A HÖLDER ESTIMATE FOR QUASICONFORMAL MAPS BETWEEN SURFACES IN EUCLIDEAN SPACE

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In [2] C. B. Morrey proved a Hölder estimate for quasiconformal mappings in the plane. Such a Hölder estimate was a fundamental development in the theory of quasiconformal mappings, and had very important applications to partial differential equations. L. Nirenberg in [3] made significant simplifications and improvements to Morrey's work (in particular, the restriction that the mappings involved be $1-1$ was removed), and he was consequently able to develop a rather complete theory for second order elliptic equation with 2 independent variables.

In Theorem (2.2) of the present paper we obtain a Hölder estimate which is analogous to that obtained by Nirenberg in [3] but which is applicable to quasiconformal mappings between *surfaces* in Euclidean space. The methods used in the proof are quite analogous to those of [3], although there are of course some technical difficulties to be overcome because of the more general setting adopted here.

In § 3 and § 4 we discuss applications to graphs with quasiconformal Gauss map. In this case Theorem (2.2) gives a Hölder estimate for the unit normal of the graph. One rather striking consequence is given in Theorem (4.1), which establishes the linearity of any $C^2(\mathbf{R}^2)$ function having a graph with quasiconformal Gauss map. This result includes as a special case the classical theorem of Bernstein concerning $C^2(\mathbf{R}^2)$ solutions of the minimal surface equation, and the analogous theorem of Jenkins [1] for a special class of variational equations. There are also in § 3 and § 4 a number of other results for graphs with quasiconformal Gauss map, including some gradient estimates and a global estimate of Hölder continuity. § 4 concludes with an application to the minimal surface system.

One of the main reasons for studying graphs satisfying the condition that the Gauss map is quasiconformal (or (Λ_1, Λ_2)-quasiconformal in the sense of (1.8) below) is that such

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a condition must automatically be satisfied by the graph of a solution of any equation of mean curvature type (see (1.9) (ii) below). However we here only briefly discuss the application of the results of § 3 and § 4 to such equations; a more complete discussion will appear in [7].

§ 1. Terminology

M, N will denote oriented 2-dimensional C^2 submanifolds of $\mathbf{R}^n$, $\mathbf{R}^m$ respectively, $n, m \geq 2$. Given $X \in M$(¹) and $Y \in N$ we let $T_X(M)$, $T_Y(N)$ denote the tangent spaces (considered as subspaces of $\mathbf{R}^n$ and $\mathbf{R}^m$) of M at X and N at Y respectively. δ will denote the gradient operator on M; that is, if $h \in C^1(M)$, then

$$\delta h(X) = (\delta_1 h(X), \ldots, \delta_n h(X)) \in T_X(M)$$

is defined by

$$\delta_i h(X) = \sum_{j=1}^{n} \tilde{g}^{ij}(X) D_j \bar{h}(X), \tag{1.1}$$

where $\bar{h}$ is any C^1 function defined in a neighbourhood of M with $\bar{h}|_M = h$, and where $(\tilde{g}^{ij}(X))$ is the matrix of the orthogonal projection of $\mathbf{R}^n$ onto $T_X(M)$.

We note that of course the definition (1.1) is independent of the particular C^1 extension $\bar{h}$ of h that one chooses to use. We note also that in the special case $n=3$ we can represent $\tilde{g}^{ij}(X)$ explicitly in terms of the unit normal $\nu(X) = (\nu_1(X), \nu_2(X), \nu_3(X))$ of M at X according to the formula

$$\tilde{g}^{ij}(X) = \delta_{ij} - \nu_i(X)\nu_j(X), \quad i, j = 1, 2, 3. \tag{1.2}$$

η, θ will denote area forms for M, N respectively; that is, η and θ are C^1 differential 2-forms on M and N respectively such that

$$\int_A \eta = \text{area } (A), \quad \int_B \theta = \text{area } (B)$$

whenever $A \subset M$ and $B \subset N$ are Borel subsets of finite area.

(1.3) *Remark.* We can always take a C^1 2-form ζ on M to be the restriction to M of a C^1 form $\tilde{\zeta}$ defined in a neighbourhood of $M \subset \mathbf{R}^n$, so that $\tilde{\zeta}(X) \in \Lambda^2(\mathbf{R}^n)$ for each $X \in M$. Thus in case $n=3$, we can write

$$\tilde{\zeta}(X) = \zeta_1(X)\, dx_2 \wedge dx_3 + \zeta_2(X)\, dx_1 \wedge dx_3 + \zeta_3(X)\, dx_1 \wedge dx_2,$$

(¹) We will use $X = (x_1, \ldots, x_n)$ to denote points in M; the symbol x will be reserved to denote points $(x_1, x_2) \in \mathbf{R}^2$.

where $\zeta_1, \zeta_2, \zeta_3$ are C^1 in some neighbourhood of M. Using the notation $*\tilde{\zeta}(X)=(\zeta_1(X), -\zeta_2(X), \zeta_3(X))$ ($*$ is the usual linear isometry of $\Lambda^2(\mathbf{R}^3)$ onto $\mathbf{R}^3$) we then have

$$\int_A \zeta = \int_A \nu \cdot (*\tilde{\zeta})\, d\mathcal{H}^2, \quad A \subset M,$$

where ν is the appropriately oriented unit normal for M and $\mathcal{H}^2$ denotes 2-dimensional Hausdorff measure in $\mathbf{R}^3$. In particular, we see that ζ is an area form for M if and only if $(\zeta_1(X), -\zeta_2(X), \zeta_3(X))$ is a unit normal for M at each point $X \in M$. Thus there is no difficulty in recognizing an area form in case $n=3$. (Of course one can give an analogous, but not quite so convenient, characterization of area forms for arbitrary n.)

Our basic assumption concerning N is that there is a 1-form $\omega(X)=\sum_{i=1}^m \omega_i(X)dx_i$ which is C^2 in a neighbourhood of N and such that

$$(1.4) \qquad d\omega_N = \theta, \quad \sup_N \left\{\sum_{i=1}^m \omega_i^2\right\}^{1/2} + \sup_N \left\{\sum_{i,j=1}^m (D_j\omega_i)^2\right\}^{1/2} \leqslant \Lambda_0 < \infty.$$

Here Λ_0 is a constant and ω_N denotes the restriction of ω to N; henceforth we will not distinguish notationally between ω and ω_N.

(1.5) *Examples.* (i) If N is an open ball of radius R and centre 0 in $\mathbf{R}^2$, we can take $\omega = -\frac{1}{2}x_2dx_1 + \frac{1}{2}x_1dx_2$ and $\Lambda_0 = R+1$.

(ii) If N is the upper hemisphere S^2_+ of the unit sphere $S^2 \subset \mathbf{R}^3$, we can take $\omega = (-x_2/(1+x_3))dx_1 + (x_1/(1+x_3))dx_2 + 0dx_3$ and $\Lambda_0=4$. One can easily check this by directly computing $d\omega$ and using the relation $\sum_{i=1}^3 x_i^2=1$ on S^2; to check that $d\omega$ is an area form for S^2_+ it is convenient to use the characterization of area forms given in Remark (1.3) above. (Alternatively one obtains $d\omega$ as an area form by using an elementary computation involving example (i) above and stereographic projection of S^2_+ into $\mathbf{R}^2$.)

(iii) More generally, we can let N be the surface obtained from a compact surface $L \subset \mathbf{R}^m$ by deleting a compact neighbourhood of an arbitrary chosen point $y_0 \in L$. There will then always exist ω as in (1.4) because the 2-dimensional de Rham chomology group $H^2(L \sim \{y_0\})$ is zero. (And this of course guarantees that *any* 2-form ζ on $L\sim\{y_0\}$ can be written in the form $d\omega$ for some 1-form ω on $L\sim\{y_0\}$.) To check that $H^2(L\sim\{y_0\})=0$ we first note that de Rahm's theorem gives an isomorphism $H^2(L\sim\{y_0\} \cong H^2(L\sim\{y_0\}, \mathbf{R})$, where $H^2(L\sim\{y_0\}, \mathbf{R})$ denotes the 2-dimensional singular cohomology group with real coefficients. Next we note the duality isomorphism $H^2(L\sim\{y_0\}, \mathbf{R}) \cong \operatorname{Hom}(H_2(L\sim\{y_0\}), \mathbf{R})$, where $H_2(L\sim\{y_0\})$ denotes the 2-dimensional singular homology group with integer coefficients. Finally we note that $H_2(L\sim\{y_0\})=0$. This follows from the exactness of the homology sequence for the pair $(L, L\sim\{y_0\})$, together with the fact that the inclusion map $(L, \phi)\subset(L, L\sim\{y_0\})$ induces an isomorphism $H_2(L)\cong H_2(L, L\sim\{y_0\})$ (see [9]).

We now consider a C^1 mapping

$$\varphi = (\varphi_1, \ldots, \varphi_m): M\to N.$$

In order to formulate the concept of quasiconformality for φ we need to introduce some terminology. Firstly, for $X\in M$ we let

$$\delta\varphi(X): T_X(M)\to T_{\varphi(X)}(N)$$

denote the linear map between tangent spaces induced by φ. We note that the matrix $(\delta_i\varphi_j(X))$ represents $\delta\varphi(X)$ in the sense that if $v=(v_1, \ldots, v_n)\in T_X(M)$, $w=(w_1, \ldots, w_m)$ $\in T_{\varphi(X)}(N)$ and $w=\delta\varphi(X)(v)$, then

$$w_j=\sum_{i=1}^{n}\delta_i\varphi_j(X)v_i, \quad j=1, \ldots, m.$$

(Here $\delta_i\varphi_j(X)$ is defined by (1.1)). The adjoint transformation $(\delta\varphi(X))^*$ is represented in a similar way by the transposed matrix $(\delta_j\varphi_i(X))$. We define

$$|\delta\varphi(X)|=\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}(\delta_i\varphi_j(X))^2\right\}^{1/2};$$

thus $|\delta\varphi(X)|$ is just the inner product norm $\{\text{trace}\,((\delta\varphi(X))^*\delta\varphi(X))\}^{\frac{1}{2}}$. Next, we let $J\varphi(X)$ denote the signed area magnification factor of φ computed relative to the given area forms η, θ. That is, letting

$$\Lambda^2(\delta\varphi(X)): \Lambda^2(T_{\varphi(X)}(N))\to\Lambda^2(T_X(M))$$

be the linear map of 2-forms induced by $\delta\varphi(X)$, we define the real number $J\varphi(X)$ by

$$\Lambda^2(\delta\varphi(X))d\omega(\varphi(X)) = J\varphi(X)\eta(X), \quad X\in M. \tag{1.7}$$

Notice that this makes sense as a definition for $J\varphi(X)$ because $\Lambda^2(T_X(M))$ and $\Lambda^2(T_{\varphi(X)}(N))$ are 1-dimensional vector spaces spanned by the unit vectors $\eta(X)$ and $d\omega(\varphi(X))$ respectively. Notice also that $|J\varphi(X)| = \|\Lambda^2(\delta\varphi(X))\|$. In fact,

$$J\varphi(X) = \pm\|\Lambda^2(\delta\varphi(X))\|,$$

with $+$ or $-$ according as φ preserves or reverses orientation at X.

(1.8) *Definition.* We say φ is (Λ_1, Λ_2)-quasiconformal on M if Λ_1, Λ_2 are constants with $\Lambda_2 \geqslant 0$, and if

$$|\delta\varphi(X)|^2 \leqslant \Lambda_1 J\varphi(X) + \Lambda_2$$

at each point $X \in M$.[2]

The geometric interpretation of this condition is well known:

$$\delta\varphi(X)\colon T_X(M) \to T_{\varphi(X)}(N)$$

maps the unit circle of $T_X(M)$ onto an ellipse with semi-axes a and b, $a \geqslant b$, in $T_{\varphi(X)}(N)$, and

$$|\delta\varphi(X)|^2 = a^2 + b^2, \quad |J\varphi(X)| = ab.$$

Thus the definition (1.8), with $\Lambda_2 = 0$, implies

$$a^2 + b^2 \leqslant |\Lambda_1| ab,$$

which implies $|\Lambda_1| \geqslant 2$ and

$$a \leqslant \left(\frac{|\Lambda_1|}{2} + \sqrt{\frac{\Lambda_1^2}{4} - 1}\right) b.$$

Furthermore, (1.8) can hold with $|\Lambda_1| = 2$ if and only if $a = b$; that is, either $\delta\varphi(X) = 0$ or $\delta\varphi(X)$ takes circles into circles. This latter property holds if and only if φ is *conformal* at X.

In case $\Lambda_2 \neq 0$ a similiar interpretation holds if $a^2 + b^2$ is sufficiently large relative to Λ_2; an important point however is that in this case condition (1.8) imposes no restriction on the mapping φ at points X where $|\delta\varphi(X)|$ is sufficiently small relative to Λ_2.

(1.9) *Examples.* (i) A classical example considered by Morrey [2] and Nirenberg [3] involves equations

$$\sum_{i,j=1}^{2} a_{ij}(x)\, D_{ij} u = b(x)$$

on a domain $\Omega \subset \mathbf{R}^2$, with conditions

$$|\xi|^2 \leqslant \sum_{i,j=1}^{2} a_{ij}(x)\, \xi_i \xi_j \leqslant \lambda_1 |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbf{R}^2$$

$$|b(x)| \leqslant \lambda_2, \quad x \in \Omega.$$

Provided that $\sup_\Omega |Du| < \infty$, we can define M, N, φ by $M = \Omega$, $N = \{x \in \mathbf{R}^2 \colon |x| < \sup_\Omega |Du|\}$ (see example (1.5)(i)) and

$$\varphi = Du\colon M \to N.$$

(2) Notice that we do *not* require φ to be 1–1.

In this case we have $J(\varphi)=(D_{11}u)(D_{22}u)-(D_{12}u)^2$, $|\delta\varphi|^2\equiv|D\varphi|^2=\sum_{i,j=1}^2(D_{ij}u)^2$, and φ is (Λ_1,Λ_2)-quasiconformal with $\Lambda_1=-2\lambda_1$, $\Lambda_2=\lambda_1\lambda_2^2$. To prove this last assertion we choose coordinates which diagonalize $(D_{ij}u(x))$ at a given point $x=x_0$; in these coordinates the equation takes the form

$$\alpha_1 D_{11}\tilde{u}+\alpha_2 D_{22}\tilde{u}=\beta,$$

where

$$1\leqslant\alpha_i\leqslant\lambda_1,\quad i=1,2,\quad |\beta|\leqslant\lambda_2.$$

Squaring and dividing by $\alpha_1\alpha_2$ then gives

$$\frac{1}{\lambda_1}((D_{11}\tilde{u})^2+(D_{22}\tilde{u}^2))\leqslant-2(D_{11}\tilde{u})(D_{22}\tilde{u})+\lambda_2^2.$$

In the original coordinates, this gives

$$\sum_{i,j=1}^{2}(D_{ij}u)^2\leqslant-2\lambda_1((D_{11}u)(D_{22}u)-(D_{12}u)^2)+\lambda_1\lambda_2^2$$

as asserted.

(ii) Another important example of a quasiconformal map arises by considering the *equations of mean curvature type*; that is, any equation of the form

$$\sum_{i,j=1}^{2}a_{ij}(x,u,Du)D_{ij}u=b(x,u,Du),\quad x\in\Omega,$$

where the following conditions (see [7] for a discussion) are satisfied:

(a) $$\sum_{i,j=1}^{2}g^{ij}\xi_i\xi_j\leqslant\sum_{i,j=1}^{2}a_{ij}(x,u,Du)\xi_i\xi_j\leqslant\lambda_1\sum_{i,j=1}^{2}g^{ij}\xi_i\xi_j$$

where

$$g^{ij}=\delta_{ij}-\nu_i\nu_j,\quad \nu_i=-D_iu/\sqrt{1+|Du|^2},$$

(b) $$|b(x,u,Du)|\leqslant\lambda_2\sqrt{1+|Du|^2}.$$

It is shown in [7] that (a), (b) imply that the graph $M=\{X=(x_1,x_2,x_3): x_3=u(x_1,x_2)\}$ has principal curvatures $\varkappa_1,\varkappa_2$ which satisfy, at each point of M, an equation of the form

$$\alpha_1\varkappa_1+\alpha_2\varkappa_2=\beta,$$

where

$$1\leqslant\alpha_i\leqslant\lambda_1,\quad i=1,2,\quad |\beta|\leqslant\lambda_2.$$

Squaring, we obtain

$$\frac{\alpha_1}{\alpha_2}\varkappa_1^2+\frac{\alpha_2}{\alpha_1}\varkappa_2^2=-2\varkappa_1\varkappa_2+\frac{\beta}{\alpha_1\alpha_2},$$

so that

$$\varkappa_1^2+\varkappa_2^2\leqslant\Lambda_1\varkappa_1\varkappa_2+\Lambda_2, \tag{1.10}$$

where

$$\Lambda_1=-2\lambda_1,\quad \Lambda_2=\lambda_1\lambda\ .$$

We now let $N=S_+^2$ (see example (1.5) (ii)) and we let $\varphi: M\to N$ be the *Gauss map* ν, defined by setting $\nu(X)$ equal to the upward unit normal of M at X; that is,

$$\nu(X)=(-Du(x),1)/\sqrt{1+|Du(x)|^2},\quad X=(x,u(x)),\quad x\in\Omega.$$

Then, as is well known,

$$J\nu=K\equiv\varkappa_1\varkappa_2\ \ (=\text{Gauss curvature of } M).$$

(This is easily checked by working with a "principal coordinate system at X"; that is, a coordinate system with origin at X and with coordinate axes in the directions $e_1(X)$, $e_2(X)$, $\nu(X)$, where $e_1(X)$, $e_2(X)$ are principal directions of M at X.)

Furthermore (and again one can easily check this by working with a principal coordinate system at X)

$$|\delta\nu|^2=\varkappa_1^2+\varkappa_2^2.$$

Thus the inequality (1.10) above asserts that the Gauss map ν is (Λ_1,Λ_2)-quasiconformal with $\Lambda_1=-2\lambda_1$, $\Lambda_2=\lambda_1\lambda_2^2$.

Thus the main Hölder continuity result we are to obtain below (Theorem (2.2)) will apply to the gradient map $x\to Du(x)$, $x\in\Omega$, in the case of uniformly elliptic equations (as in (i)) and to the Gauss map $X\to\nu(X)$, $X\in\operatorname{graph}(u)$, in the case of equations of mean curvature type. In the former case one obtains the classical estimate of Morrey-Nirenberg concerning Hölder continuity of first derivatives for uniformly elliptic equations; in the latter case we obtain a new Hölder continuity result for the unit normal of the graph of the solution of an equation of mean curvature type. (See the remarks at the beginning of § 4 below and the reference [7] for further discussion and applications.)

We conclude this section with some notations concerning the subsets obtained by intersecting the surface M with an n-dimensional ball. We write

$$S_\varrho(X_1)=\{X\in M: |X-X_1|<\varrho\}$$

whenever $X_1\in M$ and $\varrho>0$. $X_0\in M$ and $R>0$ will be such that

$$(\overline{M}\sim M)\cap\{X\in\mathbf{R}^n: |X-X_0|\leqslant R\}=\emptyset$$

(here $\bar{M}$ denotes the closure of M taken in $\mathbf{R}^n$), so that $\bar{S}_R(X_0)$ is a compact subset of M. Λ_3 will denote a constant such that

$$(3R/4)^{-2}|S_{3R/4}(X_0)| \leq \Lambda_3. \tag{1.11}$$

Here and subsequently we let $|S_\varrho(X_1)|$ denote the 2-dimensional Hausdorff measure of $S_\varrho(X_1)$.

In the important special case when M is a graph with (Λ_1, Λ_2)-quasiconformal Gauss map, we will show in § 3 that Λ_3 can be chosen to depend only on Λ_1 and $\Lambda_2 R^2$.

It will be proved in the appendix that

$$\sigma^{-2}|S_\sigma(X_1)| \leq 40\left\{\varrho^{-2}|S_\varrho(X_1)| + \int_{S_\varrho(X_1)} H^2 dA\right\} \tag{1.12}$$

for any $X_1 \in S_R(X_0)$ and any σ, ϱ with $0 < \sigma \leq \varrho < R - |X_1 - X_0|$. (Here H denotes the mean curvature vector of M.)

§ 2. The Hölder estimate

The main Hölder continuity result (Theorem (2.2) below) will be obtained as a consequence of estimates for the *Dirichlet integral* corresponding to the map $\varphi: M \to N$ (cf. the original method of Morrey [2].) For a given $X_1 \in S_{R/2}(X_0)$ and $\varrho \in (0, R/2)$, the Dirichlet integral is denoted $\mathcal{D}(X_1, \varrho)$, and is defined by

$$\mathcal{D}(X_1, \varrho) = \int_{S_\varrho(X_1)} |\delta\varphi|^2 dA.$$

Before deriving the estimates for these integrals, some preliminary remarks are needed. We are going to adopt the standard terminology that if ζ is a k-form on $N (k = 1, 2)$ then $\varphi^{\#}\zeta$ denotes the "pulled-back" k-form on M ,defined by

$$(\varphi^{\#}\zeta)(X) = \Lambda^k(\delta\varphi(X))\zeta(\varphi(X)), \quad X \in M.$$

Thus, letting h be an arbitrary C^1 function on M, and using the definition (1.7) together with the relation

$$\varphi^{\#}d = d\varphi^{\#},$$

we have

$$d(h\varphi^{\#}N) = dh \wedge \varphi^{\#}\omega + hJ\varphi\, dA, \tag{2.1}$$

where dA denotes the area form η for M. We also need to note that if $X_1 \in S_R(X_0)$ and if r_{X_1} is the Euclidean distance function defined by

$$r_{X_1}(X) = |X - X_1|, \quad X \in \mathbf{R}^n, \tag{2.2}$$

then, by Sard's theorem, we have that, for almost all $\varrho \in (0, R - |X_1 - X_0|)$, δr_{X_1} *vanishes at no point of* $\partial S_\varrho(X_1)$. For such values of ϱ we can write

$$\partial S_\varrho(X_1) = \bigcup_{j=1}^{N(\varrho)} \Gamma_\varrho^{(j)}, \tag{2.3}$$

where $N(\varrho)$ is a positive integer and $\Gamma_\varrho^{(j)}$, $j = 1, \ldots, N(\varrho)$, are C^2 Jordan curves in M. Thus, by Stoke's theorem, for almost all $\varrho \in (0, R - |X_1 - X_0|)$ (2.1) will imply

$$\int_{S_\varrho(X_1)} hJ\varphi\, dA = -\int_{S_\varrho(X_1)} dh \wedge \varphi^{\#}\omega + \sum_{j=1}^{N(\varrho)} \int_{\Gamma_\varrho^{(j)}} h\varphi^{\#}\omega. \tag{2.4}$$

(We are assuming that the $\Gamma_\varrho^{(j)}$ are appropriated oriented.) In case h has compact support in $S_\varrho(X_1)$ we can write

$$\int_{S_\varrho(X_1)} hJ\varphi\, dA = -\int_{S_\varrho(X_1)} dh \wedge \varphi^{\#}\omega, \tag{2.5}$$

and of course this holds for *all* $\varrho \in (0, R - |X_1 - X_0|)$.

The following lemma gives a preliminary bound for $\mathcal{D}(X_0, R/2)$.

LEMMA (2.1). *If φ is (Λ_1, Λ_2)-quasiconformal, then*

$$\mathcal{D}(X_0, R/2) \leq c,$$

where c depends only on Λ_0, Λ_1, $\Lambda_2 R^2$ and Λ_3.

Proof. We let ψ be a C^1 "cut-off function" satisfying $0 \leq \psi \leq 1$ on M, $\psi \equiv 1$ on $S_{R/2}(X_0)$, $\psi \equiv 0$ outside $S_{3R/4}(X_0)$ and $\sup_M |\delta\psi| \leq 5/R$. (Such a function is obtained by defining $\psi(X) = \gamma(|X - X_1|)$, where γ is a suitably chosen $C^1(\mathbf{R})$ function.)

Since

$$\varphi^{\#}\omega = \sum_{i=1}^{m} \omega_i \circ \varphi\, d\varphi_i, \tag{2.6}$$

we can easily check, by using (1.4), that

$$|(d\psi \wedge \varphi^{\#}\omega)(X)| \leq \Lambda_0 |\delta\psi(X)|\, |\delta\varphi(X)| \leq 5R^{-1}\Lambda_0 |\delta\varphi(X)|, \quad X \in M.$$

(Here, on the left, $|\ |$ denotes the usual inner product norm for forms on $T_X(M)$.) Then by using (2.5) with $X_1 = X_0$, $\varrho = R$ and $h = \psi^2$, we easily obtain

$$\left|\int_{S_R(X_0)} \psi^2 J\varphi\, dA\right| \leq 10R^{-1}\Lambda_0 \int_{S_R(X_0)} \psi |\delta\varphi|\, dA.$$

The quasiconformal condition (1.8) then implies

$$\int_{S_R(X_0)} \psi^2 |\delta\varphi|^2 dA \leq 10R^{-1}\Lambda_0|\Lambda_1| \int_{S_R(X_0)} \psi|\delta\varphi| dA + \Lambda_2 \int_{S_R(X_0)} \psi^2 dA.$$

Using the Cauchy inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and the definition of Λ_3, we then obtain

$$\int_{S_R(X_0)} \psi^2 |\delta\varphi|^2 dA \leq \tfrac{1}{2} \int_{S_R(X_0)} \psi^2 |\delta\varphi|^2 dA + (50\,\Lambda_0^2\Lambda_1^2 + \Lambda_2 R^2)\,\Lambda_3.$$

Since $\psi \equiv 1$ on $S_{R/2}(X_0)$, the required inequality then follows (with $c = (100\Lambda_0^2\Lambda_1^2 + 2\Lambda_2 R^2)\Lambda_3$).

The next theorem contains the main estimate for $\mathcal{D}(X_1, \varrho)$. In the statement of the theorem, and subsequently, Λ_4 denotes a constant such that

$$\int_{S_{R/2}(X_0)} H^2 dA \leq \Lambda_4,$$

where H is the mean-curvature vector of M. (See [4].)

THEOREM (2.1). *If φ is (Λ_1, Λ_2)-quasiconformal, then*

$$\mathcal{D}(X_1, \varrho) \leq c(\varrho/R)^\alpha$$

for all $X_1 \in S_{R/4}(x_0)$ and all $\varrho \in (0, R/4)$, where $c > 0$ and $\alpha \in (0, 1)$ are constants depending only on Λ_0, Λ_1, $\Lambda_2 R^2$, Λ_3 and Λ_4.

Proof. Since the curves $\Gamma_\varrho^{(j)}$ of (2.3) are closed we have $\int \Gamma_\varrho^{(j)} d\varphi_i = \int \Gamma_\varrho^{(j)} d\varphi_i/ds = 0$. (Here ds denotes integration with respect to arc-length and $d\varphi_i/ds$ denotes directional differentation in the direction of the appropriate unit tangent T of $\Gamma_\varrho^{(j)}$; that is $d\varphi_i/ds = T \cdot \delta\varphi_i$.) Then by (2.6) we have

$$\int_{\Gamma_\varrho^{(j)}} \varphi^{\#}\omega \sum_{i=1}^{m} \int_{\Gamma_\varrho^{(j)}} (\omega_i \circ \varphi - \omega_i \circ \varphi(X^{(j)})) \frac{d\varphi_i}{ds}\, ds,$$

where $X^{(j)}$ denotes an initial point (corresponding to arc-length $= 0$) of $\Gamma_\varrho^{(j)}$. Then using (2.4) with $h \equiv 1$, we obtain

$$(2.7) \qquad \left|\int_{S_\varrho(X_1)} J\varphi\, dA\right| = \left|\sum_{j=1}^{N(\varrho)} \int_{\Gamma_\varrho^{(j)}} \sum_{i=1}^{m} (\omega_i \circ \varphi - \omega_i \circ \varphi(X^{(j)})) \frac{d\varphi_i}{ds}\, ds\right|$$

$$\leq \sum_{j=1}^{N(\varrho)} \left\{\sup_{\Gamma_\varrho^{(j)}} |\omega \circ \varphi - \omega \circ \varphi(X^{(j)})| \int_{\Gamma_\varrho^{(j)}} |\delta\varphi|\, ds\right\}.$$

But clearly

$$\sup_{\Gamma_\varrho^{(j)}} |\omega\circ\varphi - \omega\circ\varphi(X^{(j)})| \leqslant \int_{\Gamma_\varrho^{(j)}} \left|\frac{d\omega\circ\varphi}{ds}\right| ds \leqslant \int_{\Gamma_\varrho^{(j)}} |\delta\omega\circ\varphi|\, ds. \tag{2.8}$$

Since

$$|\delta\omega\circ\varphi| \leqslant \sup_N |D\omega|\,|\delta\varphi| \leqslant \Lambda_0 |\delta\varphi|,$$

(2.7) and (2.8) clearly imply

$$\begin{aligned} \left|\int_{S_\varrho(X_1)} J\varphi\, dA\right| &\leqslant \Lambda_0 \sum_{j=1}^{N(\varrho)} \left\{\int_{\Gamma_\varrho^{(j)}} |\delta\varphi|\, ds\right\}^2 \\ &\leqslant \Lambda_0 \left\{\sum_{j=1}^{N(\varrho)} \int_{\Gamma_\varrho^{(j)}} |\delta\varphi|\, ds\right\}^2 = \Lambda_0 \left(\int_{\partial S_\varrho(X_1)} |\delta\varphi|\, ds\right)^2 \\ &= \Lambda_0 \left(\int_{\partial S_\varrho(X_1)} (|\delta\varphi|\,|\delta r_{X_1}|^{-1/2})\,(|\delta r_{X_1}|^{1/2})\, ds\right)^2 \\ &\leqslant \Lambda_0 \left\{\int_{\partial S_\varrho(X_1)} |\delta\varphi|^2 |\delta r_{X_1}|^{-1}\, ds\right\} \left\{\int_{\partial S_\varrho(X_1)} |\delta r_{X_1}|\, ds\right\} \\ &= \Lambda_0 \left(\frac{d}{d\varrho}\int_{S_\varrho(X_1)} |\delta\varphi|^2 dA\right) \left(\frac{d}{d\varrho}\int_{S_\varrho(X_1)} |\delta r_{X_1}|^2 dA\right). \end{aligned} \tag{2.9}$$

Here r_{X_1} is as in (2.2) and in the last equality we have used the differentiated version of the co-area formula:

$$\frac{d}{d\varrho}\int_{S_\varrho(X_1)} h\, dA = \int_{\partial S_\varrho(X_1)} h\,|\delta r_{X_1}|\, ds$$

whenever h is a continuous function on M.

Now by using (1.12) and the identity (A.2) with $h \equiv 1$, it is easily seen that

$$\frac{d}{d\varrho}\int_{S_\varrho(X_1)} |\delta r_{X_1}|^2 dA \leqslant c_1 \varrho, \tag{2.10}$$

where c_1 depends only on Λ_3 and Λ_4. Hence, by combining (2.9) and (2.10) we have

$$\left|\int_{S_\varrho(X_1)} J\varphi\, dA\right| \leqslant c_1 \Lambda_0 \varrho \frac{d}{d\varrho} \mathcal{D}(X_1, \varrho).$$

The condition (1.8) then implies (after using (1.11), (1.12))

$$\mathcal{D}(X_1, \varrho) \leqslant c_1' \left(|\Lambda_1| \Lambda_0 \varrho \frac{d}{d\varrho} \mathcal{D}(X_1, \varrho) + \Lambda_2 \varrho^2\right)$$

for almost all $\varrho \in (0, R/4)$. If we now define

$$\mathcal{E}(\varrho) = \mathcal{D}(X_1, \varrho) + \Lambda_2 \varrho^2$$

we see that this last inequality implies

$$\mathcal{E}(\varrho) \leqslant c_2 \varrho \mathcal{E}'(\varrho), \quad \text{a.e. } \varrho \in (0, R/4),$$

where c_2 depends only on Λ_0, Λ_1, Λ_3 and Λ_4. This can be written

$$\frac{d}{d\varrho} \log \mathcal{E}(\varrho) \geqslant c_2^{-1} \varrho^{-1}, \quad \text{a.e. } \varrho \in (0, R/4).$$

Since $\mathcal{E}(\varrho)$ is increasing in ϱ, we can integrate to obtain

$$\log (\mathcal{E}(\varrho)/\mathcal{E}(R/4)) \leqslant c_2^{-1} \log (4\varrho/R), \quad \varrho \leqslant R/4;$$

that is

$$\mathcal{E}(\varrho) \leqslant 4^\alpha \mathcal{E}(R/4)(\varrho/R)^\alpha, \quad \alpha = c_2^{-1}, \varrho \in (0, R/4). \tag{2.11}$$

Since $S_{R/4}(X_1) \subset S_{R/2}(X_0)$, we must have

$$\mathcal{E}(R/4) \leqslant \mathcal{D}(X_0, R/2) + \Lambda_2 (R/4)^2. \tag{2.12}$$

The required estimate for $\mathcal{D}(X_1, \varrho)$ now follows from (2.11), (2.12) and Lemma (2.1); note that the exponent α is actually independent of Λ_2.

We next need an analogue of the Morrey lemma ([2], Lemma 1) for surfaces; this will enable us to deduce a Hölder estimate for φ from Theorem (2.1) (cf. the orginal method of Morrey [2].)

LEMMA (2.2). *Suppose h is C^1 on M and suppose $K > 0$, $\beta \in (0, 1)$ are such that*

$$\int_{S_\varrho(X_1)} |\delta h| \, dA \leqslant K \varrho (\varrho/R)^\beta$$

for all $X_1 \in S_{R/4}(X_0)$ and all $\varrho \in (0, R/4)$. Then

$$\sup_{X \in S_\varrho^*(X_0)} |h(X) - h(X_0)| \leqslant cK(\varrho/R)^\beta, \quad \varrho \in (0, R/4),$$

where c depends on Λ_3 and Λ_4, and where $S_\varrho^*(X_0)$ denotes the component of $S_\varrho(X_0)$ which contains X_0.

This lemma is proven in the appendix.

We can now finally deduce the Hölder estimate for quasiconformal maps.

THEOREM (2.2). *If φ is (Λ_1, Λ_2)-quasiconformal, then*

$$\sup_{X\in S_\varrho^*(X_0)} |\varphi(X)-\varphi(X_0)| \leq c(\varrho/R)^{\alpha/2}, \quad \varrho\in(0, R/4),$$

where $c>0$ depends only on Λ_0, Λ_1, $\Lambda_2 R^2$, Λ_3 and Λ_4 and where $\alpha\in(0,1)$ is as in Theorem (2.1); $S_\varrho^*(X_0)$ is as in Lemma (2.2).

Proof. Let X_1 be an arbitrary point of $S_{R/4}(X_0)$. By the Hölder inequality, (1.12) and Theorem (2.1) we have

$$\int_{S_\varrho(X_1)} |\delta\varphi_i|\,dA \leq c'(c)^{1/2}\varrho(\varrho/R)^{\alpha/2}, \quad \varrho\in(0, R/4), \quad i=1, \dots, m$$

where c, α are as in Theorem (2.1) and c' depends on Λ_3, Λ_4. Hence the hypotheses of Lemma (2.2) are satisfied, with $\beta=\alpha/2$ and $K=c'c^{1/2}$.

§ 3. Graphs with (Λ_1, Λ_2)-quasiconformal Gauss map

In this section M will denote the graph $\{X=(x,z)\colon x\in\Omega,\ z=u(x)\}$ of a $C^2(\Omega)$ function u, where $\Omega\subset\mathbf{R}^2$ is an arbitrary open set. x_0 will denote a fixed point of Ω, and it will be assumed that Ω contains the disc $D_R(x_0)=\{x\in\mathbf{R}^2\colon |x-x_0|<R\}$. X_0 will denote the point $(x_0, u(x_0))$ of M and ν will denote the Gauss map of M into S_+^2 defined (as in (1.9) (ii)) by setting $\nu(X)$ equal to the upward unit normal at X; that is,

$$\nu(X) \equiv \nu(x) = (1+|Du(x)|^2)^{-\frac{1}{2}}(-Du(x), 1), \quad X=(x, u(x)), x\in\Omega. \tag{3.1}$$

We already mentioned in (1.9) (ii) that $J\nu=K=\varkappa_1\varkappa_2$ and $|\delta\nu|^2=\varkappa_1^2+\varkappa_2^2$, where $\varkappa_1$, $\varkappa_2$ are the principal curvatures of M. Hence the Gauss map ν is (Λ_1, Λ_2)-quasiconformal if and only if

$$\varkappa_1^2+\varkappa_2^2 \leq \Lambda_1 K+\Lambda_2; \tag{3.2}$$

this inequality will be assumed throughout this section. The remaining notation and terminology will be as in § 1 and § 2.

In order to effectively apply Theorem (2.2) to the Gauss map, we first need to discuss appropriate choices for the constants Λ_0, Λ_3 and Λ_4.

To begin with, we have already seen in (1.5) (ii) that in case $N=S_+^2$ we can take $\Lambda_0=4$. Next we notice that, since $|\delta\nu|^2=\varkappa_1^2+\varkappa_2^2$, Lemma (2.1) with $\varphi=\nu$ gives $\int_{S_{R/2}(X_0)}(\varkappa_1^2+\varkappa_2^2)\,dA \leq c$, where c depends only on $\Lambda_1, \Lambda_2 R^2$ and Λ_3. Thus since $\varkappa_1^2+\varkappa_2^2\geq\frac{1}{2}(\varkappa_1+\varkappa_2)^2=\frac{1}{2}H^2$ we can in this case make the choice $\Lambda_4=2c$. The next lemma shows that we can choose Λ_3 to depend only on Λ_1, $\Lambda_2 R^2$.

LEMMA (3.1). *If $X_1 \in S_R(X_0)$ and $\varrho \in (0, \frac{1}{2}(R-|X_1-X_0|))$, then*

$$|S_\varrho(X_1)| \leqslant c\varrho^2,$$

where c is a constant depending only on Λ_1 and $\Lambda_2 R^2$.

Proof. We will use the well-known identities

$$\Delta\nu_l + \nu_l(\varkappa_1^2+\varkappa_2^2) = \delta_l H, \quad l=1,2,3, \tag{3.3}$$

where $H=\varkappa_1+\varkappa_2$ is the mean curvature of M and $\Delta=\sum_{i=1}^3 \delta_i\delta_i$ is the Laplace-Beltrami operator on M. We will also need the first variation formula:

$$\int_M \delta_i h\,dA = \int_M \nu_i H h\,dA, \quad i=1,2,3, \tag{3.4}$$

which is valid whenever h is a C^1 function with compact support on M. Finally, we will need to use the fact that if $\zeta \in C^2(\Omega\times\mathbf{R})$, then

$$\Delta(\zeta|M) = \sum_{i,j=1}^3 (\delta_{ij}-\nu_i\nu_j)D_{ij}\zeta + H\sum_{i=1}^3 \nu_i D_i\zeta \tag{3.5}$$

on M; one easily checks this by direct computation together with (1.2).

We now let $h\geqslant 0$ be a $C^2(M)$ function with compact support in M. Multiplying by h in (3.3), with $i=3$, and integrating by parts with the aid of (3.4), we obtain

$$\int_M \{(\varkappa_1^2+\varkappa_2^2)h + \Delta h\}\nu_3\,dA = \int_M \{\nu_3(\varkappa_1+\varkappa_2)^2 h - (\varkappa_1+\varkappa_2)\delta_3 h\}\,dA.$$

that is, since $\varkappa_1^2+\varkappa_2^2-(\varkappa_1+\varkappa_2)^2 = -2\varkappa_1\varkappa_2 = -2K$,

$$-2\int_M Kh\nu_3\,dA = \int_M (-\nu_3\Delta h - (\varkappa_1+\varkappa_2)\delta_3 h)\,dA.$$

Choosing h of the form $h(X)\equiv\zeta(x)$, $X=(x,u(x))$, $x\in\Omega$, where $\zeta\in C^2(\Omega)$ has compact support we then deduce, with the aid of (3.5) and (1.1)–(1.2),

$$2\int_M K\zeta(x)\nu_3\,dA = \int_M \nu_3\left\{\sum_{i,j=1}^2 (\delta_{ij}-\nu_i\nu_j)D_{ij}\zeta(x) + 2(\varkappa_1+\varkappa_2)\sum_{i=1}^2 \nu_j D_j\zeta(x)\right\}dA. \tag{3.6}$$

Replacing ζ by ζ^2 and using (3.2), it is easily seen that this implies

$$\begin{aligned}\int_M &(\varkappa_1^2+\varkappa_2^2)\zeta^2(x)\nu_3\,dA \\ &\leqslant \int_M 2|\Lambda_1|\left\{|D\zeta(x)|^2 + \zeta(x)\sum_{i,j=1}^2 |D_{ij}\zeta(x)| + |\varkappa_1+\varkappa_2|\,\zeta(x)\,|D\zeta(x)|\right\}\nu_3\,dA + \Lambda_2\int_M \zeta^2(x)\nu_3\,dA.\end{aligned}$$

Since we have

$$2|\Lambda_1|\,|\varkappa_1+\varkappa_2|\zeta|D\zeta| \leqslant \tfrac{1}{2}(\varkappa_1^2+\varkappa_2^2)\zeta^2+\tfrac{1}{2}(2\Lambda_1|D\zeta|)^2,$$

this gives

$$\text{(3.7)}\quad \frac{1}{2}\int_M (\varkappa_1^2+\varkappa_2^2)\,\zeta^2(x)\,\nu_3\,dA \leqslant \int_M \left\{c_1(|D\zeta(x)|^2+\zeta(x)\sum_{i,j=1}^{2}|D_{ij}\zeta(x)|)+\Lambda_2\zeta^2(x)\right\}\nu_3\,dA,$$

where c_1 depends only on Λ_1.

Now let $x^{(1)}\in\Omega$ be such that $X_1=(x^{(1)}, u(x^{(1)}))$, note that $D_{2\varrho}(x^{(1)})\subset\Omega$ and choose ζ such that

$$0\leqslant\zeta\leqslant 1 \text{ on } \Omega,\ \zeta\equiv 1 \text{ on } D_\varrho(x^{(1)}),\ \zeta\equiv 0 \text{ on } \mathbf{R}^3-D_{2\varrho}(x^{(1)}),$$

$$\sup_\Omega|D\zeta|\leqslant c_2/\varrho,\quad \sup_\Omega\sum_{i,j=1}^{2}|D_{ij}\zeta|\leqslant c_2/\varrho^2,$$

where c_2 is an absolute constant. Then, since $\Lambda_2\leqslant(\Lambda_2R^2)/\varrho^2$, (3.7) implies

$$\text{(3.8)}\qquad \int_M(\varkappa_1^2+\varkappa_2^2)\,\zeta^2(x)\,\nu_3\,dA\leqslant c_3\varrho^{-2}\int_{M\cap(D_{2\varrho}(x^{(1)})\times\mathbf{R})}\nu_3\,dA,$$

where c_3 depends only on Λ_1 and Λ_2R^2.

Next we notice that, since M is the graph of u, if f is any given continuous function on M then

$$\int_M f\,dA=\int_\Omega \tilde f(x)\sqrt{1+|Du(x)|^2}\,dx,$$

where $\tilde f$ is defined on Ω by $\tilde f(x)=f(x,u(x))$. In particular since $\sqrt{1+|Du(x)|^2}=(\nu_3(x))^{-1}$, we have

$$\text{(3.9)}\qquad \int_M f\nu_3\,dA=\int_\Omega\tilde f(x)\,dx.$$

Hence (3.8) can be written

$$\text{(3.8)}'\qquad \int_\Omega(\tilde\varkappa_1^2+\tilde\varkappa_2^2)\,\zeta^2dx\leqslant c_3\varrho^{-2}\int_{D_{2\varrho}(x^{(1)})}dx=4c_3\varrho^{-2}\pi\varrho^2=4c_3\pi$$

where

$$\tilde\varkappa_i(x)=\varkappa_i(x,u(x)),\quad x\in\Omega,\quad i=1,2.$$

Writing $\tilde H=\tilde\varkappa_1+\tilde\varkappa_2$, noting that $\tilde H^2\leqslant 2(\tilde\varkappa_1^2+\tilde\varkappa_2^2)$ and using Hölder's inequality, we then have

$$\text{(3.10)}\qquad \int_\Omega|\tilde H|\,\zeta\,dx\leqslant\left\{\int_\Omega\tilde H^2\zeta^2dx\right\}^{1/2}|D_{2\varrho}(x^{(1)})|^{1/2}\leqslant(8c_3\pi)^{1/2}(4\pi\varrho^2)^{1/2}<8\sqrt{c_3}\,\pi\varrho.$$

We now let M_- denote the region below the graph of u; that is,

$$M_- = \{X = (x, z): x \in \Omega, z < u(x)\}.$$

Also, letting $B_\sigma = \{X \in \mathbf{R}^3: |X - X_1| < \sigma\}$, we take γ to be a C^1 function on $\mathbf{R}^3$ such that

$$0 \leqslant \gamma \leqslant 1 \text{ on } \mathbf{R}^3, \gamma \equiv 1 \text{ on } B_\varrho, \gamma \equiv 0 \text{ on } \mathbf{R}^3 - B_{2\varrho}, \sup_\Omega |D\gamma| \leqslant c_2/\varrho.$$

Applying the divergence theorem on M_- we have

$$\int_M \gamma\zeta(x)\, \nu \cdot \nu\, dA = \int_{M-} \operatorname{div}(\gamma\zeta(x)\,\nu)\, dx\, dz.$$

Here we take ν to be a $C^1(\Omega \times \mathbf{R})$ function defined by

$$\nu(x, z) \equiv \nu(x) = (1 + |Du(x)|^2)^{-1/2}(-Du(x), 1), \quad x \in \Omega, z \in \mathbf{R}.$$

Hence we obtain

$$|S_\varrho(X_1)| \leqslant \left| \int_{M-} \{\gamma\zeta(x) \operatorname{div} \nu + \nu \cdot D(\gamma\zeta(x))\}\, dx\, dz \right|. \tag{3.11}$$

Finally, noting that

$$\operatorname{div} \nu(X) = \sum_{i=1}^{2} D_i \nu_i(X) = \sum_{i=1}^{2} D_i \nu_i(x) = \tilde{H}(x), \quad X = (x, z) \in \Omega \times \mathbf{R}, \tag{3.12}$$

and using (3.11) together with the fact that $|D(\gamma\zeta(x))| \leqslant 2c_2\varrho^{-1}$, we easily deduce the required area bound from (3.10).

Thus we have shown that Λ_3, Λ_4 can both be chosen to depend only on $\Lambda_1, \Lambda_2 R^2$. Hence Theorem (2.2) gives the Hölder estimate

$$\sup_{X \in S_\varrho^*(X_0)} |\nu(X) - \nu(X_0)| \leqslant c(\varrho/R)^\alpha, \quad \varrho \in (0, R), \tag{3.13}$$

where $c > 0$ and $\alpha \in (0, 1)$ depend only on $\Lambda_1, \Lambda_2 R^2$. Notice that we assert (3.13) for all $\varrho \in (0, R)$ rather than $\varrho \in (0, R/4)$ as in Theorem (2.2). We can do this because $|\nu| = 1$ (which means an inequality of the form (3.13) trivially holds for $\varrho \in (R/4, R)$).

We now wish to show that an inequality of the form (3.13) holds with $S_\varrho(X_0)$ in place of $S_\varrho^*(X_0)$; we will in fact prove that there is a constant $\theta \in (0, 1)$, depending only on $\Lambda_1, \Lambda_2 R^2$ such that $S_\varrho^*(X_0) = S_\varrho(X_0)$ for all $\varrho \leqslant \theta R$.

We first use (3.13) to deduce some facts about local non-parametric representations for M. Let

$$S = S_{\theta R}^*(X_0)$$

$$\tilde{S} = \{(\xi, \zeta): (\xi, \zeta) = (x - x_0, z - z_0)Q, \quad (x, z) \in S\}$$

where $\theta \in (0, 1)$, $z_0 = u(x_0)$ and Q is the 3×3 orthogonal matrix with rows e_1, e_2, $\nu(X_0)$, where e_1, e_2 are principal directions of M at X_0. Since M is a C^2 surface we of course know that for *small enough* θ there is a neighbourhood U of $0 \in \mathbf{R}^2$ and a $C^2(U)$ function $\tilde{u}$ with $D\tilde{u}(0) = 0$ and

$$\tilde{S} = \operatorname{graph} \tilde{u} = \{(\xi, \zeta)\colon \xi \in U, \zeta = \tilde{u}(\xi)\}. \tag{3.14}$$

Furthermore, letting

$$\tilde{\nu}(\xi) = (1 + |D\tilde{u}(\xi)|^2)^{-\frac{1}{2}}(-D\tilde{u}(\xi), 1), \quad \xi \in U, \tag{3.15}$$

we have by (3.13) that

$$|\tilde{\nu}(\xi) - \tilde{\nu}(0)| \leqslant c\theta^\alpha, \quad \xi \in U,$$

where c, α are as in (3.13). That is, by (3.15),

$$(1 + |D\tilde{u}(\xi)|^2)^{-1}|D\tilde{u}(\xi)|^2 + ((1 + |D\tilde{u}(\xi)|^2)^{-\frac{1}{2}} - 1)^2 \leqslant (c\theta^\alpha)^2, \quad \xi \in U,$$

which implies

$$|D\tilde{u}(\xi)| \leqslant (1 - (c\theta^\alpha)^2)^{-\frac{1}{2}} c\theta^\alpha < \tfrac{1}{2}, \quad \xi \in U, \tag{3.16}$$

provided θ is such that

$$c\theta^\alpha \leqslant 1/4. \tag{3.17}$$

Because of (3.16), we can infer that a representation of the form (3.14) holds for any θ satisfying (3.17).

For later reference we also note that (3.16) implies

$$D_{\theta R/2}(0) \subset U. \tag{3.18}$$

The next lemma contains the connectivity result referred to above.

LEMMA (3.2). *There is a constant $\theta \in (0, 1)$, depending only on Λ_1, $\Lambda_2 R^2$, such that $S_\varrho(X_0)$ is connected for each $\varrho \leqslant \theta R$.*

Proof. In the proof we will let $c_1, c_2 \ldots$ denote constants depending only on Λ_1, $\Lambda_2 R^2$. B_σ, for $\sigma > 0$, will denote the open ball $\{X \in \mathbf{R}^3\colon |X - X_0| < \sigma\}$.

Let $\theta \in (0, 1)$ satisfy (3.17), let $\varrho = \theta R/2$, let $\beta \in (0, \frac{1}{4})$ and define $\mathcal{S}_\beta$ to be the collection of those components of $S_{\varrho/2}(X_0)$ which intersect the ball $B_{\beta\varrho}$. For each $S \in \mathcal{S}_\beta$ we can find $X_1 \in S \cap B_{\varrho/4}$ such that

$$S \subset S^*_\varrho(X_1), \tag{3.19}$$

and hence, replacing X_0 by X_1 and R by $R/2$ in the discussion preceding the lemma, we see that S can be represented in the form (3.14), (3.16). Using such a non-parametric representation for each $S \in \mathcal{S}_\beta$ and also using the fact that no two elements of $\mathcal{S}_\beta$ can intersect, it follows that the *union* of all the components $S \in \mathcal{S}_\beta$ is contained in a region bounded between two parallel planes π_1, π_2 with

$$d(\pi_1, \pi_2) \leqslant c_1(\beta + \theta^\alpha)\varrho. \tag{3.20}$$

Here $d(\pi_1 \pi_2)$ denotes the distance between π_1 and π_2 and α is as in (3.17).

Our aim now is to show that, for suitable choices of β and θ depending only on Λ_1 and $\Lambda_2 R^2$, there is only one element (viz. $S^*_{\varrho/2}(X_0)$) in $\mathcal{S}_\beta$. Suppose that in fact there are two distinct elements $S_1, S_2 \in \mathcal{S}_\beta$. We can clearly choose S_1, S_2 to be adjacent in the sense that the volume V enclosed by S_1, S_2 and $\partial B_{\varrho/2}$ intersects no other elements $S \in \mathcal{S}_\beta$. Thus $V \cap B_{\beta\varrho}$ consists either entirely of points above the graph M or entirely of points below M; it is then evident that if the unit normal ν points out of (into) V on S_1, then it also points out of (into) V on S_2. Furthermore by (3.20) we have

$$\text{volume } (V) \leqslant c_2(\beta + \theta^\alpha)\varrho^2, \tag{3.21}$$

$$\text{area } (V \cap \partial B_{\varrho/2}) \leqslant c_3(\beta + \theta^\alpha)\varrho^2. \tag{3.22}$$

An application of the divergence theorem over V then gives

$$\int_{S_1} \nu \cdot \nu\, dA + \int_{S_2} \nu \cdot \nu\, dA = \pm \left\{ \int_V \operatorname{div} \nu\, dx\, dz - \int_{\partial B_{\varrho/2} \cap V} \eta \cdot \nu\, dA \right\},$$

where η is the outward unit normal of $\partial B_{\varrho/2}$. By (3.22) and (3.12) this gives

$$\text{area } (S_1) + \text{area } (S_2) \leqslant \int_V |\bar{H}(x)|\, dx\, dz + c_3(\beta + \theta^\alpha)\, \varrho^2. \tag{3.23}$$

Also, by (3.8)′ and (3.21),

$$\begin{aligned} \int_V |\bar{H}(x)|\, dx\, dz &\leqslant \left(\int_V \bar{H}^2(x)\, dx\, dz \right)^{1/2} \{\text{volume } (V)\}^{1/2} \\ &\leqslant \left(\int_{B_{\varrho/2}} \bar{H}^2(x)\, dx\, dz \right)^{1/2} \{c_2(\beta + \theta^\alpha)\, \varrho^3\}^{1/2} \\ &\leqslant (c_4 \varrho)^{1/2} \{c_2(\beta + \theta^\alpha)\, \varrho^3\}^{1/2} = \sqrt{c_4 c_2(\beta + \theta^\alpha)}\, \varrho^2. \end{aligned}$$

Hence (3.23 gives

$$\text{(3.24)} \qquad \text{area } (S_1) + \text{area } (S_2) \leqslant c_5\sqrt{\beta+\theta^\alpha}\varrho^2.$$

On the other hand by using a non-parametric representation as in (3.14), (3.16) we infer that

$$\text{(3.25)} \qquad \text{area } (S) \geqslant c_6\varrho^2$$

for each $S \in \mathcal{S}_\beta$, where $c_6 > 0$ is an absolute constant.

(3.24) and (3.25) are clearly contradictory if we choose β, θ small enough (but depending only on Λ_1 and $\Lambda_2 R^2$). For such a choice of β, θ we thus have

$$S_{\beta\varrho}(X_0) = M \cap B_{\beta\varrho} = S_{\varrho/2}(X_0) \cap B_{\beta\varrho} = S^*_{\varrho/2}(X_0) \cap B_{\beta\varrho}.$$

But by using a representation of the form (3.14), (3.16) for $S^*_{\varrho/2}(X_0)$, we clearly have $S^*_{\varrho/2}(X_0) \cap B_{\beta\varrho}$ connected. Thus $S_{\beta\varrho}(X_0) = S_{\beta\theta R/2}(X_0)$ is connected. The lemma follows because the choice of β, θ depended only on Λ_1, $\Lambda_2 R^2$.

Because of the above connectivity result we can replace $S^*_\varrho(X_0)$ in (3.13) by $S_\varrho(X_0)$ for $\varrho \leqslant \theta R$. However since $|\nu| = 1$, an inequality of the form (3.13) is trivial for $\varrho > \theta R$. Hence we have the result of the following theorem.

THEOREM (3.1). *For each $\varrho \in (0, R)$ we have*

$$\sup_{X \in S_\varrho(X_0)} |\nu(X) - \nu(X_0)| \leqslant c(\varrho/R)^\alpha,$$

where $c > 0$ and $\alpha \in (0, 1)$ depend only on Λ_1, $\Lambda_2 R^2$.

Remark. The above inequality implies

$$\text{(3.26)} \qquad |\nu(X) - \nu(\tilde{X})| \leqslant c'(|X - \tilde{X}|/R)^\alpha, \quad X, \tilde{X} \in S_{R/2}(X_0)$$

($c' = 4^\alpha c$). This is seen by using $\tilde{X}$ in place of X_0 and $R/4$ in place of R.

§ 4. Graphs with (Λ_1, 0)-quasiconformal Gauss map

Here the notation will be as in § 3, except that we take $\Lambda_2 = 0$ always; that is, we assume that the graph M of u has (Λ_1, 0)-quasiconformal Gauss map. It will be shown that there are a number of special results which can be established in this case.

We note that, in particular, the graph of a solution of any *homogeneous* equation of mean curvature type (i.e. an equation as in (1.9) (ii) with $b \equiv 0$) is (Λ_1, 0)-quasiconformal.

Hence the results of this section apply in particular to these equations. (See [7] for further discussion.)

Our first observation is that if $\Omega = \mathbf{R}^2$, then we can let $R \to \infty$ in (3.26) to obtain $\nu \equiv$ const.; that is, u is linear. Thus we have

THEOREM (4.1). *Suppose $\Omega = \mathbf{R}^2$ and ν is $(\Lambda_1, 0)$-quasiconformal. Then u is a linear function.*

Remark. Actually this theorem can be deduced directly from Theorem (2.1) (by letting $R \to \infty$) without first proving (3.26) (or even (3.13)). However note that Lemma (3.1) is still needed to show that Λ_3 can be chosen to depend only on Λ_1.

Before proceeding further, we want to establish an interesting integral identity (equation (4.5) below) involving the Gauss curvature K of the graph M.

Recall first that K is the area magnification factor for the Gauss map; hence since the area form for S^2_+ is $d\omega$, where $\omega(X) = (1 + x_3)^{-1}(-x_2 dx_1 + x_1 dx_2)$ (see (1.5) (ii)), we have the identity

$$KdA = d\omega^*, \quad \omega^* = \nu^{\#}\omega = (1 + \nu_3)^{-1}(-\nu_2 d\nu_1 + \nu_1 d\nu_2), \tag{4.1}$$

where dA is the area form for M. Since $\sum_{i=1}^3 \nu_i^2 = |\nu|^2 = 1$, we have

$$d\nu_3 = -\nu_3^{-1}(\nu_1 d\nu_1 + \nu_2 d\nu_2), \tag{4.2}$$

and using this in (4.1) yields the identity

$$KdA = \nu_3^{-1} d\nu_1 \wedge d\nu_2. \tag{4.3}$$

Now, by using (4.1) together with Stoke's theorem, we deduce

$$\int_M \zeta K\, dA = -\int_M d\zeta \wedge \omega^* \tag{4.4}$$

for any $\zeta \in C^1(M)$ with compact support in M. In particular, choosing ζ of the form $\zeta = \gamma(\nu_3)\zeta_1$, where γ is a $C^1(\mathbf{R})$ function and $\zeta_1 \in C^1(M)$ has compact support in M, it can be checked, by using (4.2) and (4.3), that (4.4) implies

$$\int_M \zeta_1(\gamma(\nu_3) - (1 - \nu_3)\gamma'(\nu_3))\, K\, dA = -\int_M \gamma(\nu_3)\, d\zeta_1 \wedge \omega^*,$$

which can be written

$$\int_M \zeta_1((1 - \nu_3)\gamma(\nu_3))' K\, dA = \int_M \gamma(\nu_3)\, d\zeta_1 \wedge \omega^*. \tag{4.5}$$

We will subsequently need the following inequalities for the principal curvatures $\varkappa_1, \varkappa_2$ of M:

$$(1-\nu_3^2)\min\{\varkappa_1^2, \varkappa_2^2\} \leqslant |\delta\nu_3|^2 \leqslant (1-\nu_3^2)\max\{\varkappa_1^2, \varkappa_2^2\}. \tag{4.6}$$

To prove this, first recall that the 3×3 matrix $(\delta_i \nu_j)$ is the second fundamental form for M in the sense that there are orthogonal tangent vectors (principal directions) $e^{(i)} = (e_1^{(i)}, e_2^{(i)}, e_3^{(i)})$, $i=1, 2$, such that

$$\sum_{j=1}^{3} (\delta_j \nu_k)\, e_j^{(i)} = \varkappa_i e_k^{(i)}, \quad i=1,2, \quad k=1,2,3.$$

Since $\sum_{j=1}^{3}(\delta_j \nu_k)\nu_j = 0$, $k=1, 2, 3$, we can set $k=3$ in these identities to give

$$(\delta\nu_3) = (\varkappa_1 e_3^{(1)}, \varkappa_2 e_3^{(2)}, 0)\, Q,$$

where Q is the orthogonal matrix with rows $e^{(1)}$, $e^{(2)}$, ν. Thus

$$|\delta\nu_3|^2 = \varkappa_1^2 (e_3^{(1)})^2 + \varkappa_2^2 (e_3^{(2)})^2.$$

(4.6) now easily follows by noting that $(e_3^{(1)})^2 + (e_3^{(2)})^2 = 1-\nu_3^2$, because $(e_3^{(1)}, e_3^{(2)}, \nu_3)$ is the third column of the orthogonal matrix Q.

Now we are assuming the Gauss map of M is $(\Lambda_1, 0)$ quasiconformal; that is

$$|\delta\nu|^2 = \varkappa_1^2 + \varkappa_2^2 \leqslant \Lambda_1 \varkappa_1 \varkappa_2 = \Lambda_1 K. \tag{4.7}$$

This implies

$$\max\{\varkappa_1^2, \varkappa_2^2\} \leqslant \Lambda_1^2 \min\{\varkappa_1^2, \varkappa_2^2\},$$

and hence, since $|\delta\nu|^2 = \varkappa_1^2 + \varkappa_2^2$, (4.6) implies

$$\tfrac{1}{2}(1-\nu_3^2)\Lambda_1^{-2}|\delta\nu|^2 \leqslant |\delta\nu_3|^2 \leqslant (1-\nu_3^2)|\delta\nu|^2. \tag{4.8}$$

This inequality will be needed in the proof of the following theorem, which gives an interesting Harnack inequality for the quantity $v(X)$, defined by

$$v(X) = \sqrt{1+|Du(x)|^2}, \quad X = (x, u(x)), \quad x \in \Omega.$$

(Note that $v = \nu_3^{-1}$ on M.)

THEOREM (4.2). *If ν is $(\Lambda_1, 0)$-quasiconformal, then*

$$\sup_{S_{\frac{1}{2}R}(X_0)} v \leqslant c \inf_{S_{\frac{1}{2}R}(X_0)} v,$$

where v is as defined above and c is a constant depending only on Λ_1.

Before giving the proof of this theorem we note the following corollary.

COROLLARY. *If $u \geq 0$ on the disc $D_R(x_0)$, then*

$$|Du(x_0)| \leq c_1 \exp\{c_2 u(x_0)/R\},$$

where c_1 and c_2 depend only on Λ_1.

Proof of Corollary. Let

$$G = \{x \in \bar{D}_{R/2}(x_0) = u(x) \leq u(x_0)\}$$

and let $y \in G$ be such that

$$|Du(y)| = \inf_G |Du|.$$

Now take a sequence $X_0, X_1, ..., X_N$ of points in $M \cap (G \times \mathbf{R})$ with $|X_i - X_{i-1}| \leq \frac{1}{4}R$, $i = 1, ..., N$, and with $X_N = (y, u(y))$. Clearly, repeated applications of Theorem (4.2) imply

$$\sqrt{1 + |Du(x_0)|^2} \leq c^N \sqrt{1 + |Du(y)|^2}. \tag{4.9}$$

Also, it is not difficult to see that it is possible to choose N such that

$$N \leq c_1(1 + u(x_0)/R), \tag{4.10}$$

where c_1 is an absolute constant. The required result now follows from (4.9) and (4.10), because $|Du(y)| \leq 2u(x_0)/R$. (To see this, we note that either $Du(y) = 0$, or else one can apply the mean value theorem to the function $\varphi(s) = u(x(s))$, $s \in [0, R/2]$, where $x(s)$ is the solution of the ordinary differential equation $dx(s)/ds = -Du(x(s))/|Du(x(s))|$, $s \in [0, R/2]$, with $x(0) = x_0$.)

Proof of Theorem (4.2). Since we can vary X_0, it suffices to prove the lemma with θR in place of R, where $\theta \in (0, 1)$, provided the eventual choice of θ depends only on Λ_1.

We first consider the case when $\nu_3(X) > \frac{1}{2}$ at some point of $S_{\theta R}(X_0)$. Then provided θ is small enough to ensure $c\theta^\alpha < \frac{1}{2}$, where c and α are as in Theorem (3.1), we can use Theorem (3.1) to deduce $\nu_3(X) \geq c_1 > 0$ at each point X of $S_{\theta R}(X_0)$, where c_1 depends only on Λ_1. Then, since $v = \nu_3^{-1}$, the required result is established in this case. Hence we can assume $\nu_3(X) < \frac{1}{2}$ at each point of $S_{\theta R}(X_0)$. In this case we can replace $\gamma(\nu_3)$ in (4.3) by $\gamma(\nu_3)/(1 - \nu_3)$, provided $\gamma(\nu_3)\zeta_1$ has support contained in $S_{\theta R}(X_0)$. This gives

$$\int_M \zeta_1 \gamma'(\nu_3) K \, dA = \int_M \frac{\gamma(\nu_3)}{1 - \nu_3} d\zeta_1 \wedge \omega^*. \tag{4.11}$$

Now one easily checks that

$$(4.12) \qquad |d\zeta_1 \wedge \omega^*| \leq |\delta\zeta_1| \, |\delta\nu|,$$

and, by the quasiconformal condition (4.7) we can use (4.8) to deduce

$$(4.13) \qquad \int_M \zeta_1 \gamma'(\nu_3) |\delta\nu_3|^2 dA \leq c \int_M \gamma(\nu_3) |\delta\zeta_1| |\delta\nu_3| dA$$

whenever $\zeta_1 \gamma(\nu_3)$ has support contained in $S_{\theta R}(X_0)$, where c depends only on Λ_1.

Now if we also take θ small enough (depending on Λ_1) to ensure that (3.17) and the conclusion of Lemma (3.2) both hold, then $S_{\theta R}(X_0)$ is topologically a disc, and one can easily check that (4.13) implies that ν_3 satisfies a maximum and a minimum principle on each open subset of $S_{\theta R}(X_0)$. (If, for example, $\nu_3(X_1) > \sup_{\partial U} \nu_3$ for some $X_1 \in U \subset S_{\theta R}(X_0)$, U open, then we choose γ such that $\gamma(t) \equiv 0$ for $t < \frac{1}{2}\{\nu_3(X_1) + \sup_{\partial U} \nu_3\}$, $\gamma'(t) > 0$ for $t > \frac{1}{2}\{\nu_3(X_1) + \sup_{\partial U} \nu_3\}$ (so that $\gamma(\nu_3(X_1)) > 0$) and choose $\zeta_1 \equiv 0$ on $S_{\theta R}(X_0) \sim U$ and $\zeta_1 \equiv 1$ on $\{X \in U : \nu_3(X) > \frac{1}{2}(\nu_3(X_1) + \sup_{\partial U} \nu_3)\}$. Then $\delta\zeta_1 \equiv 0$ when $\gamma(\nu_3) \neq 0$, and hence (4.13) gives

$$\int_U \gamma'(\nu_3) |\delta\nu_3|^2 dA = 0;$$

that is, $\nu_3 \equiv$ const. on each component of $\{X : \nu_3(X) > \frac{1}{2}(\nu_3(X_1) + \sup_{\partial U} \nu_3)\}$, which is clearly absurd. Similarly one proves that ν_3 satisfies a minimum principle on U.)

We now choose ζ_1 in (4.13) such that $\zeta_1 \equiv 1$ on $S_{3\theta R/4}(X_0)$, $\zeta_1 \equiv 0$ outside $S_{\theta R}(X_0)$ and $\sup_M |\delta\zeta_1| \leq 5/(\theta R)$. Also we choose $\gamma(\nu_3) \equiv \nu_3^{-1}$. Then using the Cauchy inequality, Lemma (3.1) and (4.13) we can prove

$$\int_{S_{3\theta R/4}(X_0)} |\delta w|^2 dA \leq c,$$

where $w = \log \nu_3^{-1}$ (so that $\delta w = -\nu_3^{-1} \delta\nu_3$) and where c depends only on Λ_1. Thus, again using Cauchy's inequality and Lemma (3.1), we have

$$(4.14) \qquad \int_{S_{3\theta R/4}(X_0)} |\delta w| \, dA \leq c' R,$$

with c' depending only on Λ_1.

Now let

$$\bar{w} = \sup_{S_{\theta R/2}(X_0)} w, \quad \underline{w} = \inf_{S_{\theta R/2}(X_0)} w,$$

and, for $\lambda \in (\underline{w}, \bar{w})$, define

$$E_\lambda = \{X \in S_{3\theta R/4}(X_0): w(X) > \lambda\},$$

$$C_\lambda = \{X \in S_{3\theta R/4}(X_0): w(X) = \lambda\}.$$

By the co-area formula

$$(4.15) \qquad \int_{\underline{w}}^{\bar{w}} \mathcal{H}^1(C_\lambda)\, d\lambda = \int_{E_{\underline{w}} \sim E_{\bar{w}}} |\delta w|\, dA \leqslant \int_{S_{3\theta R/4}(X_0)} |\delta w|\, dA.$$

However we note that

$$C_\lambda \cap \partial S_\varrho(X_0) \neq \varnothing$$

for each $\varrho \in (\frac{1}{2}\theta R, \frac{3}{4}\theta R)$, $\lambda \in (w, \bar{w})$. (Otherwise either E_λ or $\sim \bar{E}_\lambda$ has a component contained in $S_\varrho(X_0)$, which contradicts the maximum/minimum principle for ν_3 on open subsets of $S_{\theta R}(X_0)$.) Hence

$$(4.16) \qquad \mathcal{H}^1(C_\lambda) \geqslant \frac{\theta R}{4}, \quad \lambda \in (w, \bar{w}).$$

Combining (4.14), (4.15) and (4.16) we then have

$$\bar{w} - \underline{w} \leqslant \bar{c},$$

i.e.

$$\sup_{S_{\theta R/2}(X_0)} \nu_3 \leqslant e^{\bar{c}} \inf_{S_{\theta R/2}(X_0)} \nu_3,$$

where $\bar{c}$ depends only on Λ_1. This is the required result because $v = \nu_3^{-1}$.

We can use the Harnack inequality of Theorem (4.2) to prove the following strengthened version of (3.26)

THEOREM (4.3). *Suppose ν is $(\Lambda_1, 0)$-quasiconformal. Then*

$$|\nu(X) - \nu(\bar{X})| \leqslant c\{\inf_{S_{R/2}(X_0)} \nu_3\} \left\{\frac{|X - \bar{X}|}{R}\right\}^\alpha, \quad X, \bar{X} \in S_{R/2}(X_0),$$

where $c > 0$ and $\alpha \in (0, 1)$ depend only on Λ_1.

Proof. Supposing that $\nu_3 > \frac{1}{2}$ at some point of $S_{R/2}(X_0)$, the theorem is a trivial consequence of Theorem (4.2) and (3.26). Hence we assume that $\nu_3 \leqslant \frac{1}{2}$ at each point of $S_{R/2}(X_0)$. We can then use (4.5) with $\gamma(\nu_3) \equiv \nu_3/(1 - \nu_3)$, thus giving (by (4.12))

$$\left|\int_M \zeta_1 K\, dA\right| \leqslant c \int_M \nu_3 |\delta \zeta_1| |\delta \nu|\, dA,$$

where c depends only on Λ_1 and ζ_1 has support in $S_{R/2}(X_0)$. Then by Theorem (4.2) we obtain

$$\left|\int_M \zeta_1 K\,dA\right| \leqslant c'\{\inf_{S_{R/2}(X_0)} \nu_3\} \int_M |\delta\zeta_1||\delta\nu|\,dA, \tag{4.17}$$

where c' depends only on Λ_1. Then by an argument almost identical to that used in the proof of Lemma (2.1), we see that (4.17) implies

$$\int_{S_{R/4}(X_0)} |\delta\nu|^2 dA \leqslant c''\{\inf_{S_{R/2}(X_0)} \nu_3\}^2,$$

where c'' depends only on Λ_1. Thus in the case $\varphi = \nu$, with ν $(\Lambda_1, 0)$-quasiconformal, we see that the inequality (2.12) can be improved by the addition of the factor $\{\inf_{S_{R/2}(X_0)} \nu_3\}^2$ on the right. (Note however that we must now use $R/2$ in place of R in (2.12).) Then Theorem (2.1) gives in this case

$$\int_{S_{R/8}(X_1)} |\delta\nu|^2 dA \leqslant c\{\inf_{S_{R/2}(X_0)} \nu_3\}^2(\varrho/R)^\alpha$$

whenever $X_1 \in S_{R/8}(X_0)$ and $\varrho \in (0, R/8)$, where $c > 0$ and $\alpha \in (0, 1)$ depend only on Λ_1. Then applying Lemma (2.2) as before, we obtain an inequality of the required form.

Next we wish to point out the following global Hölder continuity result for graphs with $(\Lambda_1, 0)$-quasiconformal Gauss map.

THEOREM (4.4). *Suppose u is continuous on $\bar{\Omega}$, graph $(u|\Omega)$ has $(\Lambda_1, 0)$-quasiconformal Gauss map ν, and let φ be a Lipschitz function on $\mathbf{R}^2$ with $|D\varphi(x)| \leqslant L$, $x \in \mathbf{R}^2$. Then, if $u \equiv \varphi$ on $\partial\Omega$, we have*

$$|u(\bar{x}) - u(x)| \leqslant c\{M^{1-\alpha} + |x - \bar{x}|^{1-\alpha}\}|x - \bar{x}|^\alpha, \quad x, \bar{x} \in \bar{\Omega},$$

where $M = \sup_\Omega |u - \varphi|$ and where $c > 0$ and $\alpha \in (0, 1)$ are constants depending only on L.

Remarks. 1. Note that there is no dependence in this estimate on Ω.

2. Using the above estimate as a starting point, various local estimates for the modulus of continuity of u can be obtained near boundary points at which u is continuous. (See Theorems 3 and 4 of [8].)

Proof of Theorem (4.4). As described in § 1 of [8], it suffices to establish the gradient bound

$$\sup_{\Omega_{x_0,\varrho/2}} |D(u-\varphi)^{\varkappa+1}| \leqslant \{c_1(1+L)^{1+1/n} M\}^\varkappa, \quad \varkappa = c_2(1 + L + M/\varrho),$$

where

$$\Omega_{x_0,\sigma} = \{x \in \Omega : |x - x_0| < \sigma\}\ (x_0 \in \bar{\Omega}), \quad M = \sup_{\Omega_{x_0,}} |u - \varphi|,$$

and where c_1, c_2 depend only on Λ_1. This can be proved by a method similar to the method used in the proof of Theorem 1 of [8]. Two main modifications are necessary to adapt the proof to the present setting:

(i) In the proof of Lemmas 1 and 2 of [8] we need an inequality of the form [8], (3.12). Such an inequality can be obtained in the present setting by choosing $\gamma(\nu_3)=\nu_3^{-1}\chi(w)$ (where $w=\log \nu_3^{-1}$ and χ is non-decreasing on $(0, \infty)$) in (4.5). By (4.7) and the right-hand inequality in (4.8) this gives (since $\chi(w)$ is a decreasing function of ν_3)

$$\int_M \chi(w)\,(\nu_3^{-1}|\delta\nu|^2+(1-\nu_3)|\delta w|^2)\,\zeta_1\,dA \tag{4.18}$$
$$\leqslant -|\Lambda_1|\int_M \chi(w)\,\nu_3^{-1}d\zeta_1\wedge\omega^* \leqslant |\Lambda_1|\int_M \chi(w)\,\nu_3^{-1}|\delta\zeta_1|\,|\delta\nu|\,dA$$

by (4.12). Now for $\nu_3>\frac{1}{2}$ we have $|\delta w|^2=\nu_3^{-2}|\delta\nu_3|^2\leqslant 4|\delta\nu|^2$, while for $\nu_3<\frac{1}{2}$ we have by (4.8) that $|\delta\nu|^2\leqslant 3\Lambda_1^2|\delta\nu_3|^2$. One easily sees that then (4.18) implies

$$\int_M \chi(w)\,(\nu_3^{-1}|\delta\nu|^2+|dw|^2)\,\zeta_1\,dA\leqslant c\int_M \chi(w)\,|\delta\zeta_1|(|\delta\nu|+|\delta w|)\,dA, \tag{4.19}$$

where c depends only on Λ_1. Replacing ζ by ζ_1^2 and using Cauchy's inequality on the right, we then deduce

$$\int_M \chi(w)\,(\nu_3^{-1}|\delta\nu|^2+|\delta w|^2)\,\zeta_1^2\,dA\leqslant c'\int_M \chi(w)\,|\delta\zeta_1|^2dA, \tag{4.20}$$

which gives

$$\int_M \chi(w)\,(|\delta\nu|^2+|\delta w|^2)\,\zeta_1^2\,dA\leqslant c'\int_M \chi(w)\,|\delta\zeta_1|^2dA, \tag{4.21}$$

where c' depends only on Λ_1. This is precisely an inequality of the form [8], (3.12).

(ii) The only other essential modification required is in the proof of Lemma 2 of [8]. In this proof equation (0.1) of [8] was used. In place of this equation we can in the present setting use the mean curvature equation (3.12). It is necessary to note however the bound

$$\int_M \nu_3^{-1}|\delta\nu|^2\,\zeta_1^2\,dA\leqslant c'\int_M |\delta\zeta_1|^2dA$$

(which is true by (4.20)). Using this bound we can easily see that

$$\int_\Omega (1+|Du|^2)\,\tilde{H}^2\tilde{\zeta}_1^2\,dx\leqslant c'\int_M |\delta\zeta_1|^2dA,$$

where $\tilde{H}$ is as in (3.12) and $\tilde{\zeta}_1$ is defined by $\tilde{\zeta}_1(x)=\zeta_1(x, u(x))$, $x\in\Omega$. This is sufficient to ensure that the argument of Lemma 2 of [8] can be successfully modified (in such a way that (3.12) can be used in place of equation (0.1) of [8].).

It should be pointed out that there is an error in equality (3.3) of [8]; the correct inequality has $\sup_\Omega (u-\varphi)$ in place of Δ^* on the right. (This is obtained by making the choice $\varrho=\infty$ in (3.2).) This causes no essential change in the proof of Theorem 1 on pp. 270–271 of [8].

We have already pointed out that the above theory applies to any solution u of a homogeneous equation of mean curvature type; we wish to conclude this section with an application to the minimal surface *system* with 2 independent variables.

We suppose that $u=(u^3, \ldots, u^n)$ $(n\geq 3)$ is a C^2 solution of the minimal surface system

$$\sum_{i,j=1}^{2} b^{ij} D_{ij} u^\alpha = 0, \quad \alpha = 3, \ldots, n, \tag{4.22}$$

on $\Omega \supset D_R(0) = \{x \in \mathbf{R}^2 : |x| < R\}$, where

$$b^{ij} = \delta_{ij} - \frac{D_i u \cdot D_j u}{1+|Du|^2}, \quad i,j = 1,2. \tag{4.23}$$

Suppose also that we have an a-priori bound for the gradient of each component u^α of u, *except possibly for* u^3; thus

$$\sup_\Omega |Du^\alpha| \leq \Gamma_1, \quad \alpha = 4, \ldots, n, \tag{4.24}$$

where Γ, is some given constant.

We claim that, because of (4.24), setting $\alpha=3$ in (4.22) gives (after multiplication by a suitable constant) a homogeneous equation of mean curvature type for u^3, with λ_1 in (1.9) (ii) (a) depending only on Γ_1 (and with (1.9) (ii) (b) holding with $\lambda_2=0$). This clearly follows from the fact that

$$c_0 \sum_{i,j=1}^{2} g^{ij}\xi_i\xi_j \leq \sum_{i,j=1}^{2} b^{ij}\xi_i\xi_j \leq c_1 \sum_{i,j=1}^{2} g^{ij}\xi_i\xi_j, \quad \xi \in \mathbf{R}^2, \tag{4.25}$$

where (b^{ij}) is as in (4.23) and (g^{ij}) is given by

$$g^{ij} = \delta_{ij} - \frac{D_i u^3 D_j u^3}{1+|Du^3|^2}, \quad i,j=1,2,$$

and where c_0, c_1 are positive constants determined by Γ_1. The inequality (4.25) is proved by first noting that

$$|b^{ij} - g^{ij}| \leq c(1+|Du^3|^2)^{-1}, \quad i,j=1,2,$$

with c depending only on Γ_1, and then using the facts that (b^{ij}), (g^{ij}) are both positive definite, with (g^{ij}) having eigenvalues 1, $(1+|Du^3|^2)^{-1}$.

We thus have the following theorem.

THEOREM (4.5). *The results of Theorem* (4.2), *and its corollary, and Theorem* (4.4) *are applicable to the component u^3 of the vector solution u of* (4.22), (4.24), *with constants c, α, c_1, c_2 depending only on Γ_1.*

One can of course also prove that the graph of u^3 satisfies an estimate like that in Theorem (4.3). It then follows that each of the components u^α, $\alpha=3, \dots, n$, of the vector solution u of (4.22), (4.24) satisfies the estimate of the following theorem.

THEOREM (4.6). *Let M_α denote the graph $\{X=(x_1, x_2\, x_3)\colon x_3=u^\alpha(x_1, x_2), (x_1, x_2)\in D_R(0)\}$ and let $\nu^\alpha=1+|Du^\alpha|^2)^{-\frac{1}{2}}(-Du^\alpha, 1)$ denote the upward unit normal. Then, writing $S_{R/2}=\{X\in M_\alpha\colon |X-(0, u^\alpha(0))|<R/2\}$, we have*

$$|\nu^\alpha(X)-\nu^\alpha(\bar{X})|\leqslant c\left\{\frac{|X-\bar{X}|}{R}\right\}^\beta, \quad X, \bar{X}\in S_{R/2},$$

where $c>0$, $\beta\in(0, 1)$ depend only on Γ_1.

If (4.22), (4.24) hold over the whole of $\mathbf{R}^2$, then we can let $R\to\infty$ in the above, thus giving the following corollary.

COROLLARY. *Suppose* (4.22), (4.24) *hold over the whole of* $\mathbf{R}^2$. *Then u is linear.*

It is appropriate here to point out a result of R. Osserman [6] concerning removability of isolated singularities of solutions of (4.22). As we have done above, Osserman also considers the case when all but one component of u satisfies an *a*-priori restriction (in [6] continuity is the restriction imposed).

§ 5. Concluding Remarks

We wish to conclude this paper with some remarks about the extension of the results of § 3 and § 4 to *parametric* surfaces M. This can be partly achieved provided there is a constant $\gamma>-1$ such that the Gauss map ν of M maps into $S^2_\gamma=\{X=(x_1, x_2, x_3)\in S^2\colon x_3>\gamma\}$; that is, provided $\nu_3(X)>\gamma>-1$ for each $X\in M$. If this is assumed then the proof of the main Hölder estimate carries over in a straightforward manner, giving

$$\sup_{X\in S^*_\varrho(X_0)}|\nu(X)-\nu(X_0)|\leqslant c\{\varrho/R\}^\alpha, \tag{5.1}$$

where $c>0$ and $\alpha\in(0, 1)$ depend on γ, Λ_1, $\Lambda_2 R^2$ and $R^{-2}|S_R(X_0)|$. However no appropriate analogues of Lemmas (3.1), (3.2) are known, even if M is assumed to be simply connected. Hence $S^*_\varrho(X_0)$ cannot be replaced by $S_\varrho(X_0)$ in (5.1), and the constants c, α depend on $R^{-2}|S_R(X)|$. In case $\Lambda_2=0$ Theorem (4.3) also has an analogue for the parametric surface M. In fact one can prove, by a straightforward modification of the method of § 4, that

$$\sup_{X\in S^*_\varrho(X_0)} |\nu(X)-\nu(X_0)| \leqslant c \inf_{S^*_{R/2}(X_0)} (\nu_3-\gamma)\{\varrho/R\}^\alpha \tag{5.2}$$

for $\varrho\in(0, R/2)$. However the constants c, α again depend on $R^{-2}|S_R(X_0)|$.

In the case when the principal curvatures $\varkappa_1$, $\varkappa_2$ of the surface M satisfy a relation

$$\alpha_1(X, \nu(X))\varkappa_1+\alpha_2(X, \nu(X))\varkappa_2=\beta(X, \nu(X)) \tag{5.3}$$

at each point $X\in M$ (cf. (1.9) (ii)), where α_1, α_2, β are Hölder continuous functions on $M\times S^2$ with

$$1\leqslant \alpha_i(X, \nu)\leqslant\lambda_1, \quad i=1, 2, \ |\beta(X, \nu)|\leqslant\lambda_2, \ (X, \nu)\in M\times S^2,$$

one can easily show (by using a non-parametric representation near X_0, cf. the argument of [1]) that (5.1) implies

$$(\varkappa_1^2+\varkappa_2^2)(X_0)\leqslant c/R^2,$$

where c depends on γ, $R^{-2}|S_R(X_0)|$, λ_1 and $\lambda_2 R$. As far as the author is aware, the only other result of this type previously obtained, in case $\lambda_2\neq0$, was the result of Spruck [10] for the case $\alpha_1=\alpha_2\equiv1$, $\beta\equiv$constant. In the case $\beta\equiv0$ we can use (5.2) instead of (5.1) to obtain the stronger inequality

$$(\varkappa_1^2+\varkappa_2^2)(X_0)\leqslant c(\nu_3(X_0)-\gamma)^2/R^2.$$

Such an inequality was proved by Osserman [5] in the minimal case ($\alpha_1=\alpha_2\equiv1$, $\beta\equiv0$) and by Jenkins [1] for the case when the surface M is stationary with respect to a "constant coefficient" parametric elliptic functional (such surfaces always satisfy an equation of the form (5.3) with $\alpha_i(X, \nu)\equiv\alpha_i(\nu)$ and $\beta\equiv0$; see [1] and [7] for further details). The results in [5] and [7] are obtained with constant c *independent* of $R^{-2}|S_R(X_0)|$, unlike the inequality above. (We should mention that of course one can obtain a bound for $R^{-2}|S_R(X_0)|$ if M globally minimizes a suitable elliptic parametric functional.)

Appendix. Area bounds and a proof of the Morrey-type lemma for 2 dimensional surfaces

The first variation formula for M (cf. (3.4)) is

$$\int_M \delta\cdot f\,dA=\int_M f\cdot H\,dA,$$

valid for any C^1 vector function $f=(f_1, ..., f_n)$ with compact support in M, where H is the mean curvature vector (see [4]) of M and $\delta\cdot f=\sum_{i=1}^n \delta_i f_i$ (=divergence of f on M). We begin by replacing f by $\varphi(r)(X-X_1)h$, where φ, h are non-negative functions, where $X_1=(x_1', ..., x_n')\in S_R(X_0)$, and where $r(X)\equiv r_{X_1}(X)=|X-X_1|$. Since, by (1.1),

$$\delta\cdot X = \text{trace}\,(\tilde{g}^{ij}(X)) = 2$$

and

$$(X-X_1)\cdot\delta\varphi(r) = r^{-1}\varphi'(r)\sum_{i,j=1}^{n}(x_i-x_i')\,\tilde{g}^{ij}(X)\,(x_j-x_j') = r\varphi'(r)\,|\delta r|^2,$$

this gives

$$2\int_M \varphi(r)\,h\,dA + \int_M r\varphi'(r)\,h\,|\delta r|^2 dA = \int_M \varphi(r)\,(X-X_1)\cdot(-\delta h + Hh)\,dA. \tag{A.1}$$

Now one easily checks that this holds if φ is merely continuous and *piecewise* C^1 (rather than C^1) on $\mathbf{R}$, provided we define $\varphi'(r(X))$ is some arbitrary way (e.g. $\varphi'(r(X))=0$) for those X such that φ is not differentiable at $r(X)$. (The proof of this is easily based on the fact that the set $\{X\in M:\ r(X)=\varrho \text{ and } \delta r(X)\neq 0\}$ has zero $\mathcal{H}^2$-measure for each $\varrho\in(0, R-|X_1-X_0|)$. Hence we can replace φ by the function φ_ε, defined by $\varphi_\varepsilon(t)=1$ for $t<\varrho-\varepsilon$, $\varphi_\varepsilon(t)=0$ for $t<\varrho$, and $\varphi_\varepsilon(t)=\varepsilon^{-1}(\varrho-t)$ for $\varrho-\varepsilon\leqslant t\leqslant\varrho$. Substituting this in (A.1) and letting $\varepsilon\to 0_+$, we obtain

$$2\int_{S_\varrho} h\,dA - \varrho\frac{d}{d\varrho}\int_{S_\varrho}|\delta r|^2 dA = \int_{S_\varrho}(X-X_1)\cdot\{-\delta h + hH\}\,dA. \tag{A.2}$$

Here and subsequently $S_\varrho=S_\varrho(X_1)$ and $\varrho\in(0, R-|X_1-X_0|)$.

Noting that $H\cdot\delta=0$ (since H is normal to M), we have from Cauchy's inequality that

$$(X-X_1)\cdot H = r\left(\frac{X-X_1}{r}-\delta r\right)\cdot H \leqslant 2\left|\frac{X-X_1}{r}-\delta r\right|^2 + \tfrac{1}{8}r^2H^2$$
$$= 2(1-|\delta r|^2) + \tfrac{1}{8}r^2H^2.$$

(The work of Trudinger [11] suggests handling the term $(X-X_1)\cdot H$ in this manner.) Hence we deduce from (A.2) that

$$2\int_{S_\varrho}|\delta r|^2 h\,dA - \varrho\frac{d}{d\varrho}\int_{S_\varrho}|\delta r|^2 h\,dA \leqslant \int_{S_\varrho}(\tfrac{1}{8}r^2H^2h + r|\delta h|)\,dA.$$

This last inequality can be written

$$-\frac{d}{d\varrho}\left\{\varrho^{-2}\int_{S_\varrho}|\delta r|^2 h\,dA\right\} \leqslant \varrho^{-3}\int_{S_\varrho}(\tfrac{1}{8}r^2H^2h + r|\delta h|)\,dA,$$

and hence, integrating from σ to ϱ, we have

$$\text{(A.3)} \qquad \sigma^{-2}\int_{S_\sigma} h|\delta r|^2 dA \leqslant \varrho^{-2}\int_{S_\varrho} h|\delta r|^2 dA + \int_0^\varrho \left\{\tau^{-3}\int_{S_\tau} (\tfrac{1}{8} r^2 H^2 h + r|\delta h|)\, dA\right\} d\tau.$$

But

$$\int_0^\varrho \tau^{-3}\left(\int_{S_\tau} r^2 H^2 h\, dA\right) d\tau = \tfrac{1}{2}\int_{S_\varrho} (1 - r^2/\varrho^2) H^2 h\, dA \leqslant \tfrac{1}{2}\int_{S_\varrho} H^2 h\, dA,$$

and hence (A.3) implies

$$\text{(A.4)} \qquad \sigma^{-2}\int_{S_\sigma} h|\delta r|^2 dA \leqslant \varrho^{-2}\int_{S_\varrho} h|\delta r|^2 dA + 2^{-4}\int_{S_\varrho} H^2 h\, dA + \int_0^\varrho \tau^{-3}\left(\int_{S_\tau} r|\delta h|\, dA\right) d\tau$$

$$\leqslant \varrho^{-2}\int_{S_\varrho} h\, dA + 2^{-4}\int_{S_\varrho} H^2 h\, dA + \int_0^\varrho \tau^{-2}\left(\int_{S_\tau} |\delta h|\, dA\right) d\tau.$$

We can also see from (A.2), by again using Cauchy's inequality,

$$2\int_{S_\varrho} h\, dA - \varrho\frac{d}{d\varrho}\int_{S_\varrho} h|\delta r|^2 dA \leqslant \int_{S_\varrho} (r|H|h + r|\delta h|)\, dA$$

$$\leqslant \int_{S_\varrho} (1 + \tfrac{1}{4} r^2 H^2) h + r|\delta h|)\, dA,$$

so that

$$\int_{S_\varrho} h\, dA \leqslant \varrho\frac{d}{d\varrho}\int_{S_\varrho} h|\delta r|^2 dA + \int_{S_\varrho} (\tfrac{1}{4} r^2 H^2 h + r|\delta h|)\, dA.$$

Integrating this over $\varrho \in (\sigma/2, \sigma)$, we deduce that

$$\int_{\sigma/2}^{\sigma}\left(\int_{S_\varrho} h\, dA\right) d\varrho \leqslant \int_{\sigma/2}^{\sigma}\left(\varrho\frac{d}{d\varrho}\int_{S_\varrho} h|\delta r|^2 dA\right) d\varrho + \int_{\sigma/2}^{\sigma}\left(\int_{S_\varrho} (\tfrac{1}{4} r^2 H^2 h + r|\delta h|)\, dA\right) d\varrho$$

$$\leqslant \sigma\int_0^\sigma\left(\frac{d}{d\varrho}\int_{S_\varrho} h|\delta r|^2 dA\right) d\varrho + \frac{\sigma}{2}\int_{S_\sigma} \tfrac{1}{4} r^2 H^2 h\, dA + \sigma\int_{\sigma/2}^{\sigma}\left(\int_{S_\varrho} |\delta h|\, dA\right) d\varrho$$

$$\leqslant \sigma\int_{S_\varrho} h|\delta r|^2 dA + \frac{\sigma^3}{8}\int_{S_\sigma} H^2 dA + 4\sigma^3\int_0^\sigma\left(\varrho^{-2}\int_{S_\varrho} |\delta h|\, dA\right) d\varrho.$$

In obtaining the last term on the right here, we have used the inequality $\sigma^{-2} \leqslant 4\varrho^{-2}$ for $\varrho \in (\sigma/2, \sigma)$. Multiplication by $8\sigma^{-3}$ now yields

$$\text{(A.5)} \qquad (\sigma/2)^{-2}\int_{S_{\sigma/2}} h\, dA \leqslant 8\sigma^{-2}\int_{S_\sigma} h|\delta r|^2 dA + \int_{S_\sigma} H^2 dA + 32\int_0^\sigma\left(\varrho^{-2}\int_{S_\varrho} |\delta h|\, dA\right) d\varrho.$$

Combining this with (A.4) gives

$$\sigma^{-2}\int_{S_\sigma} h\,dA \leqslant 40\left\{\int_{S_\varrho}(\varrho^{-2}+H^2)\,h\,dA+\int_0^\varrho\left(\tau^{-2}\int_{S_\tau}|\delta h|\,dA\right)d\tau\right\} \tag{A.6}$$

for each σ, ϱ with $0<\sigma\leqslant\varrho<R-|X_1-X_0|$. Notice that (A.4) and (A.5) initially only yield (A.6) for $\sigma\leqslant\varrho/2$; however (A.6) holds trivially for $\sigma\in(\varrho/2,\varrho)$ because of the term $40\varrho^{-2}\int_{S_\varrho} h dA$ on the right.

It clearly follows from this (by setting $h\equiv 1$) that (1.12) holds, as claimed in § 1.

If we let $\sigma\to 0$ in (A.6), then we have

$$h(X_1)\leqslant\frac{40}{\pi}\left\{\int_{S_\varrho}(\varrho^{-2}+H^2)\,h\,dA+\int_0^\varrho\left(\tau^{-2}\int_{S_\tau}|\delta h|\,dA\right)d\tau\right)\Big\}. \tag{A.7}$$

Next we note that if h is of arbitrary sign and if we apply (A.7) with $\psi\circ h$ in place of h (where ψ is a non-negative C^1 function on $\mathbf{R}$), then we obtain

$$\psi(h(X_1))\leqslant\frac{40}{\pi}\left\{\int_{S_\varrho}(\varrho^{-2}+H^2)\,\psi(h)\,dA+\sup_{\mathbf{R}}|\psi'|\int_0^\varrho\left(\tau^{-2}\int_{S_\tau}|\delta h|\,dA\right)d\tau\right\}. \tag{A.8}$$

Using this inequality we can prove the Morrey-type lemma, Lemma (2.2), for the surface M. In fact, if h is as in Lemma (2.2), then (A.8) implies

$$\psi(h(X_1))\leqslant\frac{40}{\pi}\int_{S_\varrho}(\varrho^{-2}+H^2)\,\psi(h)\,dA+\frac{40}{\pi}\sup_{\mathbf{R}}|\psi'|\,K\beta^{-1}(\varrho/R)^\beta. \tag{A.9}$$

We now suppose $\varrho\in(0,R/4)$ and $X_1\in S_\varrho(X_0)$, and we define

$$\bar{h}=\sup_{S^*_\varrho(X_0)} h,\quad \underline{h}=\inf_{S^*_\varrho(X_0)} h,$$

and

$$\gamma=\tfrac{1}{2}\{40\,K\beta^{-1}(\varrho/R)^\beta\}^{-1}.$$

If $\bar{h}-\underline{h}<2\gamma^{-1}$, then Lemma (2.2) is established with $c=160$. If on the other hand $\bar{h}-\underline{h}\geqslant 2\gamma^{-1}$, then we let N be the largest integer less than $(\bar{h}-\underline{h})\gamma$. Thus we have

$$N\geqslant\tfrac{1}{2}(\bar{h}-\underline{h})\gamma, \tag{A.10}$$

and, furthermore, we can subdivide the interval $[\underline{h},\bar{h}]$ into N pairwise disjoint intervals $I_1, I_2, \ldots, I_N$, each of length $\geqslant\gamma^{-1}$. For each $j=1,\ldots,N$ we then let ψ_j be a non-negative $C^1(\mathbf{R})$ function with support contained in I_j, $\max_{\mathbf{R}}\psi_j=1$ and $\max_{\mathbf{R}}|\psi_j'|\leqslant 3\gamma$. (It is clear that such a function ψ_j exists because length $I_j \geqslant \gamma^{-1}$.) Since $S_\varrho^*(X_0)$ is connected, we know that for each $j=1, \ldots, N$ we can find a point $X^{(j)} \in S_\varrho^*(X_0)$ such that $\psi_j(h(X^{(j)}))=1$. Then, assuming $\varrho < R/4$, we can use (A.9) with $X^{(j)}$ in place of X_1 and w ith ψ_j in place of ψ, thu giving

$$1 \leqslant \frac{40}{\pi} \int_{S_\varrho(X^{(j)})} (\varrho^{-2} + H^2)\, \psi_j(h)\, dA + \pi^{-1}\gamma^{-1} 3\gamma/2$$

$$\leqslant \frac{40}{\pi} \int_{S_{2\varrho}(X_0)} (\varrho^{-2} + H^2)\, \psi_j(h)\, dA + \tfrac{1}{2};$$

that is,

$$1 \leqslant \frac{80}{\pi} \int_{S_{2\varrho}(X_0)} (\varrho^{-2} + H^2)\, \psi_j(h)\, dA.$$

Summing over $j=1, \ldots, N$, noting that $\sum_{j=1}^N \psi_j(t) \leqslant 1$ for each $t \in \mathbf{R}$, we then deduce

$$N \leqslant \frac{80}{\pi} \int_{S_{2\varrho}(X_0)} (\varrho^{-2} + H^2)\, dA \leqslant c(\Lambda_3 + \Lambda_4).$$

Lemma (2.2) now follows from (A.10).

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