A Higher-Order Water-Wave Equation and the Method for Solving It

D. J. KAUP[*]

*Department of Physics, Clarkson College of Technology
Potsdam, New York 13676*

By a new technique, we have found another nonlinear evolution equation which can be solved exactly by inverse scattering techniques. This equation has a cubic nonlinearity added to the Boussinesq equation and can also be derived from the water-wave equations. This eigenvalue problem differs from any studied before, but in some aspects it is similar to the sine-Gordon eigenvalue problem in laboratory coordinates. Also, the solution to the inverse scattering problem is given.

§ 1. Introduction

In a recent paper, Zakharov[1] has demonstrated that the Boussinesq equation[2],[3] might be solvable by an "inverse scattering transform".[4] He has found an eigenvalue problem and a time evolution operator which has the Boussinesq equation as the integrability condition. However, since this eigenvalue problem is third order, the inverse scattering problem is much more complicated and is still to be solved. The Boussinesq equation is derivable from the water-wave equations. It is indeed interesting (and perhaps significant) that if we include one more order of nonlinearity in the derivation of the Boussinesq equation from the water-wave equations,[3] we obtain another equation which *can be solved exactly* by present techniques with an inverse scattering transform. Furthermore, this eigenvalue problem is *second-order*, and the inverse scattering problem is readily solvable.

Originally, this new equation was found by "scanning" possible eigenvalue problems by a linearizing technique developed by the author[5] for an eigenvalue problem which would solve the Boussinesq equation. Then it was later noted that this new equation could also be derived from the water-wave equations. To show this, we consider the case of water waves propagating in an infinite narrow channel of constant mean depth, h.[6] The free surface conditions are

$$\eta_T + U\eta_X - V = 0, \quad \text{(at } Y = h + \eta) \tag{1.1a}$$

$$U_T + U_X U + V_X V + \eta_X g - \rho^{-1}\tau\eta_{XXX} = 0, \quad \text{(at } Y = h + \eta) \tag{1.1b}$$

where subscripts indicate partial differentiation, Y is the vertical coordinate, X is the coordinate parallel to the channel, η is the amplitude of the wave, (U, V)

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are the velocity components, τ is the surface tension and ρ is the mean density of the inviscid, incompressible, irrotational fluid in the channel. One scales (1·1) by letting δ be the ratio of depth/wavelength, and ε be the ratio of the wave amplitude/depth. Upon introducing the velocity potential by $\boldsymbol{\nabla}\phi = (U, V)$, taking units where $h = g = 1$, requiring $\boldsymbol{\nabla}^2\Phi = 0$ and expanding Eq. (1·1) in a power series of ε and δ, one finds that the velocity potential (evaluated at $Y = 0$) will satisfy the equation

$$\pi_t = \Phi_{xx} + \delta^2 (\tfrac{1}{3} - \sigma)\, \Phi_{xxxx} - \varepsilon\, (\Phi_x \pi)_x + O\,(\delta^4,\, \varepsilon\delta^2), \qquad (1\cdot2a)$$

$$\pi \equiv \Phi_t + \tfrac{1}{2}\varepsilon\Phi_x^2, \qquad (1\cdot2b)$$

$$\sigma \equiv \tau/(\rho g h^2) \qquad (1\cdot3)$$

and (x, t) are the scaled, unitless (X, T) coordinates. Of course, due to the ε term in (1·2b), (1·2a) contains a cubic term of order ε^2, which means we have "too much" nonlinearity.[7] This shows up later in that some solutions of this equation can be singular.

Still, there are properties of this equation which merit consideration: i) This is the first example of a nonlinear differential evolution equation which allows waves to travel in both directions, has a quadratic linear dispersion relation ($\omega = \pm k\sqrt{1-k^2}$) and can be solved exactly by an inverse scattering transform; ii) The inverse scattering problem is different from any considered before, and its solution indicates techniques which may be useful in other problems; iii) For $\delta^2 = \varepsilon \to 0$, Eq. (1·2) can be transformed into the Boussinesq equation, and the direct and inverse scattering problems can be decomposed into the right- and left-going KdV[8] scattering problems.

In §2, we will define the scattering data for this new eigenvalue problem, and in many respects, the analytical properties of the eigenfunctions follow from similar results for the Schrödinger equation, with appropriate modifications. However, for the inverse scattering problem, done in §3, it is necessary to significantly alter the form of the "transformation kernel". Once this is done, it is then relatively straightforward to obtain the Marchenko equations. The simple soliton solutions will be given in §4 and the transformation from this equation to the Boussinesq equation will be given in §5, along with the decomposition of this new scattering problem into the right- and left-going KdV scattering problems.

§2. The direct scattering problem

As shown in,[5] if we take our eigenvalue problem to be

$$\Psi_{xx} + (k^2 + \tfrac{1}{4}\beta^{-2} + ikq + r)\Psi = 0, \qquad (2\cdot1)$$

where k is the eigenvalue, q and r are the potentials, β is a constant, and if we take the time evolution of Ψ to be of the form

$$\Psi_t = A\Psi + B\Psi_x, \qquad (2\cdot2)$$

then the integrability condition for (2·1), (2·2) gives (1·2) for a suitable choice of q and r. We shall work with a slightly more general equation than (1·2). We take

$$q = \tfrac{1}{2}\beta^{-1}\varepsilon\mathcal{O}_x, \qquad (2\cdot3a)$$

$$r = -\tfrac{1}{4}\varepsilon\beta^{-2}(\pi + \tfrac{1}{4}\varepsilon\mathcal{O}_x{}^2), \qquad (2\cdot3b)$$

$$A = \alpha + \tfrac{1}{4}\varepsilon\mathcal{O}_{xx}, \qquad (2\cdot4a)$$

$$B = -2i\beta k - \Gamma - \tfrac{1}{2}\varepsilon\mathcal{O}_x, \qquad (2\cdot4b)$$

where α, β, ε and Γ are constants. Then the integrability condition for (2·1), (2·2) is

$$\pi_t + \Gamma\pi_x = \mathcal{O}_{xx} + \beta^2\mathcal{O}_{xxxx} - \varepsilon(\mathcal{O}_x\pi)_x, \qquad (2\cdot5a)$$

where

$$\pi = \mathcal{O}_t + \Gamma\mathcal{O}_x + \tfrac{1}{2}\varepsilon\mathcal{O}_x{}^2. \qquad (2\cdot5b)$$

As can be seen from (2·5), Γ could be transformed to zero by a simple Galilean transformation. In the water-wave problem, Γ is simply the velocity of the observer relative to the water surface. To obtain Eq. (1·2), we set

$$\Gamma = 0, \qquad (2\cdot6a)$$

$$\beta^2 = \delta^2(\tfrac{1}{3} - \sigma). \qquad (2\cdot6b)$$

Thus, β^2 may be positive, zero or negative, depending on the value of σ, Eq. (1·3).

We will now consider the direct scattering problem for (2·1). First, we eliminate the branch points in the k-plane at $k = \pm(1/2)i\beta$ by going to a ζ-plane where

$$k = \tfrac{1}{4}[\zeta - 1/(\beta^2\zeta)]. \qquad (2\cdot7a)$$

Define

$$E = \tfrac{1}{4}[\zeta + 1/(\beta^2\zeta)], \qquad (2\cdot7b)$$

so that

$$E^2 = k^2 + \tfrac{1}{4}\beta^{-2}. \qquad (2\cdot8)$$

Now, as functions of ζ, the eigenstates will not have branch points in the ζ-plane, although they will have an essential singularity at $\zeta = 0$, as well as at $\zeta = \infty$. Note that for $\beta^2 > 0$, $E(\zeta)$ is real when either ζ is real or $|\zeta| = 1/|\beta|$, while for $\beta^2 < 0$, $E(\zeta)$ is real only when ζ is real.

For $E(\zeta)$ real, we define the right and left eigenstates of (2·1) by

$$\phi(\zeta, x) \to e^{-iEx} \Big\} \quad \text{as} \quad x \to -\infty \qquad (2\cdot9a)$$

$$\bar{\phi}(\zeta, x) \to e^{iEx} \Big\} \qquad\qquad\qquad\qquad\quad (2\cdot9b)$$

and

$$\left.\begin{array}{l} \phi(\zeta, x) \to e^{iEx} \\ \overline{\psi}(\zeta, x) \to e^{-iEx} \end{array}\right\} \quad \text{as} \quad x \to +\infty$$

(2·9c)

(2·9d)

with $E(\zeta)$ given by (2·7b) and providing that r and q both satisfy the Faddeev conditions

$$\int_{-\infty}^{\infty} |r|(1+|x|) dx < \infty,$$

(2·10a)

$$\int_{-\infty}^{\infty} |q|(1+|x|) dx < \infty.$$

(2·10b)

From the Wronskian, one finds that ψ and $\overline{\psi}$ are linearly independent solutions, thus ϕ and $\overline{\phi}$ must be linearly dependent on ψ and $\overline{\psi}$, or

$$\phi(\zeta, x) = a(\zeta)\overline{\psi}(\zeta, x) + b(\zeta)\psi(\zeta, x),$$

(2·11a)

$$\overline{\phi}(\zeta, x) = \overline{a}(\zeta)\psi(\zeta, x) + \overline{b}(\zeta)\overline{\psi}(\zeta, x).$$

(2·11b)

From the Wronskian of ϕ and $\overline{\phi}$, we find that

$$a(\zeta)\overline{a}(\zeta) - b(\zeta)\overline{b}(\zeta) = 1,$$

(2·12)

and thus the inverse of (2·11) is

$$\psi = a\overline{\phi} - \overline{b}\phi,$$

(2·13a)

$$\overline{\psi} = \overline{a}\phi - b\overline{\phi}.$$

(2·13b)

Equation (2·1) is invariant under a symmetry transformation on ζ. If $\Psi(\zeta, x)$ is a solution of (2·1), then $\Psi(\overline{\zeta}, x)$ is also a solution where

$$\overline{\zeta} = -(\beta^2 \zeta)^{-1}.$$

(2·14)

Thus, we have

$$\overline{\phi}(\zeta, x) = \phi(\overline{\zeta}, x),$$

(2·15a)

$$\overline{\psi}(\zeta, x) = \psi(\overline{\zeta}, x),$$

(2·15b)

$$\overline{a}(\zeta) = a(\overline{\zeta}),$$

(2·15c)

$$\overline{b}(\zeta) = b(\overline{\zeta}).$$

(2·15d)

By well-known techniques,[9] one can prove the following theorem, which allows us to extend ϕ, ψ and a into the upper half E-plane. (By ζ being in the upper half E-plane, we are referring to those values of ζ which satisfy Img $E(\zeta) > 0$ where $E(\zeta)$ is given by (2·7b). See Figs. 1 and 2.)

If r and q both satisfy the Faddeev conditions, (2·10), then $\phi(\zeta, x) e^{iE(\zeta)x}$, $\psi(\zeta, x) e^{-iE(\zeta)x}$ and $a(\zeta)$ are analytic functions of ζ for ζ in the upper half E-plane $[Img\, E(\zeta) > 0]$, while for ζ on the real E-axis $[Img\, E(\zeta) = 0]$, they, as well as $b(\zeta)$, are bounded except at the point $E = 0$. At $E = 0$, $a(\zeta)$ and $b(\zeta)$
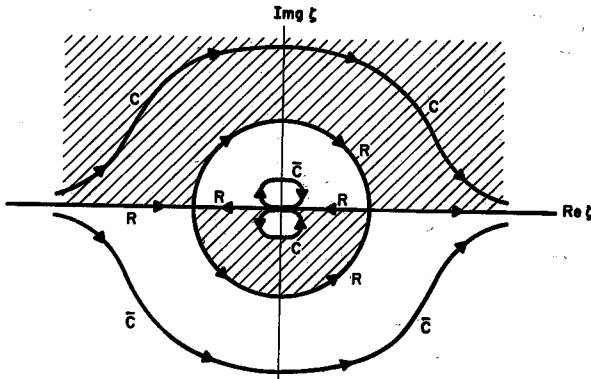
Fig. 1. The contours C, $\overline{C}$ and R in the complex ζ-plane when $\beta^2 > 0$. Note the two segments of C and $\overline{C}$. The radius of the circle for the R contour is $|1/\beta|$. The shaded region corresponds to the upper half E-plane (double-sheeted).

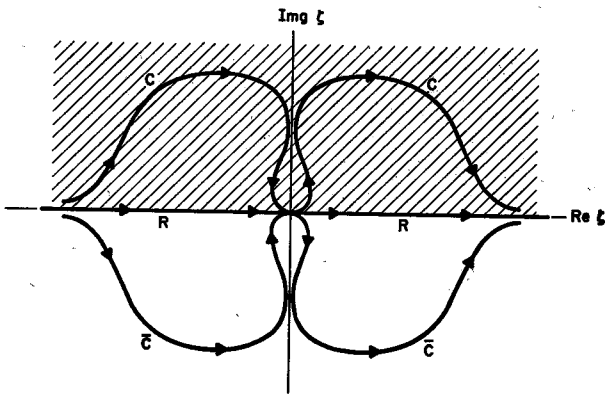Fig. 2. The contours C, $\overline{C}$ and R in the complex ζ-plane when $\beta^2 < 0$. The shaded region corresponds to the upper half E-plane (double-sheeted). When a zero of $a(\zeta)$ or $\overline{a}(\zeta)$ lies on the imaginary axis, it is immaterial which segment is distorted to enclose the zero.

will have, at worst, simple poles.

When q and r are on compact support, by identical techniques we also have the following theorem.

When q and r are on compact support, all the above functions are entire functions of ζ in the E-plane, except at $E = 0$. Furthermore, at $E = 0$, $a(\zeta)$ and $b(\zeta)$ will each have, at worst, a simple pole.

Equation (2·1) can also have bound states. These occur whenever $a(\zeta) = 0$ for ζ in the upper half E-plane. We designate these zeros of $a(\zeta)$ by $[\zeta_n]_{n=1}^N$ and assume N to be finite. At $\zeta = \zeta_n$, we have

$$\phi(\zeta_n, x) = b_n \psi(\zeta_n, x), \qquad (2\cdot16)$$

which completes the definition of the scattering data. Note that when r and q are on compact support,

$$b_n = b(\zeta_n). \qquad (2\cdot17)$$

When r and q are given by (2·3), then βq and r are real, and therefore, if $\psi(\zeta, x)$ is a solution of (2·1), $\psi^*(-\beta \zeta^*/\beta^*, x)$ is also a solution. For $\beta^2 > 0$, this gives

$$a^*(-\zeta^*) = a(\zeta), \tag{2.18a}$$

$$b^*(-\zeta^*) = b(\zeta), \tag{2.18b}$$

and we see that the bound state eigenvalues must either be pure imaginary or occur in complex conjugate pairs $(\zeta_2 = -\zeta_1^*)$, as for the "breather" state of the Sine-Gordon equation.[10] If $\beta^2 < 0$, then instead

$$a^*(\zeta^*) = \bar{a}(\zeta), \tag{2.19a}$$

$$b^*(\zeta^*) = \bar{b}(\zeta), \tag{2.19b}$$

and by (2.15c), we have that the bound state eigenvalues must either lie on the upper half of the *circle* of radius $1/|\beta|$ or occur in complex conjugate "inverse" pairs $(\zeta_2 = -\beta^{-2}/\zeta_1^*)$.

The time dependence of the scattering data follows directly from (2.2~4). For $E(\zeta)$ real,

$$a_t(\zeta) = 0, \tag{2.20a}$$

$$b_t(\zeta) = 2E(\zeta)[2\beta k(\zeta) - i\Gamma]b(\zeta), \tag{2.20b}$$

while for the bound state parameters

$$(\zeta_n)_t = 0, \tag{2.21a}$$

$$(b_n)_t = 2E(\zeta_n)[2\beta k(\zeta_n) - i\Gamma]b_n \tag{2.21b}$$

for $n = 1, 2, \cdots, N$.

For solving the inverse scattering problem, we need the asymptotic forms of ϕe^{iEx}, ψe^{-iEx} and $a(\zeta)$ as $|E| \to \infty$ in the upper half E-plane. These are

$$\phi(\zeta, x)e^{iEx} \to \exp\left[\frac{1}{2}\frac{k}{E}\int_{-\infty}^{x} q(y)dy\right] \cdot \left[1 + 0(E^{-1})\right], \tag{2.22a}$$

$$\psi(\zeta, x)e^{-iEx} \to \exp\left[\frac{1}{2}\frac{k}{E}\int_{x}^{\infty} q(y)dy\right] \cdot \left[1 + 0(E^{-1})\right], \tag{2.22b}$$

where $k(\zeta)$ and $E(\zeta)$ are given by (2.7) and similar expressions hold for $\bar{\phi}$ and $\bar{\psi}$ in the lower half E-plane. From (2.22) and the Wronskian relation

$$2iEa(\zeta) = W[\phi(\zeta, x), \psi(\zeta, x)], \tag{2.23}$$

we have

$$a(\zeta) \to e^{(k/2E)\Delta Q}[1 + 0(E^{-1})] \tag{2.24}$$

for $|E| \to \infty$ in the upper half E-plane, and where

$$\Delta Q = \int_{-\infty}^{\infty} q(y)dy. \tag{2.25}$$

§ 3. The inverse scattering problem

To solve the inverse scattering problem for (2.1), we will assume r and

q to be on compact support so that we can make full use of our second theorem, and use the power and simplicity of contour integrals. As one could note, our derivation will still be valid for the case of noncompact support, provided one decomposes all contour integrals into integrals along the real E-axis and any contributions from all poles.

We define three contours in the ζ-plane C, $\overline{C}$ and R, which will depend on the sign of β^2. In general, they are always defined so that in the double-sheeted E-plane, C lies in the upper half E-plane and passes above all zeros of $a(\zeta)$, $\overline{C}$ lies in the lower half E-plane and passes under all zeros of $\bar{a}(\zeta)$, and R will always be along the real E-axis. These contours are illustrated in Fig. 1 for the case $\beta^2>0$, and in Fig. 2 for the case $\beta^2<0$. In Fig. 1, the radius of the circle is $|\beta|$, and note that C and $\overline{C}$ consist of two segments. The shaded part in each figure indicates the regions in the ζ-plane which map into the upper half of the double-sheeted E-plane.

We shall proceed by first constructing the integral representations for ψ and $\bar{\psi}$, and then shall show that there exists "transformation kernels" for ψ and $\bar{\psi}$. From these, we can then obtain the inverse scattering equations of the Marchenko type.[11]

Consider the contour integral

$$I(\zeta, x) = \int_C \frac{\phi(\zeta, x) e^{iE(\zeta')x}}{(\zeta'-\zeta) a(\zeta')} d\zeta' \qquad (3\cdot1)$$

for r and q on compact support and for ζ "under" C. [By "under", we mean that if $\beta^2>0$, ζ must lie under the upper part of C (see Fig. 1) as well as to be exterior to the part of C passing through the origin. And if $\beta^2<0$, ζ must simply be under C (see Fig. 2).] By $(2\cdot22a)$, $(2\cdot24)$, its value is

$$I(\zeta, x) = -i\pi e^{-1/2Q(x)} , \qquad (3\cdot2)$$

where

$$Q(x) \equiv \int_x^\infty q(x) dy . \qquad (3\cdot3)$$

From $(2\cdot11a)$, we also have

$$I(\zeta, x) = \int_C \frac{\bar{\psi}(\zeta', x) e^{iE(\zeta')x}}{\zeta'-\zeta} d\zeta' + \int_C \rho(\zeta') \frac{\psi(\zeta', x) e^{iE(\zeta')x}}{\zeta'-\zeta} d\zeta' , \qquad (3\cdot4)$$

where

$$\rho(\zeta') \equiv b(\zeta')/a(\zeta') \qquad (3\cdot5)$$

and is the "reflection coefficient". Now, by continuously distorting the contour C in the first integral, we can reach the contour R, and from R, the contour $\overline{C}$, and eventually encircling the pole at $\zeta'=\zeta$ as well. Then by $(2\cdot14)$, $(2\cdot15)$, $(2\cdot22b)$, we finally obtain

$$\bar{\psi}(\zeta, x)\, e^{iE(\zeta)x} = e^{-1/2Q(x)} + \frac{1}{2\pi i} \int_c \rho(\zeta') \frac{d\zeta'}{\zeta' - \zeta} \psi(\zeta', x)\, e^{iE(\zeta')x}, \qquad (3\cdot6)$$

for ζ "under" C. Similar considerations will give an integral representation of ψe^{-iEx}, but due to $(2\cdot15)$, it will simply be the transformed form, $(2\cdot14)$, of $(3\cdot6)$.

Now, let us assume that the two "transformation kernels", $K(x, s)$ and $L(x, s)$, exist such that

$$\psi(\zeta, x) = e^{G(x)} \left\{ e^{iEx} + \int_x^\infty [K(x, s) + \frac{i}{\beta^2 \zeta} L(x, s)] e^{iEs}\, ds \right\}, \qquad (3\cdot7)$$

where $G(x)$ is to be determined. Requiring $(3\cdot7)$ to satisfy $(2\cdot1)$ gives as necessary and sufficient conditions

$$\lim_{s \to \infty} K(x, s) = 0, \qquad (3\cdot8a)$$

$$\lim_{s \to \infty} L(x, s) = 0, \qquad (3\cdot8b)$$

$$G(x) = \tfrac{1}{2} Q(x), \qquad (3\cdot9a)$$

$$L(x, x) = \tfrac{1}{4}(1 - e^{-Q(x)}), \qquad (3\cdot9b)$$

$$K(x, x) = -\tfrac{1}{2} R(x), \qquad (3\cdot9c)$$

$$[\partial_x^2 - \partial_s^2 - q(x)(\partial_x + \partial_s) - R_x(x)] K(x, s) - \tfrac{1}{2}\beta^{-2} q(x) L(x, s) = 0, \quad (3\cdot10a)$$

$$[\partial_x^2 - \partial_s^2 - q(x)(\partial_x - \partial_s) - R_x(x)] L(x, s) - \tfrac{1}{2} q(x) K(x, s) = 0, \qquad (3\cdot10b)$$

where $Q(x)$ is defined by $(3\cdot3)$, and

$$R(x) = \int_x^\infty [r(y) - \tfrac{1}{2} q_y(y) + \tfrac{1}{4} q^2(y)]\, dy. \qquad (3\cdot11)$$

To show that K and L exist and are unique, we use $(3\cdot9)$ as the initial data along the characteristic $x - s = $ zero. Then $(3\cdot8)$, $(3\cdot10)$ will allow us to find K and L for all $s > x$ and the solution will be unique. We note that while it is clear that $(3\cdot8 \sim 10)$ are clearly sufficient conditions, to prove necessity, it is necessary to take a transform with respect to $e^{iE(\zeta)z}\, d\zeta$ along the contour R and use

$$\int_R e^{iE(\zeta)z}\, d\zeta = 8\pi\delta(z), \qquad (3\cdot12a)$$

$$\int_R e^{iE(\zeta)z} \frac{d\zeta}{\zeta} = 0, \qquad (3\cdot12b)$$

$$\int_R e^{iE(\zeta)z} \frac{d\zeta}{\zeta^2} = -8\pi\beta^2\delta(z), \qquad (3\cdot12c)$$

where $\delta(z)$ is the Dirac delta function.[12]

To obtain the Marchenko equations, we insert $(3\cdot7)$ into $(3\cdot6)$, using $(2\cdot15b)$

to determine $\bar{\psi}$ from (3·7). Then, upon taking a transform with respect to $e^{iE(\zeta)x}d\zeta$, using (3·12) and

$$\frac{1}{2\pi i}\int_R\frac{e^{iE(\zeta)z}}{\zeta-\zeta'}d\zeta=e^{iE(\zeta')z}\theta(z),\qquad(3\cdot13\text{a})$$

$$\frac{1}{2\pi i}\int_R\frac{e^{iE(\zeta)z}}{\zeta(\zeta-\zeta')}d\zeta=\frac{1}{\zeta'}e^{iE(\zeta')z}\theta(z),\qquad(3\cdot13\text{b})$$

which are valid for ζ' along C, we have

$$L(x,y)+F^{(1)}(x+y)+\int_x^\infty[K(x,s)F^{(1)}(s+y)+L(x,s)F^{(2)}(s+y)]ds=0\,,\quad(3\cdot14\text{a})$$

$$K(x,y)+F^{(2)}(x+y)+\int_x^\infty[K(x,s)F^{(2)}(s+y)+L(x,s)F^{(3)}(s+y)]ds=0\,,\quad(3\cdot14\text{b})$$

where

$$F^{(m)}(z)\equiv\frac{\beta^2}{8\pi}\int_C\left(\frac{i}{\beta^2\zeta}\right)^m\rho(\zeta)e^{iE(\zeta)z}d\zeta\,.\qquad(3\cdot15)$$

Once a solution of the integral equation (3·14) is obtained, the potentials can be recovered from K and L by (3·9).

When βq and r are real, from (2·14, 15, 18, 19) we have

$$[F^{(m)}(z)]^*=F^{(m)}(z)\quad\text{if}\quad\beta^2>0\qquad(3\cdot16\text{a})$$

and

$$[F^{(m)}(z)]^*=-(-\beta^2)^{1-m}F^{(2-m)}(z)\quad\text{if}\quad\beta^2<0\,.\qquad(3\cdot16\text{b})$$

We note that when $m>0$ or $m<2$, (3·15) may be ill defined unless $\rho(\zeta)$ vanishes sufficiently rapidly as $\zeta\to0$ or $\zeta\to\infty$, although (3·15) does exist for $m=0$, 1 and 2. In these other cases, one may instead use the recursion relation

$$F^{(m+1)}(z)-\beta^{-2}F^{(m-1)}(z)=4\frac{dF^{(m)}(z)}{dz}\,.\qquad(3\cdot17)$$

§4. Soliton solutions

When ρ is zero along R, a closed form solution of (3·14) is possible, since all kernels are degenerate. When βq and r are real, we have from (3·15),

$$F^{(m)}(z)=\sum_{n=1}^N\left(\frac{i}{\beta\zeta_n}\right)^{m-1}\beta C_n\kappa_n e^{-\kappa_n z},\qquad(4\cdot1)$$

where $[\zeta_n]_{n=1}^N$ are the bound state eigenvalues,

$$\kappa_n=-\frac{i}{4}\left(\zeta_n+\frac{\beta^{-2}}{\zeta_n}\right),\qquad(4\cdot2)$$

and C_n is a constant. For βq and r real, C_n and κ_n are real when $\zeta_n=-\zeta_n^*$ for

$\beta^2 > 0$ or when $\zeta_n^* \zeta_n = -\beta^{-2}$ for $\beta^2 < 0$.

The one-soliton solution corresponds to $N=1$ and the solution of (3·14) is then (since C_1 and κ_1 must be real)

$$L(x, y) = -\frac{\kappa_1 C_1 \beta}{D(x)} e^{-\kappa_1(x+y)}, \qquad (4\cdot3\text{a})$$

$$K(x, y) = -\frac{iC_1 \kappa_1}{\zeta_1 \beta D(x)} e^{-\kappa_1(x+y)}, \qquad (4\cdot3\text{b})$$

where

$$D(x) = 1 + \frac{iC_1}{\beta \zeta_1} e^{-2\kappa_1 x}. \qquad (4\cdot4\text{a})$$

Defining

$$G(x) = 1 - i\zeta_1 \beta C_1 e^{-2i\kappa_1 x}, \qquad (4\cdot4\text{b})$$

then the solution for q and r is

$$q = 2\kappa_1(G^{-1} - D^{-1}), \qquad (4\cdot5\text{a})$$

$$r = \kappa_1^2 \left[\frac{2}{G} + \frac{2}{D} - \frac{3}{G^2} - \frac{3}{D^2} + \frac{2}{GD} \right]. \qquad (4\cdot5\text{b})$$

Solving (2·3) for Φ_x and Φ_t gives

$$\Phi_x = \kappa_1 \chi, \qquad (4\cdot6\text{a})$$

$$\Phi_t + \Gamma \Phi_x = \omega \chi, \qquad (4\cdot6\text{b})$$

where

$$\chi = \frac{4\beta}{\varepsilon} \left(\frac{1}{G} - \frac{1}{D} \right), \qquad (4\cdot7\text{a})$$

$$\omega = -\frac{\beta}{8} \left(\zeta_1^2 - \frac{1}{\beta^4 \zeta_1^2} \right), \qquad (4\cdot7\text{b})$$

and from (2·21), the time dependence of C_1 is

$$C_1(t) = C_{10} e^{-2t(\omega - \Gamma \kappa_1)}. \qquad (4\cdot8)$$

Let us now look at the solution when $\beta^2 > 0$. Scaling both β and ε to unity, and letting

$$\zeta_1 = i\eta, \qquad (4\cdot9\text{a})$$

$$C_{10} = \pm e^{2\kappa_1 x_0} \qquad (4\cdot9\text{b})$$

(since C_{10} may be positive or negative), we have

$$\kappa_1 = \frac{1}{4}(\eta - \eta^{-1}), \qquad (4\cdot10\text{a})$$

$$\omega = \frac{1}{8}(\eta^2 - \eta^{-2}),\qquad\qquad (4\cdot10b)$$

$$\chi = \frac{\mp 8\kappa_1}{\cosh[2\kappa_1(x-x_0-\Gamma t)+2t\omega]\pm\frac{1}{2}(\eta+\eta^{-1})}.\qquad (4\cdot10c)$$

When $1<\eta<\infty$, the soliton is moving to the left, and when $-1<\eta<0$ (see Fig. 1), the soliton is moving to the right, if $\Gamma=0$. If C_{10} is negative for $1<\eta<\infty$, or positive for $-1<\eta<0$, then $(4\cdot10c)$ is singular.[13] In this case, we have an exact balance between the highest nonlinearity (cubic) and the $\mathcal{O}_{xxxx}$ term in $(2\cdot5)$ at these singularities. Of course, for the Boussinesq equation, this cubic term is not present, so this balance is not possible, and singular solutions of this type will not occur. Furthermore, in the case of water waves $(1\cdot2)$, they are unphysical.

On the other hand, when $\beta^2<0$, the one-soliton solution does not have a singular solution. If we scale $\beta=i$ and $\varepsilon=1$, we have

$$\zeta_1 = e^{i\theta},\quad (0<\theta<\pi)\qquad\qquad (4\cdot11a)$$

$$C_{10} = \pm e^{2\kappa_1 x_0},\qquad\qquad (4\cdot11b)$$

and

$$\kappa_1 = \frac{1}{2}\sin\theta,\qquad\qquad (4\cdot12a)$$

$$\omega = \frac{1}{4}\sin 2\theta,\qquad\qquad (4\cdot12b)$$

$$\chi = \frac{4\sin\theta}{\cosh[2\kappa_1(x-x_0-\Gamma t)+2\omega t]+\cos\theta}.\qquad (4\cdot12c)$$

Lastly, we should note that for case $\beta^2>0$, when ζ is real, the time dependence of $b(\zeta)$ is exponentially increasing if $\zeta^2>1/\beta^2$, and exponentially decreasing if $\zeta^2<1/\beta^2$. Thus, the solution appears to be unstable if $b(\zeta)\neq0$ when $\zeta^2>1/\beta^2$. Of couse, the same problem is also present in the linearized form of $(2\cdot5)$.

§ 5. The Boussinesq and the KdV limit

Since the Boussinesq equation is given in Lagrangian coordinates,[3] and not local coordinates (which we are using), it is necessary to transform our coordinates, as well as the function $\mathcal{O}$, to obtain the Boussinesq equation. If we let

$$z = x,\qquad\qquad (5\cdot1a)$$

$$\tau = t+\varepsilon\mathcal{O}\qquad\qquad (5\cdot1b)$$

and define θ by

$$\mathcal{O}_t = \theta - \frac{1}{2}\varepsilon\theta^2,\qquad\qquad (5\cdot1c)$$

then, upon taking a time derivative, $(1\cdot2)$ becomes

$$\theta_{\tau\tau} = \theta_{zz} + \delta^2 (\tfrac{1}{3} - \sigma) \theta_{zzzz} - \tfrac{3}{2}\varepsilon (\theta^2)_{zz} + O(\varepsilon^2, \varepsilon\delta^2, \delta^4),\qquad(5\cdot2)$$

which is the Boussinesq equation.

Although our scattering problem cannot be reduced to the scattering problem for the Boussinesq equation,[1] it can be reduced to the scattering problem for the right- and left-going KdV scattering problems.[8] To show this, we let $\Gamma = 0$ and assume a solution to $(2\cdot5)$ of the form

$$\varPhi(x, t) = U(x \mp t, \varepsilon t),\qquad(5\cdot3)$$

$$\beta^2 = O(\varepsilon).\qquad(5\cdot4)$$

Then, we find

$$U_{x\tau} \pm \left(\frac{\beta^2}{2\varepsilon}\right) U_{xxxx} + U_{xx} U_x = O(\varepsilon),\qquad(5\cdot5)$$

which is the KdV equation where

$$\chi = x \mp t,\qquad(5\cdot6a)$$

$$\tau = \varepsilon t.\qquad(5\cdot6b)$$

To obtain the limit for $(2\cdot1)$, $(2\cdot2)$, we let

$$k = \pm \frac{(1 - 2\beta^2\lambda)}{2i\beta}\qquad(5\cdot7)$$

and from $(2\cdot3)$, $(2\cdot4)$, we have

$$\psi_{xx} + \left[\lambda \pm \frac{1}{2} U_x\left(\frac{\varepsilon}{\beta^2}\right) + O(\varepsilon)\right]\psi = 0\qquad(5\cdot8a)$$

and

$$\psi_\tau = \left(\frac{\alpha}{\varepsilon} + \frac{1}{4} U_{xx}\right)\psi + \left[\pm 2\left(\frac{\beta^2}{\varepsilon}\right)\lambda - \frac{1}{2} U_x\right]\psi_x,\qquad(5\cdot8b)$$

which are the eigenvalue problem and time evolution equation for the KdV equation.[8]

From $(2\cdot7)$, $(5\cdot8a)$, $(5\cdot7)$, we can see how the ζ-planes in Figs. 1 and 2 break up into the right- and left-going KdV planes. For β small

$$\zeta = \mp \frac{i}{\beta} + 2\sqrt{\lambda} + O(\beta).\qquad(5\cdot9)$$

For $\beta^2 > 0$, the right-going (left-going) KdV plane is the region [of a radius of order unity for $\lambda = O\,(1)$] around $\zeta = -i/\beta(\zeta = i/\beta)$ (see Fig. 1). For $\beta^2 < 0$, the right-going (left-going) KdV plane is the region [again of order unity] around the point $\zeta = -1/\beta$ $(\zeta = +1/\beta)$.

Finally, to decompose the Marchenko equations into the left- and right-going

KdV problems, we simply note that for small β, (3·15) becomes

$$F^{(m)}(z) \simeq \beta^2 \left(\frac{\pm 1}{\beta}\right)^m F_\pm(z), \tag{5·10}$$

where

$$F_\pm(z) = \frac{1}{4\pi} \int_{\sigma_\pm} \rho_\pm(\sqrt{\lambda}) e^{iz\sqrt{\lambda}} d\sqrt{\lambda} \tag{5·11}$$

with

$$\rho_\pm(\sqrt{\lambda}) \equiv \rho(\mp i/\beta + 2\sqrt{\lambda}), \tag{5·12}$$

and $C_\pm$ as the appropriate part of the contour C which passes "above" the point $\zeta - \mp i/\beta$. From (5·10) and (3·14), we obtain

$$K_\pm(x, y) + F_\pm(x+y) + \int_x^\infty K_\pm(x, s) F_\pm(s+y) ds \simeq 0, \tag{5·13}$$

where

$$K_\pm(x, y) = K(x, y) \mp L(x, y)/\beta \tag{5·14}$$

and U_χ is recovered by

$$U_\chi = \frac{4\beta^2}{\varepsilon} \frac{dK_\pm(\chi, \chi)}{d\chi}. \tag{5·15}$$

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References

1) V. E. Zakharov, Zh. Eksp. i Teor. Fiz. **65** (1973), 219.
2) J. Boussinesq, J. Pures Appl. **17** (1872), 55.
3) F. Ursell, Proc. Camb. Phil. Soc. **49** (1953), 685.
4) M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Stud. App. Math. **53** (1974), 249.
5) D. J. Kaup, Prog. Theor. Phys. **54** (1975), 72.
6) See for example, Benny and Luke, J. Math. Phys. **43** (1964), 309.
7) H. Segur, private communication.
8) C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Phys. Rev. Letters **19** (1967), 1095; Comm. Pure Appl. Math. **27** (1974), 97.
9) L. D. Faddeev, Dokl. Akad. Nauk SSSR **121** (1958), 63.
10) M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Phys. Rev. Letters **30** (1973), 1262.
11) V. A. Marchenko, Dokl. Akad. Nauk SSSR **104** (1955), 695.
12) P. A. M. Dirac, *Quantum Mechanics*, 4th ed. (Oxford Press, 1958).
13) For other examples of singular solitons, see Ref. 4) and V. E. Zakharov and S. V. Manakov, JETP Letters **18** (1973), 243.