CHAPTER 33

A HIGHER ORDER THEORY FOR SYMMETRICAL GRAVITY WAVES

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ABSTRACT

A higher order theory is presented for symmetrical, non-linear gravity waves As a consequence of the generality employed, the theory includes the full range of possible wave lengths, water depths and wave heights that may be encountered, and brings them into one unified formulation Thus, the theory encompasses both linear and non-linear waves, including Airy waves, Stokes waves, cnoidal waves and the solitary wave

Based on the work of Nekrasov, a complex potential in the form of an infinite series is developed to describe the flow field The potential satisfies the bottom (horizontal) condition as well as the kinematic surface condition exactly Furthermore, the dynamic surface condition is satisfied by numerical calculation of the series coefficients which appear in the complex potential The calculation of these coefficients is accomplished by solving a set of non-linear algebraic equations, with the aid of a Newton-Raphson iteration procedure and matrix inversion

Coefficients of the complex potential have been obtained for a fifth order analysis and preliminary results are presented in tabular form A brief discussion of the characteristics of the waves, including wave speed, wave shape and the height of the highest possible wave follows

INTRODUCTION

Water waves and their characteristics have received a great deal of attention by mathematicians, geophysicists and engineers over the past century and a half In particular, numerous theories have been developed to describe the characteristics of symmetrical, periodic, progressive waves Among the more classical papers are those by Stokes (1847,1880), Rayleigh (1876), Boussinesq (1872), Korteweg and DeVries (1895), Levi-Civita (1925), and Struik (1926) Reviews of some of these works, as well as many more recent publications may be found in publications by Stoker (1957), Wiegel (1964) Kinsman (1965), Ippen (1966) and Neumann and Pierson (1966) No attempt will be made here to review the many recent contributions However, of particular interest, especially for applications of the theory, are the works of Mash and Wiegel (1961), Skjelbreia and Hendrickson (1962), Laitone (1963) and Dean (1965)

The wealth of literature on the subject of periodic water waves reflects to some extent the lack of a unified approach An effort to resolve this problem was made by Nekrasov (1951), followed by Milne-Thomson (1969) and Thomas (1968) Nekrasov first formulated the wave problem in general terms, and concluded his analysis with a non-linear integral equation

The present study reexamines the work of Nekrasov and his successors and presents it in a manner which should be more useful in practice More specifically, a method is developed to compute coefficients which may be used to calculate the various characteristics of the waves

Since the theory presented herein is general, it covers the entire range of possible wave lengths, water depths and wave heights that may be encountered Thus it encompasses both linear and non-linear waves including Airy waves, Stokes waves, cnoidal waves and the solitary wave As a consequence it gives promise of simplifying the choice of the appropriate theory - a problem which currently faces the practitioner

SOLUTION OF THE WAVE PROBLEM

DEVELOPMENT OF THE THEORY

The wave theory which will be developed herein applies to progressive, symmetrical, gravity waves moving over the free surface of an inviscid, incompressible liquid, in an oscillatory manner Furthermore the waves are two-dimensional and, except for the special case of infinite depth, they move over a horizontal bottom No restriction is placed on liquid depth, D, wave length, L, or wave height, H Hence the theory is comprehensive and includes the full range of constant-profile waves, from Stokes waves to cnoidal waves and the solitary wave, as well as from small-amplitude waves to large-amplitude waves and the so-called "highest wave"

A train of oscillatory waves is moving from right to left over the surface of the liquid in question with wave speed, c By superimposing a uniform flow from left to right of the same speed as that of the waves, the wave profiles are brought to rest The net effect is to provide a steady flow from left to right, bounded by the fixed profile formed at the free surface and the impervious boundary at the bottom The steady flow-field will be seen to be considerably more amenable to study than would be the unsteady, progressive-wave field

In Fig 1 the steady wave is depicted and the more important constants are defined For convenience in the development the coordinates are described in complex terms and the physical plane is the z-plane, where z is the complex variable and x and y are the real and imaginary axes respectively The y-axis is chosen to pass through the crest of the wave, C, in order to assure symmetry The free surface is defined by $y_o = y_o(x_o)$ and the still water level is located at $y = y_s$, a distance which remains to be determined It should be noted that the depth, d, usually defined as the distance from the still water level to the bottom will equal the sum of y_s and D

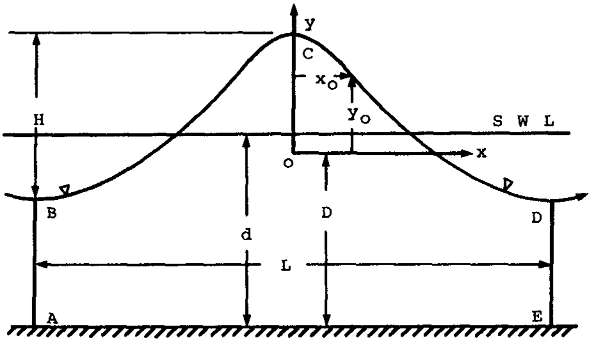

Fig 1 - The z-Plane

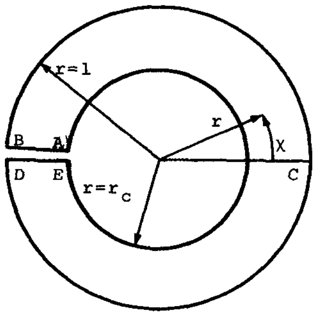

Fig 2 - The ζ-Plane

Since the flow is irrotational, an assumption which has been discussed by Stokes (1847), the complex potential is given by

$$w = \phi + i\psi \tag{1}$$

where ϕ is the potential function and ψ is the stream function Furthermore, the complex potential is analytic and so the Cauchy-Riemann equations, which may be related to the velocity components, u and v, are given by

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{2}$$

Substitution of these expressions into the continuity and irrotational conditions results in Laplace's equation for each function,

$$\nabla^2 \phi = 0 \qquad \nabla^2 \psi = 0 \tag{3}$$

respectively

At the free surface the kinematic boundary condition requires that the surface be a streamline For convenience, this bounding streamline is defined as

$$\psi = 0 \qquad \text{at } y = y_o \tag{4}$$

The dynamic boundary condition at the free surface is expressed by the Bernoulli equation with pressure equal to zero,

$$q_o^2 + 2gy_o = K_o \qquad \text{at } y = y_o \tag{5}$$

where q_o is the speed of a surface particle and K_o is a constant (twice the so-called "Bernoulli constant")

The lower boundary condition is kinematic and requires that the horizontal bottom be a streamline In order to reflect the volume rate of flow between the two bounding streamlines,

$$\psi = -cD \qquad \text{at } y = -D \tag{6}$$

where "cD' equals the two-dimensional flow rate observed in the z-plane

Up to this point the relevant differential equations, together with the appropriate boundary conditions which describe the flow, have been presented The two fundamental problems which immediately present themselves are that the location of the free surface is unknown and the dynamic boundary condition is non-linear In order to deal with the first problem a conformal transformation will be employed The purpose of this transformation is to redefine the problem in an auxiliary plane, the ζ-plane, where the location of the free surface is known

The particular conformal transformation to be used, is an outgrowth of the work of Nekrasov (1951), as well as a subsequent analysis by Milne-Thomson (1969) and Thomas (1968) More specifically, it is an extension and generalization of the special deep-water case considered by Monkmeyer and Kutzbach (1965) The transformation is given as follows

$$z = \frac{iL}{2\pi}\left[\ln(\zeta) + \frac{\pi}{2K}\sum_{j=1}^{\infty}\frac{a_j}{j}\exp[i\,\mathrm{am}(-\frac{2jK}{\pi}\ln\zeta)]\right] \qquad (7)$$

where K is the complete elliptic integral of the first kind and the a_j's are a set of real coefficients which are as yet unknown Moreover, ζ, which is the complex variable describing the coordinates of the ζ-plane is given in polar form by

$$\zeta = r\exp(i\chi) \qquad (8)$$

where r is the radial coordinate and χ the angular coordinate of the ζ-plane as shown in Fig 2 Finally, am( ) is the amplitude of the elliptic integral of the first kind

One may verify, by application of the mapping function, Eq 7, that the region bounded by ABCDE in the z-plane is a mapping of the equivalent region inside the unit circle in the ζ-plane, subject only to the proper evaluation of the constant coefficients, a_j In fact it may be shown that the boundaries AB, DE and EA are mapped exactly from the z-plane to the ζ-plane, regardless of the values of the coefficients, a_j The boundaries AB and DE transform exactly as a consequence of the periodicity of the transformation The exact transformation of the lower boundary DE follows from the characteristics of the elliptic function, am( ) The choice of this particular conformal transformation was essentially dictated by the exact transformation of the bottom boundary As a by-product of the bottom transformation it is required that

$$\frac{K'}{K} = \frac{4D}{L} \qquad (9)$$

where $K'(m) = K(1-m)$, and m is the parameter of the complete elliptic integral of the first kind This is a convenient formula since it permits a consideration of the entire range of waves for all wave lengths and depths In particular it facilitates inclusion of the two limiting cases, $L\to\infty$ and $D\to\infty$, since $K\to\infty$ in the first instance and $K'\to\infty$ in the second Eq 9 may therefore be used to convert Eq 7 and many of the following equations, if the limiting case of the solitary wave is of interest or if cnoidal waves are to be expressed in terms of depth rather than wave length

The transformation can also be applied to the boundary conditions, Eqs 4, 5 and 6, to generate the equivalent conditions in the ζ-plane,

$$\psi = 0 \qquad \text{at } r = 1 \qquad (10)$$

$$q_o^2 + 2gy_o = K_o \qquad \text{at } r = 1 \tag{11}$$

and

$$\psi = -cD \qquad \text{at } r = r_c \tag{12}$$

where

$$r_c = \exp\left(-\frac{2\pi D}{L}\right) \tag{13}$$

The validity of these representations of the boundary conditions is verified in Fig 2 Moreover it should be noted that Eqs 10 and 12 are precisely the boundary conditions for a portion of a clockwise irrotational vortex in the ζ-plane, so that the complex potential for the flow in this plane may be written

$$w = \frac{icL}{2\pi} \ln\zeta \tag{14}$$

By separating real and imaginary parts of this expression and rearranging terms,

$$\chi = -\frac{2\pi\phi}{cL} \tag{15}$$

and

$$r = \exp\left(\frac{2\pi\psi}{cL}\right) \tag{16}$$

It is therefore apparent that χ is a normalized form of the potential function and r is the exponential of the normalized stream function

To this point the physical problem in the z-plane has been transformed to one in the ζ-plane and two of the boundary conditions, Eqs 10 and 12, have been satisfied by the complex potential, Eq 14 It therefore remains to satisfy the dynamic surface condition, Eq 11, and this will be done by a proper choice of the coefficients in the conformal transformation The remaining portion of this analysis is devoted to a method for calculating these coefficients so that the dynamic condition will be satisfied, approximately The degree of approximation will depend on the truncation of the infinite series which makes up the conformal transformation The greater the number of terms retained, the more nearly the dynamic condition will be satisfied

Before proceeding to a calculation of the complex velocity, which will be needed in an examination of the Bernoulli condition on the free surface, it is necessary to substitute the complex potential in the ζ-plane, Eq 14, into the conformal transformation, Eq 7, in order to obtain the complex potential for the z-plane The result may be regarded as the <u>general wave equation</u>,

$$z = \frac{w}{c} + \frac{iL}{4K} \sum_{j=1}^{\infty} \frac{a_j}{j} \exp\left[i\, \mathrm{am}\left(-\frac{4jK}{cL} w\right)\right] \tag{17}$$

For the limiting case of infinite depth, K approaches $\pi/2$ and am( ) approaches its argument Therefore Eq 17 reduces to the <u>deep-water equation</u>,

$$z = \frac{w}{c} + \frac{iL}{2\pi} \sum_{j=1}^{\infty} \frac{a_j}{j} \exp\left[-i\frac{2j\pi}{cL} w\right] \tag{18}$$

On the other hand for the limiting case of infinite wave length, it should be noted that, after Eq 9 is introduced, K' approaches $\pi/2$ and am( ) approaches gd( ), the gudermannian Therefore, Eq 17 reduces to the solitary wave equation,

$$z = \frac{w}{c} + \frac{i2D}{\pi} \sum_{j=1}^{\infty} \frac{a_j}{j} \exp[i\ gd(-\frac{j\pi}{2cD} w)] \qquad (19)$$

Now along the free surface, $\psi = 0$, where we wish to apply the dynamic boundary condition, Eq 17 becomes

$$z_o = \frac{\phi}{c} + \frac{iL}{4K} \sum_{j=1}^{\infty} \frac{a_j}{j} \exp[i\ am(-\frac{4jK}{cL} \phi)] \qquad (20)$$

or in terms of the dimensionless potential function, χ,

$$z_o = -\frac{L\chi}{2\pi} + \frac{iL}{4K} \sum_{j=1}^{\infty} \frac{a_j}{j} \exp[i\ am(\frac{2jK}{\pi} \chi)] \qquad (21)$$

Taking real and imaginary parts of Eq 21 one obtains

$$x_o = -\frac{L\chi}{2\pi} - \frac{L}{4K} \sum_{j=1}^{\infty} \frac{a_j}{j} \operatorname{sn}(\frac{2jK}{\pi} \chi) \qquad (22)$$

and

$$y_o = \frac{L}{4K} \sum_{j=1}^{\infty} \frac{a_j}{j} \operatorname{cn}(\frac{2jK}{\pi} \chi) \qquad (23)$$

where sn( ) and cn( ) are Jacobian elliptic functions These two equations are parametrically related through χ to define the shape of the free surface In order to make them somewhat more tractable, it is convenient to replace the elliptic functions by their expansions in sine and cosine series respectively (see Milne-Thomson (1950)) Eqs 22 and 23 therefore become

$$x_o = -\frac{L}{4K} (\frac{2K\chi}{\pi} + \sum_{j=1}^{\infty} \frac{b_j}{j} \sin j\chi) \qquad (24)$$

and

$$y_o = \frac{L}{4K} \sum_{j=1}^{\infty} \frac{c_j}{j} \cos j\chi \qquad (25)$$

where

$$b_j = \frac{2\pi}{m^{1/2}K} \sum_{k=1}^{j} \frac{\frac{j}{k} a_k q^{j/2k}}{(1-q^{j/k})} \qquad (26)$$

and

$$c_j = \frac{2\pi}{m^{1/2}K} \sum_{k=1}^{j} \frac{\frac{j}{k} a_k q^{j/2k}}{(1+q^{j/k})} \qquad (27)$$

where

$$k = j, \frac{j}{3}, \frac{j}{5} \quad 1$$

and only integer values of k are included in the summation, and where

$$q = \exp(-\pi \frac{K'}{K}) \quad (28)$$

For the limiting case of deep water waves

$$b_j = c_j = a_j \quad (29)$$

whereas for the limiting case of the solitary wave

$$b_j = \frac{4}{\pi} \sum_{k=1}^{j} a_k \quad (30)$$

and

$$Kc_j = \pi \sum_{k=1}^{j} \frac{j}{k} a_k \quad (31)$$

with k defined as for Eqs 26 and 27

In view of Eqs 24 and 25 it is now possible to reexpress the comples variable, z_o, that was given in Eq 21, as follows

$$z_o = \frac{iL}{2\pi} \left[\ln\zeta_o + \frac{\pi}{2K} \sum_{j=1}^{\infty} \left[\frac{c_j+b_j}{2} \zeta_o^{\,j} + \frac{c_j-b_j}{2} \zeta_o^{\,-j} \right] \right] \quad (32)$$

where, on the free surface, Eq 8 reduces to

$$\zeta_o = \exp(i\chi) \quad (33)$$

By differentiating Eq 32, the complex operator

$$\frac{dz_o}{d\zeta_o} = \frac{iL}{2\pi} \frac{f(\zeta_o)}{\zeta_o} \quad (34)$$

is obtained, where

$$f(\zeta_o) = \frac{\pi}{2K} \sum_{j=0}^{\infty} (\frac{c_j+b_j}{2} \zeta_o^{\,j} - \frac{c_j-b_j}{2} \zeta_o^{\,-j}) \quad (35)$$

$$= R_o \exp(i\theta_o), \text{ say} \quad (36)$$

and where

$$b_o = c_o = \frac{2K}{\pi} \quad (37)$$

Moreover, the modulus, R_o, of the function $f(\zeta_o)$ is given by

$$R_o^{\,2} = \left[\frac{\pi}{2K} \sum_{j=0}^{\infty} b_j \cos j\chi\right]^2 + \left[\frac{\pi}{2K} \sum_{j=0}^{\infty} c_j \sin j\chi\right]^2 \quad (38)$$

and the argument, θ_o, of $f(\zeta_o)$ is given by

$$\theta_o = \cos^{-1}\left[\frac{\frac{\pi}{2K}\sum_{j=0}^{\infty} b_j \cos j\chi}{R_o}\right] \tag{39}$$

The above relations may now be employed, together with Eq 14, to derive an expression for the complex velocity, as follows

$$u_o - i v_o = \frac{dw_o}{dz_o} = \frac{dw_o}{d\zeta_o}\frac{d\zeta_o}{dz_o} = \frac{c}{f(\zeta_o)} = \frac{c}{R_o}\exp(-i\theta_o) \tag{40}$$

and therefore

$$u_o = \frac{c}{R_o}\cos\theta_o \quad \text{and} \quad v_o = \frac{c}{R_o}\sin\theta_o \tag{41}$$

and furthermore

$$q_o^2 = \frac{c^2}{R_o^2} \tag{42}$$

The dynamic free surface condition, Eq 11, may now be written

$$\frac{c^2}{R_o^2} + 2gy_o = K_o \tag{43}$$

an expression which was apparently first derived by Nekrasov (1951) who proceeded to derive a non-linear integral equation In addition, with the aid of the following dimensionless terms

$$x'_o \equiv \frac{4K}{L}x_o = \frac{K'}{D}x_o \qquad y'_o \equiv \frac{4K}{L}y_o = \frac{K'}{D}y_o$$
$$c' \equiv \frac{c}{\sqrt{\frac{gL}{4K}}} = \frac{c}{\sqrt{\frac{gD}{K'}}} \qquad \hat{K}_o \equiv \frac{K_o}{\frac{gL}{4K}} = \frac{K_o}{\frac{gD}{K'}} \tag{44}$$

Eq 43 becomes

$$c'^2 + 2y'_o R_o^2 = \hat{K}_o R_o^2 \tag{45}$$

where

$$y'_o = \sum_{j=1}^{\infty}\frac{c_j}{j}\cos j\chi \tag{46}$$

and, with the aid of trigonometric identities,

$$R_o^2 = \left(\frac{\pi}{2K}\right)^2\left[D_o + 2\sum_{j=1}^{\infty} D_j \cos j\chi\right] \tag{47}$$

where

$$D_o = b_o^2 + \sum_{k=1}^{\infty}\frac{b_k^2 + c_k^2}{2} \tag{48}$$

$$D_j = A_j + B_j + C_j \qquad j = 1,2,3,4 \qquad (49)$$

$$A_j = \sum_{k=j}^{\infty} \frac{b_{k-j}\, b_k + c_{k-j}\, c_k}{2} \qquad j = 1,2,3,4 \qquad (50)$$

$$B_j = \sum_{k=0}^{(j-1)/2} \frac{b_k\, b_{j-k} - c_k\, c_{j-k}}{2} \qquad j = 1,2,3,4 \qquad (51)$$

and

$$C_j = \frac{1}{4}\left(b_{j/2}^2 - c_{j/2}^2\right) \qquad \text{for } j = 2,4,6,8$$
$$= 0 \qquad \text{for } j = 1,3,5,7 \qquad (52)$$

By substituting Eqs 46 and 47 into Eq 45, the problem of the general symmetrical wave of finite amplitude is reduced to one of finding the solution to the equation,

$$c'^2 + 2\left[\left(\sum_{j=1}^{\infty} \frac{c_j}{j} \cos j\chi\right) - \frac{\hat{K}_o}{2}\right]\left[\left(\frac{\pi}{2K}\right)^2 \left(D_o + 2\sum_{j=1}^{\infty} (A_j + B_j + C_j)\cos j\chi\right)\right] = 0 \qquad (53)$$

It is of interest to note that to this point no approximations have been made Therefore Eq 53 is an exact representation of the problem

In order to solve Eq 53 for a finite number of coefficients, it will be necessary to truncate the infinite trigonometric series which appear in the equation Therefore Eqs 46, 47, 48 and 50 become

$$y_o' \simeq \sum_{j=1}^{n} \frac{c_j}{j} \cos j\chi \qquad (54)$$

$$R_o^2 \simeq \left(\frac{\pi}{2K}\right)^2 \left[D_o + 2\sum_{j=1}^{n} D_j \cos j\chi\right] \qquad (55)$$

$$D_o \simeq b_o^2 + \sum_{k=1}^{n} \frac{b_k^2 + c_k^2}{2} \qquad (56)$$

$$A_j \simeq \sum_{k=j}^{n} \frac{b_{k-j}\, b_k + c_{k-j}\, c_k}{2} \qquad j = 1,2,3,4 \quad n \qquad (57)$$

Furthermore, by combining and expanding these equations, one obtains

$$y_o' R_o^2 \simeq \left(\frac{\pi}{2K}\right)^2 \left[\sum_{k=1}^{n} \frac{c_k}{k} D_k + \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{c_k}{k} (D_{k-j} + D_{k+j}) \cos j\chi\right] \qquad (58)$$

where absolute value signs are omitted on the subscripts of D_{k-j}, and furthermore, $D_\ell = 0$ if $|\ell| > n$ Since harmonics higher than the nth have been omitted, Eq 58 is not exact

The expressions for R_o^2, Eq 55, and $y' R_o^2$, Eq 58, may now be substituted into Eq 45 By equating the coefficients of the harmonics, one obtains

$$c'^2 + 2\left(\frac{\pi}{2K}\right)^2 \sum_{k=1}^{n} \frac{c_k}{k} D_k = \hat{K}_o \left(\frac{\pi}{2K}\right)^2 D_o \quad \text{(0th harmonic)} \tag{59}$$

$$\sum_{k=1}^{n} \frac{c_k}{k} (D_{k-j} + D_{k+j}) = \hat{K}_o D_j \quad (j\text{th harmonic}, j=1,2,3 \quad n) \tag{60}$$

where absolute value signs are omitted on the subscripts of D_{k-j}, and furthermore D_ℓ vanishes if $|\ell| > n$

Since the unknown terms in Eqs 59 and 60 are all functions of the height of the wave, it is appropriate to add an equation for wave height The wave height is seen to be equal to the sum of the displacements of the crest and trough from the x-axis Therefore, using Eqs 23 and 9,

$$H = (y_o)_{\chi=0} + (-y_o)_{\chi=\pi} = \frac{L}{2K} \sum_{j=1}^{n} \frac{a_j}{j} = \frac{2D}{K'} \sum_{j=1}^{n} \frac{a_j}{j} \tag{61}$$

$$j = 1,3,5,7$$

and in dimensionless form,

$$H' = 2 \sum_{j=1}^{n} \frac{a_j}{j} \qquad j = 1,3,5,7 \tag{62}$$

Eqs 59, 60 and 62 are therefore seen to constitute a set of (n+2) equations in (n+2) unknowns ($c', \hat{K}_o, a_1, a_2, a_3 \quad a_n$) for any desired value of the dimensionless wave height, H'

COMPUTER SOLUTION

In setting up the equations for computer solution, the coefficient, $\hat{K}_o$, is eliminated between the first of Eqs 60 (j-1) and each succeeding equation (j=2,3 n), thereby reducing Eqs 60 to (n-1) equations in (n-1) unknowns ($a_1, a_2, \quad a_{n-1}$) for a fixed value of a_n After the unknown coefficients are assumed, the simultaneous solution of these (n-1) non-linear equations is accomplished with the aid of a Newton-Raphson iteration By this technique the problem is reduced to one of obtaining the solution of a set of (n-1) linear equations at each iteration The matrix is then inverted using triangular decomposition and a solution of the set of equations is obtained for corrections on the assumed values of a_1, a_2, a_{n-2} and a_{n-1} The entire procedure is repeated until the corrections are small enough to be neglected After the coefficients have been computed, Eqs 61, 60 (first harmonic) and 59 are solved to yield H', $\hat{K}_o$ and c' respectively The entire procedure is repeated iteratively until the coefficients $\hat{K}_o$ and c' are evaluated for uniform-interval values of H/L, which are appropriate for tabular presentation

Computations were made on the University of Wisconsin UNIVAC 1108 computer A fifth order solution was undertaken and some preliminary results are shown in Tables 1 and 2 In these tables the dimensionless wave height, H/L, is calculated correct to ± 000001, while all other terms appearing in the tables were computed correct to at least the last place shown It should be noted that this precision is significant for the lower values of wave height where the convergence of the a_j series is rapid and a fifth order solution is sufficient As the wave height increases, however, the truncation of the trigonometric series which replace the elliptic functions as well as the omission of higher harmonics in the development of $(y_o' R_o^2)$, result in a less accurate satisfaction of Eq 53 As a consequence the dynamic boundary condition, Eq 5, is only satisfied approximately By developing higher order solutions, greater than the fifth order solution considered here, the accuracy can be improved

DISCUSSION

No specific attempt will be made at this point to compare the new theory with those which exist The primary objective at the present is to develop the method of solution and to prepare sample tables of the coefficients

Nevertheless it is already possible to indicate some agreement with the existing theories In an earlier paper, Monkmeyer and Kutzbach (1965) compare the theory with that of Stokes (1880) to reveal that the basic equation for deep water, Eq 18, is common to both theories Stokes was, of course, limited in his ability to carry out computations to higher orders, and so restricted his attention to the well-known fifth-order theory The success of this theory in deep water has been the prime stimulus for using a fifth order approach in the present work Following Stokes, Wilton (1914) developed a twelfth order solution for deep water waves and Monkmeyer and Kutzbach (1965) developed a fifteenth order solution These higher order computations resulted in little deviation from the fifth and third order theories, especially in the prediction of wave speed Only in wave shape did the fifteenth order theory diverge from the lower order theories, as might be expected

Wave speed may be computed with the aid of Eq 59 However, since this equation demands considerable computation, a more convenient approach is to print out the wave speed together with the wave coefficients as shown in Tables 1 and 2 The wave speeds obtained in Table 1 show excellent agreement with those of Stokes' third and fifth order theories for deep water waves (see Monkmeyer and Kutzbach, 1965)

In order to describe the wave shape or profile of a wave, one may choose to use the parametric set of equations,

$$x_o = -\frac{L\chi}{2\pi} - \frac{L}{4K}\sum_{j=1}^{n}\frac{a_j}{j}\,\mathrm{sn}\left(\frac{2jK}{\pi}\chi\right) \qquad (22)$$

Table 1

FIFTH ORDER WAVE COEFFICIENTS

L/D = .00000

H/L	H/D	C'	$\hat{K}$.	A(1) B(1) C(1)	A(2) B(2) C(2)	A(3) B(3) C(3)	A(4) B(4) C(4)	A(5) B(5) C(5)
.140	.000	1.1063	1.3825	.33811	.25489	.21308	.18215	.15343
				.33811	.25489	.21308	.18215	.15343
				.33811	.25489	.21308	.18215	.15343
.130	.000	1.0900	1.3304	.32414	.23004	.18174	.14753	.11848
				.32414	.23004	.18174	.14753	.11848
				.32414	.23004	.18174	.14753	.11848
.120	.000	1.0756	1.2816	.30833	.20519	.15253	.11702	.08910
				.30833	.20519	.15253	.11702	.08910
				.30833	.20519	.15253	.11702	.08910
.110	.000	1.0628	1.2366	.29086	.18038	.12534	.09024	.06467
				.29086	.18038	.12534	.09024	.06467
				.29086	.18038	.12534	.09024	.06467
.100	.000	1.0513	1.1954	.27177	.15575	.10026	.06712	.04487
				.27177	.15575	.10026	.06712	.04487
				.27177	.15575	.10026	.06712	.04487
.090	.000	1.0412	1.1582	.25101	.13154	.07756	.04772	.02943
				.25101	.13154	.07756	.04772	.02943
				.25101	.13154	.07756	.04772	.02943
.080	.000	1.0323	1.1250	.22855	.10806	.05755	.03207	.01799
				.22855	.10806	.05755	.03207	.01799
				.22855	.10806	.05755	.03207	.01799
.070	.000	1.0246	1.0958	.20440	.08573	.04051	.02009	.01006
				.20440	.08573	.04051	.02009	.01006
				.20440	.08573	.04051	.02009	.01006
.060	.000	1.0180	1.0705	.17861	.06500	.02665	.01149	.00502
				.17861	.06500	.02665	.01149	.00502
				.17861	.06500	.02665	.01149	.00502
.050	.000	1.0124	1.0491	.15131	.04638	.01601	.00582	.00215
				.15131	.04638	.01601	.00582	.00215
				.15131	.04638	.01601	.00582	.00215
.040	.000	1.0079	1.0315	.12270	.03035	.00846	.00248	.00074
				.12270	.03035	.00846	.00248	.00074
				.12270	.03035	.00846	.00248	.00074
.030	.000	1.0045	1.0177	.09299	.01737	.00365	.00081	.00018
				.09299	.01737	.00365	.00081	.00018
				.09299	.01737	.00365	.00081	.00018
.020	.000	1.0020	1.0079	.06246	.00782	.00110	.00016	.00002
				.06246	.00782	.00110	.00016	.00002
				.06246	.00782	.00110	.00016	.00002
.010	.000	1.0005	1.0020	.03137	.00197	.00014	.00001	.00000
				.03137	.00197	.00014	.00001	.00000
				.03137	.00197	.00014	.00001	.00000

VALUES OF THE COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

K= 1.5707963 K'=INF. M= .0000000 K'/K=INF.

Table 2

FIFTH ORDER WAVE COEFFICIENTS

L/D = 2.00000

H/L	H/D	C'	$\hat{K}_o$	A(1) B(1) C(1)	A(2) B(2) C(2)	A(3) B(3) C(3)	A(4) B(4) C(4)	A(5) B(5) C(5)
.140	.280	1.1090	1.3908	.34108	.25736	.21331	.18393	.15467
				.34172	.25784	.21562	.18427	.15496
				.34044	.25688	.21482	.18358	.15438
.130	.260	1.0926	1.3381	.32696	.23229	.18179	.14906	.11954
				.32757	.23273	.18396	.14934	.11977
				.32635	.23186	.18328	.14878	.11932
.120	.240	1.0781	1.2888	.31101	.20722	.15243	.11830	.08997
				.31159	.20761	.15445	.11852	.09014
				.31043	.20683	.15388	.11808	.08981
.110	.220	1.0652	1.2433	.29340	.18219	.12508	.09128	.06536
				.29394	.18253	.12696	.09145	.06549
				.29285	.18185	.12649	.09111	.06525
.100	.200	1.0537	1.2016	.27415	.15734	.09986	.06793	.04540
				.27466	.15763	.10159	.06806	.04548
				.27363	.15704	.10121	.06781	.04532
.090	.180	1.0434	1.1639	.25322	.13290	.07705	.04833	.02979
				.25369	.13314	.07861	.04842	.02985
				.25275	.13265	.07832	.04824	.02974
.080	.160	1.0344	1.1304	.23058	.10919	.05695	.03250	.01823
				.23101	.10940	.05834	.03256	.01826
				.23015	.10899	.05813	.03244	.01820
.070	.140	1.0266	1.1008	.20623	.08664	.03986	.02036	.01020
				.20662	.08680	.04109	.02040	.01023
				.20585	.08648	.04094	.02033	.01019
.060	.120	1.0200	1.0752	.18023	.06570	.02598	.01165	.00510
				.18056	.06583	.02704	.01167	.00511
				.17989	.06558	.02694	.01163	.00509
.050	.100	1.0144	1.0534	.15269	.04688	.01536	.00590	.00218
				.15298	.04697	.01625	.00591	.00219
				.15241	.04680	.01619	.00589	.00218
.040	.080	1.0099	1.0356	.12383	.03069	.00787	.00252	.00075
				.12406	.03075	.00858	.00252	.00076
				.12360	.03063	.00855	.00251	.00075
.030	.060	1.0064	1.0217	.09386	.01757	.00318	.00082	.00019
				.09403	.01760	.00371	.00082	.00019
				.09368	.01753	.00370	.00082	.00019
.020	.040	1.0039	1.0117	.06304	.00791	.00076	.00017	.00002
				.06316	.00792	.00112	.00017	.00003
				.06292	.00789	.00112	.00017	.00003

VALUES OF THE COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

K= 1.5825517 K'= 3.1651034 M= .0294372 K'/K= 2.0000000

$$y_o = \frac{L}{4K} \sum_{j=1}^{n} \frac{a_j}{j} \operatorname{cn}(\frac{2jK}{\pi}\chi) \qquad (23)$$

or to avoid the elliptic functions, but at the expense of some accuracy,

$$x_o = -\frac{L\chi}{2\pi} - \frac{L}{4K} \sum_{j=1}^{n} \frac{b_j}{j} \sin j\chi \qquad (24)$$

$$y_o = \frac{L}{4K} \sum_{j=1}^{n} \frac{c_j}{j} \cos j\chi \qquad (25)$$

Numerical values relating x_o and y_o may be obtained by substituting arbitrary values of the normalized potential function, χ This procedure is adequate for a graphical presentation of the wave shape Stokes (1880) suggests that, with the aid of Lagrange's theorem (Whittaker and Watson, 1963), the two equations may be reduced to one

It should be observed that the depth, D, differs from the depth to stillwater, d, by the elevation of the stillwater level, y_s (see Fig 1) The computation of y_s, and therefore, d, may be accomplished by noting that the net area bounded by the free surface and the stillwater level vanishes Therefore

$$0 = \int_{o}^{L/2} (y_o - y_s) dx_o \qquad (63)$$

By substituting the parametric profile expressions, Eqs 24 and 25, the equation may be solved for the dimensionless stillwater elevation

$$y_s' = \frac{\pi^2}{4K} \sum_{k=1}^{n} \frac{b_k c_k}{k} \qquad (64)$$

Sample profiles for deep water waves are presented by Monkmeyer and Kutzbach

In order to study the characteristics of the highest possible wave it will be necessary to add one further restriction to those imposed by Eqs 59, 60 and 61 This restriction, which was first suggested by Stokes, affects the Bernoulli equation which describes the dynamic upper boundary condition In effect Stokes suggests that for a fluid particle on the surface to reach the highest possible point above the surface, the wave crest, it must give up all of its kinetic energy Hence at this point it has no velocity and the crest is a stagnation point The dynamic boundary condition, Eq 45, therefore reduces to

$$2y_o' = \hat{K}_o \qquad \text{at } r = 1, \chi = 0 \qquad (65)$$

or substituting for y_o' with the aid of Eq 23

$$2 \sum_{k=1}^{n} \frac{a_k}{k} = \hat{K}_o \qquad (66)$$

Since this equation adds no new unknowns to those already appearing in Eqs 59, 60 and 62 its inclusion results in a set of (n + 3) equations with (n + 3) unknowns, H' no longer being arbitrary but now considered as an unknown

For the deep water case using the fifteenth order theory Monkmeyer and Kutzbach show that

$$(H/L)_{max} = 0\ 1442$$

This compares well with Michell's (1893) result of

$$(H/L)_{max} = 0\ 142$$

and Havelock's (1919) conclusion that

$$(H/L)_{max} = 0\ 1418$$

No precise computations of the highest wave have been made for the finite waves considered herein However, Eq 57 has been used to determine whether or not the wave data obtained in the preliminary computations includes waves that exceed the highest This check showed for example that for L/D = 6 0, the maximum wave lies between H/L = 13 and 14, which is in good agreement with the breaking index curve of Reid and Bretschneider (1953)

CONCLUSION

A higher order wave theory has been developed for the full range of waves from Stokes waves to cnoidal waves to the solitary wave and from small amplitude waves to finite amplitude waves and the "highest wave' By means of a conformal transformation the problem is reduced to obtaining a solution to a non-linear set of equations Solutions of these equations using a high speed digital machine have been obtained to fifth order for L/D values of 0 0, 2 0, 4 0 and 6 0, and samples of this data are presented in tabular form

A consideration of some preliminary results as well as earlier results of the deep water case, suggest that the theory is in good agreement with existing theories Furthermore, it appears that this theory may provide a comprehensive practical means for wave analysis of the full-range of symmetrical waves from deep-water to shallow-water

ACKNOWLEDGEMENTS

The derivation of the basic theory was completed while the writer was a guest researcher at the River and Harbor Laboratory of the Technical University of Norway, Trondheim, Norway In particular the writer wishes to thank Director H Berge and Dr T Carstens for providing him with the opportunity to complete the theoretical analysis

Also, this research was supported in part by the National Science Foundation and the State of Wisconsin under the University of Wisconsin Sea Grant Program

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APPENDIX - NOTATION

A_j	see Eq 50
a_j	coefficient of the jth harmonic - see Eq 7
am( )	amplitude of the elliptic integral of the first kind
B_j	see Eq 51
b_j	coefficient - see Eq 26
C_j	see Eq 52
c	wave speed
c'	the dimensionless wave speed - see Eq 44
c_j	coefficient - see Eq 27
cn( )	Jacobian elliptic function
D	depth measured from the origin
D_j	see Eqs 48 and 49
d	depth measured from stillwater
$f(\zeta)$	see Eq 35
g	acceleration due to gravity
gd( )	gudermannian
H	wave height
H'	dimensionless wave height - see Eq 62
i	$\sqrt{-1}$
j	integer which identifies the jth harmonic - see Eq 8
K	complete elliptic integral of the first kind (parameter - m)
K'	complete elliptic integral of the first kind (K'(m) = K(1-m))
K_o	Bernoulli constant for free surface streamline
$\hat{K}_o$	see Eq 44
k	integer which identifies the kth harmonic
L	wave length
ℓ	integer which identifies the ℓth harmonic
m	parameter of the complete elliptic integral of the first kind, K

n	integer which identifies the highest harmonic and the order of the analysis
q_o	magnitude of the particle velocity at the free surface
R_o	modulus of $f(\zeta_o)$ - see Eq 38, also = c/q_o
r	radial coordinate in the ζ-plane
r_c	radius of AE in the ζ-plane
sn( )	Jacobian elliptic function
u	x-component of the particle velocity
u_o	x-component of the particle velocity at the free surface
v	y-component of the particle velocity
v_o	y-component of the particle velocity at the free surface
w	complex potential $\equiv \phi + i\psi$
x	horizontal coordinate in the z-plane
x'	dimensionless horizontal coordinate
x_o	horizontal free surface coordinate in the z-plane
x'_o	dimensionless horizontal free surface coordinate
y	vertical coordinate in the z-plane
y'	dimensionless vertical coordinate
y_o	vertical free surface coordinate in the z-plane
y'_o	dimensionless vertical free surface coordinate
y_s	still water elevation in the z-plane
y'_s	dimensionless still water elevation
z	$\equiv x + iy$ and refers to the physical plane
z_o	complex variable at the free surface
ζ	$\equiv r \exp(i\chi)$ and refers to the auxiliary plane
ζ_o	= $\exp(i\chi)$, or ζ at the free surface
θ_o	argument of $f(\zeta_o)$ - see Eq 39, also local slope angle of the free surface in the z-plane
π	$\equiv$ 3 1415927
ϕ	potential function
χ	tangential coordinate in the ζ-plane, also a normalized form of the potential function
ψ	stream function