}{\partial x_1 \partial x_2} \right)
\end{aligned} \tag{2.11}$$

Here  $x_i$  denote the cartesian coordinates ( $dx_i = \alpha_i d\varepsilon_i$ ,  $i = 1, 2$ ).

#### 2.4 Constitutive Relations

The stress-strain relations for the  $k$ -th lamina in the lamina coordinates are given by

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{pmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11}^{(k)} & \bar{Q}_{12}^{(k)} & 0 & 0 & 0 \\ & \bar{Q}_{22}^{(k)} & 0 & 0 & 0 \\ & & \bar{Q}_{66}^{(k)} & 0 & 0 \\ \text{symm.} & & & \bar{Q}_{44}^{(k)} & \bar{Q}_{45}^{(k)} \\ & & & & \bar{Q}_{55}^{(k)} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{pmatrix}^{(k)} \tag{2.12}$$

where  $\bar{Q}_{ij}^{(k)}$  are the plane stress reduced stiffnesses of the  $k$ -th lamina

in the lamina coordinate system. The coefficients  $\bar{Q}_{ij}$  can be expressed in terms of the engineering constants of a lamina:

$$\bar{Q}_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} \quad \bar{Q}_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}$$

$$\bar{Q}_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \quad (2.13)$$

$$\bar{Q}_{44} = G_{23} \quad \bar{Q}_{55} = G_{13} \quad \bar{Q}_{66} = G_{12}$$

To determine the laminate constitutive equations, Eq. (2.12) should be transformed to the laminate coordinates. We obtain

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}_{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}_{(k)} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix}_{(k)}, \quad \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix}_{(k)} = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix}_{(k)} \begin{Bmatrix} \epsilon_4 \\ \epsilon_5 \end{Bmatrix}_{(k)} \quad (2.14)$$

where

$$Q_{11} = \bar{Q}_{11} \cos^4 \theta + 2(\bar{Q}_{12} + 2\bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{22} \sin^4 \theta$$

$$Q_{12} = (\bar{Q}_{11} + \bar{Q}_{22} - 4\bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{12} (\sin^4 \theta + \cos^4 \theta)$$

$$Q_{22} = \bar{Q}_{11} \sin^4 \theta + 2(\bar{Q}_{12} + 2\bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{22} \cos^4 \theta$$

$$Q_{16} = (\bar{Q}_{11} - \bar{Q}_{12} - 2\bar{Q}_{66}) \sin \theta \cos^3 \theta + (\bar{Q}_{12} - \bar{Q}_{22} + 2\bar{Q}_{66}) \sin^3 \theta \cos \theta$$

$$Q_{26} = (\bar{Q}_{11} - \bar{Q}_{12} - 2\bar{Q}_{66}) \sin^3 \theta \cos \theta + (\bar{Q}_{12} - \bar{Q}_{22} + 2\bar{Q}_{66}) \sin \theta \cos^3 \theta$$

$$Q_{66} = (\bar{Q}_{11} + \bar{Q}_{22} - 2\bar{Q}_{12} - 2\bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{66} (\sin^4 \theta + \cos^4 \theta)$$

$$Q_{44} = \bar{Q}_{44} \cos^2 \theta + \bar{Q}_{55} \sin^2 \theta$$

$$Q_{45} = (\bar{Q}_{55} - \bar{Q}_{44}) \cos \theta \sin \theta$$

$$Q_{55} = \bar{Q}_{55} \cos^2 \theta + \bar{Q}_{44} \sin^2 \theta \quad (2.15)$$

### 2.5 Equations of Motion

The dynamic version of the principle of virtual work for the present case yields

$$\begin{aligned} 0 = & \int_0^t \left\{ \int_{-h/2}^{h/2} \int_{\Omega} \{ \int_{\Omega}^{(k)} [\sigma_1 \delta \epsilon_1^{(k)} + \sigma_2 \delta \epsilon_2^{(k)} + \sigma_6 \delta \epsilon_6^{(k)} + \sigma_4 \delta \epsilon_4^{(k)} + \sigma_5 \delta \epsilon_5^{(k)}] \right. \\ & \left. dx_1 dx_2 \right\} d\tau - \int_{\Omega} q \delta w dx_1 dx_2 - \delta \left( \int_{-h/2}^{h/2} \int_{\Omega} \{ \int_{\Omega}^{(k)} \rho [(\dot{u})^2 + (\dot{v})^2 + (\dot{w})^2] dx_1 dx_2 \} d\tau \right) dt \\ = & \int_0^t \left\{ \int_{\Omega} \left[ [N_1 \delta \epsilon_1^0 + M_1 \delta \kappa_1^0 + P_1 \delta \kappa_1^2 + N_2 \delta \epsilon_2^0 + M_2 \delta \kappa_2^0 \right. \right. \\ & \left. \left. + P_2 \delta \kappa_2^2 + N_6 \delta \epsilon_6^0 + M_6 \delta \kappa_6^0 + P_6 \delta \kappa_6^2 + Q_2 \delta \epsilon_4^0 + K_2 \delta \kappa_4^1 + Q_1 \delta \epsilon_5^0 + K_1 \delta \kappa_5^1 - q \delta w \right] \right. \\ & \left. + \left[ (\bar{I}_1 \ddot{u} + \bar{I}_2 \ddot{\phi}_1 - \bar{I}_3 \frac{\partial \ddot{w}}{\partial x_1}) \delta u + (\bar{I}_1' \ddot{v} + \bar{I}_2' \ddot{\phi}_2 - \bar{I}_3' \frac{\partial \ddot{w}}{\partial x_2}) \delta v \right. \right. \\ & \left. \left. + (\bar{I}_3 \frac{\partial \ddot{u}}{\partial x_1} + \bar{I}_5 \frac{\partial \ddot{\phi}_1}{\partial x_1} + \bar{I}_3' \frac{\partial \ddot{v}}{\partial x_2} + \bar{I}_5' \frac{\partial \ddot{\phi}_2}{\partial x_2} - \frac{16I_7}{9h^4} (\frac{\partial^2 \ddot{w}}{\partial x_1^2} + \frac{\partial^2 \ddot{w}}{\partial x_2^2}) + I_1 \ddot{w}) \delta w \right. \right. \\ & \left. \left. + (\bar{I}_2 \ddot{u} + \bar{I}_4 \ddot{\phi}_1 - \bar{I}_5 \frac{\partial \ddot{w}}{\partial x_1}) \delta \phi_1 + (\bar{I}_2' \ddot{v} + \bar{I}_4' \ddot{\phi}_2 - \bar{I}_5' \frac{\partial \ddot{w}}{\partial x_2}) \delta \phi_2 \right] dx_1 dx_2 \right\} dt \end{aligned} \quad (2.16)$$

where  $q$  is the distributed transverse load,  $N_i$ ,  $M_i$ , etc. are the resultants,

$$\begin{aligned} (N_i, M_i, P_i) &= \sum_{k=1}^n \int_{\zeta_{k-1}}^{\zeta_k} \sigma_i^{(k)}(1, \zeta, \zeta^3) d\zeta \quad (i = 1, 2, 6) \\ (Q_1, K_1) &= \sum_{k=1}^n \int_{\zeta_{k-1}}^{\zeta_k} \sigma_5^{(k)}(1, \zeta^2) d\zeta \end{aligned}$$

$$(Q_2, K_2) = \sum_{k=1}^n \int_{\zeta_{k-1}}^{\zeta_k} \sigma_4^{(k)}(1, \zeta^2) d\zeta \quad (2.17)$$

The inertias  $\bar{I}_i$  and  $\bar{I}'_i$  ( $i = 1, 2, 3, 4, 5$ ) are defined by the equations,

$$\bar{I}_1 = I_1 + \frac{2}{R_1} I_2 ,$$

$$\bar{I}'_2 = I_1 + \frac{2}{R_2} I_2$$

$$\bar{I}_2 = I_2 + \frac{1}{R_1} I_3 - \frac{4}{3h^2} I_4 - \frac{4}{3h^2 R_1} I_5 ,$$

$$\bar{I}'_2 = I_2 + \frac{1}{R_2} I_3 - \frac{4}{3h^2} I_4 - \frac{4}{3h^2 R_2} I_5$$

$$\bar{I}_3 = \frac{4}{3h^2} I_4 + \frac{4}{3h^2 R_1} I_5 ,$$

$$\bar{I}'_3 = \frac{4}{3h^2} I_4 + \frac{4}{3h^2 R_2} I_5$$

$$\bar{I}_4 = I_3 - \frac{8}{3h^2} I_5 + \frac{16}{9h^4} I_7$$

$$\bar{I}'_4 = I_3 - \frac{8}{3h^2} I_5 + \frac{16}{9h^4} I_7$$

$$\bar{I}_5 = \frac{4}{3h^2} I_5 - \frac{16}{9h^4} I_7 ,$$

$$\bar{I}'_5 = \frac{4}{3h^2} I_5 - \frac{16}{9h^4} I_7 \quad (2.18a)$$

and



$$(I_1, I_2, I_3, I_4, I_5, I_7) = \sum_{k=1}^n \int_{\zeta_{k-1}}^{\zeta_k} \rho^{(k)}(1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^6) d\zeta \quad (2.18b)$$

The governing equations of motion can be derived from Eq. (2.16) by integrating the displacement gradients by parts and setting the coefficients of  $\delta u, \delta v, \delta w$  and  $\delta \phi_i$  ( $i = 1, 2$ ) to zero separately:

$$\begin{aligned} \delta u: \quad \frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} &= \bar{I}_1 \ddot{u} + \bar{I}_2 \ddot{\phi}_1 - \bar{I}_3 \frac{\partial \ddot{w}}{\partial x_1} \\ \delta v: \quad \frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} &= \bar{I}_1' \ddot{v} + \bar{I}_2' \ddot{\phi}_2 - \bar{I}_3' \frac{\partial \ddot{w}}{\partial x_2} \\ \delta w: \quad \frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} - \frac{4}{h^2} \left( \frac{\partial K_1}{\partial x_1} + \frac{\partial K_2}{\partial x_2} \right) + \frac{4}{3h^2} \left( \frac{\partial^2 P_1}{\partial x_1^2} + \frac{\partial^2 P_2}{\partial x_2^2} + 2 \frac{\partial^2 P_6}{\partial x_1 \partial x_2} \right) \\ &\quad - \frac{N_1}{R_1} - \frac{N_2}{R_2} + N(w) = \bar{I}_3 \frac{\partial \ddot{u}}{\partial x_1} - \bar{I}_5 \frac{\partial \ddot{\phi}_1}{\partial x_1} + \bar{I}_3' \frac{\partial \ddot{v}}{\partial x_2} + \bar{I}_5' \frac{\partial \ddot{\phi}_2}{\partial x_2} + \bar{I}_1 \ddot{w} \\ &\quad - \frac{16I_7}{9h^4} \left( \frac{\partial^2 \ddot{w}}{\partial x_1^2} + \frac{\partial^2 \ddot{w}}{\partial x_2^2} \right) - q \\ \delta \phi_1: \quad \frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} - Q_1 + \frac{4}{h^2} K_1 - \frac{4}{3h^2} \left( \frac{\partial P_1}{\partial x_1} + \frac{\partial P_6}{\partial x_2} \right) &= \bar{I}_2 \ddot{u} + \bar{I}_4 \ddot{\phi}_1 - \bar{I}_5 \frac{\partial \ddot{w}}{\partial x_1} \\ \delta \phi_2: \quad \frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} - Q_2 + \frac{4}{h^2} K_2 - \frac{4}{3h^2} \left( \frac{\partial P_6}{\partial x_1} + \frac{\partial P_2}{\partial x_2} \right) &= \bar{I}_2' \ddot{v} + \bar{I}_4' \ddot{\phi}_2 - \bar{I}_5' \frac{\partial \ddot{w}}{\partial x_2} \end{aligned} \quad (2.19)$$

where

$$N(w) = \frac{\partial}{\partial x_1} \left( N_1 \frac{\partial w}{\partial x_1} + N_6 \frac{\partial w}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( N_6 \frac{\partial w}{\partial x_1} + N_2 \frac{\partial w}{\partial x_2} \right) \quad (2.20)$$

The essential (i.e., geometric) and natural boundary conditions of the theory are given by:

$$u_n, u_{ns}, w, \frac{\partial w}{\partial n}, \frac{\partial w}{\partial s}, \phi_n, \phi_{ns} \quad (\text{essential})$$

$$N_n, N_{ns}, \tilde{Q}_n, \tilde{P}_n, \tilde{P}_{ns}, \tilde{M}_n, \tilde{M}_{ns} \quad (\text{natural}) \quad (2.21)$$

where

$$N_n = n_x^2 N_1 + n_y^2 N_2 + 2n_x n_y N_6$$

$$N_{ns} = (N_2 - N_1)n_x n_y + N_6(n_x^2 - n_y^2)$$

$$P_n = n_x^2 P_1 + n_y^2 P_2 + 2n_x n_y P_6$$

$$P_{ns} = (P_2 - P_1)n_x n_y + P_6(n_x^2 - n_y^2)$$

$$M_n = n_x^2 M_1 + n_y^2 M_2 + 2n_x n_y M_6$$

$$M_{ns} = (M_2 - M_1)n_x n_y + M_6(n_x^2 - n_y^2)$$

$$\tilde{Q}_n = N_n \frac{\partial w}{\partial n} + N_{ns} \frac{\partial w}{\partial s} + \frac{4}{3h^2} \left( \frac{\partial P_n}{\partial n} + \frac{\partial P_{ns}}{\partial s} \right) + Q_n - \frac{4}{h^2} K_n$$

$$Q_n = n_x Q_1 + n_y Q_2$$

$$K_n = n_x K_1 + n_y K_2$$

$$\tilde{P}_n = -\frac{4}{3h^2} P_n$$

$$\tilde{P}_{ns} = -\frac{4}{3h^2} P_{ns}$$

$$\tilde{M}_n = M_n - \frac{4}{3h^2} P_n$$

$$\tilde{M}_{ns} = M_{ns} - \frac{4}{3h^2} P_{ns} \quad (2.22)$$

and  $n_x$  and  $n_y$  are the direction cosines of the unit normal on the boundary of the laminate.

The resultants can be expressed in terms of the strain components using Eqs. (2.10) and (2.12) in Eq. (2.14). We get

$$\begin{aligned} N_i &= A_{ij}\epsilon_j^0 + B_{ij}\kappa_j^0 + E_{ij}\kappa_j^2 \\ M_i &= B_{ij}\epsilon_j^0 + D_{ij}\kappa_j^0 + F_{ij}\kappa_j^2 \\ P_i &= E_{ij}\epsilon_j^0 + F_{ij}\kappa_j^0 + H_{ij}\kappa_j^2 \end{aligned} \quad (i, j = 1, 2, 6) \quad (2.23)$$

$$\begin{aligned} Q_2 &= A_{4j}\epsilon_j^0 + D_{4j}\kappa_j^1 \\ Q_1 &= A_{5j}\epsilon_j^0 + D_{5j}\kappa_j^1 \\ K_2 &= D_{4j}\epsilon_j^0 + F_{4j}\kappa_j^1 \\ K_1 &= D_{5j}\epsilon_j^0 + F_{5j}\kappa_j^1 \end{aligned} \quad (j = 4, 5) \quad (2.24)$$

where  $A_{ij}$ ,  $B_{ij}$ , etc. are the laminate stiffnesses,

$$\begin{aligned} &(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) \\ &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} Q_{ij}^{(k)} (1, z, z^2, z^3, z^4, z^6) dz \end{aligned} \quad (2.25)$$

for  $i, j = 1, 2, 4, 5, 6$ .

### 3. THE NAVIER SOLUTIONS

#### 3.1 Introduction

Exact solutions of the partial differential equations (2.19) on arbitrary domains and for general conditions is not possible. However, for simply supported shells whose projection in the  $x_1x_2$ -plane is a rectangle, the linear version of Eq. (2.19) can be solved exactly, provided the lamination scheme is of antisymmetric cross-ply  $[0^\circ/90^\circ/0^\circ/90^\circ\dots]$  or symmetric cross-ply  $[0^\circ/90^\circ\dots]_s$  type. The Navier solution exists if the following stiffness coefficients are zero [21]:

$$\begin{aligned} A_{i6} = B_{i6} = D_{i6} = E_{i6} = F_{i6} = H_{i6} = 0 \quad , \quad (i = 1,2) \quad (3.1) \\ A_{45} = D_{45} = F_{45} = 0 \end{aligned}$$

The boundary conditions are assumed to be of the form,

$$\begin{aligned} u(x_1,0) = u(x_1,b) = v(0,x_2) = v(a,x_2) = 0 \\ w(x_1,0) = w(x_1,b) = w(0,x_2) = w(a,x_2) = 0 \\ \frac{\partial w}{\partial x_1}(x_1,0) = \frac{\partial w}{\partial x_1}(x_1,b) = \frac{\partial w}{\partial x_2}(0,x_2) = \frac{\partial w}{\partial x_2}(a,x_2) = 0 \\ N_2(x_1,0) = N_2(x_1,b) = N_1(0,x_2) = N_1(a,x_2) = 0 \\ M_2(x_1,0) = M_2(x_1,b) = M_1(0,x_2) = M_1(a,x_2) = 0 \quad (3.2) \\ P_2(x_1,0) = P_2(x_1,b) = P_1(0,x_2) = P_1(a,x_2) = 0 \\ \phi_1(x_1,0) = \phi_1(x_1,b) = \phi_2(0,x_2) = \phi_2(a,x_2) = 0 \end{aligned}$$

where  $a$  and  $b$  denote the lengths along the  $x_1$ - and  $x_2$ -directions, respectively (see Fig. 3.1).

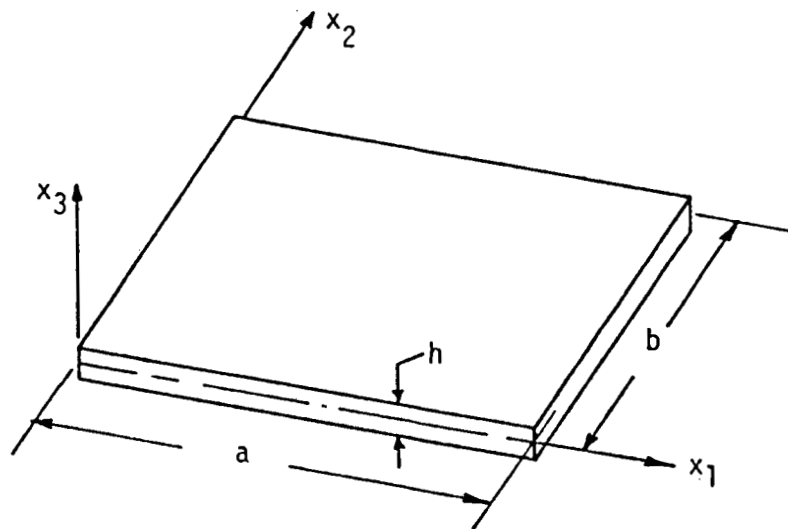


Figure 3.1 The geometry and the coordinate system for a projected area of shell element

### 3.2 The Navier Solution

Following the Navier solution procedure (see Reddy [21]), we assume the following solution form that satisfies the boundary conditions in Eq. (3.2):

$$\begin{aligned}
 u &= \sum_{m,n=1}^{\infty} U_{mn} f_1(x_1, x_2) \\
 v &= \sum_{m,n=1}^{\infty} V_{mn} f_2(x_1, x_2) \\
 w &= \sum_{m,n=1}^{\infty} W_{mn} f_3(x_1, x_2) \\
 \phi_1 &= \sum_{m,n=1}^{\infty} \phi_{mn}^1 f_1(x_1, x_2) \\
 \phi_2 &= \sum_{m,n=1}^{\infty} \phi_{mn}^2 f_2(x_1, x_2)
 \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
 f_1(x_1, x_2) &= \cos \alpha x_1 \sin \beta x_2, \quad f_2(x_1, x_2) = \sin \alpha x_1 \cos \beta x_2 \\
 f_3(x_1, x_2) &= \sin \alpha x_1 \sin \beta x_2, \quad \alpha = m\pi/a, \quad \beta = n\pi/b
 \end{aligned} \tag{3.4}$$

Substituting Eq. (3.3) into Eq. (2.19), we obtain

$$[M] \begin{Bmatrix} \ddot{U}_{mn} \\ \ddot{V}_{mn} \\ \ddot{W}_{mn} \\ \ddot{\phi}_{mn}^1 \\ \ddot{\phi}_{mn}^2 \end{Bmatrix} + [C] \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ \phi_{mn}^1 \\ \phi_{mn}^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ 0 \\ 0 \end{Bmatrix}, \quad \text{for any } m, n \tag{3.5}$$

where  $Q_{mn}$  are the coefficients in the double Fourier expansion of the transverse load,

$$q(x_1, x_2) = \sum_{m,n=1}^{\infty} Q_{mn} f_3(x_1, x_2) \quad (3.6)$$

and the coefficients  $M_{ij}$  and  $C_{ij}$  ( $i, j = 1, 2, \dots, 5$ ) are given in Appendix A.

Equation (3.5) can be solved for  $U_{mn}, V_{mn}$ , etc., for each  $m$  and  $n$ , and then the solution is given by Eq. (3.3). The series in Eq. (3.3) are evaluated using a finite number of terms in the series. For free vibration analysis, Eq. (3.5) can be expressed as an eigenvalue equation,

$$([C] - \omega^2 [M])\{\Delta\} = \{0\} \quad (3.7)$$

where  $\{\Delta\} = \{U_{mn}, V_{mn}, W_{mn}, \phi_{mn}^1, \phi_{mn}^2\}^T$ , and  $\omega$  is the frequency of natural vibration. For static bending analysis, Eq. (3.5) becomes

$$[C]\{\Delta\} = \begin{pmatrix} 0 \\ 0 \\ Q_{mn} \\ 0 \\ 0 \end{pmatrix} \quad (3.8)$$

## 4. MIXED VARIATIONAL PRINCIPLES

### 4.1 Introduction

The variational formulations used for developing plate and shell elements can be classified into three major categories:

(i) formulations based on the principle of virtual displacements (or the principle of total potential energy), (ii) formulations based on the principle of virtual forces (or the principle of complementary energy) and (iii) formulations based on mixed variational principles (Hu-Washizu-Reissner's principles). The finite-element models based on these formulations are called, respectively, the displacement models, equilibrium models and mixed models.

In the principle of virtual displacements, one assumes that the kinematic relations (i.e., strain-displacement relations and geometric boundary conditions) are satisfied exactly (i.e., point-wise) by the displacement field, and the equilibrium equations and force boundary conditions are derived as the Euler equations. No a-priori assumption concerning the constitutive behavior (i.e., stress-strain relations) is necessary in using the principle. The principle of (the minimum) total potential energy is a special case of the principle of virtual displacements applied to solid bodies that are characterized by the strain energy function  $U$  such that

$$\frac{\partial U(\epsilon_{ij})}{\partial \epsilon_{ij}} = \sigma_{ij}$$

where  $\epsilon_{ij}$  and  $\sigma_{ij}$  are the components of strain and stress tensors, respectively. When the principle of total potential energy is used to



develop a finite-element model of a shell theory, the kinematic relations are satisfied point-wise but the equilibrium equations are met only in an integral (or variational) sense. The principle of complementary potential energy, a special case of the principle of virtual forces, can be used to develop equilibrium models that satisfy the equilibrium equations point-wise but meet the strain compatibility only in a variational sense.

There are a number of mixed variational principles in elasticity (see [64-71]). The phrase 'mixed' is used to imply the fact that both the displacement (or primal) variables and (some of the) force (or dual) variables are given equal importance in the variational formulations. The associated finite-element models use independent approximations of dependent variables appearing in the variational formulation. There are two kinds of mixed models: (i) models in which both primal and dual variables are interpolated independently in the interior and on the boundary of an element, (ii) models in which the variables are interpolated inside the element and their values on the boundary are interpolated by the boundary values of the interpolation. The first kind are often termed hybrid models, and the second kind are known simply as the mixed models. In the present study the second kind (i.e., mixed model) will be discussed.

#### 4.2 Variational Principles

To develop a mixed variational statement of the third-order laminate theory, we rewrite Eq. (2.23) in matrix notation as follows:

$$\{N\} = [A^*]\{\varepsilon^0\} + [B^*]\{M\} + [E^*]\{P\}$$

$$\begin{aligned}
 -\{\kappa^S\} &= [B^*]^t \{\epsilon^0\} - [D^*]\{M\} - [F^*]\{P\} \\
 -\{\kappa\} &= [E^*]^t \{\epsilon^0\} - [F^*]^t \{M\} - [H^*]\{P\}
 \end{aligned} \tag{4.1}$$

where

$$\{\kappa\} = -c_1(\{\kappa^S\} + \{\kappa^C\}), \quad \{\kappa^S\} = \begin{pmatrix} \frac{\partial \phi_1}{\partial x} \\ \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \end{pmatrix}$$

$$\{\kappa^C\} = \begin{pmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{pmatrix}, \quad \{\epsilon^0\} = \begin{pmatrix} \frac{\partial u}{\partial x} + \frac{w}{R_1} + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{w}{R_2} + \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{pmatrix}$$

$$c_1 = \frac{4}{3h^2} \text{ and } c_2 = \frac{4}{h^2}$$

$$[A^*] = [A] - \begin{Bmatrix} [B] \\ [E] \end{Bmatrix}^t \begin{bmatrix} [D] & [F] \\ [F] & [H] \end{bmatrix}^{-1} \begin{Bmatrix} [B] \\ [E] \end{Bmatrix} \quad (\text{symmetric})$$

$$[B^*] = [B][D^*] + [E][F^*] \quad (\text{not symmetric})$$

$$[E^*] = [B][F^*] + [E][H^*] \quad (\text{not symmetric})$$

$$\begin{bmatrix} [D^*] & [F^*] \\ [F^*] & [H^*] \end{bmatrix} \equiv \begin{bmatrix} [D] & [F] \\ [F] & [H] \end{bmatrix}^{-1} \quad (\text{symmetric}) \tag{4.2}$$

Note that in this part the notation  $x = x_1$  and  $y = x_2$  is used, and  $[ ]^t$  denotes the transpose of a vector or matrix.

To develop a mixed variational statement of the third-order laminate theory, the generalized displacements ( $u, v, w, \phi_1, \phi_2$ ) and generalized moments ( $M_1, M_2, M_6, P_1, P_2, P_6$ ) are treated as the dependent variables.

The mixed functional associated with the static version of Eqs. (2.19), (4.1) is given by  $[\Lambda = (u, v, w, \phi_1, \phi_2, M_1, M_2, M_6, P_1, P_2, P_6)]$ ,

$$\begin{aligned} \Pi_{H1}(\Lambda) = & \int_{\Omega} \left[ \frac{1}{2} \{\epsilon^0\}^t [A^*] \{\epsilon^0\} + \{\epsilon^0\}^t ([B^*] \{M\} + [E^*] \{P\}) \right. \\ & + \{M\}^t \{k^s\} - \{M\}^t [F^*] \{P\} - \frac{1}{2} \{M\}^t [D^*] \{M\} \\ & + \{P\}^t \{k\} - \frac{1}{2} \{P\}^t [H^*] \{P\} \\ & \left. + \frac{1}{2} \{\bar{\epsilon}^0\}^t [\hat{A}] \{\bar{\epsilon}^0\} - qw \right] dx dy \\ & - \int_{\Gamma} (\hat{N}_n u_n + \hat{N}_s u_s + \hat{Q}_n w + \hat{M}_n \phi_n + \hat{M}_s \phi_s) ds \end{aligned} \quad (4.3)$$

where the quantities with a hat over them denote specified values, and

$$Q_n = Q_1 n_x + Q_2 n_y$$

$$u_n = u n_x + v n_y, \quad u_s = -u n_y + v n_x$$

$$\frac{\partial}{\partial n} = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial s} = n_x \frac{\partial}{\partial y} - n_y \frac{\partial}{\partial x}$$

$$N_n = N_1 n_x^2 + 2N_6 n_x n_y + N_2 n_y^2, \quad N_s = (N_2 - N_1) n_x n_y + N_6 (n_x^2 - n_y^2) \quad (4.4)$$

$$\{\bar{\epsilon}_0\} = \begin{Bmatrix} \phi_1 + \frac{\partial w}{\partial x} \\ \phi_2 + \frac{\partial w}{\partial y} \end{Bmatrix}$$

$$[\hat{A}] = [\bar{A}] - 2c_2 [\bar{D}] + c_2^2 [\bar{F}] \quad (4.5)$$

with

$$[\bar{A}] = \begin{bmatrix} A_{55} & A_{45} \\ A_{45} & A_{44} \end{bmatrix} \quad [\bar{D}] = \begin{bmatrix} D_{55} & D_{45} \\ D_{45} & D_{44} \end{bmatrix} \quad [\bar{F}] = \begin{bmatrix} F_{55} & F_{45} \\ F_{45} & F_{44} \end{bmatrix}$$

Note that the bending moments ( $M_1, M_2, M_6$ ) do not enter the set of essential boundary conditions, and the resultants ( $P_1, P_2, P_6$ ) do not enter the set of natural boundary conditions also.

The (mixed) variational principle corresponding to the functional in Eq. (4.3) can be stated as follows. Of all possible configurations a laminate can assume, the one that renders the functional  $\pi_{H1}$  stationary also satisfies the equations of equilibrium (2.19) and the kinematic-constitutive equations [i.e., the last two equations of Eq. (4.3)].

We note that the variational statement in Eq. (4.3) contains the second-order derivatives of the transverse deflection  $w$  (through  $\{\kappa\}$ ). To relax the continuity requirements on  $w$ , we integrate the term  $\{P\}^t \{\kappa\}$  by parts to trade the differentiation to  $\{P\}$ . We obtain

$$\begin{aligned}
 \pi_{H2}(\Lambda) = & \int_{\Omega} \left\{ \frac{1}{2} \{\epsilon^0\}^t [A^*] \{\epsilon^0\} + \{\epsilon^0\}^t ([B^*] \{M\} + [E^*] \{P\}) \right. \\
 & - \{M\}^t [F^*] \{P\} - \frac{1}{2} \{M\}^t [D^*] \{M\} \\
 & + \{\kappa^s\}^t (\{M\} - c_1 \{P\}) - \frac{1}{2} \{P\}^t [H^*] \{P\} \\
 & + \frac{1}{2} \{\bar{\epsilon}^0\}^t [\hat{A}] \{\bar{\epsilon}^0\} + c_1 \left[ \left( \frac{\partial P_1}{\partial x} + \frac{\partial P_6}{\partial y} \right) \frac{\partial w}{\partial x} \right. \\
 & \left. + \left( \frac{\partial P_6}{\partial x} + \frac{\partial P_2}{\partial y} \right) \frac{\partial w}{\partial y} \right] - qw \} dx dy \\
 & - \int_{\Gamma} [\hat{N}_n u_n + \hat{N}_s u_s + \hat{Q}_n w + \hat{M}_n \phi_n + \hat{M}_s \phi_s \\
 & + c_1 (\hat{\theta}_n P_n + \hat{\theta}_s P_s)] ds
 \end{aligned} \tag{4.6}$$

$$\theta_n = \frac{\partial w}{\partial n}, \quad \theta_s = \frac{\partial w}{\partial s}$$

We note that the stress resultants  $P_n$  and  $P_s$  (hence  $P_1$ ,  $P_2$  and  $P_6$ ) enter the set of essential boundary conditions in the case of the second mixed formulation of the higher-order theory. As special cases, the mixed variational statements for the classical and the first-order theories can be obtained from Eq. (4.6) by setting appropriate terms to zero.

## 5. FINITE-ELEMENT MODEL

### 5.1 Introduction

The mixed variational principles presented in Chapter 4 can be used to develop mixed finite-element models, which contain the bending moments as the primary variables along with the generalized displacements. Historically, at least with regard to bending of plates, the first mixed formulation is attributed to Herrmann [72,73] and Hellan [74].

Since the bending moments do not enter the set of essential boundary conditions, they are not required to be continuous across interelement boundaries. Thus, one can develop mixed models with discontinuous (between elements) bilinear or higher-order approximation of the bending moments. The present section deals with the development of a mixed model based on  $\Pi_{H^2}$ .

### 5.2 Finite-Element Model

Let  $\Omega$ , the domain (i.e., the midplane) of the laminate, be represented by a collection of finite elements

$$\bar{\Omega} = \bigcup_{e=1}^E \bar{\Omega}^e, \quad \Omega^e \cap \Omega^f = \text{empty for } e \neq f \quad (5.1)$$

where  $\bar{\Omega} = \Omega \cup \Gamma$  is the closure of the open domain  $\Omega$  and  $\Gamma$  is its boundary. Over a typical element  $\Omega^e$ , each of the variables  $u$ ,  $v$ ,  $w$ , etc. are interpolated by expressions of the form

$$u = \sum_{j=1}^N u_j \psi_j, \quad v = \sum_{j=1}^N v_j \psi_j, \quad w = \sum_{j=1}^N w_j \psi_j \quad (5.2)$$

etc., where  $\{\psi_j\}$  denote the set of admissible interpolation functions and  $j$  denotes the number of functions (see Reddy and Putchu [64]). Although the same set of interpolation functions is used, for simplicity, to interpolate each of the dependent variables, in general, different sets of interpolation can be used for  $(u,v)$ ,  $w$ ,  $(\phi_x, \phi_y)$ ,  $(M_1, M_2, M_6)$  and  $(P_1, P_2, P_6)$ . From an examination of the variational statement in Eq. (4.6), it is clear that the linear, quadratic, etc., interpolation functions of the Lagrange type are admissible. The resulting element is called a  $C^0$ -element, because no derivative of the dependent variable is required to be continuous across interelement boundaries.

To develop the finite-element equations of a typical element, we first compute the strains, rotations, and resultants in terms of their finite element approximations. We have

$$\{\epsilon^O\} = \{\epsilon^{OL}\} + \{\epsilon^{ON}\} \quad (5.3)$$

$$\begin{aligned} \{\epsilon^{OL}\} &= \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{w}{R_1} \\ \frac{\partial v}{\partial y} + \frac{w}{R_2} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \{\psi_{i,x}\} & \{0\} \\ \{0\} & \{\psi_{i,y}\} \\ \{\psi_{i,y}\} & \{\psi_{i,x}\} \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{v\} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{R_1} \{\psi_i\} \\ \frac{1}{R_2} \{\psi_i\} \\ \{0\} \end{Bmatrix} \{w\} \\ &\equiv [H^L] \{\Delta^1\} + [H^O] \{\Delta^2\} \quad (5.4) \end{aligned}$$

$$\begin{aligned} \{\epsilon^{ON}\} &= \frac{1}{2} \begin{Bmatrix} \left(\frac{\partial w}{\partial x}\right)^2 \\ \left(\frac{\partial w}{\partial y}\right)^2 \\ 2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \{f_x \psi_{i,x}\} \\ \{f_y \psi_{i,y}\} \\ \{f_x \psi_{i,y} + f_y \psi_{i,x}\} \end{Bmatrix} \{w\} \\ &\equiv \frac{1}{2} [H^N] \{\Delta^2\} \end{aligned} \quad (5.5)$$

$$\{\delta \epsilon^{ON}\} = [H^N] \{\delta \Delta^2\}$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{bmatrix} \{\psi_i\} & \{0\} \\ \{0\} & \{\psi_i\} \end{bmatrix} \begin{Bmatrix} \{\phi_1\} \\ \{\phi_2\} \end{Bmatrix} \equiv [H^2] \{\Delta^3\} \quad (5.6)$$

$$\{\kappa^S\} = \begin{Bmatrix} \frac{\partial \phi_1}{\partial x} \\ \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \end{Bmatrix} \equiv [H^L] \{\Delta^3\} \quad (5.7)$$

$$\{\theta\} = \begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \{\psi_{i,x}\} \\ \{\psi_{i,y}\} \end{Bmatrix} \{w\} \equiv [H^1] \{\Delta^2\} \quad (5.8)$$

$$\{\bar{\epsilon}^O\} = \{\theta\} + \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = [[H^1] \quad [H^2]] \begin{Bmatrix} \{\Delta^2\} \\ \{\Delta^3\} \end{Bmatrix} \quad (5.9)$$

$$\{M\} = \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} \{\psi_i\} & \{0\} & \{0\} \\ \{0\} & \{\psi_i\} & \{0\} \\ \{0\} & \{0\} & \{\psi_i\} \end{bmatrix} \begin{Bmatrix} \{M_1\} \\ \{M_2\} \\ \{M_6\} \end{Bmatrix} \equiv [H^3] \{\Delta^4\} \quad (5.10)$$



$$\{Q\} = \begin{pmatrix} \frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} \\ \frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} \end{pmatrix} = \begin{bmatrix} \{\psi_{i,x}\} & \{0\} & \{\psi_{i,y}\} \\ \{0\} & \{\psi_{i,y}\} & \{\psi_{i,x}\} \end{bmatrix} \begin{pmatrix} \{M_1\} \\ \{M_2\} \\ \{M_6\} \end{pmatrix} \equiv [H^4]\{\Delta^4\} \quad (5.11)$$

$$\{P\} = [H^3] \begin{pmatrix} \{P_1\} \\ \{P_2\} \\ \{P_6\} \end{pmatrix} \equiv [H^3]\{\Delta^5\} \quad (5.12)$$

$$\begin{pmatrix} \frac{\partial P_1}{\partial x} + \frac{\partial P_6}{\partial y} \\ \frac{\partial P_6}{\partial x} + \frac{\partial P_2}{\partial y} \end{pmatrix} \equiv [H^4]\{\Delta^5\} \quad (5.13)$$

Here,  $f_x = \frac{\partial w}{\partial x}$ ,  $f_y = \frac{\partial w}{\partial y}$  and

$$\{\psi_i\} = \{\psi_1 \ \psi_2 \ \dots \ \psi_N\}, \quad \{\psi_{i,x}\} = \{\psi_{1,x} \ \psi_{2,x} \ \dots \ \psi_{N,x}\}, \text{ etc.} \quad (5.14)$$

and  $\{u\}$ , for example, denotes the column of the nodal values of  $u$ .

The finite-element model for the refined shear deformation theory is derived from the variational statement in Eq. (4.6) over an element. The first variation of the functional in (4.6) for a typical element  $\Omega^e$  is given by

$$0 = \int_{\Omega^e} \{ \{\delta \epsilon^0\}^t ([A^*]\{\epsilon^0\} + [B^*]\{M\} + [E^*]\{P\}) + \{\delta M\}^t (\{\kappa^s\} + [B^*]^t \{\epsilon^0\} - [D^*]\{M\} - [F^*]\{P\}) \}$$

$$\begin{aligned}
& + \{\delta P\}^t (-c_1 \{K^S\} + [E^*] \{\epsilon^0\} - [F^*] \{M\} - [H^*] \{P\}) \\
& + \{\delta \epsilon^0\}^t [\hat{A}] \{\bar{\epsilon}^0\} + \{\delta K^S\}^t (\{M\} - c_1 \{P\}) \\
& + c_1 \left[ \left( \frac{\partial \delta P_1}{\partial x} + \frac{\partial \delta P_6}{\partial y} \right) \frac{\partial w}{\partial x} + \left( \frac{\partial \delta P_6}{\partial x} + \frac{\partial \delta P_2}{\partial y} \right) \frac{\partial w}{\partial y} + \left( \frac{\partial P_1}{\partial x} + \frac{\partial P_6}{\partial y} \right) \frac{\partial \delta w}{\partial x} \right. \\
& \left. + \left( \frac{\partial P_6}{\partial x} + \frac{\partial P_2}{\partial y} \right) \frac{\partial \delta w}{\partial y} \right] - q \delta w \} dx dy \\
& - \int_{\Gamma^e} (N_n \delta u_n + N_s \delta u_s + Q_n \delta w + M_n \delta \phi_n + M_s \delta \phi_s \\
& \quad + c_1 \theta_n \delta P_n + c_1 \theta_s \delta P_s) ds
\end{aligned} \tag{5.15}$$

Substitution of Eqs. (5.2)-(5.14) into the variational statement in Eq. (5.15), we obtain

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] \\ [K^{21}] & [\hat{K}^{22}] & [K^{23}] & [K^{24}] & [K^{25}] \\ [K^{31}] & [K^{32}] & [\bar{K}^{33}] & [K^{34}] & [K^{35}] \\ [K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] \\ [K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] \end{bmatrix} \begin{Bmatrix} \{\Delta^1\} \\ \{\Delta^2\} \\ \{\Delta^3\} \\ \{\Delta^4\} \\ \{\Delta^5\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \\ \{F^5\} \end{Bmatrix} \tag{5.16}$$

where

$$[K^{11}] = \int_{\Omega^e} [H^L]^t [A^*] [H^L] dA, \quad \{F^1\} = \int_{\Gamma^e} [H^2]^t \begin{Bmatrix} N_n \\ N_s \end{Bmatrix} ds$$

$$[K^{12}] = \frac{1}{2} \int_{\Omega^e} [H^L]^t [A^*] ([H^N] + 2[H^0]) dA$$

$$[K^{21}] = \int_{\Omega^e} ([H^N] + H^0)^t [A^*] [H^L] dA$$

$$[K^{14}] = \int_{\Omega^e} [H^L]^t [B^*] [H^3] dA = [K^{41}]^t$$

$$[K^{15}] = \int_{\Omega^e} [H^L]^t [E^*] [H^3] dA = [K^{51}]^t$$

$$[\hat{K}^{22}] = \int_{\Omega^e} ([H^0] + [H^N])^t [A^*] ([H^0] + \frac{1}{2} [H^N]) dA + \int_{\Omega^e} [H^1]^t [\hat{A}] [H^1] dA$$

$$[K^{24}] = \int_{\Omega^e} (([H^0] + [H^N])^t [B^*] [H^3]) dA$$

$$[K^{42}] = \int_{\Omega^e} (\frac{1}{2} [H^3]^t [B^*]^t ([H^N] + 2[H^0])) dA$$

$$[K^{25}] = \int_{\Omega^e} ([H^0] + [H^N])^t [E^*] [H^3] + c_1 [H^1]^t [H^L]^t dA$$

$$[K^{52}] = \int_{\Omega^e} ([H^3]^t [E^*] ([H^0] + \frac{1}{2} [H^N]) + c_1 [H^L] [H^1]) dA$$

$$\{F^2\} = \int_{\Omega^e} \{\psi_i\}^t q dA + \int_{\Gamma^e} \{\psi_i\}^t Q_n ds$$

$$[\bar{K}^{33}] = \int_{\Omega^e} [H^2]^t [\hat{A}] [H^2] dA, [\bar{K}^{23}] = \int_{\Omega^e} [H^1]^t [\hat{A}] [H^2] dA = [\bar{K}^{32}]^t$$

$$[K^{34}] = \int_{\Omega^e} [H^L]^t [H^3] dA = [K^{43}]^t$$

$$[K^{35}] = - \int_{\Omega^e} c_1 [H^L]^t [H^3] dA = [K^{53}]^t$$

$$\{F^3\} = \int_{\Gamma^e} [H^2]^t [R]^t \begin{Bmatrix} M_n \\ M_s \end{Bmatrix} ds, [R] = \begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix}$$

$$[K^{44}] = - \int_{\Omega^e} [H^3]^t [D^*] [H^3] dA,$$

$$[K^{45}] = - \int_{\Omega^e} [H^3]^t [F^*] [H^3] dA = [K^{54}]^t$$

$$\begin{aligned}
 [K^{55}] &= - \int_{\Omega^e} [H^3]^t [H^*] [H^3] dA \\
 [T] &= \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{bmatrix} \\
 \{F^5\} &= \int_{\Omega^e} [H^3]^t [T]^t \begin{Bmatrix} \theta_n \\ \theta_s \end{Bmatrix} ds \quad (5.17)
 \end{aligned}$$

Clearly, the element stiffness matrix is not symmetric. For the linear case (i.e.,  $[H^N] = [0]$ ), the element stiffness matrix is symmetric. The finite-element models for the classical and first-order theories can be obtained as special cases from Eq. (5.16). An explicit form of the coefficients of the stiffness matrix in Eq. (5.16) is given in Appendix B.

### 5.3 Solution Procedure

The assembled finite-element equations are of the form

$$[K^D(\Delta)]\{\Delta\} = \{F\} \quad (5.18)$$

where  $[K^D]$  is the assembled (direct) stiffness matrix, unsymmetric in general, and  $\{\Delta\}$  is the global solution vector. Because the stiffness matrix is a nonlinear function of the unknown solution, Eq. (5.18) should be solved iteratively. Two iterative methods of analysis are quite commonly used in the finite-element analysis of nonlinear problems: the Picard-type direction iteration and the Newton-Raphson iteration methods. Here the Newton-Raphson method is used to solve the nonlinear equations.



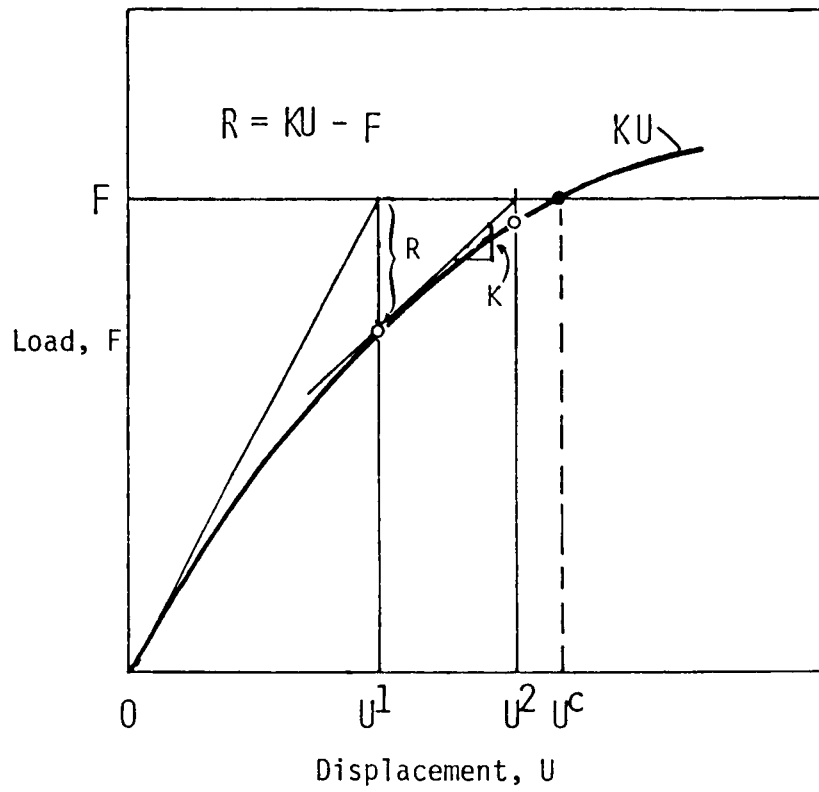


Figure 5.1 Geometric interpretation of the Newton-Raphson iteration for the solution of one-parameter problems.

$$\begin{aligned}
& + \int_{\Omega^e} ([\bar{H}^N]^t [A^*] [H^N] + [H^N]^t [A^*] [\bar{H}^N]) dA \{\Delta^2\} \\
& + \left( \int_{\Omega^e} [\bar{H}^N]^t [B^*] [H^3] dA \right) \{\Delta^4\} \\
& + \left( \int_{\Omega^e} [\bar{H}^N]^t [E^*] [H^3] dA \right) \{\Delta^5\}
\end{aligned}$$

$$[K^{31T}] = [0], [K^{32T}] = [K^{32}], [K^{33T}] = [K^{33}]$$

$$[K^{41T}] = [K^{41}], [K^{42T}] = [K^{24}]^t, [K^{43T}] = [K^{43}],$$

$$[K^{44T}] = [K^{44}], [K^{51T}] = [K^{51}], [K^{52T}] = [K^{25}]^t,$$

$$[K^{53T}] = [K^{53}], [K^{54T}] = [K^{54}], [K^{55T}] = [K^{55}]$$

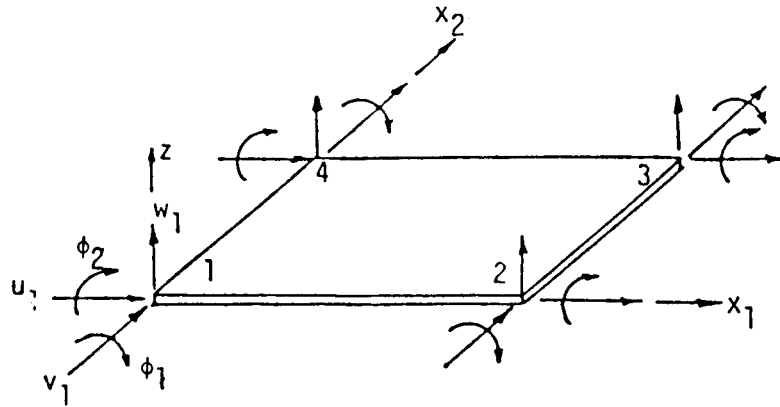
$$[\bar{H}^N] = \left\{ \begin{array}{l} \{\psi_{i,x}\}^t \{\psi_{k,x}\} \\ \{\psi_{i,y}\}^t \{\psi_{k,y}\} \\ \{\psi_{i,x}\}^t \{\psi_{k,y}\} + \{\psi_{i,y}\}^t \{\psi_{k,x}\} \end{array} \right\} \quad (5.23)$$

For all coupled (i.e., bending-stretching coupling) problems, the mixed models of classical, first-order and third-order theories have six, eight and eleven degrees of freedom per node, respectively. If the bending moments are eliminated at the element level, the element degrees of freedom can be reduced by 3 (see Fig. 5.2).

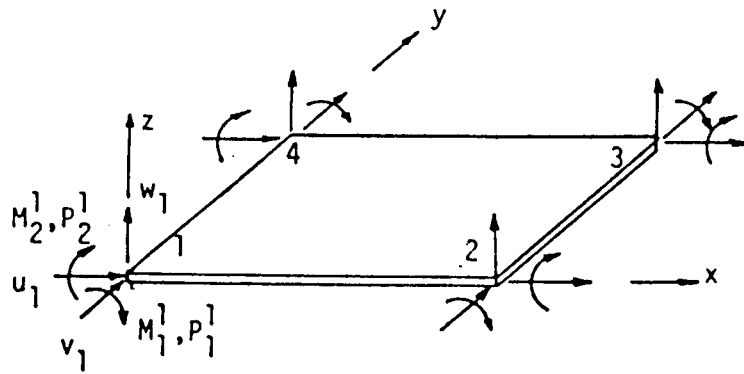
The finite-element model (5.16) for the dynamic case is of the form,

$$[K]\{\Delta\} + [M]\{\ddot{\Delta}\} = \{F\} \quad (5.24)$$

where  $[M]$  is the mass matrix (see Appendix B). For free vibration, Eq. (5.24) can be written as



(a) Displacement model



(b) Mixed model

Figure 5.2 The displacement and mixed finite elements for the third-order shear deformation theory



$$[K]\{\Delta\} = \lambda[M]\{\Delta\} \quad (5.25)$$

where  $\lambda$  is the square of the frequency.

The evaluation of the element matrices requires numerical integration. Reduced integration is used to evaluate the stiffness coefficients associated with the shear energy terms. More specifically, the 2 x 2 Gauss rule is used for shear terms and the standard 3 x 3 Gauss rule is used for the bending terms when the nine node quadratic isoparametric element is considered.

## 6. SAMPLE APPLICATIONS

### 6.1 Introduction

A number of representative problems are analyzed using the higher-order theory developed in the present study. The first few problems included here illustrate the accuracy of the present theory. These problems are solved using the closed-form solutions presented in Section 3. Then problems that do not allow closed-form solutions are solved by the mixed finite-element model described in Section 5.

The geometries of typical cylindrical and spherical shell panels are shown in Fig. 6.1. Of course, plates are derived as special cases from cylindrical or spherical shell panels.

### 6.2 Exact Solutions

It is well known that the series solution in Eq. (3.3) converges faster for uniform load than for a point load. For a sinusoidal distribution of the transverse load, the series reduces to a single term.

#### 1. Four-layer, cross-ply (0/90/90/0) square laminated flat plate under sinusoidal load

This example demonstrates the relative accuracy of the present higher-order theory when compared to three-dimensional elasticity theory and to the first-order theory. Square, cross-ply laminates under simply-supported boundary conditions [see Eq. (3.2)] and a sinusoidal distribution of the transverse load are studied for deflections. The lamina properties are assumed to be

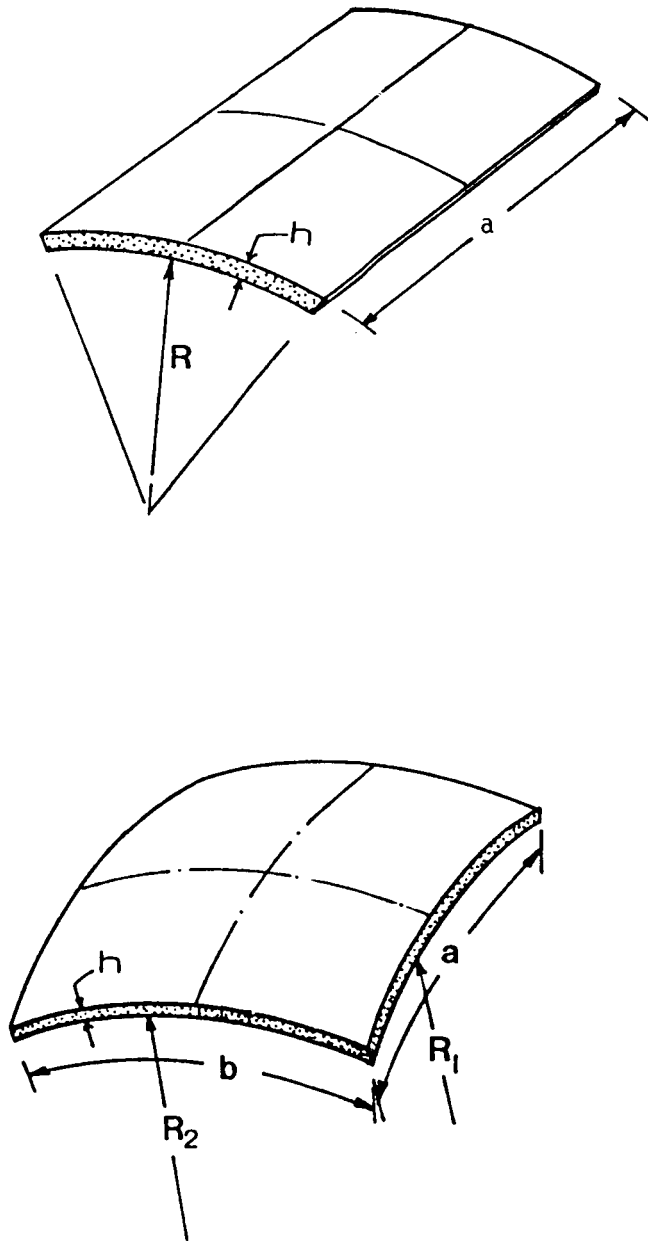


Figure 6.1 Geometry of a typical cylindrical and spherical shell

$$E_1 = 25E_2, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

Plots of the nondimensional center deflection  $[\bar{w} = wE_2h^3)/q_0a^4]$  versus the side to thickness ratio ( $a/h$ ) obtained using various theories are shown in Figure 6.2. The present third-order theory gives the closest solution to the three-dimensional elasticity solution [58] than either the first-order theory or the classical theory.

## 2. An isotropic spherical shell segment under point load at the center.

The problem data are (see Fig. 6.1 for the geometry)

$$R_1 = R_2 = 96.0 \text{ in.}, a = b = 32.0 \text{ in.}, h = 0.1 \text{ in.},$$

$$E_1 = E_2 = 10^7 \text{ psi}, \nu = 0.3, \text{ intensity of load} = 100 \text{ lbs.}$$

A comparison of the center transverse deflection of the present theory (HSDT) with that obtained using the first-order shear deformation theory (FSDT) and classical shell theory (CST) for various terms in the series is presented in Table 1 (for simply supported boundary conditions). It should be noted that Vlasov [76] did not consider transverse shearing strains in his study. The difference between the values predicted by HSDT and FSDT is not significant for the thin isotropic shell problem considered here.

## 3. Cross-ply spherical shell segments under sinusoidal, uniform, and point loads.

The geometric parameters used are the same as those used in Problem 2, and the material parameters used are the same as those used in Problem 1. The shell segments are assumed to be simply supported. Nondimensionalized center deflection of various cross-ply shells under sinusoidal, uniform, and point loads are presented in Tables 2 through

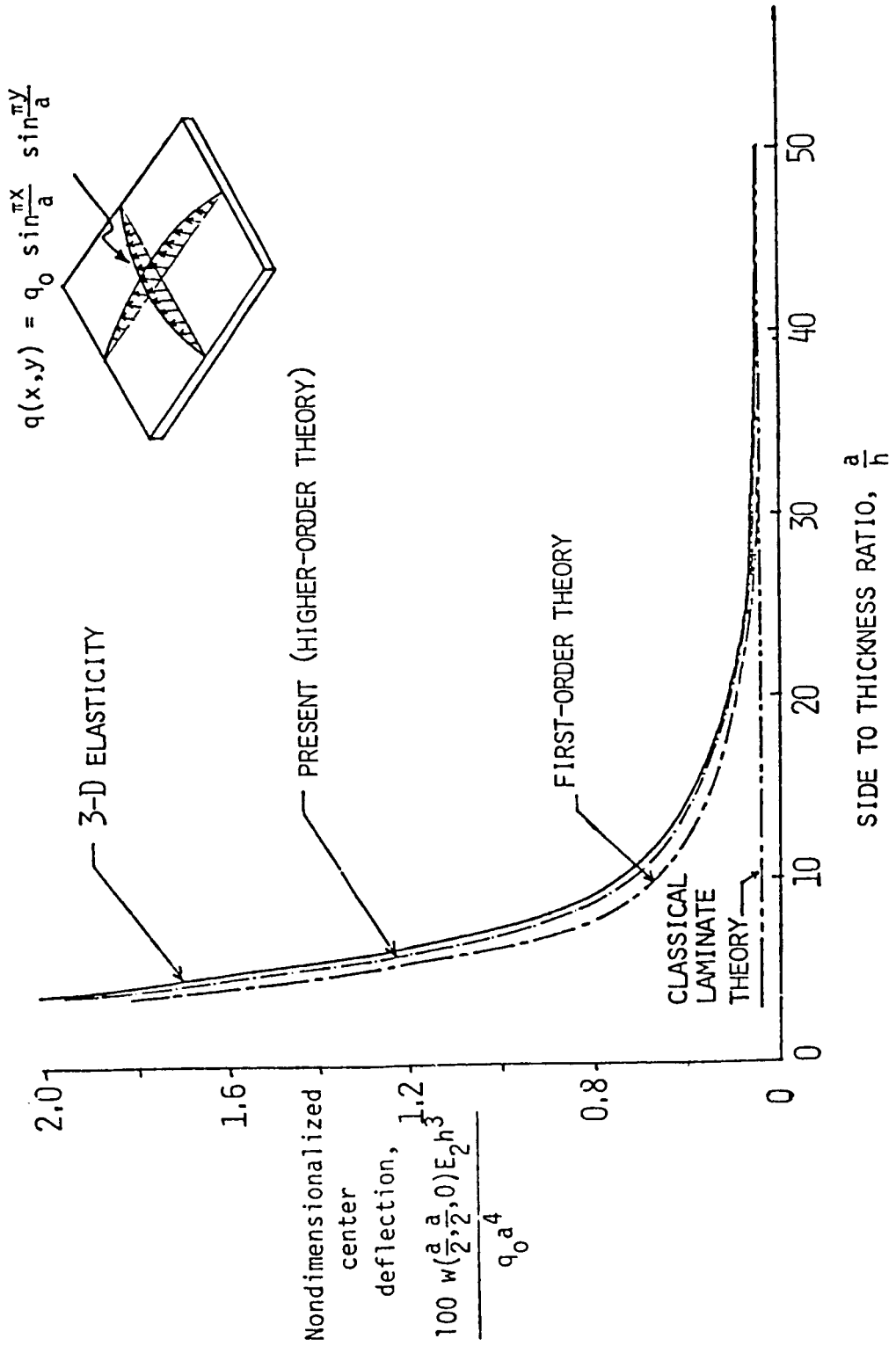


Figure 6.2 Nondimensionalized center deflection versus side to thickness ratio for four-layer, symmetric cross-ply [0 /90 /90 /0 ] square laminated plate.

Table 1. Center deflection ( $-w \times 10^3$ ) of a simply supported spherical shell segment under point load at the center (see Fig. 6.1)

h	Theory	$\bar{w}$				
		Number of terms in the series				
		N = 9	N = 49	N = 99	N = 149	N = 199
	CST			39.591		39.647
	Vlasov [79]	-	-	39.560	-	-
0.1	FSDT [40]	32.594	39.469	39.724	39.786	39.814
	HSDT	32.584	39.458	39.714	39.775	39.803
0.32	FSDT	3.664	3.902	3.919	3.927	3.932
	HSDT	3.661	3.899	3.916	3.923	3.927
1.6	FSDT	0.165	0.171	0.174	0.175	0.176
	HSDT	0.164	0.170	0.172	0.172	0.172
3.2	FSDT	0.035	0.038	0.039	0.039	0.040
	HSDT	0.035	0.037	0.037	0.037	0.037
6.4	FSDT	0.007	0.008	0.009	0.009	0.009
	HSDT	0.007	0.007	0.007	0.007	0.007

4, respectively. The difference between the solutions predicted by the higher-order theory and the first-order theory increases with increasing values of  $R/a$ . For  $a/h = 10$ , the difference between the deflections given by FSDT and HSDT is larger than those for  $a/h = 100$ . For unsymmetric laminates ( $0/90$ ), FSDT yields higher deflections than HSDT, whereas for symmetric laminates HSDT yields higher deflections than FSDT for values of  $a/h = 10$ . Note that for point-loaded shells the difference between the solution predicted by the first-order theory and the higher-order theory is more significant, with FSDT results higher than HSDT, especially for antisymmetric cross-ply laminates.

#### 4. Natural vibration of cross-ply cylindrical shell segments

Nondimensionalized fundamental frequencies of cross-ply cylindrical shells are presented in Table 5 for three lamination schemes:  $[0^\circ/90^\circ]$ ,  $[0^\circ/90^\circ/0^\circ]$ , and  $[0^\circ/90^\circ/90^\circ/0^\circ]$ . For thin antisymmetric cross-ply shells, the first-order theory underpredicts the natural frequencies when compared to the higher-order theory. However, for symmetric cross-ply shells, the trend reverses.

#### 5. Natural vibration of cross-ply spherical shell segments

Nondimensionalized natural frequencies obtained using the first- and higher-order theories are presented in Table 6 for various cross-ply spherical shell segments. Analogous to cylindrical shells, the first-order theory underpredicts fundamental natural frequencies of antisymmetric cross-ply shells; for symmetric thick shells and symmetric shallow thin shells the trend reverses.

Table 2. Nondimensionalized center deflections,  $\bar{w} = (-wh^3E_2/q_0 a^4)10^3$ , of cross-ply laminated spherical shell segments under sinusoidally distributed load ( $a/b = 1$ ,  $R_1 = R_2 = R$ ,  $q_0 = 100$ )

R/a	Theory	$\bar{w}$					
		0°/90°		0°/90°/0°		0°/90°/90°/0°	
		a/h=100	a/h=10	a/h=100	a/h=10	a/h=100	a/h=10
5	FSDT	1.1948	11.429	1.0337	6.4253	1.0279	6.3623
	HSDT	1.1937	11.166	1.0321	6.7688	1.0264	6.7865
10	FSDT	3.5760	12.123	2.4109	6.6247	2.4030	6.5595
	HSDT	3.5733	11.896	2.4099	7.0325	2.4024	7.0536
20	FSDT	7.1270	12.309	3.6150	6.6756	3.6104	6.6099
	HSDT	7.1236	12.094	3.6170	7.1016	3.6133	7.1237
50	FSDT	9.8717	12.362	4.2027	6.6902	4.2015	6.6244
	FSDT	9.8692	12.150	4.2071	7.1212	4.2071	7.1436
100	FSDT	10.4460	12.370	4.3026	6.6923	4.3021	6.6264
	HSDT	10.4440	12.158	4.3074	7.1240	4.3082	7.1464
Plate	FSDT	10.6530	12.373	4.3370	6.6939	4.3368	6.6280
R/a=∞	FSDT	10.6510	12.161	4.3420	7.1250	4.3430	7.1474



Table 3. Nondimensionalized center deflections,  $\bar{w} = (-wE_2h^3/q_0a^4) \times 10^3$ , of cross-ply laminated spherical shell segments under uniformly distributed load

$\frac{R}{a}$	Theory	$\bar{w}$					
		$0^\circ/90^\circ$		$0^\circ/90^\circ/0^\circ$		$0^\circ/90^\circ/90^\circ/0^\circ$	
		$a/h=100$	$a/h=10$	$a/h=100$	$a/h=10$	$a/h=100$	$a/h=10$
5	FSDT	1.7535	19.944	1.5118	9.794	1.5358	9.825
	HSDT	1.7519	17.566	1.5092	10.332	1.5332	10.476
10	FSDT	5.5428	19.065	3.6445	10.110	3.7208	10.141
	HSDT	5.5388	18.744	3.6426	10.752	3.7195	10.904
20	FSDT	11.273	19.365	5.5473	10.191	5.6618	10.222
	HSDT	11.268	19.064	5.5503	10.862	5.666	11.017
50	FSDT	15.714	19.452	6.4827	10.214	6.6148	10.245
	HSDT	15.711	19.155	6.4895	10.893	6.6234	11.049
100	FSDT	16.645	19.464	6.6421	10.218	6.7772	10.294
	HSDT	16.642	19.168	6.6496	10.898	6.7866	11.053
Plate	FSDT	16.980	19.469	6.6970	10.220	6.8331	10.251
$\infty$		16.977	19.172	6.7047	10.899	6.8427	11.055

Table 4. Nondimensionalized center deflection of cross-ply spherical shell segments under point load at the center

$$\bar{w} = - \left( \frac{wh^3E_2}{Pa^2} \right) 10^2, \quad a/b = 1, \quad a/h = 10$$

R/a	Theory	Center deflection, $\bar{w}$		
		0°/90°	0°/90°/0°	0°/90°/90°/0°
5	FSDT	7.1015	5.1410	4.9360
	HSDT	5.8953	4.4340	4.3574
10	FSDT	7.3836	5.2273	5.0186
	HSDT	6.1913	4.5470	4.4690
20	FSDT	7.4692	5.2594	5.0496
	HSDT	6.2714	4.5765	4.4982
50	FSDT	7.4909	5.2657	5.0557
	HSDT	6.2943	4.5849	4.5065
100	FSDT	7.4940	5.2666	5.0565
	HSDT	6.2976	4.5861	4.5077
Plate	FSDT	7.4853	5.2572	5.0472
	HSDT	6.2987	4.5865	4.5081

Table 5 Nondimensionalized fundamental frequencies of cross-ply cylindrical shell panels (see Fig. 6.1 for geometry).

$$\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\rho/E_2}$$

R/a	Theory	0°/90°		0°/90°/0°		0°/90°/90°/0°	
		a/h=100	a/h=10	a/h=100	a/h=10	a/h=100	a/h=10
5	FSDT	16.668	8.9082	20.332	12.207	20.361	12.267
	HSDT	16.690	9.0230	20.330	11.850	20.360	11.830
10	FSDT	11.831	8.8879	16.625	12.173	16.634	12.236
	HSDT	11.840	8.9790	16.620	11.800	16.630	11.790
20	FSDT	10.265	8.8900	15.556	12.166	15.559	12.230
	HSDT	10.270	8.9720	15.550	11.791	15.550	11.780
50	FSDT	9.7816	8.8951	15.244	12.163	15.245	12.228
	HSDT	9.7830	8.9730	15.240	11.790	15.230	11.780
100	FSDT	9.7108	8.8974	15.198	12.163	15.199	12.227
	HSDT	9.7120	8.9750	15.190	11.790	15.190	11.780
Plate $\infty$	FSDT	9.6873	8.8998	15.183	12.162	15.184	12.226
	HSDT	9.6880	8.9760	15.170	11.790	15.170	11.780

Table 6. Nondimensionalized fundamental frequencies of cross-ply laminated spherical shell segments

$$\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\rho/E_2}$$

R/a	Theory	0°/90°		0°/90°/0°		0°/90°/90°/0°	
		a/h=100	a/h=10	a/h=100	a/h=10	a/h=100	a/h=10
5	FSDT	28.825	9.2309	30.993	12.372	31.079	12.437
	HSDT	28.840	9.3370	31.020	12.060	31.100	12.040
10	FSDT	16.706	8.9841	20.347	12.215	20.380	12.280
	HSDT	16.710	9.0680	20.350	11.860	20.380	11.840
20	FSDT	11.841	8.9212	16.627	12.176	16.638	12.240
	HSDT	11.84	8.999	16.62	11.81	16.63	11.79
50	FSDT	10.063	8.9034	15.424	12.165	15.426	12.229
	HSDT	10.06	8.980	15.42	11.79	15.42	11.78
100	FSDT	9.7826	8.9009	15.244	12.163	15.245	12.228
	HSDT	9.784	8.977	15.24	11.79	15.23	11.78
Plate	FSDT	9.6873	8.8998	15.183	12.162	15.184	12.226
	HSDT	9.6880	8.9760	15.170	11.790	15.170	11.780

## 6.3 Approximate (Finite-Element) Solutions

### 6.3.1 Bending Analysis

#### 1. Linear analysis of a rectangular plate under uniformly distributed load.

The geometry and boundary conditions for the problem are shown in Fig. 6.3. The plate is assumed to be made of steel ( $E = 30 \times 10^6$  psi and  $\nu = 0.3$ ). The problem was also solved by Timoshenko [77], Herrmann [72] and Prato [78]. Figures 6.3-6.5 contain plots of the transverse displacement and bending moments obtained by various investigators (using the linear theory). The agreement between the present solution and others is very good, verifying the accuracy of the theory and the finite element formulation.

#### 2. Nonlinear analysis of rectangular laminates under uniformly distributed load

Figure 6.6 contains load-deflection curves for a simply supported square orthotropic plate under uniformly distributed load [23]. The following geometric and material properties are used:

$$a = b = 12 \text{ in.}, h = 0.138 \text{ in.}$$

$$E_1 = 3 \times 10^6 \text{ psi}, E_2 = 1.28 \times 10^6 \text{ psi}, G_{12} = G_{13} = G_{23} = 0.37 \times 10^6 \text{ psi}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.32$$

The experimental results and classical solutions are taken from the paper by Zaghoul and Kennedy [79]. The agreement between the present solution and the experimental solution is extremely good. It is clear that, even for thin plates, the shear deformation effect is significant in the nonlinear range.

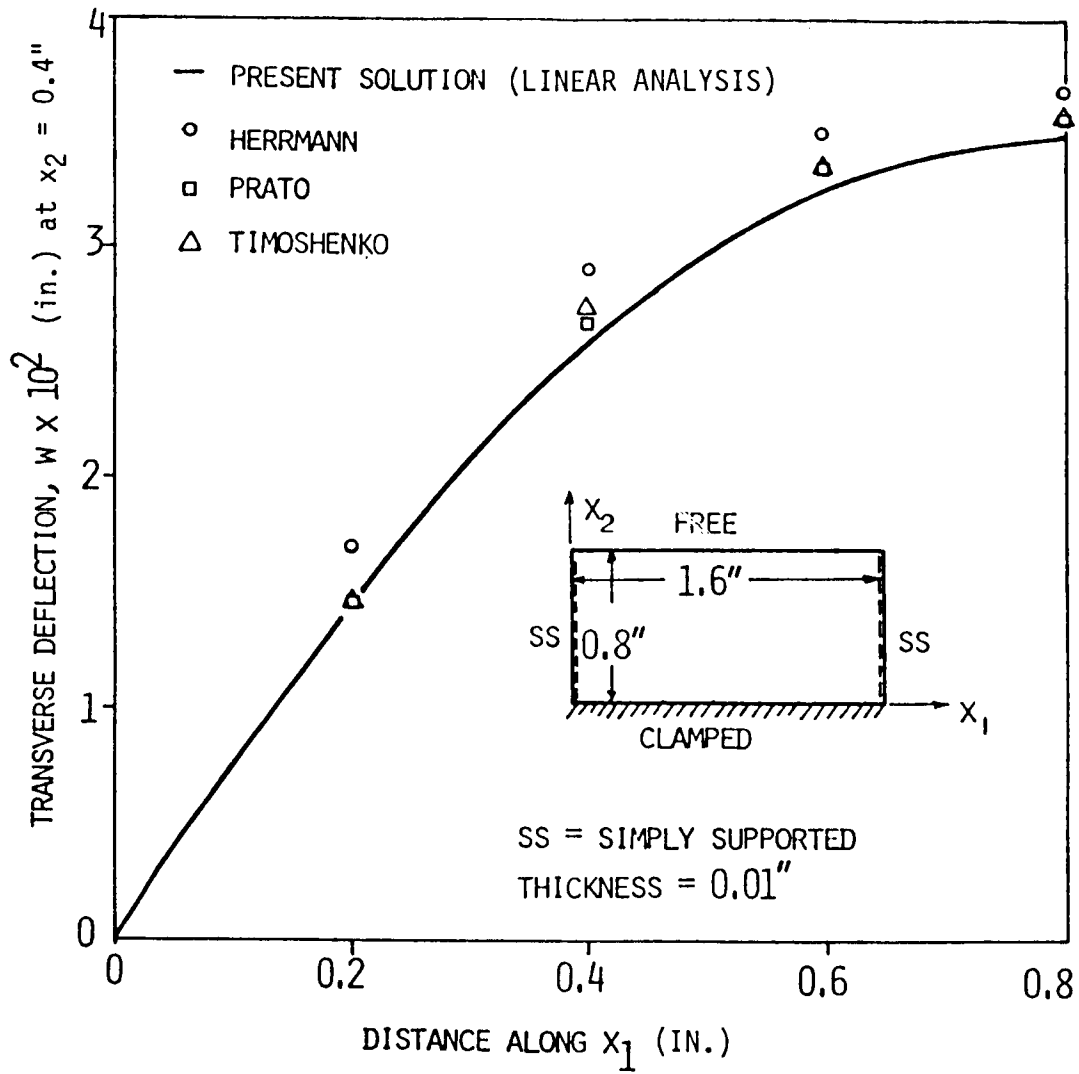


Figure 6.3 Comparison of the transverse deflection of a clamped-simply-supported-free isotropic rectangular plate under uniformly distributed transverse load.

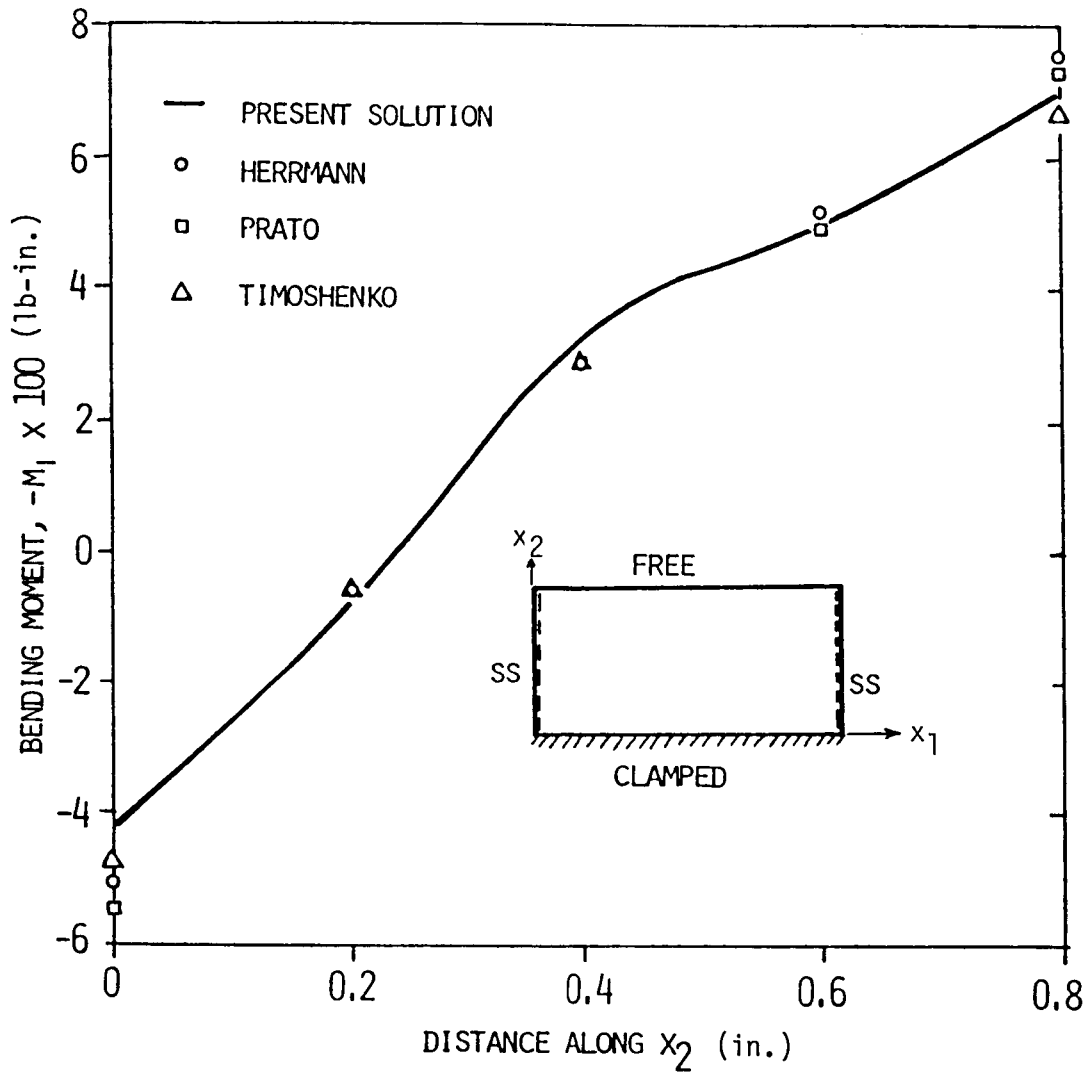


Figure 6.4 Comparison of the bending moment along the line  $x_1=0.4$ " for the problem of Figure 6.3.

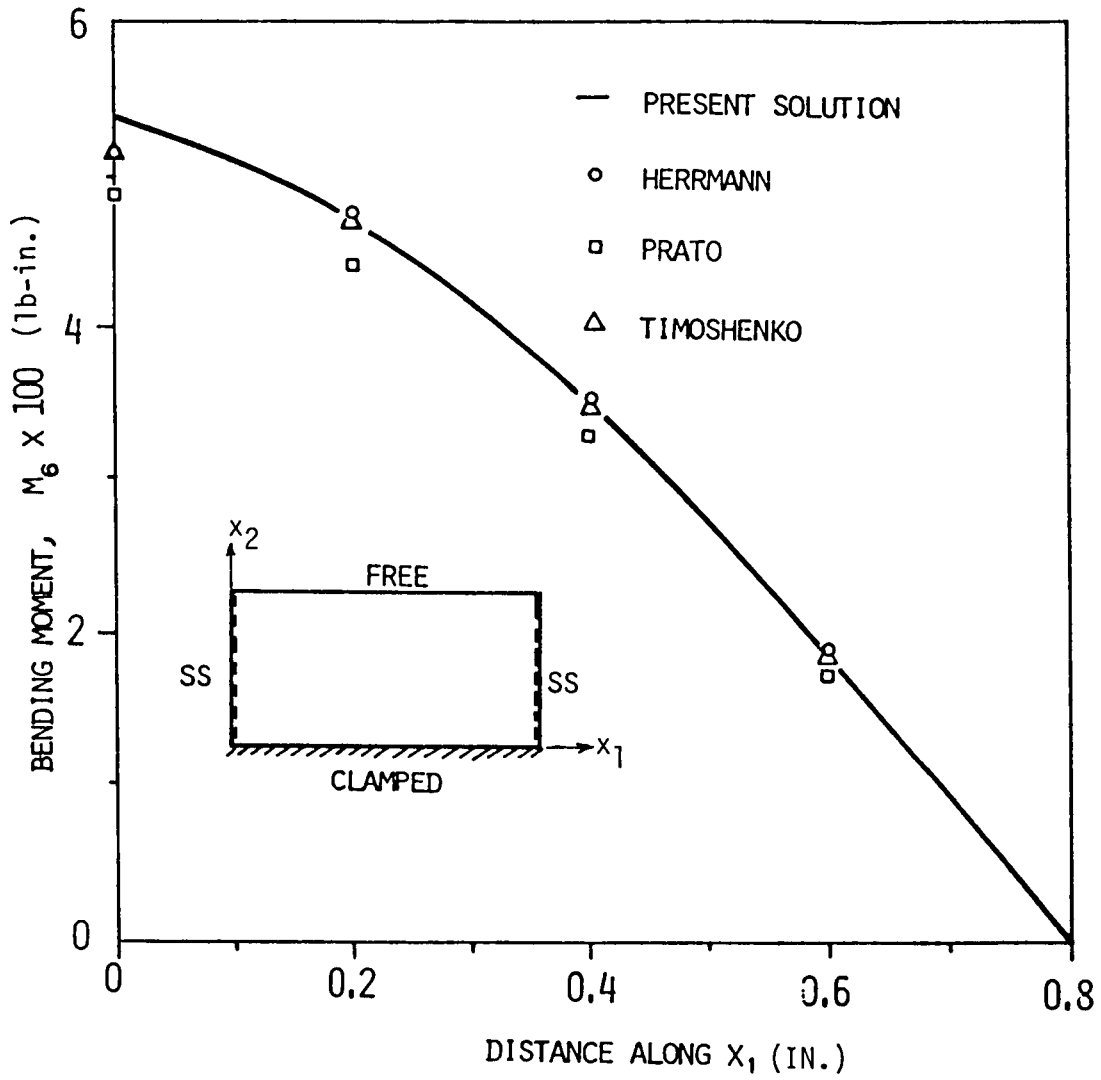


Figure 6.5 Comparison of the bending moment  $M_6$  along the line  $x_2=0.4$ " for the problem in Figure 6.3.



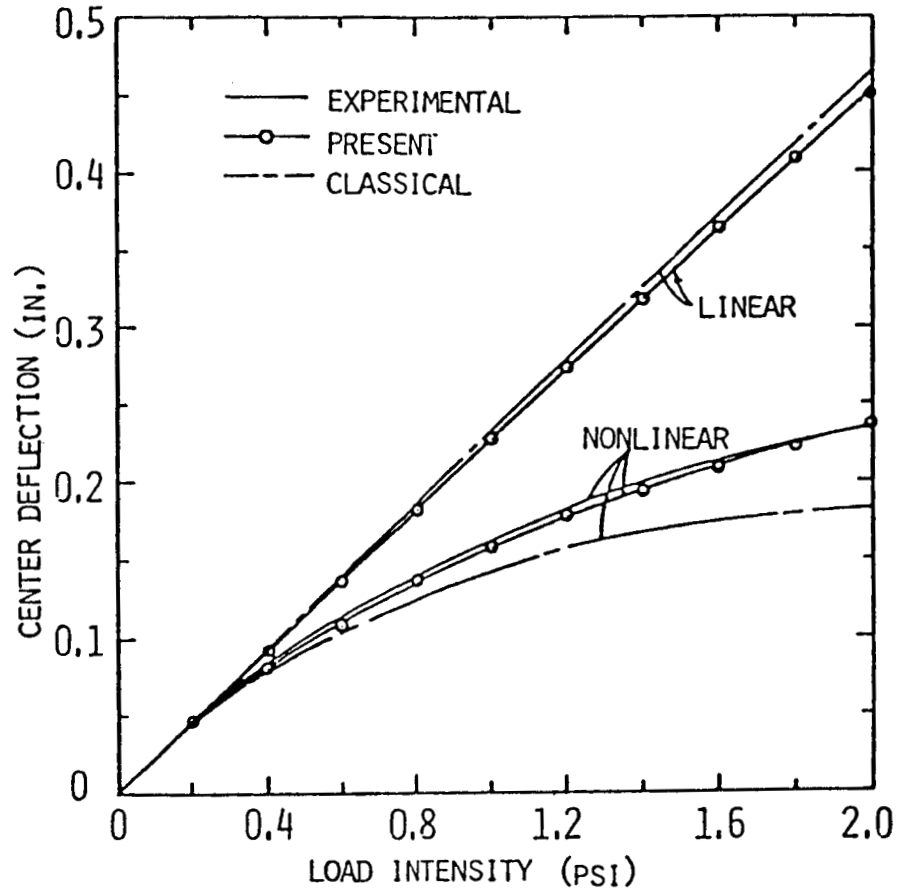


Figure 6.6 Center deflection versus load intensity for a simply supported square orthotropic plate under uniformly distributed transverse load. The following simply supported boundary conditions were used:

$$v = w = \psi_y = M_x = P_x = 0 \text{ on side } x = a$$

$$u = w = \psi_x = M_y = P_y = 0 \text{ on side } y = b$$

Figure 6.7 contains load-deflection curves for a clamped bidirectional laminate [0/90/90/0] under uniformly distributed load. The geometric parameters and layer properties are given by

$$a = b = 12 \text{ in.}, h = 0.096 \text{ in.},$$

$$E_1 = 1.8282 \times 10^6 \text{ psi}, E_2 = 1.8315 \times 10^6 \text{ psi},$$

$$G_{12} = G_{13} = G_{23} = 3.125 \times 10^6 \text{ psi}, \nu_{12} = \nu_{13} = \nu_{23} = 0.2395$$

The experimental results and the classical laminate solutions are taken from [79]. The present solution is in good agreement with the experimental results, and the difference is attributed to possible errors in the simulation of the material properties and boundary conditions.

### 3. Orthotropic cylinder subjected to internal pressure.

Consider a clamped orthotropic cylinder with the following geometric and material properties (see Fig. 6.8)

$$R_1 = 20 \text{ in.}, R_2 = \infty$$

$$E_1 = 2 \times 10^6 \text{ psi}, E_2 = 7.5 \times 10^6 \text{ psi}$$

$$G_{12} = 1.25 \times 10^6 \text{ psi}$$

$$G_{13} = G_{23} = 0.625 \times 10^6 \text{ psi}, h = 1 \text{ in.}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

$$a = 10 \text{ in.}, P = 6.41/\pi \text{ psi}$$

This problem has an analytical solution (see [77]) for the linear case, and Rao [47] used the finite-element method to solve the same problem. Both solutions are based on the classical theory. The center deflections from [77] and [47] are 0.000367 in. and 0.000366 in., respectively. Chao and Reddy [50] obtained 0.0003764 in. and 0.0003739

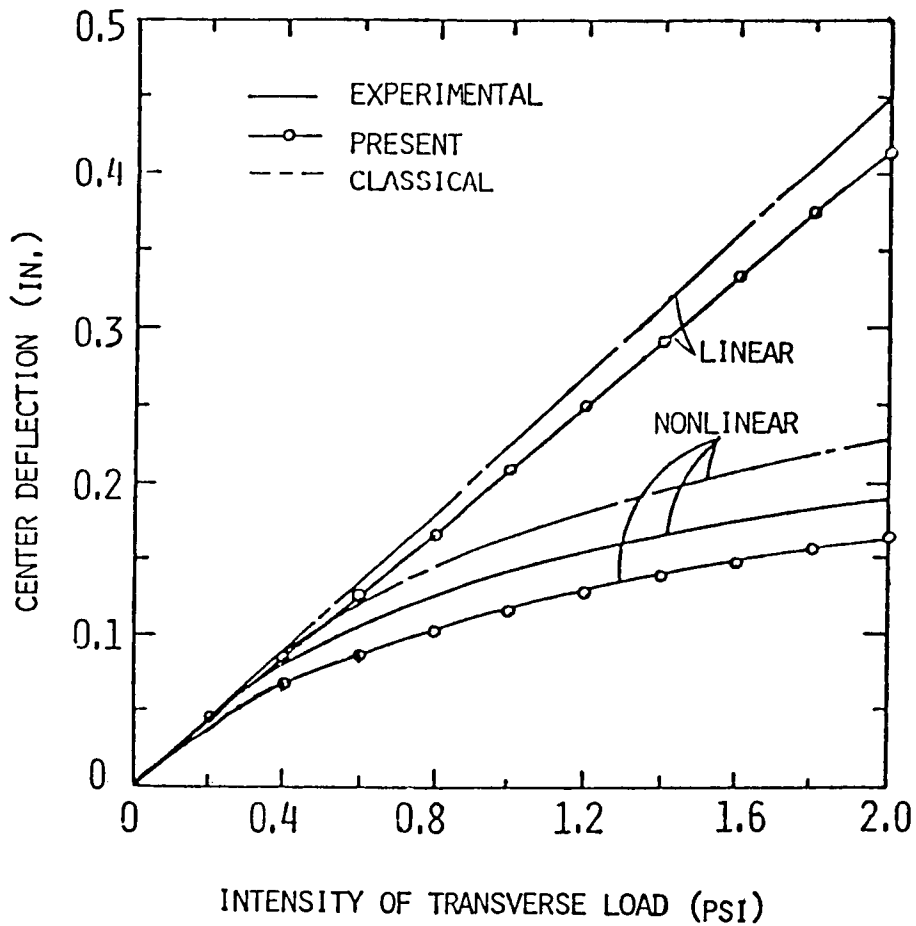


Figure 6.7 Center deflection versus load intensity for a clamped (CC-1) square laminate  $[0/90/90/0]$  under uniform transverse load (von Karman theory).

$$\text{CC-1: } u = v = w = \psi_x = \psi_y = 0 \text{ on all four clamped edges}$$

in. using the finite-element model based on the first-order shear deformation theory and 3-D degenerate element, respectively. The current result is 0.0003761 in., which is closer to Chao and Reddy [50], as expected.

For the nonlinear analysis of the same problem, the present results are compared to those of Chang and Sawamiphakdi [80] and Chao and Reddy [50] in Fig. 6.9, which contains plots of the center deflection versus the load obtained by various investigators. The agreement between the various results is very good.

#### 4. Nine-layer cross-ply spherical shell segment subjected to uniform loading.

Consider a nine-layer  $[0^\circ/90^\circ/0^\circ\dots/0^\circ]$  cross-ply laminated spherical shell segment with the following material and geometric data:

$$R_1 = R_2 = 1000 \text{ in.} , \quad a = b = 100 \text{ in.}$$

$$h = 1 \text{ in.} , \quad E_1 = 40 \times 10^6 \text{ psi}$$

$$E_2 = 10^6 \text{ psi} , \quad G_{12} = 0.6 \times 10^6 \text{ psi} , \quad G_{13} = G_{23} = 0.5 \times 10^6 \text{ psi}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

The present results are compared with those obtained by Noor and Hartley [52] and Chao and Reddy [50] in Fig. 6.10. Noor and Hartley used mixed isoparametric elements with 13 degree-of-freedom per node which is based on a shear deformation shell theory. The present results agree with both investigations.

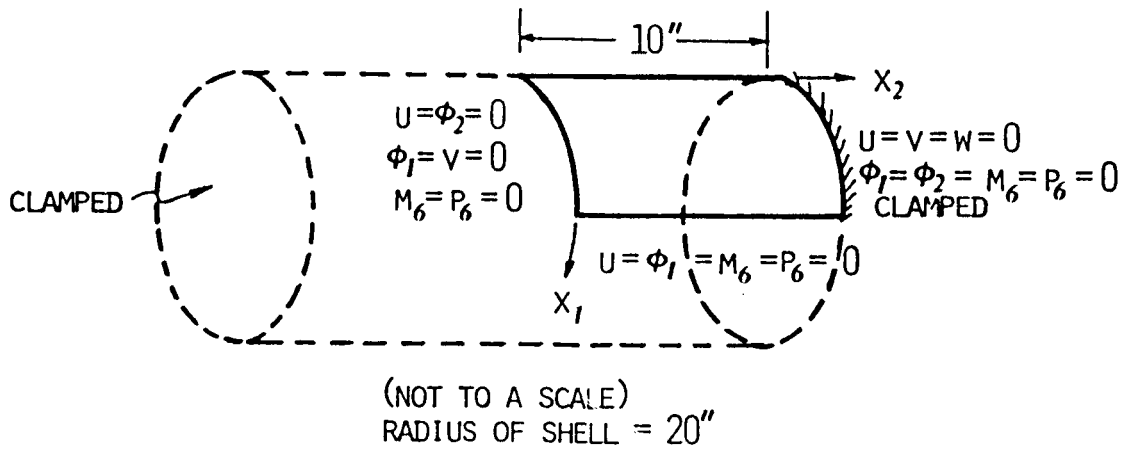


Figure 6.8 Geometry and boundary conditions for the octant of the clamped cylindrical shell.

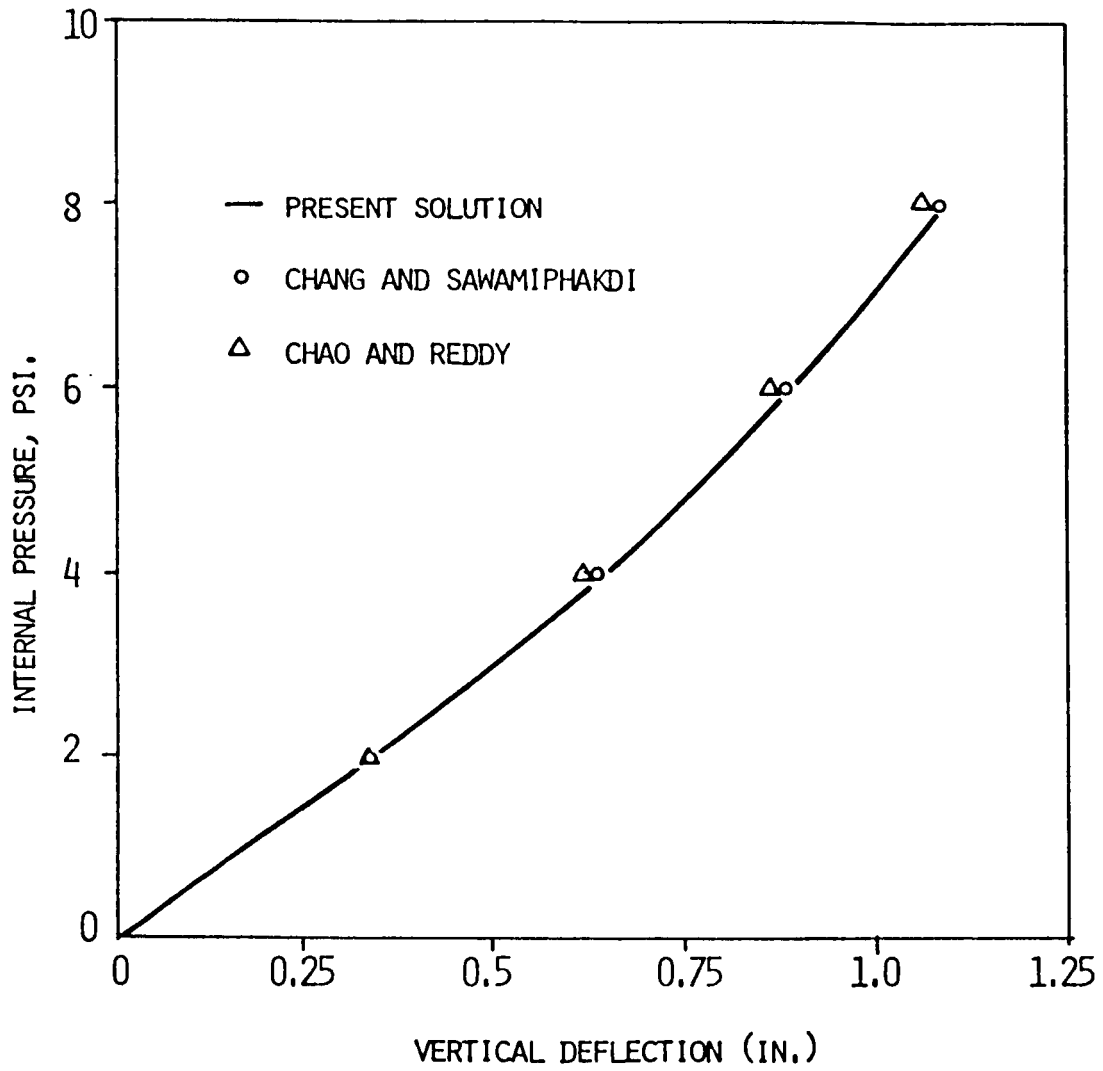


Figure 6.9 Center deflection versus load for the clamped cylinder with internal pressure.

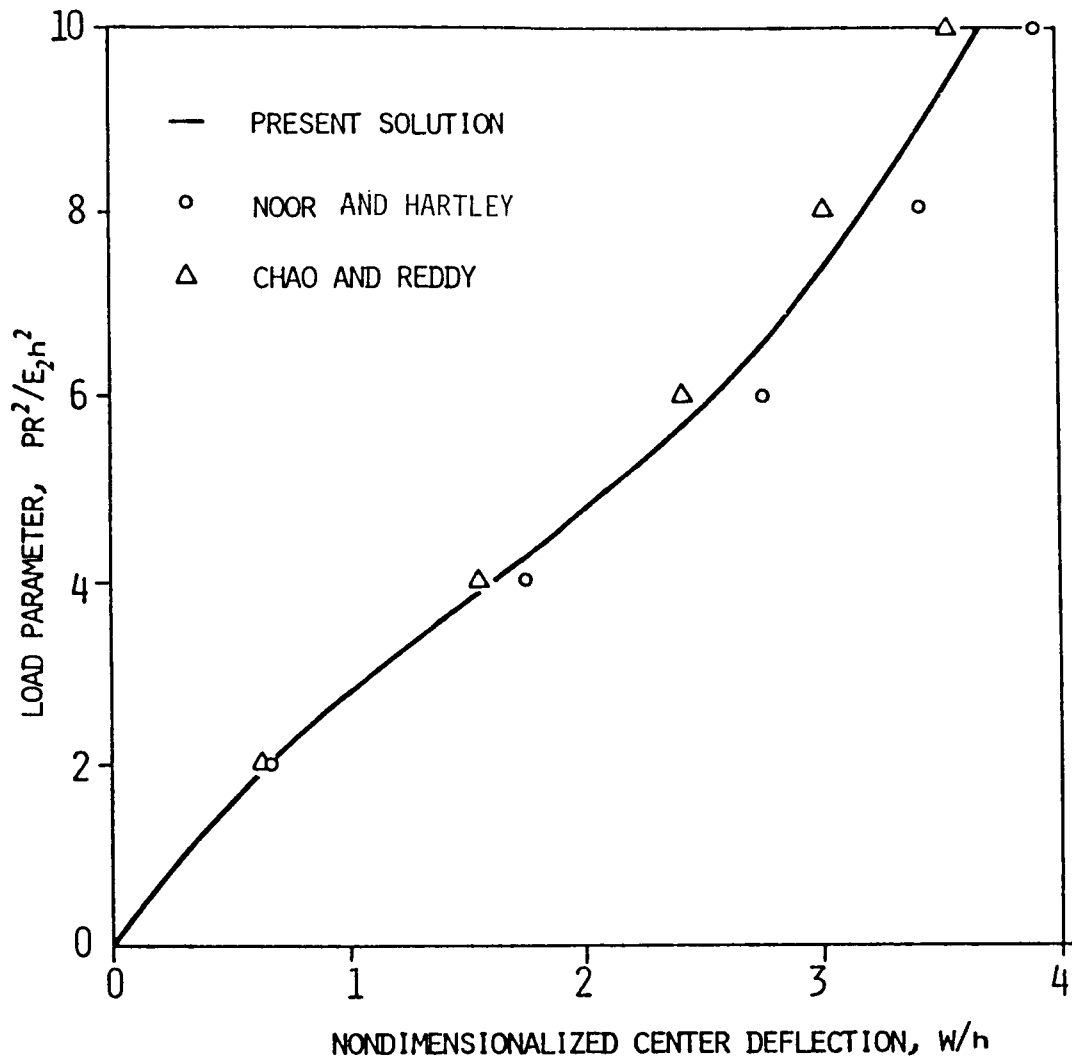


Figure 6.10 Nonlinear bending of a nine-layer cross-ply spherical shell ( $0^\circ/90^\circ/0^\circ/90^\circ/\dots/0^\circ$ ).

### 6.3.2 Vibration Analysis

Equation (5.24) can be expressed in the alternative form as

$$\begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} \begin{Bmatrix} \{\Delta_1\} \\ \{\Delta_2\} \end{Bmatrix} + \begin{bmatrix} [M_{11}] & [M_{12}] \\ [M_{21}] & [M_{22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{\Delta}_1\} \\ \{\ddot{\Delta}_2\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \quad (6.1)$$

where

$$\begin{aligned} [M_{12}] &= [M_{21}] = [M_{22}] = [0] \\ \{\Delta_1\}^T &= [\{u\}^T \{v\}^T \{w\}^T \{\phi_1\}^T \{\phi_2\}^T] \\ \{\Delta_2\}^T &= [\{M_1\}^T \{M_2\}^T \{M_6\}^T \{P_1\}^T \{P_2\}^T \{P_6\}^T] \end{aligned} \quad (6.2)$$

For free vibration analysis, we wish to eliminate  $\{\Delta_2\}$  as follows. From Eq. (6.1) we have

$$[K_{11}]\{\Delta_1\} + [K_{12}]\{\Delta_2\} = -[M_{11}]\{\ddot{\Delta}_1\} \quad (6.3)$$

$$[K_{21}]\{\Delta_1\} + [K_{22}]\{\Delta_2\} = 0 \quad (6.4)$$

From (6.4) we have,

$$\{\Delta_2\} = -[K_{22}]^{-1}[K_{21}]\{\Delta_1\} \quad (6.5)$$

Substituting Eq. (6.5) into Eq. (6.3), we obtain,

$$([K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}])\{\Delta_1\} = -[M_{11}]\{\ddot{\Delta}_1\} \quad (6.6)$$

For the free vibration case, Eq. (6.6) reduces to

$$([\bar{K}] - \omega^2[M])\{\Delta_1\} = 0 \quad (6.7)$$

where  $\omega$  is the natural frequencies of the system.

#### 1. Natural frequencies of a two-layer $[0^\circ/90^\circ]$ laminated plate.

The geometric and material properties used are

$$a = b = 100 \text{ in.} \quad , \quad h = 0.1 \text{ in.}$$

$$E_1 = 40 \times 10^6 \text{ psi} \quad , \quad E_2 = 10^6 \text{ psi.}$$

$$G_{12} = G_{13} = G_{23} = 0.5 \times 10^6 \text{ psi} \quad , \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

$$\rho = 1 \text{ lb-sec}^2/\text{in}^4$$



The boundary conditions are shown in Fig. 6.11, which also contains the plots of the ratios  $\omega_{NL}/\omega_L$  versus  $w_0/h$ . Here  $\omega_{NL}$  and  $\omega_L$  denote the nonlinear and linear natural frequencies, and  $w_0$  is the normalized center deflection of the first node. The results are compared with those of Chia and Prabhakara [81], and Reddy and Chao [82]. The present results are slightly higher compared to those obtained by the classical and first-order shear deformation theories.

## 2. Natural frequencies of two-layer [45°/-45°] angle-ply square plate.

The material and geometric parameters used are

$$\begin{aligned} E_1 &= 10 \times 10^6 \text{ psi} \quad , \quad E_2 = 10^6 \text{ psi} \\ G_{12} = G_{13} = G_{23} &= 0.3333 \times 10^6 \text{ psi} \quad , \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.3 \\ \rho &= 1 \text{ lb-sec.}^2/\text{in.}^4 \quad , \quad a = b = 100 \text{ in.} \quad , \quad h = 0.1 \text{ in.} \end{aligned}$$

The boundary conditions used are shown in Fig. 6.12, which also contains plots of  $\omega_{NL}/\omega_L$  versus  $w_0/h$ . For this case, no results are available in the literature for comparison.

## 3. Natural vibration of two-layer [0°/90°] cross-ply plates.

Consider a two-layer cross-ply plate with the following geometric and material properties:

$$\begin{aligned} E_1 &= 7.07 \times 10^6 \text{ psi} \quad , \quad E_2 = 3.58 \times 10^6 \text{ psi} \\ G_{12} = G_{23} = G_{13} &= 1.41 \times 10^6 \text{ psi} \quad , \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.3 \\ \rho &= 1 \text{ lb-sec.}^2/\text{in.}^4 \quad , \quad a/h = 1000 \quad , \quad a = 100 \text{ in.} \end{aligned}$$

The results of  $\omega_{NL}/\omega_L$  versus  $w_0/h$  are shown in Fig. 6.13. Compared to the results of Reddy [82] and Chandra and Raju [83], the results of the present study are in general a little higher. The fact that the present results for natural frequencies are higher than those predicted by the first-order theory indicates that the additional inertia terms contribute to the increase of natural frequencies.

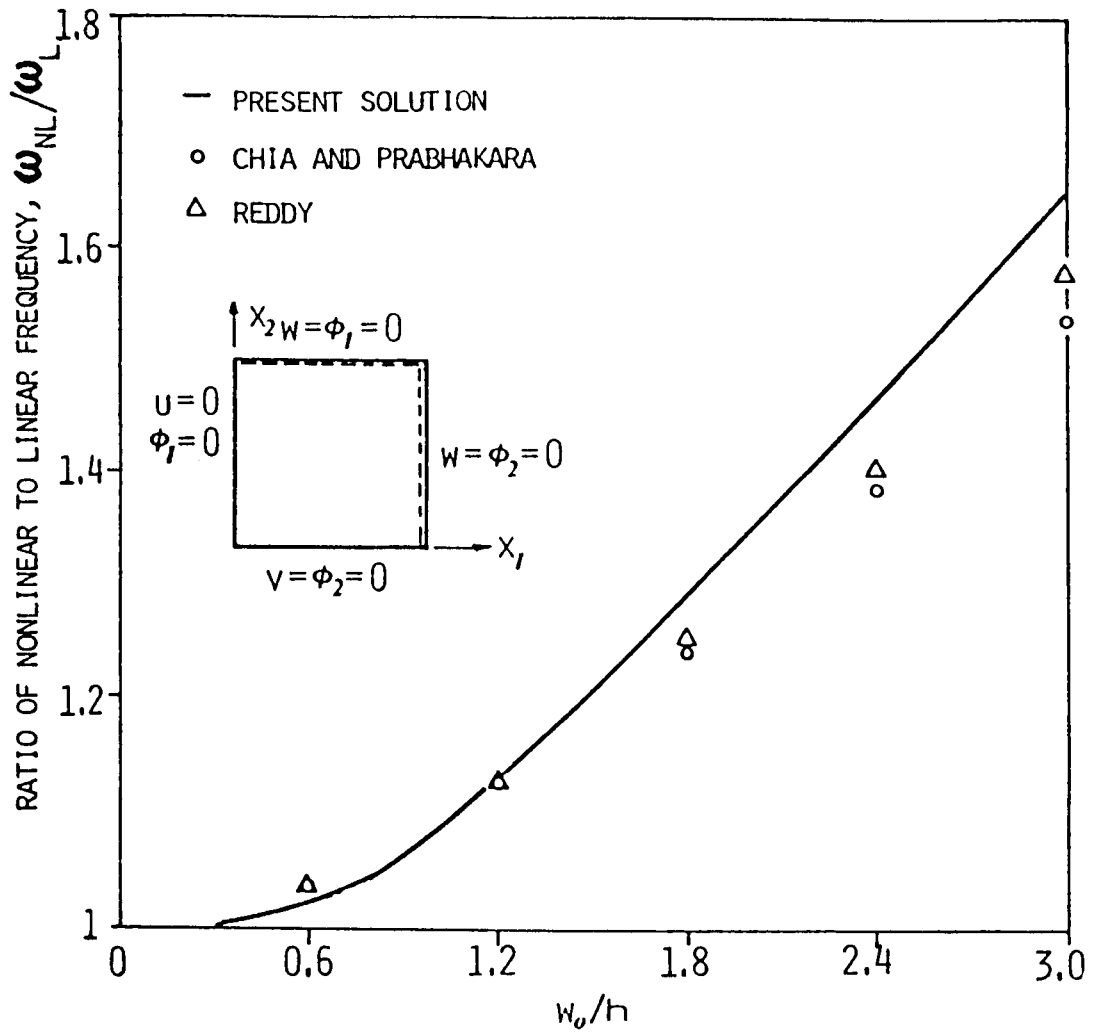


Figure 6.11 Fundamental frequencies of a two-layer cross-ply ( $0^\circ/90^\circ$ ) square laminate under simply-supported boundary conditions.

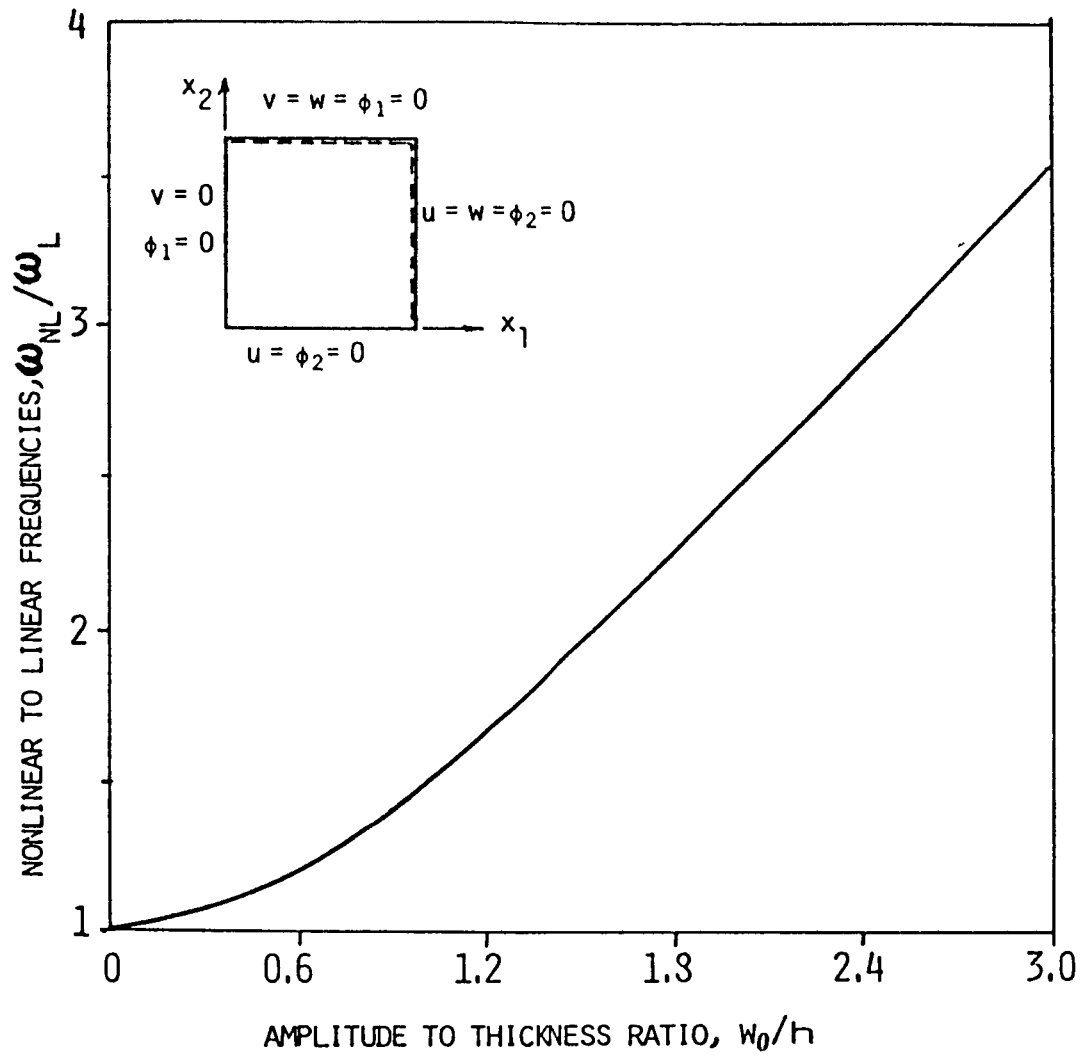


Figure 6.12 Ratio of nonlinear to linear frequency versus amplitude to thickness ratio for two-layer angle-ply square plate ( $45^\circ/-45^\circ$ ).

BC1 boundary conditions:

$$u = w = \phi_1 = 0; M_2 = P_2 = 0 \text{ along } x_2 = b/2$$

$$v = w = \phi_2 = 0; M_1 = P_1 = 0 \text{ along } x_1 = a/2$$

$$u = \phi_1 = 0; M_6 = P_6 = 0 \text{ along } x_1 = 0$$

$$v = \phi_2 = 0; M_6 = P_6 = 0 \text{ along } x_2 = 0$$

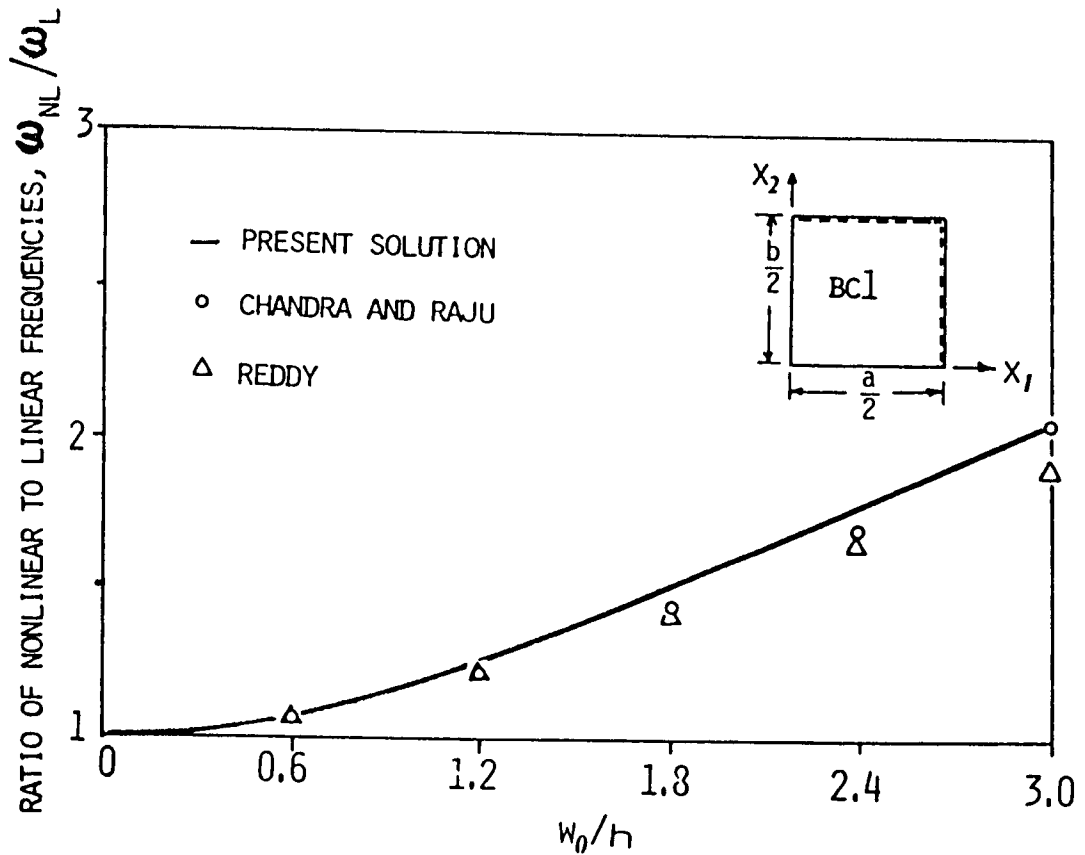


Figure 6.13 Ratio of the nonlinear to linear frequency versus the amplitude to thickness ratio of a two-layer cross-ply ( $0^\circ/90^\circ$ ) square laminate.

## 7. SUMMARY AND RECOMMENDATIONS

### 7.1 Summary and Conclusions

The present study dealt with the following major topics:

- (i) The development of a variationally-consistent, third-order shear deformation theory of laminated composite doubly-curved shells. The theory accounts for (a) the parabolic variation of the transverse shear strains, and (b) the von Karman strains. It does not require the use of the shear correction coefficients.
- (ii) The development of the closed-form solutions (for the linear theory) for the simply supported cross-ply laminates. These solutions are used as a check for the numerical analysis of shells.
- (iii) The construction of a mixed variational principle for the third-order theory that includes the classical theory and the first-order theory as special cases.
- (iv) The development and application of the finite-element model of the third-order theory for laminated composite shells, accounting for the geometric nonlinearity in the sense of von Karman (moderate rotations).

The increased accuracy of the present third-order theory (for thin as well as thick laminates) over the classical or first-order shear deformation theory is demonstrated via examples that have either the three-dimensional elasticity solution or experimental results. Many of the other results on bending and vibration analysis included here can serve as references for future investigations.

## 7.2 Some Comments on Mixed Models

The displacement model of the classical laminate theory requires the use of  $C^1$  elements, which are algebraically complex and computationally expensive. The  $C^0$  mixed elements are algebraically simple and allows the direct computation of the bending moments at the nodes. The inclusion of the bending moments as nodal degrees of freedom not only results in increased accuracy of the average stress compared to that determined from the displacement model but it allows us to determine stresses at the nodes. This feature is quite attractive in contact problems and singular problems in general. It should be noted that the bending moments are not required to be continuous across interelement boundaries, as was shown in Chapter 4. The mixed models based on the shear deformation theories also have the same advantages, except that the displacement model of the first-order shear deformation theory is also a  $C^0$  element. In general, the formulative and programming efforts are less with the mixed elements.

## 7.3 Recommendations

The theory presented here can be extended to a more general theory; for example, the development of the theory in general curvilinear coordinates, and for more general shells (than the doubly-curved shells considered here). Extension of the present theory to include nonlinear material models is awaiting. Of course, the inclusion of thermal loads and damping in the present theory is straight forward.

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APPENDIX A

COEFFICIENTS OF THE NAVIER SOLUTION

$$C(1,1) = -A_{11}\alpha^2 - A_{66}\beta^2$$

$$C(1,2) = -A_{12}\alpha\beta - A_{66}\alpha\beta$$

$$C(1,3) = A_{11}\frac{\alpha}{R_1} + A_{12}\frac{\alpha}{R_2} + \frac{4\alpha}{3h^2}(E_{11}\alpha^2 + E_{12}\beta^2 + 2E_{66}\beta^2)$$

$$C(1,4) = \alpha^2(-B_{11} + \frac{4E_{11}}{3h^2}) + \beta^2(-B_{66} + \frac{4E_{66}}{3h^2})$$

$$C(1,5) = \alpha\beta(-B_{12} - B_{66} + \frac{4E_{12}}{3h^2} + \frac{4E_{66}}{3h^2})$$

$$C(2,2) = -\alpha^2 A_{66} - \beta^2 A_{22}$$

$$C(2,3) = \beta(\frac{A_{12}}{R_1} + \frac{A_{22}}{R_2}) + \alpha^2\beta(\frac{4E_{12}}{3h^2} + \frac{8E_{66}}{3h^2}) + \frac{4\beta^3}{3h^2} E_{22}$$

$$C(2,4) = \alpha\beta(-B_{12} - B_{66} + \frac{4E_{12}}{3h^2} + \frac{4E_{66}}{3h^2})$$

$$C(2,5) = \alpha^2(-B_{66} + \frac{4E_{66}}{3h^2}) + \beta^2(-B_{22} + \frac{4E_{22}}{3h^2})$$

$$C(3,3) = \alpha^4(-\frac{16H_{11}}{9h^4}) + \beta^4(-\frac{16H_{22}}{9h^4}) - \frac{32\alpha^2\beta^2}{9h^4}(H_{12} + 2H_{66})$$

$$+ \alpha^2(-A_{55} + \frac{8D_{55}}{h^2} - \frac{16F_{55}}{h^4} - \frac{8E_{11}}{3h^2 R_1} - \frac{8E_{12}}{3h^2 R_2})$$

$$+ \beta^2(-A_{44} + \frac{8D_{44}}{h^2} - \frac{16F_{44}}{h^4} - \frac{8E_{12}}{3h^2 R_1} + \frac{8E_{22}}{3h^2 R_2})$$

$$\begin{aligned}
& -\frac{A_{11}}{R_1^2} - \frac{2A_{12}}{R_1 R_2} - \frac{A_{22}}{R_2^2} \\
C(3,4) &= \frac{4\alpha^3}{3h^2} \left( F_{11} - \frac{4H_{11}}{3h^2} \right) + \frac{4\alpha\beta^2}{3h^2} (F_{12} + 2F_{66}) - \frac{16\alpha\beta^2}{9h^4} (H_{12} \\
& + 2H_{66}) + \alpha \left[ \frac{B_{11}}{R_1} + \frac{B_{12}}{R_2} - \frac{4}{3h^2} \left( \frac{E_{11}}{R_1} + \frac{E_{12}}{R_2} \right) \right] - A_{55}\alpha \\
& + \frac{8\alpha D_{55}}{h^2} - \frac{16\alpha F_{55}}{h^4} \\
C(3,5) &= \frac{4\beta^3}{3h^2} \left( F_{22} - \frac{4H_{22}}{3h^2} \right) + \frac{4\alpha^2\beta}{3h^2} (F_{12} + 2F_{66}) - \frac{16\alpha^2\beta}{9h^4} (H_{12} \\
& + 2H_{66}) + \beta \left[ \frac{B_{12}}{R_1} + \frac{B_{22}}{R_2} - \frac{4}{3h^2} \left( \frac{E_{12}}{R_1} + \frac{E_{22}}{R_2} \right) \right] - A_{44}\beta \\
& + \frac{8\beta D_{44}}{h^2} - \frac{16\beta F_{44}}{h^4} \\
C(4,4) &= \frac{8\alpha^2}{3h^2} F_{11} + \frac{8\beta^2}{3h^2} F_{66} - \frac{16}{9h^4} (H_{11}\alpha^2 + H_{66}\beta^2) - D_{11}\alpha^2 - D_{66}\beta^2 \\
& - A_{55} + \frac{8D_{55}}{h^2} - \frac{16F_{55}}{h^4} \\
C(4,5) &= \alpha\beta \left[ -D_{12} - D_{66} + \frac{8}{3h^2} (F_{12} + F_{66}) - \frac{16}{9h^4} (H_{12} + H_{66}) \right] \\
C(5,5) &= \alpha^2 \left( -D_{66} + \frac{8F_{66}}{3h^2} - \frac{16H_{66}}{9h^4} \right) + \beta^2 \left( -D_{22} + \frac{8F_{22}}{3h^2} - \frac{16H_{22}}{9h^4} \right) - A_{44} \\
& + \frac{8D_{44}}{h^2} - \frac{16F_{44}}{h^4} \\
M(1,1) &= - \left( I_1 + \frac{2I_2}{R_1} \right)
\end{aligned}$$

$$M(1,2) = 0$$

$$M(1,3) = \left( \frac{4}{3h^2} I_4 + \frac{4}{3h^2 R_1} I_5 \right) \alpha$$

$$M(1,4) = -\left( I_2 + \frac{I_3}{R_1} - \frac{4}{3h^2} I_4 - \frac{4}{3h^2 R_1} I_5 \right)$$

$$M(1,5) = 0$$

$$M(2,2) = -\left( I_1 + \frac{2I_2}{R_2} \right)$$

$$M(2,3) = \left( \frac{4}{3h^2} I_4 + \frac{4}{3h^2 R_2} I_5 \right) \beta$$

$$M(2,4) = 0$$

$$M(2,5) = -\left( I_2 + \frac{I_3}{R_2} - \frac{4}{3h^2} I_4 - \frac{4}{3h^2 R_2} I_5 \right)$$

$$M(3,3) = -\frac{16}{9h^4} I_7 \alpha^2 - \frac{16}{9h^4} I_7 \beta^2 - I_1$$

$$M(3,4) = \left( \frac{4}{3h^2} I_5 - \frac{16}{9h^4} I_7 \right) \alpha$$

$$M(3,5) = \left( \frac{4}{3h^2} I_5 - \frac{16}{9h^4} I_7 \right) \beta$$

$$M(4,4) = -\left( I_3 - \frac{8}{3h^2} I_5 + \frac{16}{9h^4} I_7 \right)$$

$$M(4,5) = 0$$

$$M(5,5) = -\left( I_3 - \frac{8}{3h^2} I_5 + \frac{16}{9h^4} I_7 \right)$$



APPENDIX B

STIFFNESS COEFFICIENTS FOR THE MIXED MODEL

The element equation (5.16) can be written in the form

$$\begin{bmatrix} [K^{11}] & [K^{12}] & \dots\dots & [K^{1(11)}] \\ [K^{21}] & [K^{22}] & \dots\dots & [K^{2(11)}] \\ \vdots & & & \vdots \\ [K^{(11)1}] & [K^{(11)2}] & \dots & [K^{(11)(11)}] \end{bmatrix} \begin{Bmatrix} \{\Delta^1\} \\ \{\Delta^2\} \\ \vdots \\ \{\Delta^{11}\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \vdots \\ \{F^{11}\} \end{Bmatrix}$$

where  $\{\Delta^1\} = \{U\}$ ,  $\{\Delta^2\} = \{V\}$ , etc., and

$$[K^{11}] = A_{11}[S_{11}] + A_{13}[S_{12}] + A_{31}[S_{12}] + A_{33}[S_{22}]$$

$$[K^{12}] = A_{12}[S_{12}] + A_{13}[S_{11}] + A_{32}[S_{22}] + A_{33}[S_{21}]$$

$$[K^{13}] = \frac{A_{11}}{R_1} [S_{10}] + \frac{1}{2} A_{11} \left[ \frac{\partial w}{\partial x_1} S_{11} \right] + \frac{A_{12}}{R_2} [S_{10}]$$

$$+ \frac{1}{2} A_{12} \left[ \frac{\partial w}{\partial x_2} S_{12} \right] + \frac{1}{2} A_{13} \left[ \frac{\partial w}{\partial x_1} S_{12} \right] + \frac{1}{2} A_{13} \left[ \frac{\partial w}{\partial x_2} S_{11} \right]$$

$$+ \frac{A_{31}}{R_1} [S_{20}] + \frac{1}{2} A_{31} \left[ \frac{\partial w}{\partial x_1} S_{21} \right] + \frac{A_{32}}{R_2} [S_{20}]$$

$$+ \frac{1}{2} A_{32} \left[ \frac{\partial w}{\partial x_2} S_{22} \right] + \frac{1}{2} A_{33} \left[ \frac{\partial w}{\partial x_1} S_{22} \right] + \frac{1}{2} A_{33} \left[ \frac{\partial w}{\partial x_2} S_{21} \right]$$

$$[K^{14}] = [K^{15}] = 0$$

$$[K^{16}] = B_{11}[S_{10}] + B_{31}[S_{20}]$$

$$[K^{17}] = B_{12}[S_{10}] + B_{32}[S_{20}]$$

$$[K^{18}] = B_{13}[S_{10}] + B_{33}[S_{20}]$$

$$[K^{19}] = E_{11}[S_{10}] + E_{31}[S_{20}]$$

$$[K^{1,10}] = E_{12}[S_{10}] + E_{32}[S_{20}]$$

$$[K^{1,11}] = E_{13}[S_{10}] + E_{33}[S_{20}]$$

$$[K^{21}] = A_{31}[S_{11}] + A_{33}[S_{12}] + A_{21}[S_{21}] + A_{23}[S_{22}]$$

$$[K^{22}] = A_{32}[S_{12}] + A_{33}[S_{11}] + A_{22}[S_{22}] + A_{23}[S_{21}]$$

$$\begin{aligned} [K^{23}] &= \frac{A_{31}}{R_1} [S_{10}] + \frac{1}{2} A_{31} \left[ \frac{\partial w}{\partial x_1} S_{11} \right] + \frac{A_{32}}{R_2} [S_{10}] \\ &+ \frac{1}{2} A_{32} \left[ \frac{\partial w}{\partial x_2} S_{12} \right] + \frac{1}{2} A_{33} \left[ \frac{\partial w}{\partial x_1} S_{12} \right] + \frac{1}{2} A_{33} \left[ \frac{\partial w}{\partial x_2} S_{11} \right] \\ &+ \frac{A_{21}}{R_1} [S_{20}] + \frac{1}{2} A_{21} \left[ \frac{\partial w}{\partial x_1} S_{21} \right] + \frac{A_{22}}{R_2} [S_{20}] \\ &+ \frac{1}{2} A_{22} \left[ \frac{\partial w}{\partial x_2} S_{22} \right] + \frac{1}{2} A_{23} \left[ \frac{\partial w}{\partial x_1} S_{22} \right] + \frac{1}{2} A_{23} \left[ \frac{\partial w}{\partial x_2} S_{21} \right] \end{aligned}$$

$$[K^{24}] = [K^{25}] = 0$$

$$[K^{26}] = B_{31}[S_{10}] + B_{21}[S_{20}]$$

$$[K^{27}] = B_{32}[S_{10}] + B_{22}[S_{20}]$$

$$[K^{28}] = B_{33}[S_{10}] + B_{23}[S_{20}]$$

$$[K^{29}] = E_{31}[S_{10}] + E_{21}[S_{20}]$$

$$[K^{2,10}] = E_{32}[S_{10}] + E_{22}[S_{20}]$$

$$[K^{2,11}] = E_{33}[S_{10}] + E_{23}[S_{20}]$$

$$\begin{aligned} [K^{31}] &= \frac{A_{11}}{R_1} [S_{01}] + \frac{A_{13}}{R_1} [S_{02}] + A_{11} \left[ \frac{\partial w}{\partial x_1} S_{11} \right] \\ &+ A_{13} \left[ \frac{\partial w}{\partial x_1} S_{12} \right] + \frac{A_{21}}{R_2} [S_{01}] + \frac{A_{23}}{R_2} [S_{02}] + A_{21} \left[ \frac{\partial w}{\partial x_2} S_{21} \right] \\ &+ A_{23} \left[ \frac{\partial w}{\partial x_2} S_{22} \right] + A_{31} \left[ \frac{\partial w}{\partial x_1} S_{21} \right] + A_{33} \left[ \frac{\partial w}{\partial x_1} S_{22} \right] \\ &+ A_{31} \left[ \frac{\partial w}{\partial x_2} S_{11} \right] + A_{33} \left[ \frac{\partial w}{\partial x_2} S_{12} \right] \end{aligned}$$

$$\begin{aligned} [K^{32}] &= \frac{A_{12}}{R_1} [S_{02}] + \frac{A_{13}}{R_1} [S_{01}] + A_{12} \left[ \frac{\partial w}{\partial x_1} S_{12} \right] + A_{13} \left[ \frac{\partial w}{\partial x_1} S_{11} \right] \\ &+ \frac{A_{22}}{R_2} [S_{02}] + \frac{A_{23}}{R_2} [S_{01}] + A_{22} \left[ \frac{\partial w}{\partial x_2} S_{22} \right] + A_{23} \left[ \frac{\partial w}{\partial x_2} S_{21} \right] \\ &+ A_{32} \left[ \frac{\partial w}{\partial x_1} S_{22} \right] + A_{33} \left[ \frac{\partial w}{\partial x_1} S_{21} \right] + A_{32} \left[ \frac{\partial w}{\partial x_2} S_{12} \right] \\ &+ A_{33} \left[ \frac{\partial w}{\partial x_2} S_{11} \right] \end{aligned}$$

$$\begin{aligned} [K^{33}] &= \frac{A_{11}}{R_1^2} [S_{00}] + \frac{1}{2} \frac{A_{11}}{R_1} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] + \frac{A_{12}}{R_1 R_2} [S_{00}] \\ &+ \frac{1}{2} \frac{A_{12}}{R_1} \left[ \frac{\partial w}{\partial x_2} S_{02} \right] + \frac{1}{2} \frac{A_{13}}{R_1} \left[ \frac{\partial w}{\partial x_1} S_{02} \right] + \frac{1}{2} \frac{A_{13}}{R_1} \left[ \frac{\partial w}{\partial x_2} S_{01} \right] \\ &+ \frac{A_{11}}{R_1} \left[ \frac{\partial w}{\partial x_1} S_{10} \right] + \frac{1}{2} A_{11} \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{11} \right] + \frac{A_{12}}{R_2} \left[ \frac{\partial w}{\partial x_1} S_{10} \right] \\ &+ \frac{1}{2} A_{12} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{12} \right] + \frac{1}{2} A_{13} \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{12} \right] \\ &+ \frac{1}{2} A_{13} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{11} \right] + \frac{A_{21}}{R_1 R_2} [S_{00}] + \frac{1}{2} \frac{A_{21}}{R_2} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{A_{22}}{R_2^2} [S_{00}] + \frac{1}{2} \frac{A_{22}}{R_2} \left[ \frac{\partial w}{\partial x_2} S_{02} \right] + \frac{1}{2} \frac{A_{23}}{R_2} \left[ \frac{\partial w}{\partial x_1} S_{02} \right] \\
& + \frac{1}{2} \frac{A_{23}}{R_2} \left[ \frac{\partial w}{\partial x_2} S_{01} \right] + \frac{A_{21}}{R_1} \left[ \frac{\partial w}{\partial x_2} S_{20} \right] + \frac{1}{2} A_{21} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{21} \right] \\
& + \frac{A_{22}}{R_2} \left[ \frac{\partial w}{\partial x_2} S_{20} \right] + \frac{1}{2} A_{22} \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{22} \right] + \frac{1}{2} A_{23} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{22} \right] \\
& + \frac{1}{2} A_{23} \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{21} \right] + \frac{A_{31}}{R_1} \left[ \frac{\partial w}{\partial x_1} S_{20} \right] + \frac{1}{2} A_{31} \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{21} \right] \\
& + \frac{A_{32}}{R_2} \left[ \frac{\partial w}{\partial x_1} S_{20} \right] + \frac{1}{2} A_{32} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{22} \right] + \frac{1}{2} A_{33} \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{22} \right] \\
& + \frac{1}{2} A_{33} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{21} \right] + \frac{A_{31}}{R_1} \left[ \frac{\partial w}{\partial x_2} S_{10} \right] + \frac{1}{2} A_{31} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{11} \right] \\
& + \frac{A_{12}}{R_2} \left[ \frac{\partial w}{\partial x_2} S_{10} \right] + \frac{1}{2} A_{32} \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{12} \right] + \frac{1}{2} A_{33} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{12} \right] \\
& + \frac{1}{2} A_{33} \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{11} \right] + \bar{A}_{55} [S_{11}] + \bar{A}_{45} [(S_{12} + S_{21})] \\
& + \bar{A}_{44} [S_{22}]
\end{aligned}$$

$$[K^{34}] = \bar{A}_{55} [S_{10}] + \bar{A}_{45} [S_{20}]$$

$$[K^{35}] = \bar{A}_{45} [S_{10}] + \bar{A}_{44} [S_{20}]$$

$$\begin{aligned}
[K^{36}] &= \frac{B_{11}}{R_1} [S_{00}] + B_{11} \left[ \frac{\partial w}{\partial x_1} S_{10} \right] + \frac{B_{21}}{R_2} [S_{00}] + B_{21} \left[ \frac{\partial w}{\partial x_2} S_{20} \right] \\
&+ B_{31} \left[ \frac{\partial w}{\partial x_1} S_{20} \right] + B_{31} \left[ \frac{\partial w}{\partial x_2} S_{10} \right]
\end{aligned}$$

$$[K^{37}] = \frac{B_{12}}{R_1} [S_{00}] + B_{12} \left[ \frac{\partial w}{\partial x_1} S_{10} \right] + \frac{B_{22}}{R_2} [S_{00}] + B_{22} \left[ \frac{\partial w}{\partial x_2} S_{20} \right]$$

$$+ B_{32} \left[ \frac{\partial w}{\partial x_1} S_{20} \right] + B_{32} \left[ \frac{\partial w}{\partial x_2} S_{10} \right]$$

$$[K^{38}] = \frac{B_{13}}{R_1} [S_{00}] + B_{13} \left[ \frac{\partial w}{\partial x_1} S_{10} \right] + \frac{B_{23}}{R_2} [S_{00}] + B_{23} \left[ \frac{\partial w}{\partial x_2} S_{20} \right]$$

$$+ B_{33} \left[ \frac{\partial w}{\partial x_1} S_{20} \right] + B_{33} \left[ \frac{\partial w}{\partial x_2} S_{10} \right]$$

$$[K^{39}] = \frac{E_{11}}{R_1} [S_{00}] + E_{11} \left[ \frac{\partial w}{\partial x_1} S_{10} \right] + \frac{E_{21}}{R_2} [S_{00}] + E_{21} \left[ \frac{\partial w}{\partial x_2} S_{20} \right]$$

$$+ E_{31} \left[ \frac{\partial w}{\partial x_1} S_{20} \right] + E_{31} \left[ \frac{\partial w}{\partial x_2} S_{10} \right] + CP[S_{11}]$$

$$[K^{3,10}] = \frac{E_{12}}{R_1} [S_{00}] + E_{12} \left[ \frac{\partial w}{\partial x_1} S_{10} \right] + \frac{E_{22}}{R_2} [S_{00}] + E_{22} \left[ \frac{\partial w}{\partial x_2} S_{20} \right]$$

$$+ E_{32} \left[ \frac{\partial w}{\partial x_1} S_{20} \right] + E_{32} \left[ \frac{\partial w}{\partial x_2} S_{10} \right] + CP[S_{22}]$$

$$[K^{3,11}] = \frac{E_{13}}{R_1} [S_{00}] + E_{13} \left[ \frac{\partial w}{\partial x_1} S_{10} \right] + \frac{E_{23}}{R_2} [S_{00}] + E_{23} \left[ \frac{\partial w}{\partial x_2} S_{20} \right]$$

$$+ E_{33} \left[ \frac{\partial w}{\partial x_1} S_{20} \right] + E_{33} \left[ \frac{\partial w}{\partial x_2} S_{10} \right] + CP[(S_{12} + S_{21})]$$

$$[K^{41}] = [K^{42}] = 0$$

$$[K^{43}] = \bar{A}_{55}[S_{01}] + \bar{A}_{45}[S_{02}]$$

$$[K^{44}] = \bar{A}_{55}[S_{00}]$$

$$[K^{45}] = \bar{A}_{45}[S_{00}]$$

$$[K^{46}] = [S_{10}] , [K^{47}] = 0 , [K^{48}] = [S_{20}]$$

$$[K^{49}] = CP[S_{01}] , [K^{4,10}] = 0 , [K^{4,11}] = CP[S_{02}]$$

$$[K^{51}] = [K^{52}] = 0$$

$$[K^{53}] = \bar{A}_{45}[S_{01}] + \bar{A}_{44}[S_{02}]$$

$$[K^{54}] = \bar{A}_{45}[S_{00}]$$

$$[K^{55}] = \bar{A}_{44}[S_{00}]$$

$$[K^{56}] = 0 , [K^{57}] = [S_{20}] , [K^{58}] = [S_{10}]$$

$$[K^{59}] = 0 , [K^{5,10}] = CP[S_{02}] , [K^{5,11}] = CP[S_{01}]$$

$$[K^{61}] = B_{11}[S_{01}] + B_{31}[S_{02}]$$

$$[K^{62}] = B_{21}[S_{02}] + B_{31}[S_{01}]$$

$$[K^{63}] = \frac{B_{11}}{R_1} [S_{00}] + \frac{1}{2} B_{11} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] + \frac{B_{21}}{R_2} [S_{00}] + \frac{1}{2} B_{21} \left[ \frac{\partial w}{\partial x_2} S_{02} \right] \\ + \frac{1}{2} B_{31} \left[ \frac{\partial w}{\partial x_1} S_{02} \right] + \frac{1}{2} B_{31} \left[ \frac{\partial w}{\partial x_2} S_{01} \right]$$

$$[K^{64}] = [S_{01}] , [K^{65}] = 0$$

$$[K^{66}] = -D_{11}[S_{00}]$$

$$[K^{67}] = -D_{12}[S_{00}]$$

$$[K^{68}] = -D_{13}[S_{00}]$$

$$[K^{69}] = -F_{11}[S_{00}]$$

$$[K^{6,10}] = -F_{12}[S_{00}]$$

$$[K^{6,11}] = -F_{13}[S_{00}]$$

$$[K^{71}] = B_{12}[S_{01}] + B_{32}[S_{02}]$$

$$[K^{72}] = B_{22}[S_{02}] + B_{32}[S_{01}]$$

$$[K^{73}] = \frac{B_{12}}{R_1} [S_{00}] + \frac{1}{2} B_{12} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] + \frac{B_{22}}{R_2} [S_{00}] + \frac{1}{2} B_{22} \left[ \frac{\partial w}{\partial x_2} S_{02} \right] \\ + \frac{1}{2} B_{32} \left[ \frac{\partial w}{\partial x_1} S_{02} \right] + \frac{1}{2} B_{32} \left[ \frac{\partial w}{\partial x_2} S_{01} \right]$$

$$[K^{74}] = 0 \quad , \quad [K^{75}] = [S_{02}]$$

$$[K^{76}] = -D_{21}[S_{00}]$$

$$[K^{77}] = -D_{22}[S_{00}]$$

$$[K^{78}] = -D_{23}[S_{00}]$$

$$[K^{79}] = -F_{21}[S_{00}]$$

$$[K^{7,10}] = -F_{22}[S_{00}]$$

$$[K^{7,11}] = -F_{23}[S_{00}]$$

$$[K^{81}] = B_{13}[S_{01}] + B_{33}[S_{02}]$$

$$[K^{82}] = B_{23}[S_{02}] + B_{33}[S_{01}]$$

$$[K^{83}] = \frac{B_{13}}{R_1} [S_{00}] + \frac{1}{2} B_{13} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] + \frac{B_{23}}{R_2} [S_{00}] + \frac{1}{2} B_{23} \left[ \frac{\partial w}{\partial x_2} S_{02} \right] \\ + \frac{1}{2} B_{33} \left[ \frac{\partial w}{\partial x_1} S_{02} \right] + \frac{1}{2} B_{33} \left[ \frac{\partial w}{\partial x_2} S_{01} \right]$$

$$[K^{84}] = [S_{02}]$$

$$[K^{85}] = [S_{01}]$$

$$[K^{86}] = -D_{31}[S_{00}]$$

$$[K^{87}] = -D_{32}[S_{00}]$$

$$[K^{88}] = -D_{33}[S_{00}]$$

$$[K^{89}] = -F_{31}[S_{00}]$$

$$[K^{8,10}] = -F_{32}[S_{00}]$$

$$[K^{8,11}] = -F_{33}[S_{00}]$$

$$[K^{91}] = E_{11}[S_{01}] + E_{31}[S_{02}]$$

$$[K^{92}] = E_{21}[S_{02}] + E_{31}[S_{01}]$$

$$[K^{93}] = \frac{E_{11}}{R_1} [S_{00}] + \frac{1}{2} E_{11} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] + \frac{E_{21}}{R_2} [S_{00}] + \frac{1}{2} E_{21} \left[ \frac{\partial w}{\partial x_2} S_{02} \right]$$



$$+ \frac{1}{2} E_{31} \left[ \frac{\partial w}{\partial x_1} S_{02} \right] + \frac{1}{2} E_{31} \left[ \frac{\partial w}{\partial x_2} S_{01} \right] + CP[S_{11}]$$

$$[K^{94}] = CP[S_{10}] \quad [K^{95}] = 0$$

$$[K^{96}] = - F_{11}[S_{00}]$$

$$[K^{97}] = - F_{21}[S_{00}]$$

$$[K^{98}] = - F_{31}[S_{00}]$$

$$[K^{99}] = - H_{11}[S_{00}]$$

$$[K^{9,10}] = - H_{12}[S_{00}]$$

$$[K^{9,11}] = - H_{13}[S_{00}]$$

$$[K^{10,1}] = E_{12}[S_{01}] + E_{32}[S_{02}]$$

$$[K^{10,2}] = E_{22}[S_{02}] + E_{32}[S_{01}]$$

$$[K^{10,3}] = \frac{E_{12}}{R_1} [S_{00}] + \frac{1}{2} E_{12} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] + \frac{E_{22}}{R_2} [S_{00}] + \frac{1}{2} E_{22} \left[ \frac{\partial w}{\partial x_2} S_{02} \right]$$

$$+ \frac{1}{2} E_{32} \left[ \frac{\partial w}{\partial x_1} S_{02} \right] + \frac{1}{2} E_{32} \left[ \frac{\partial w}{\partial x_2} S_{01} \right] + CP[S_{22}]$$

$$[K^{10,4}] = 0, \quad [K^{10,5}] = CP[S_{20}]$$

$$[K^{10,6}] = - F_{12}[S_{00}]$$

$$[K^{10,7}] = - F_{22}[S_{00}]$$

$$[K^{10,8}] = -F_{32}[S_{00}]$$

$$[K^{10,9}] = -H_{21}[S_{00}]$$

$$[K^{10,10}] = -H_{22}[S_{00}]$$

$$[K^{10,11}] = -H_{23}[S_{00}]$$

$$[K^{11,1}] = E_{13}[S_{01}] + E_{33}[S_{02}]$$

$$[K^{11,2}] = E_{23}[S_{02}] + E_{33}[S_{01}]$$

$$[K^{11,3}] = \frac{E_{13}}{R_1} [S_{00}] + \frac{1}{2} E_{13} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] + \frac{E_{23}}{R_2} [S_{00}] + \frac{1}{2} E_{23} \left[ \frac{\partial w}{\partial x_2} S_{02} \right] \\ + \frac{1}{2} E_{33} \left[ \frac{\partial w}{\partial x_1} S_{02} \right] + \frac{1}{2} E_{33} \left[ \frac{\partial w}{\partial x_2} S_{01} \right] + CP[(S_{12} + S_{21})]$$

$$[K^{11,4}] = CP[S_{20}]$$

$$[K^{11,5}] = CP[S_{10}]$$

$$[K^{11,6}] = -F_{13}[S_{00}]$$

$$[K^{11,7}] = -F_{23}[S_{00}]$$

$$[K^{11,8}] = -F_{33}[S_{00}]$$

$$[K^{11,9}] = -H_{31}[S_{00}]$$

$$[K^{11,10}] = -H_{32}[S_{00}]$$

$$[K^{11,11}] = -H_{33}[S_{00}]$$

Mass Matrix [M] for the Mixed Model

$$[M] = \begin{bmatrix} [M^{11}] & [M^{12}] & \dots & [M^{1,11}] \\ [M^{21}] & [M^{22}] & \dots & [M^{2,11}] \\ \dots & \dots & \dots & \dots \\ [M^{11,1}] & [M^{11,2}] & \dots & [M^{11,11}] \end{bmatrix}$$

$$[M^{11}] = I_1[S_{00}]$$

$$[M^{13}] = -\frac{4}{3h^2} I_4[S_{01}] \quad [M^{31}] = -\frac{4}{3h^2} I_4[S_{10}]$$

$$[M^{14}] = I_2[S_{00}] - \frac{4}{3h^2} I_4[S_{00}] \quad [M^{41}] = I_2[S_{00}] - \frac{4}{3h^2} I_4[S_{00}]$$

$$[M^{22}] = I_1[S_{00}]$$

$$[M^{23}] = -\frac{4}{3h^2} I_4[S_{02}] \quad , \quad [M^{32}] = -\frac{4}{3h^2} I_4[S_{20}]$$

$$[M^{25}] = (I_2 - \frac{4}{3h^2} I_4)[S_{00}] \quad , \quad [M^{52}] = [M^{25}]$$

$$[M^{33}] = I_1[S_{00}] + I_7\left(\frac{4}{3h^2}\right)^2[S_{11} + S_{22}]$$

$$[M^{34}] = \left(-\frac{4}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7\right)[S_{10}]$$

$$[M^{43}] = \left(-\frac{4}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7\right)[S_{01}]$$

$$[M^{35}] = \left(-\frac{4}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7\right)[S_{20}]$$

$$[M^{53}] = \left(-\frac{4}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7\right)[S_{02}]$$

$$[M^{44}] = \left(I_3 - \frac{8}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7\right)[S_{00}]$$

$$[M^{55}] = [M^{44}]$$

all others are zero, and  $I_i$  are defined in Eq. (2.18b)

$$\bar{A}_{44} = A_{44} - 2E_{44} \frac{4}{h^2} + G_{44} \left(\frac{4}{h^2}\right)^2$$

$$\bar{A}_{45} = A_{45} - 2E_{45} \frac{4}{h^2} + G_{45} \left(\frac{4}{h^2}\right)^2$$

$$\bar{A}_{55} = A_{55} - 2E_{55} \frac{4}{h^2} + G_{55} \left(\frac{4}{h^2}\right)^2$$

$$CP = \frac{4}{3h^2}$$

$$[S_{IJ}] = \int_{\Omega} e \left[ \frac{\partial \psi_i}{\partial x_I} \frac{\partial \psi_j}{\partial x_J} \right] dx_1 dx_2$$

with  $\frac{\partial \psi_i}{\partial x_0} = \psi_i$

$$\left[ \frac{\partial w}{\partial x_2} S_{IJ} \right] = \left[ \int_{\Omega} e \frac{\partial w}{\partial x_2} \frac{\partial \psi_i}{\partial x_I} \frac{\partial \psi_j}{\partial x_J} dx_1 dx_2 \right]$$

For the tangent stiffness matrix, the coefficients are given by

$$[(K_T)^{13}] = 2[K^{13}] = [K^{31}]$$

$$[(K_T)^{23}] = 2[K^{23}] = [K^{32}]$$

$$[(K_T)^{33}] = [K^{33}]$$

$$+ A_{11} \left[ \frac{\partial u}{\partial x_1} S_{11} \right] + A_{21} \left[ \frac{\partial u}{\partial x_2} S_{22} \right] + A_{31} \left[ \frac{\partial u}{\partial x_1} (S_{21} + S_{12}) \right]$$

$$+ A_{13} \left[ \frac{\partial u}{\partial x_2} S_{11} \right] + A_{23} \left[ \frac{\partial u}{\partial x_2} S_{22} \right] + A_{33} \left[ \frac{\partial u}{\partial x_2} (S_{21} + S_{12}) \right]$$

$$\begin{aligned}
& + A_{13} \left[ \frac{\partial v}{\partial x_1} S_{11} \right] + A_{23} \left[ \frac{\partial v}{\partial x_1} S_{22} \right] + A_{33} \left[ \frac{\partial v}{\partial x_1} (S_{21} + S_{12}) \right] \\
& + A_{12} \left[ \frac{\partial v}{\partial x_2} S_{11} \right] + A_{22} \left[ \frac{\partial v}{\partial x_2} S_{22} \right] + A_{32} \left[ \frac{\partial v}{\partial x_2} (S_{21} + S_{12}) \right] \\
& + \frac{1}{2} A_{12} \left[ \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{12} \right] + \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{11} \right] \right] \\
& + \frac{1}{2} A_{13} \left[ \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{12} \right] + \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{11} \right] \right] \\
& + \frac{1}{2} A_{21} \left[ \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{21} \right] + \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{22} \right] \right] \\
& + \frac{1}{2} A_{23} \left[ \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{22} \right] + \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{21} \right] \right] \\
& + \frac{1}{2} A_{31} \left[ \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{11} \right] + \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{12} \right] \right] \\
& + \frac{1}{2} A_{33} \left[ \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{11} \right] + \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{12} \right] \right] \\
& + \frac{1}{2} A_{32} \left[ \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{22} \right] + \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{21} \right] \right] \\
& + \frac{1}{2} A_{33} \left[ \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{22} \right] + \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{21} \right] \right] \\
& + A_{11} \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{11} \right] + A_{13} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{11} \right] \\
& + \frac{1}{2R_1} \left[ A_{11} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] + A_{12} \left[ \frac{\partial w}{\partial x_2} S_{02} \right] \right]
\end{aligned}$$

$$\begin{aligned}
& + A_{13} \left( \left[ \frac{\partial w}{\partial x_2} S_{01} \right] + \left[ \frac{\partial w}{\partial x_1} S_{02} \right] \right) \\
& + \frac{1}{2R_2} \left[ A_{21} \left[ \frac{\partial w}{\partial x_1} S_{01} \right] + A_{22} \left[ \frac{\partial w}{\partial x_2} S_{02} \right] + A_{23} \left( \left[ \frac{\partial w}{\partial x_2} S_{01} \right] \right. \right. \\
& \left. \left. + \left[ \frac{\partial w}{\partial x_1} S_{02} \right] \right) \right] + \frac{1}{R_1} \left[ A_{11} [wS_{11}] + A_{21} [wS_{22}] \right] \\
& + \frac{1}{R_2} \left[ A_{12} [wS_{11}] + A_{22} [wS_{22}] \right] + \frac{1}{R_1} \left[ A_{31} [w(S_{12} + S_{21})] \right] \\
& + \frac{1}{R_2} \left[ A_{32} [(wS_{12} + S_{21})] \right] + A_{22} \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{22} \right] \\
& + A_{23} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{22} \right] + A_{32} \left[ \left( \frac{\partial w}{\partial x_2} \right)^2 S_{12} \right] + A_{33} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{12} \right] \\
& + A_{31} \left[ \left( \frac{\partial w}{\partial x_1} \right)^2 S_{21} \right] + A_{33} \left[ \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{21} \right] \\
& + M_1 [B_{11} [S_{11}] + B_{21} [S_{22}] + B_{31} [(S_{12} + S_{21})]] \\
& + M_2 [B_{12} [S_{11}] + B_{22} [S_{22}] + B_{32} [(S_{12} + S_{21})]] \\
& + M_6 [B_{13} [S_{11}] + B_{23} [S_{22}] + B_{33} [(S_{12} + S_{21})]] \\
& + P_1 [E_{11} [S_{11}] + E_{21} [S_{22}] + E_{31} [(S_{21} + S_{12})]] \\
& + P_2 [E_{12} [S_{11}] + E_{22} [S_{22}] + E_{32} [(S_{21} + S_{12})]] \\
& + P_6 [E_{13} [S_{11}] + E_{23} [S_{22}] + E_{33} [(S_{21} + S_{12})]]
\end{aligned}$$

$$[(K_T)^{63}] = [K^{36}]$$

$$[(K_T)^{73}] = [K^{37}]$$

$$[(K_T)^{83}] = [K^{38}]$$

$$[(K_T)^{93}] = [K^{39}]$$

$$[(K_T)^{10,3}] = [K^{3,10}]$$

$$[(K_T)^{11,3}] = [K^{3,11}]$$

All others are same as those in [K].

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16. Abstract A third-order shear deformation theory of laminated composite plates and shells is developed, the Navier solutions are derived, and its finite element models are developed. The theory allows parabolic description of the transverse shear stresses, and therefore the shear correction factors of the usual shear deformation theory are not required in the present theory. The theory also accounts for the von Karman nonlinear strains. Closed-form solutions of the theory for rectangular cross-ply and angle-ply plates and cross-ply shells are developed. The finite element model is based on independent approximations of the displacements and bending moments (i.e. mixed finite element model), and therefore, only $C^0$ -approximation are required. The finite element model is used to analyze cross-ply and angle-ply laminated plates and shells for bending and natural vibration. Many of the numerical results presented here should serve as references for future investigations. Three major conclusions resulted from the research: First, for thick laminates, shear deformation theories predict deflections, stresses and vibration frequencies significantly different from those predicted by classical theories. Second, even for thin laminates, shear deformation effects are significant in dynamic and geometrically nonlinear analyses. Third, the present third-order theory is more accurate compared to the classical and first-order theories in predicting static and dynamic response of laminated plates and shells made of high-modulus composite materials.					
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