2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

A Homogeneous Domination Approach for Global Output Feedback Stabilization of a Class of Nonlinear Systems

Chunjiang Qian

Abstract— **In this paper, a novel systematic design method, namely *homogeneous domination approach*, is developed for the global output feedback stabilization of nonlinear systems. The nonlinearities of the systems considered in this paper are neither linearly growing nor Lipschitz in unmeasurable states, which make the most of existing methods inapplicable to solve the problem. By utilizing the *homogeneous domination approach*, a global output feedback stabilizer is explicitly constructed in two steps: i) we first design for the nominal linear system a unique *homogeneous* output feedback controller whose construction is genuinely nonlinear, rather than linear as used in the literature; ii) then we scale the homogeneous observer and controller with an appropriate choice of gain to render the nonlinear system globally asymptotically stable. The *homogeneous domination approach* not only enables us to completely remove the linear growth condition, which has been the common assumption for global output feedback stabilization, but also provides us a new perspective to deal with the output feedback control problem for nonlinear systems.**

I. Introduction

The primary objective of this paper is to consider the problem of global output feedback stabilization of a general class of uncertain nonlinear systems described by

$$\dot{x}_i = x_{i+1} + \phi_i(t, x, u), \quad i = 1, \cdots, n \tag{1.1}$$

where $u := x_{n+1} \in \mathbb{R}$ and $y := x_1 \in \mathbb{R}$ are the system input and output respectively, and $\phi_i(t, x, u)$ is an unknown nonlinear function of all the states and control input. An important fact presently known about the global output feedback stabilization of system (1.1) is that the nonlinear functions $\phi_i(t, x, u)$ cannot grow too fast due to the finite escape time phenomenon as demonstrated in [17]. Because of this negative result, it has been assumed that it is almost impossible to use output feedback to globally stabilize systems with polynomially growing nonlinearities of unmeasurable states. Therefore, most of the existing results on global output feedback stabilization of system (1.1) are based on certain restrictive conditions imposed on the nonlinear terms $\phi_i(\cdot)$. A class of nonlinear systems called "output feedback form" in which nonlinear function is only dependent of the output was characterized and considered in [3], [13], [14], [16], [15]. Another common assumption for the global output feedback stabilization is the Lipschitz or linear condition in the unmeasurable states as discussed in the works [12], [23], [7], [2] and the references therein. For the non-Lipschitz systems, a recent result [20] developed a feedback domination method to achieve the global output feedback stabilization of (1.1) under a linear growth condition with a constant growth rate. Later in [18], [4], this condition was extended to the case when the growth rate is a polynomial of the output.

Nevertheless, a common property of all aforementioned conditions is that the unmeasurable states in the nonlinear functions $\phi_i(\cdot)$, if there is any, should be at least linearly growing. A rarely-asked but important question is *to what extent this restriction can be relaxed to achieve global stabilization using output feedback for system (1.1).* For example, is it possible to achieve global output feedback stabilization of the following system

$$\begin{aligned} \dot{x}_1 &= x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_4 + d(t)x_3 \ln(1 + x_3^2) \\ \dot{x}_4 &= u + d(t)x_2^3 + x_2^2 \sin x_4, \quad |d(t)| \le 1 \end{aligned} \tag{1.2}$$

with output $y = x_1$? Apparently, in system (1.2) the nonlinear functions of unmeasurable states satisfy neither the linear growth nor the Lipschitz condition. Therefore, all the aforementioned design methods are inapplicable to achieve global output feedback stabilization of (1.2).

In this paper, we aim to tackle this challenging question and shall provide a solution to the problem of global output feedback stabilization for nonlinear systems without requiring the linear growth restriction. To accomplish this goal, we first identify a polynomial growth condition of the nonlinearities under which the global output feedback stabilization of (1.1) is still achievable. The next important issue is that most of the existing design methods are based on the linear-like observer which is inadequate in dealing with the highly nonlinear functions of unmeasurable states. In order to overcome this obstacle, we introduce a novel recursive design method called the *homogeneous domination approach* which can be used to construct a global output feedback stabilizer for system (1.1) under the weaker growth condition.

The *homogeneous domination approach* that we employ in this paper begins with the linear domination idea proposed in [20] where a global output feedback stabilizer is constructed for uncertain system (1.1) under a linear growth condition. The underlying philosophy of the *linear feedback domination* approach is that the output feedback controller is first developed for the nominal linear system and then is utilized to dominate the uncertain nonlinear terms without knowing their precise information. To be more specific, we first design a *linear* controller and observer for the nominal system without considering the unknown nonlinear functions. Then we scale the linear output feedback controller

This work was supported in part by the U.S. NSF under grant ECS-0239105. The author is with the Department of Electrical Engineering, The University of Texas at San Antonio, San Antonio, TX 78249, USA. Email: chunjiang.qian@utsa.edu

0-7803-9098-9/05/$25.00 ©2005 AACC

using a gain which will be chosen based on the growth rate of the nonlinearities to counteract the effect of the uncertain nonlinearities. Due to this design method, the proposed output feedback controller will work for different systems satisfying the same linear growth condition. This robustness property is the most significant advantage of the domination approach over the other existing methods and ultimately enables us to handle non-triangular, non-Lipschitz and uncertain nonlinearities.

Inspired by the result [20] obtained by the *linear domination approach*, we intend to apply the same idea to systems with higher-order nonlinear terms. Apparently, the linear-based observer and controller in [20] will not have enough control authorities to dominate the higher-order terms such as those in (1.2). Therefore, a new observer/controller that ideally should be of the same order as the higher-order terms are needed. Fortunately, the homogenous system theory provides us a perfect tool to treat the traditional higher-order terms from a new point of view. This idea has been very successful in the state-feedback stabilization of a class of nonlinear systems without controllable/observerable linearization [5], [6], [11], [1], [19]. Recently, the output feedback stabilization problem of such system has also been addressed in [21], [24]. A new tool proposed in [21] is the systematic construction of a reduced-order homogenous observer. Although those results were initially developed for the high-order systems without controllable/observerable linearization, they also provide us a new point of view to deal with linear systems. In this paper, we will first extend the *adding a power integrator* [19] technique to construct for the nominal linear system of (1.1) a *nonlinear* controller which is homogenous in nature. Then, we will recursively design a reduced-order homogeneous observer which again is genuinely nonlinear even for the linear nominal system. Finally, based on the homogeneous observer/controller, we design a scaled output feedback controller which can effectively dominate the highly nonlinear terms $\phi_i(\cdot)$ by taking advantage of the homogenous structure of the controller.

The contribution of this paper is two-fold: i) we show that the linear growth condition can be removed and a weaker sufficient condition is given for the global output feedback stabilization of (1.1); and ii) we develop a novel design method called the *homogeneous domination approach* which handles the nonlinear output feedback control problem from a new perspective. The approach enables us to achieve rather general results on global output feedback stabilization of (1.1) without requiring the linear growth condition which is the once seemingly indispensable assumption. Moreover, an interesting byproduct of the paper is the unique construction of a homogenous observer and controller which will always be genuinely nonlinear even for the linear system.

II. Mathematical Preliminaries

In this section, we collect some useful definitions and lemmas which paly very important roles in this paper.

A. Homogenous Systems

The innovative idea of homogeneity was introduced for the stability analysis of a nonlinear system [8] and has led to a number of interesting results (see [9], [1], [10], [5], [11], [6]). We recall the definitions of homogeneous systems with weighted dilation (refer to [11], [9], [1] for details).

Weighted Homogeneity: For fixed coordinates $(x_1, \cdots, x_n) \in \mathbb{R}^n$ and real numbers $r_i > 0,\ i = 1, \cdots, n$,

- the dilation $\Delta_\varepsilon(x)$ is defined by $\Delta_\varepsilon(x) = (\varepsilon^{r_1}x_1, \cdots, \varepsilon^{r_n}x_n), \quad \forall \varepsilon > 0$, with r_i being called as the weights of the coordinates (For simplicity of notation, we define dilation weight $\Delta = (r_1, \cdots, r_n)$).
- a function $V \in C(\mathbb{R}^n, \mathbb{R})$ is said to be homogeneous of degree τ if there is a real number $\tau \in \mathbb{R}$ such that
$$\forall x \in \mathbb{R}^n \backslash \{0\},\ \varepsilon > 0,\ V(\Delta_\varepsilon(x)) = \varepsilon^\tau V(x_1, \cdots, x_n)$$
- a vector field $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ is said to be homogeneous of degree τ if there is a real number $\tau \in \mathbb{R}$ such that for $i = 1, \cdots, n$
$$\forall x \in \mathbb{R}^n \setminus \{0\}, \quad \varepsilon > 0, \quad f_i(\Delta_\varepsilon(x)) = \varepsilon^{\tau + r_i} f_i(x).$$
- a homogeneous p-norm is defined as $\|x\|_{\Delta,p} = \left(\sum_{i=1}^n |x_i|^{p/r_i}\right)^{1/p}, \quad \forall x \in \mathbb{R}^n$, for a constant $p \geq 1$. For the simplicity, in this paper, we choose $p = 2$ and write $\|x\|_\Delta$ for $\|x\|_{\Delta,2}$.

In what follows, we list some useful properties of homogenous function.

Lemma 2.1: Given a dilation weight $\Delta = (r_1, \cdots, r_n)$, suppose $V_1(x)$ and $V_2(x)$ are homogenous functions of degree τ_1 and τ_2, respectively. Then $V_1(x)V_2(x)$ is also homogeneous with respect to the same dilation weight Δ. Moreover, the homogeneous degree of $V_1 \cdot V_2$ is $\tau_1 + \tau_2$. ∎

Lemma 2.2: Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a homogenous function of degree τ with respect to the dilation weight Δ. Then the following holds:

1) $\dfrac{\partial V}{\partial x_i}$ is still homogeneous of degree $\tau - r_i$ with r_i being the homogeneous weight of x_i.

2) There is a constant c such that
$$V(x) \leq c\|x\|_\Delta^\tau. \tag{2.1}$$
Moreover, if $V(x)$ is positive definite,
$$\underline{c}\|x\|_\Delta^\tau \leq V(x),\ \text{for a positive constant } \underline{c} > 0. \tag{2.2}$$

B. Useful Inequalities

The next three lemmas were first introduced in [19] and will also be frequently used in this paper.

Lemma 2.3: For $x \in \mathbb{R}, y \in \mathbb{R}$, $p \geq 1$ is a constant, the following inequalities hold:
$$|x + y|^p \leq 2^{p-1}|x^p + y^p|, \tag{2.3}$$
$$(|x| + |y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}(|x| + |y|)^{\frac{1}{p}}. \tag{2.4}$$
If $p \geq 1$ is *odd*[1] then
$$|x - y|^p \leq 2^{p-1}|x^p - y^p|. \tag{2.5}$$

[1] In this paper, a real number is called “odd” if the number is an odd integer or a ratio of odd integers

Lemma 2.4: Let c, d be positive constants. Given any positive number $\gamma > 0$, the following inequality holds:

$$|x|^c|y|^d \le \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}. \qquad (2.6)$$

Lemma 2.5: Let $p \ge 1$ be an odd real number and x, y be real-valued functions. Then,

$$|x^p - y^p| \le p|x-y|(x^{p-1} + y^{p-1}). \qquad (2.7)$$

III. Homogeneous Stabilizer by State Feedback

In this section, we propose a new design method for a homogeneous state feedback stabilizer for (1.1). Unlike the controller obtained using adding a linear integrator or backstepping methods, the homogeneous controller is genuinely nonlinear even for the linear system. However, this unusual construction of the homogeneous controller is essential for the construction of the output feedback controller in this paper.

In order to obtain a homogenous state feedback controller of degree τ which globally stabilizes system (1.1), we assume the following assumption:

Assumption 3.1: There are constants $\tau \ge 0$ and $c \ge 0$ such that for $i = 1, \cdots, n$

$$|\phi_i| \le c\left[|x_1|^{i\tau+1} + |x_2|^{\frac{i\tau+1}{\tau+1}} + \cdots + |x_i|^{\frac{i\tau+1}{(i-1)\tau+1}}\right]. \qquad (3.1)$$

The following theorem shows that Assumption 3.1 guarantees a homogenous state feedback controller for (1.1).

Theorem 3.1: Under Assumption 3.1 there exists a homogeneous state-feedback controller rendering system (1.1) globally asymptotically stable.

Proof. The proof is carried out by using an inductive argument which enables one to simultaneously constructs a C^1 Lyapunov function which is positive definite and proper, as well as a homogeneous stabilizer.

For the simplicity, we assume $\tau = \frac{q}{p}$ with an even integer q and an odd integer p. We further denote

$$r_i = (i-1)\tau + 1, \quad i = 1, \cdots, n \qquad (3.2)$$

which will be always odd numbers. We will show later that similar result can be achieved when r_i is not odd.

Initial Step. Choose $V_1 = \frac{r_1}{2r_n-\tau}x_1^{(2r_n-\tau)/r_1}$. Clearly, the time derivative of V_1 along the trajectory of (1.1) is

$$\dot V_1 = x_1^{(2r_n-\tau)/r_1-1}\left[x_2 + \phi_1(t,x)\right]. \qquad (3.3)$$

By Assumption 3.1,

$$\dot V_1 \le x_1^{(2r_n-\tau)/r_1-1}\left[x_2 - x_2^* + x_2^* + x_1^{r_2/r_1}c\right].$$

Then, the virtual controller x_2^* defined by $x_2^* = -x_1^{r_2/r_1}(n+c) := -x_1^{r_2/r_1}\beta_1$, yields

$$\dot V_1(x_1) \le -nx_1^{2r_n/r_1} + x_1^{(2r_n-\tau)/r_1-1}\left[x_2 - x_2^*\right]. \qquad (3.4)$$

Inductive Step. Suppose at step $k-1$, there are a C^1 Lyapunov function $V_{k-1} : \mathbb{R}^{k-1} \to \mathbb{R}$, which is positive definite and homogeneous with respect to (3.2), and a set of C^1 virtual controllers $x_1^*, \cdots, x_k^*$, defined by

$$\begin{aligned} x_1^* &= 0, & \xi_1 &= x_1 - x_1^*, \\ x_j^* &= -\xi_{j-1}^{r_j/r_{j-1}}\beta_{j-1} & \xi_j &= x_j - x_j^*, \quad j = 2, \cdots, k \end{aligned} \qquad (3.5)$$

with constants $\beta_1 > 0, \cdots, \ \beta_{k-1} > 0$, such that

$$\begin{aligned} \dot V_{k-1} \ \le\ & -(n-k+2)\textstyle\sum_{j=1}^{k-1}\xi_j^{2r_n/r_j} \\ & +\xi_{k-1}^{(2r_n-\tau)/r_{k-1}-1}(x_k - x_k^*). \end{aligned} \qquad (3.6)$$

Obviously, (3.6) reduces to the inequality (3.4) when $k = 2$.

We claim that (3.6) also holds at step k. To prove the claim, we consider the Lyapunov function

$$V_k(x_1, \cdots, x_k) = V_{k-1} + \frac{r_k}{(2r_n-\tau)}\xi_k^{(2r_n-\tau)/r_k}. \qquad (3.7)$$

The derivative of the Lyapunov function V_k is

$$\begin{aligned} \dot V_k \ \le\ & -(n-k+2)\sum_{j=1}^{k-1}\xi_j^{2r_n/r_j} + \xi_{k-1}^{\frac{2r_n-\tau}{r_{k-1}}-1}\xi_k \\ & +\xi_k^{\frac{2r_n-\tau}{r_k}-1}\left(x_{k+1} + \phi_k(\cdot) - \sum_{l=1}^{k-1}\frac{\partial x_k^*}{\partial x_l}\dot x_l\right) \end{aligned} \qquad (3.8)$$

Next we estimate each term in the right hand side of (3.8). First, it follows from Young Inequality ($p = 2r_n - \tau - r_{k-1} = 2r_n - r_k, \ q = r_k$) that for a constant $c_k > 0$

$$\xi_{k-1}^{(2r_n-\tau)/r_{k-1}-1}\xi_k \le \frac{1}{2}\xi_{k-1}^{\frac{2r_n}{r_{k-1}}} + \xi_k^{\frac{2r_n}{r_k}}c_k. \qquad (3.9)$$

By Lemma 2.3, Assumption 3.1 can be rewritten as

$$|\phi_k| \le c\sum_{j=1}^{k}|\xi_j - \xi_{j-1}^{\frac{r_j}{r_{j-1}}}\beta_{j-1}|^{\frac{r_k+\tau}{r_j}} \le \bar c_k\sum_{j=1}^{k}|\xi_j|^{\frac{r_k+\tau}{r_j}} \qquad (3.10)$$

for a constant $\bar c_k$.

The last term in (3.8), namely $\sum_{l=1}^{k-1}\dfrac{\partial x_k^*}{\partial x_l}\dot x_l$, can be estimated as the following proposition whose proof is included in the Appendix.

Proposition 3.1: There is a constant $\tilde c_k$ such that

$$\left|\sum_{l=1}^{k-1}\frac{\partial x_k^*}{\partial x_l}\dot x_l\right| \le \tilde c_k\left(|\xi_1|^{(r_k+\tau)/r_1} + \cdots + |\xi_k|^{(r_k+\tau)/r_k}\right).$$

Hence, combining (3.10) and Proposition 3.1 yields

$$\begin{aligned} & \xi_k^{\frac{2r_n-\tau}{r_k}-1}\left(\phi_k(\cdot) - \sum_{l=1}^{k-1}\frac{\partial x_k^*}{\partial x_l}\dot x_l\right) \\ & \le \xi_k^{(2r_n-\tau)/r_k-1}(\bar c_k + \tilde c_k)\sum_{l=1}^{k}|\xi_l|^{(r_k+\tau)/r_l} \\ & \le \frac{1}{2}\sum_{l=1}^{k-1}\xi_l^{2r_n/r_l} + \xi_k^{2r_n/r_k}\hat c_k \end{aligned} \qquad (3.11)$$

where $\hat c_k$ is a constant. Note that the last inequality follows from Lemma 2.4.

Substituting (3.9) and (3.11) into (3.8) yields

$$\dot{V}_k \leq -(n-k+1)\sum_{j=1}^{k}|\xi_j|^{\frac{r_k+\tau}{r_j}} + \xi_k^{(2r_n-\tau)/r_k-1}\left(x_{k+1}+(c_k+\hat{c}_k)\xi_k^{r_{k+1}/r_k}\right).$$

Observe that a virtual controller of the form

$$x_{k+1}^* = -\xi_k^{r_{k+1}/r_k}\beta_k = -\xi_k^{r_{k+1}/r_k}\left[n-k+1+c_k+\hat{c}_k\right], \quad (3.12)$$

renders

$$\dot{V}_k \leq -(n-k+1)\sum_{j=1}^{k+1}|\xi_j|^{(r_k+\tau)/r_j} + \xi_k^{(2r_n-\tau)/r_k-1}(x_{k+1}-x_{k+1}^*).$$

This completes the inductive proof.

The inductive argument shows that (3.6) holds for $k = n+1$ with a set of virtual controllers (3.5). Hence,

$$u = x_{n+1}^* = -\xi_n^{(r_n+\tau)/r_n}\beta_n, \text{ for a constant } \beta_n > 0 \quad (3.13)$$

yields $\dot{V}_n \leq -\left(\xi_1^{2r_n/r_1}+\cdots+\xi_n^{2r_n/r_n}\right) < 0, \ \forall x \neq 0$ where $V_n(x_1,\cdots,x_n)$ is a positive definite and proper Lyapunov function of the form (3.7). As a result, (1.1)–(3.13) is *globally asymptotically stable.* ∎

Remark 3.1: In the case when τ is any nonnegative real number, we are still able to design a homogenous controller globally stabilizing the system (1.1) with necessary modification to preserve the sign of function $[\cdot]^{r_i/r_{i-1}}$. Specifically, for any real number $r_i/r_{i-1} > 0$, we define

$$[\cdot]^{r_i/r_{i-1}} = \text{sign}(\cdot)|\cdot|^{r_i/r_{i-1}}. \quad (3.14)$$

Moreover, it can be verified that the function $[\cdot]^{r_i/r_{i-1}}$ defined in (3.14) is C^1. With the help of (3.14), we are able to design the controller without requiring odd r_i/r_{i-1}.

IV. Global Stabilization of (1.1) by Homogeneous Output Feedback

In this section, we show that under Assumption 3.1, the problem of global output feedback stabilization for system (1.1) is solvable. This is accomplished by developing a novel homogeneous observer which will be combined with the homogeneous state feedback controller developed in the preceding section. To be more specific, we will first construct a homogeneous output feedback controller for the nominal linear system

$$\dot{z}_i = z_{i+1}, \quad i=1,\cdots,n-1, \quad \dot{z}_n = v, \quad y = z_1. \quad (4.1)$$

Then, based on this output feedback controller, we will develop a scaled observer and controller to render the system (1.1) globally asymptotically stable under the polynomial growth condition (3.1).

A. Homogeneous Stabilizer of Nominal Linear System

In this subsection, we design a homogeneous output feedback controller of degree τ for the nominal linear system (4.1)

Theorem 4.1: There is a homogeneous output feedback controller of degree τ rendering linear system (4.1) globally asymptotically stable.

Proof. The construction of the homogeneous output feedback controller is accomplished by three steps. First, by Theorem 3.1, a homogeneous state-feedback stabilizer is explicitly constructed. Then, we develop a novel homogeneous observer whose construction is inspired by the one used in [21], [22]. Our last step is to replace the unmeasurable states with the estimates recovered from the observer. An appropriate selection of the observer gain will render the closed-loop system globally asymptotically stable.

For the simplicity, we also assume that r_i is odd.

State Feedback Controller: For linear system (4.1), Assumption 3.1 is automatically satisfied since $\phi_i(\cdot)$ is trivial. Hence, by Theorem 3.1, there is a homogeneous (with respect to the weight (3.2)) state feedback controller globally stabilizing (4.1). Specifically, there exists

$$v^*(z) = -\beta_n\xi_n^{(r_n+\tau)/r_n} \quad (4.2)$$

where

$$\begin{aligned} z_1^* &= 0 & \xi_1 &= z_1 - z_1^*, \\ z_k^* &= -\xi_{k-1}^{r_k/r_{k-1}}\beta_{k-1} & \xi_k &= z_k - z_k^*, \quad k=2,\cdots,n \end{aligned} \quad (4.3)$$

with constants $\beta_1 > 0,\cdots,\ \beta_n > 0$, such that

$$\dot{V}_n \leq -\sum_{j=1}^{n}\xi_j^{2r_n/r_j} + \xi_n^{(2r_n-\tau)/r_n-1}(v - v^*(z)) \quad (4.4)$$

where V_n is a positive definite and proper Lyapunov function of the form $V_n(z_1,\cdots,z_n) = \sum_{i=1}^{n}\frac{r_i}{(2r_n-\tau)}\xi_i^{(2r_n-\tau)/r_i}$.

Homogeneous Observer Design: Next, we construct a homogeneous observer inspired by the one used in [21]

$$\begin{aligned} \dot{\eta}_k &= f_{n+k-1}(z_1,\eta_2,\cdots,\eta_k) \\ &= -\ell_{n+k-1}\left[\eta_k + \ell_{k-1}\hat{z}_{k-1}\right]^{r_k/r_{k-1}} \\ \hat{z}_k &= \left[\eta_k + \ell_{k-1}\hat{z}_{k-1}\right]^{r_k/r_{k-1}} \quad k=2,\cdots,n, \hat{z}_1 = z_1 \end{aligned} \quad (4.5)$$

with gains $l_i > 0, 1 \leq i \leq n-1$ to be determined later.

Based on the estimated states, we design an output feedback controller of the form

$$v(\hat{z}) = -\beta_n[\hat{z}_n + \beta_{n-1}[\hat{z}_{n-1} + \cdots + \beta_2[\hat{z}_2 + \beta_1 z_1^{r_2/r_1}]^{r_3/r_2}\cdots]^{r_n/r_{n-1}}]^{(r_n+\tau)/r_n}. \quad (4.6)$$

For $i = 2,\cdots,n$, we construct

$$U_i = \int_{(\eta_i+\ell_{i-1}z_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}}}^{z_i^{\frac{2r_n-r_i}{r_i}}} \left(s^{\frac{r_{i-1}}{2r_n-r_i}} - (\eta_i+\ell_{i-1}z_{i-1})\right)ds.$$

By construction, it can be verified that U_i is C^1. As a matter of fact, we have the following

$$\begin{aligned} \frac{\partial U_i}{\partial z_i} &= \frac{2r_n-r_i}{r_i}z_i^{\frac{2r_n-2r_i}{r_i}}\left(z_i^{\frac{r_{i-1}}{r_i}} - (\eta_i+\ell_{i-1}z_{i-1})\right) \\ \frac{\partial U_i}{\partial \eta_i} &= -\left(z_i^{\frac{2r_n-r_i}{r_i}} - (\eta_i+\ell_{i-1}z_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}}\right), \\ \frac{\partial U_i}{\partial z_{i-1}} &= -\ell_{i-1}\left(z_i^{\frac{2r_n-r_i}{r_i}} - (\eta_i+\ell_{i-1}z_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}}\right). \end{aligned}$$

Hence, the derivative of U_i along (4.1)-(4.5) is

$$\begin{aligned}\dot{U}_i &= z_{i+1}\frac{2r_n - r_i}{r_i} z_i^{\frac{2r_n-2r_i}{r_i}} \left(z_i^{\frac{r_{i-1}}{r_i}} - (\eta_i + \ell_{i-1}z_{i-1}) \right) \\ &\quad -\ell_{i-1}e_i \left(z_i^{\frac{2r_n-r_i}{r_i}} - \hat{z}_i^{\frac{2r_n-r_i}{r_i}} \right) \\ &\quad -\ell_{i-1}e_i \left(\hat{z}_i^{\frac{2r_n-r_i}{r_i}} - (\eta_i + \ell_{i-1}z_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}} \right) \quad (4.7)\end{aligned}$$

where $e_i = z_i - \hat{z}_i,\ i = 2, \cdots, n$, and $z_{n+1} = v(\hat{z})$.

In what follows, we estimate the terms in (4.7). First, by Lemma 2.3 ($|a-b|^p \le 2^{p-1}|a^p - b^p|$)

$$-\ell_{i-1}e_i \left(z_i^{\frac{2r_n-r_i}{r_i}} - \hat{z}_i^{\frac{2r_n-r_i}{r_i}} \right) \le -\ell_{i-1} e_i^{\frac{2r_n}{r_i}} 2^{\frac{2r_i-2r_n}{r_i}}. \quad (4.8)$$

The remaining terms in (4.7) can be estimated using the following propositions that are proven in the Appendix.

Proposition 4.1: For $i = 2, \cdots, n-1$

$$\begin{aligned}& z_{i+1}\frac{2r_n - r_i}{r_i} z_i^{\frac{2r_n-2r_i}{r_i}} \left(z_i^{\frac{r_{i-1}}{r_i}} - (\eta_i + \ell_{i-1}z_{i-1}) \right) \\ &\le \frac{1}{12}\left(\xi_{i-1}^{2r_n/r_{i-1}} + \xi_i^{2r_n/r_i} + \xi_{i+1}^{2r_n/r_{i+1}} \right) \\ &\quad + \alpha_i e_i^{2r_n/r_i} + g_i(\ell_{i-1}) e_{i-1}^{2r_n/r_{i-1}} \quad (4.9)\end{aligned}$$

where α_i is a constant and g_i is a continuous function of ℓ_{i-1} with $g_2(\cdot) = 0$.

Proposition 4.2: For the controller $v(\hat{z})$ obtained by substituting estimated states $\hat{z}$ into (4.2), we have

$$\begin{aligned}& v(\hat{z}) r_n \left(z_n^{\frac{r_{n-1}}{r_n}} - (\eta_n + \ell_{n-1}z_{n-1}) \right) \le \frac{1}{8}\sum_{i=1}^{n} \xi_i^{2r_n/r_i} \\ &+ \bar{\alpha}\sum_{i=2}^{n} e_i^{2r_n/r_i} + g_n(\ell_{n-1}) e_{n-1}^{2r_n/r_{n-1}},\ \text{for a constant } \bar{\alpha}. \quad (4.10)\end{aligned}$$

Proposition 4.3: For $i = 3, \cdots, n$

$$\begin{aligned}& -\ell_{i-1}e_i \left(\hat{z}_i^{\frac{2r_n-r_i}{r_i}} - (\eta_i + \ell_{i-1}z_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}} \right) \le e_i^{2r_n/r_r} \\ &\quad + \frac{1}{16}\xi_i^{2r_n/r_i} + \xi_{i-1}^{2r_n/r_{i-1}} + h_i(l_{i-1}) e_{i-1}^{2r_n/r_i} \quad (4.11)\end{aligned}$$

where $h_i(l_{i-1})$ is a continuous function.

With the help of the above propositions, it can be shown that the derivative of $U = \sum_{i=2}^{n} U_i$

$$\begin{aligned}\dot{U} &\le \frac{1}{2}\sum_{i=1}^{n} \xi_i^{\frac{2r_n}{r_i}} - (\ell_1 2^{\frac{2r_1-2r_n}{r_1}} - \alpha_2 - \bar{\alpha} - g_3(\ell_2) \\ &\quad - h_3(\ell_2)) e_2^{\frac{2r_n}{r_2}} - \sum_{i=3}^{n-1} (\ell_{i-1} 2^{\frac{2r_i-2r_n}{r_i}} - \alpha_i - 1 - \bar{\alpha} \\ &\quad - g_{i+1}(\ell_i) - h_{i+1}(\ell_i)) e_i^{2r_n/r_i} - (\ell_{n-1} - 1 - \bar{\alpha}) e_n^2. \quad (4.12)\end{aligned}$$

Determination of the Observer Gain ℓ_i: Due to the fact that states $z_2, \cdots, z_n$ are not measurable, the controller $v = v(\hat{z})$ results in a redundant term $\xi_n^{(2r_n-\tau)/r_n - 1}(v(\hat{z}) - v^*(z))$ in (4.4). To deal with this term, we have the following proposition.

Proposition 4.4: There is a constant $\tilde{\alpha} \ge 0$ such that

$$\xi_n^{\frac{2r_n-\tau}{r_n}-1}[v(\hat{z}) - v^*(z)] \le \frac{1}{4}\sum_{i=1}^{n} \xi_i^{\frac{2r_n}{r_i}} + \tilde{\alpha}\sum_{i=2}^{n} e_i^{\frac{2r_n}{r_i}}. \quad (4.13)$$

Combining (4.12), (4.4) and (4.13) together yields

$$\begin{aligned}\dot{W} &\le -\frac{1}{4}\sum_{i=1}^{n} \xi_i^{\frac{2r_n}{r_i}} - (\ell_1 2^{\frac{2r_1-2r_n}{r_1}} - \alpha_2 - \tilde{\alpha} - \bar{\alpha} - g_3(\ell_2) \\ &\quad - h_3(\ell_2)) e_2^{\frac{2r_n}{r_2}} - \sum_{i=3}^{n-1} (\ell_{i-1} 2^{\frac{2r_i-2r_n}{r_i}} - \alpha_i - 1 - \tilde{\alpha} - \bar{\alpha} \\ &\quad - g_{i+1}(\ell_i) - h_{i+1}(\ell_i)) e_i^{\frac{2r_n}{r_i}} - (\ell_{n-1} - 1 - \tilde{\alpha} - \bar{\alpha}) e_n^2 \quad (4.14)\end{aligned}$$

where the Lyapunov function $W = V_n + U$.

Clearly, by choosing

$\ell_{n-1} = \frac{1}{4} + 1 + \tilde{\alpha} + \bar{\alpha}$, and for $i = n-1, \cdots, 3$

$\ell_{i-1} = 2^{\frac{2r_n-2r_i}{r_i}} \left[\frac{5}{4} + \alpha_i + \tilde{\alpha} + \bar{\alpha} + g_{i+1}(\ell_i) + h_{i+1}(\ell_i) \right]$

$\ell_1 = 2^{\frac{2r_n-2r_1}{r_1}} \left[\frac{1}{4} + \alpha_2 + \tilde{\alpha} + \bar{\alpha} + g_3(\ell_2) + h_3(\ell_2) \right]$,

(4.14) becomes

$$\dot{W} \le -\frac{1}{4}\left(\xi_1^{2r_n/r_1} + \textstyle\sum_{j=2}^{n} (\xi_j^{2r_n/r_j} + e_j^{2r_n/r_j}) \right). \quad (4.15)$$

Note that from the construction of W, it is not difficult to verify that W is positive definite and proper with respect to

$$(z_1, \cdots, z_n, \eta_2, \cdots, \eta_n)^T =: \mathcal{Z}. \quad (4.16)$$

Similarly, from the construction of the ξ_i and e_i, the right hand side of (4.15) is negative definite. Therefore, the closed-loop system is globally asymptotically stable.

Moreover, it is straightforward to verify that the closed-loop system (4.1)-(4.5)-(4.6), which can be rewritten as the following compact form

$$\dot{\mathcal{Z}} = F(\mathcal{Z}) = (z_2, \cdots, z_n, v, f_{n+1}, \cdots, f_{2n-1})^T, \quad (4.17)$$

is homogeneous. In fact, choosing the dilation weight

$$\begin{aligned}\Delta &= (r_1, r_2, \cdots, r_{2n-1}) = (\Delta_z, \Delta_\eta) \\ \Delta_z &= (1,\ \tau+1, \cdots, (n-1)\tau + 1) \\ \Delta_\eta &= (1,\ \tau+1, \cdots, (n-2)\tau + 1) \quad (4.18)\end{aligned}$$

it can be shown that (4.17) is homogeneous of degree τ. In addition, W is homogeneous of degree $2r_n - \tau$ and the right hand side of (4.15) is homogeneous of degree $2r_n$. ∎

Remark 4.1: Note that the right hand side of (4.15) is negative definite and homogenous of degree $2r_n$. Hence, it can be concluded by Lemma 2.2 that there is a constant $c_1 > 0$ such that

$$\frac{\partial W(\mathcal{Z})}{\partial \mathcal{Z}} F(\mathcal{Z}) \le -c_1 \|\mathcal{Z}\|_\Delta^{2r_n},\ \|\mathcal{Z}\|_\Delta = \sqrt{\textstyle\sum_{i=1}^{2n-1} \|\mathcal{Z}_i\|^{2/r_i}} \quad (4.19)$$

B. Global Output Feedback Stabilization for System (1.1)

With the help of the homogeneous controller and observer established in the preceding sections, we are ready to use the homogeneous domination approach to globally stabilize nonlinear system (1.1) by its output feedback.

Theorem 4.2: Under Assumption 3.1, the problem of global output feedback stabilization of (1.1) can be solved.

Proof: Under the new coordinates $z_i = \frac{x_i}{L^{i-1}}, \quad i = 1, \cdots, n, \ v = \frac{u}{L^n}$ with $L > 1$, the system (1.1) can be rewritten as the following system

$$\begin{aligned} \dot{z}_i &= Lz_{i+1} + \frac{\phi_i(\cdot)}{L^{i-1}}, \quad i = 1, \cdots, n-1 \\ \dot{z}_n &= \frac{u}{L^{n-1}} + \frac{\phi_n(\cdot)}{L^{n-1}} = Lv + \frac{\phi_n(\cdot)}{L^{n-1}}. \end{aligned} \quad (4.20)$$

Next, we construct an observer with a scaling gain L

$$\begin{aligned} \dot{\eta}_k &= Lf_{n+k-1}(z_1, \eta_2, \cdots, \eta_k) \\ &= -L\ell_{n+k-1}\left[\eta_k + \ell_{k-1}\hat{z}_{k-1}\right]^{r_k/r_{k-1}} \\ \hat{z}_k &= \left[\eta_k + \ell_{k-1}\hat{z}_{k-1}\right]^{r_k/r_{k-1}} \quad k = 2, \cdots, n, \hat{z}_1 = z_1 \end{aligned} \quad (4.21)$$

where ℓ_i, $i = 1, \cdots, n-1$ are the gains selected by (4.14) in Theorem 4.1. In addition, we design v using the same construction of (4.6). Using the same notations (4.16) and (4.17), the closed-loop system (4.20)-(4.21)-(4.6) can be written as

$$\dot{\mathcal{Z}} = LF(\mathcal{Z}) + (\phi_1(\cdot), \frac{\phi_2(\cdot)}{L}, \cdots \frac{\phi_n(\cdot)}{L^{n-1}}, 0 \cdots, 0)^T. \quad (4.22)$$

Note that the $F(\mathcal{Z})$ in (4.22) has the exactly same structure as (4.17) due to the use of same gains ℓ_i and β_i. Hence, adopting the same Lyapunov function $W(\mathcal{Z})$ used in preceding subsection, it can be concluded from Remark 4.1 that

$$\begin{aligned} \dot{W}|_{(4.20)-(4.21)-(4.6)} &\leq -Lc_1\|\mathcal{Z}\|_\Delta^{2r_n} \\ &+ \frac{\partial W(\mathcal{Z})}{\partial \mathcal{Z}}(\phi_1(\cdot), \frac{\phi_2(\cdot)}{L}, \cdots \frac{\phi_n(\cdot)}{L^{n-1}}, 0 \cdots, 0)^T. \end{aligned} \quad (4.23)$$

Under the changes of coordinates $x_i = L^{i-1}z_i$ and $u = L^n v$, we deduce the following from Assumption 3.1

$$|\phi_i(t, x, u)| \leq c\textstyle\sum_{l=1}^{i}|L^{l-1}z_l|^{\frac{i\tau+1}{(l-1)\tau+1}}.$$

Due to the fact that $L > 1$, we can conclude that

$$\left|\frac{\phi_i(t, x, u)}{L^{i-1}}\right| \leq cL^{1-\frac{1}{(i-1)\tau+1}}\textstyle\sum_{j=1}^{i}|z_j|^{\frac{r_i+\tau}{r_j}}. \quad (4.24)$$

Recall that for $i = 1, \cdots, n-1$, $\partial W/\partial \mathcal{Z}_i$ is homogeneous of degree $2r_n - \tau - r_i$. By Lemma 2.1, we know that

$$\left|\frac{\partial W}{\partial \mathcal{Z}_i}\right|\left(|z_1|^{\frac{r_i+\tau}{r_1}} + |z_2|^{\frac{r_i+\tau}{r_2}} + \cdots + |z_i|^{\frac{r_i+\tau}{r_i}}\right) \quad (4.25)$$

is homogeneous of degree $2r_n$.

With (4.24) and (4.25) in mind, by Lemma 2.2 we can find a constant ρ_i such that

$$\frac{\partial W}{\partial \mathcal{Z}_i}\frac{\phi_i(\cdot)}{L^{i-1}} \leq \rho_i L^{1-\frac{1}{(i-1)\tau+1}}\|\mathcal{Z}\|_\Delta^{2r_n}. \quad (4.26)$$

Substituting (4.26) into (4.23) yields

$$\dot{W} \leq -L(c_1 - \textstyle\sum_{i=1}^{n}\rho_i L^{-\frac{1}{(i-1)\tau+1}})\|\mathcal{Z}\|_\Delta^{2r_n}. \quad (4.27)$$

Apparently, when L is large enough the right hand side of the (4.27) is negative definite. Consequently, the closed-loop system is globally asymptotically stable. ∎

The *homogeneous domination approach* is a systematic design method and provides us a new perspective to deal with nonlinear output feedback stabilization. With the help of Theorem 4.2, we are able to provide a systematic design method for the global output feedback stabilization of uncertain nonlinear systems with higher-order unmeasurable states, which was previously considered to be almost impossible to be globally stabilizable via output feedback. For instance, the motivating example (1.2) now can be globally stabilized via output feedback since its higher-order nonlinearities satisfy the growth condition (3.1). As a matter of fact, by choosing $\tau = 2$, it can be verified that $|d(t)x_3\ln(1 + x_3^2)| \leq c_1|x_3|^{7/5}$ and $|d(t)x_2^3 + x_2^2\sin x_4| \leq c_2(|x_2|^{9/3} + |x_4|^{9/7})$.

V. Extensions

In this section, we show that the *homogeneous domination approach* can be further extended to solve the global output feedback stabilization problem of more general nonlinear systems than those discussed in Section IV.

A. Non-Triangular Systems

In this subsection, we consider the problem of global output feedback stabilization of a more general class of nonlinear systems which are not necessarily bounded by a triangular form.

Assumption 5.1: For $i = 1, \cdots, n$, there is a constant $\mu_i > 0$ such that

$$\left|\frac{\phi_i(t, x, u)}{L^{i-1}}\right| \leq cL^{1-\mu_i}\left(\sum_{l=1}^{n}|z_l|^{\frac{i\tau+1}{(i-1)\tau+1}} + |v|^{\frac{i\tau+1}{n\tau+1}}\right), (5.1)$$

where $x_k = L^{k-1}z_k, \ u = L^n v$ and $L \geq 1$ is an arbitrary real number.

By the relation (4.24), it is obvious that Assumption 5.1 includes Assumption 3.1 as its specifical case. The following theorem is a more general result on the global output feedback stabilization of the non-triangular systems.

Theorem 5.1: Under Assumption 5.1, the problem of global output feedback stabilization of non-triangular system (1.1) can be solved by a homogeneous output feedback controller of the form (4.21)-(4.6).

Proof. We use the exactly same observer (4.21) and the controller (4.6). Although the nonlinear function is not in the triangular form, Assumption 5.1 will directly lead to relation (4.26). ∎

Example 5.1: Consider a non-triangular system

$$\begin{aligned} \dot{x}_1 &= x_2, \quad \dot{x}_2 = x_3 + x_2\ln(1 + x_4^2) \\ \dot{x}_3 &= x_4, \quad \dot{x}_4 = u + d(t)x_2^3, \quad y = x_1, \end{aligned} \quad (5.2)$$

where $d(t)$ is a bounded disturbance. Apparently, $\phi_4(\cdot)$ satisfies Assumption 3.1 which is a special case of Assumption 5.1. However, $\phi_2(x_2, x_4) = x_2\ln(1 + x_4^2)$ cannot be bounded by a lower-triangular function and therefore doesn't satisfy Assumption 3.1. Hence, Theorem 4.2 is inapplicable to (5.2). On the other hand, it can be verified that there is a constant $\bar{c}$ such that $|x_2 \ln(1 + x_4^2)| \leq \bar{c}|x_2 x_4^{2/7}|$. Then, letting $x_2 = Lz_2, x_4 = L^3 z_4$ we have

$$L^{-1}|\phi_2(x_2, x_4)| \leq \bar{c}L^{1-1/7}\left(|z_2|^{5/3} + |z_4|^{5/7}\right),$$

which implies that Assumption 5.1 holds. Therefore, by Theorem 5.1, there is an output feedback controller globally stabilizing the system (5.2).

B. Nonlinear Systems without Controllable Linearization

The *homogeneous domination approach* can also be extended to nonlinear systems without controllable/observable linearization. One interesting new result achieved in this subsection is based on a recent result on the output feedback stabilization of a chain of power integrators [24]. Particularly, we consider a class of nonlinear systems without controllable/observable linearization.

$$\begin{aligned} \dot{x}_i &= x_{i+1}^p + \phi_i(t, x, u), \quad i = 1, \cdots, n-1 \\ \dot{x}_n &= u + \phi_n(t, x, u), \quad y = x_1, \end{aligned} \tag{5.3}$$

where $p \geq 1$ is an odd real number.

In this paper, we show that the homogeneous domination approach allows us to achieve, by using a relatively simpler design procedure, the global output feedback stabilization of (5.3) under a weaker condition as follows.

Assumption 5.2: There are constants $c \geq 0$ and $\mu_i > 0$, $i = 1, \cdots, n$ such that

$$L^{-\frac{1+p+\cdots+p^{i-2}}{p^{i-1}}}|\phi_i(t, x, u)| \leq L^{1-\mu_i} c(\textstyle\sum_{l=1}^n |z_l|^p + |v|)$$

where $x_1 = z_1$, $x_i = z_i L^{(1+p+\cdots+p^{i-2})/p^{i-1}}$, $i = 2, \cdots, n$, $u = vL^{(1+p+\cdots+p^{n-1})/p^{n-1}}$ for $L \geq 1$.

Theorem 5.2: Under Assumption 5.2, there is an output feedback controller of the form

$$\begin{aligned} \dot{\hat{z}}_1 &= L\,[\hat{z}_2^p + \ell_1(x_1^p - \hat{z}_1^p)] \\ &\vdots \\ \dot{\hat{z}}_{n-1} &= L\,[\hat{z}_n^p + \ell_{n-1}\cdots\ell_1(x_1^p - \hat{z}_1^p)] \\ \dot{\hat{z}}_n &= L\,[-(b_n\hat{z}_n + \cdots + b_1\hat{z}_1)^p + \ell_n\cdots\ell_1(x_1^p - \hat{z}_1^p)] \\ u &= -L^{(1+p+\cdots+p^{n-1})/p^{n-1}}\,[b_n\hat{z}_n + \cdots + b_2\hat{z}_2 + b_1\hat{z}_1]^p \end{aligned}$$

globally stabilizing (5.3). ∎

The proof of Theorem 5.2 is very similar to the proofs used in the preceding sections and hence is omitted.

Example 5.2: Consider a high-order nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2^3 + d(t)x_1x_2x_3, \quad \dot{x}_2 = x_3^3 + x_2^3 \sin x_2 \\ \dot{x}_3 &= u, \quad y = x_1, \end{aligned} \tag{5.4}$$

where $|d(t)| \leq 1$ is a bounded disturbance. Apparently, even when $d(t)$ is known, $\phi_1(t, x_1, x_2, x_3)$ doesn't satisfy conditions in [24] due to the non-triangular structure. On the other hand, by choosing $z_1 = x_1,\ z_2 = x_2/L^{1/3},\ z_3 = x_3/L^{4/9}$, it can be easily verified that $|\phi_1(\cdot)| \leq L^{1-2/9}|z_1z_2z_3| \leq L^{1-2/9}\left(|z_1|^3 + |z_2|^3 + |z_3|^3\right)$. Similarly, $L^{-1/3}|\phi_2(x_2)| \leq L^{-1/3}|x_2^3| = L^{1-1/3}|z_2|^3$. Therefore, Assumption 5.2 holds and there exists a global output feedback stabilizer for (5.4).

APPENDIX

In this appendix, we collect some technical details of the proofs. Throughout the appendix we use a generic constant c which stands for any finite constant value and may be implicitly changed in different places. Nevertheless, the constant c is always independent of ℓ_i.

Proofs of Proposition 3.1: By definition of x_k^*, it can be shown that for $l = 1, \cdots, k-1$

$$\frac{\partial x_k^*}{\partial x_l}\dot{x}_l = c\xi_{k-1}^{\tau/r_{k-1}} \cdots \xi_l^{\tau/r_l}\left(x_{l+1} + \phi_l(\cdot)\right).$$

This, together with (3.10)

$$\left|\frac{\partial x_k^*}{\partial x_l}\dot{x}_l\right| \leq c|\xi_{k-1}^{\tau/r_{k-1}} \cdots \xi_l^{\tau/r_l}| \textstyle\sum_{j=1}^{l+1} |\xi_j|^{(r_l+\tau)/r_j}$$

By Young's Inequality and the fact that $(k-l)\tau + r_l = r_k$,

$$\left|\frac{\partial x_k^*}{\partial x_l}\dot{x}_l\right| \leq c\sum_{j=1}^{l+1}|\xi_j|^{(r_k+\tau)/r_j}, \text{for } l = 1, \cdots, k-1. \tag{A.1}$$

Clearly, Proposition 3.1 follows immediately from (A.1). ∎

Proof of Proposition 4.1: By the definition of $\hat{z}_i$, it can be shown that

$$\begin{aligned} & z_{i+1}\frac{2r_n - r_i}{r_i} z_i^{\frac{2r_n - 2r_i}{r_i}}\left(z_i^{\frac{r_{i-1}}{r_i}} - (\eta_i + \ell_{i-1}z_{i-1})\right) \\ &= cz_{i+1}z_i^{\frac{2r_n-2r_i}{r_i}}\left[z_i^{\frac{r_{i-1}}{r_i}} - [z_i - e_i]^{\frac{r_{i-1}}{r_i}} - \ell_{i-1}e_{i-1}\right] \end{aligned} \tag{A.2}$$

Note that $r_{i-1}/r_i \leq 1$. By (2.5) with $p = r_i/r_{i-1}$

$$\left|z_i^{\frac{r_{i-1}}{r_i}} - (z_i - e_i)^{\frac{r_{i-1}}{r_i}}\right| \leq 2^{1-\frac{r_{i-1}}{r_i}}|e_i|^{\frac{r_{i-1}}{r_i}}. \tag{A.3}$$

On the other hand,

$$|z_i| \leq |\xi_i| + \beta_{i-1}|\xi_{i-1}|^{r_i/r_{i-1}}, \ |z_{i+1}| \leq |\xi_{i+1}| + \beta_i|\xi_i|^{r_{i+1}/r_i}$$

With these in mind, we have

$$\begin{aligned} \text{(A.2)} \leq c & \left[|\xi_{i-1}|^{\frac{2r_n - r_{i-1}}{r_{i-1}}} + |\xi_i|^{\frac{2r_n - r_{i-1}}{r_i}} + |\xi_{i+1}|^{\frac{2r_n - r_{i-1}}{r_{i+1}}}\right] \\ & \times \left[c|e_i|^{\frac{r_{i-1}}{r_i}} + \ell_{i-1}|e_{i-1}|\right]. \end{aligned}$$

Applying Young's Inequality to each terms in the above relation will lead to (4.9). In the case when $i = 2$, $e_1 = 0$ since $\hat{z}_1 := z_1$. Hence, we can simply set $g_2 = 0$. ∎

Proof of Proposition 4.2: Similar to (A.2) and (A.3),

$$\begin{aligned} & v(\hat{z})r_n\left(z_n^{\frac{r_{n-1}}{r_n}} - (\eta_n + \ell_{n-1}z_{n-1})\right) \\ & \quad \leq c|v(\hat{z})|\left(2^{\frac{1-r_{n-1}}{r_n}}|e_n|^{\frac{r_{n-1}}{r_n}} + \ell_{n-1}|e_{n-1}|\right) \end{aligned} \tag{A.4}$$

By the homogeneity of v,

$$|v(\hat{z})| \leq c\,\|\hat{z}\|_{\Delta_z}^{r_n+\tau} \leq c\,\|z\|_{\Delta_z}^{r_n+\tau} + c\,\|e\|_{\Delta_z}^{r_n+\tau} \tag{A.5}$$

where $\|\hat{z}\|_{\Delta_z} = \left(\sum_{i=1}^n |\hat{z}_i|^{2/r_i}\right)^{1/2}$, $\quad \Delta_z = (r_1, \cdots, r_n)$.

In addition, by the definition of the homogeneous norm,

$$\|z\|_{\Delta_z} = \left(|\xi_1|^{2/r_1} + \sum_{i=2}^{n} |\xi_i - \beta_{i-1}\xi_{i-1}^{r_i/r_{i-1}}|^{2/r_i}\right)^{1/2} \leq c\left(\sum_{i=1}^{n}|\xi_i|^{2/r_i}\right)^{1/2} = c\|\xi\|_{\Delta_z}. \quad \text{(A.6)}$$

This, together with (A.5), implies that

$$|v(\hat{z})| \leq c\,\|\xi\|_{\Delta_z}^{r_n+\tau} + c\,\|e\|_{\Delta_z}^{r_n+\tau}. \quad \text{(A.7)}$$

Applying (A.7) to (A.4) yields,

$$\begin{aligned} v(\hat{z})r_n&\left(z_n^{\frac{r_{n-1}}{r_n}} - (\eta_n + \ell_{n-1}z_{n-1})\right) \\ &\leq c\left(\|\xi\|_{\Delta_z}^{r_n+\tau} + \|e\|_{\Delta_z}^{r_n+\tau}\right)\left(c|e_n|^{\frac{r_{n-1}}{r_n}} + \ell_{n-1}|e_{n-1}|\right) \\ &\leq \frac{1}{8}\sum_{i=1}^{n}\xi_i^{2r_n/r_i} + \bar{\alpha}\sum_{i=1}^{n}e_i^{2r_n/r_i} + g_n(\ell_{n-1})e_{n-1}^{2r_n/r_{n-1}} \end{aligned}$$

for a constant $\bar{\alpha} > 0$. The last relation is obtained by applying Lemma 2.4 to each individual terms in the above inequality. ∎

Proof of Proposition 4.3: By definition of $\hat{z}_i$ we have

$$-\ell_{i-1}e_i\left(\hat{z}_i^{\frac{2r_n-r_i}{r_i}} - (\eta_i + \ell_{i-1}z_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}}\right) \leq \ell_{i-1}|e_i| \times \left|(\eta_i + \ell_{i-1}\hat{z}_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}} - (\eta_i + \ell_{i-1}\hat{z}_{i-1} + \ell_{i-1}e_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}}\right|$$

By Lemma 2.5 ($p = \frac{2r_n-r_i}{r_i} > 1$), we have

$$\begin{aligned} -\ell_{i-1}e_i&\left(\hat{z}_i^{\frac{2r_n-r_i}{r_i}} - (\eta_i + \ell_{i-1}z_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}}\right) \leq c\ell_{i-1}^2|e_ie_{i-1}| \\ &\times\left|z_i^{\frac{2r_n-r_i-r_{i-1}}{r_i}} + e_i^{\frac{2r_n-r_i-r_{i-1}}{r_i}} + [\ell_{i-1}e_{i-1}]^{\frac{2r_n-r_i}{r_{i-1}}-1}\right| \\ &\leq c|e_{i-1}|\ell_{i-1}^2|e_i|\left|\xi_i^{\frac{2r_n-r_i-r_{i-1}}{r_i}} + c\xi_{i-1}^{\frac{2r_n-r_i-r_{i-1}}{r_{i-1}}}\right. \\ &\quad \left. + e_i^{\frac{2r_n-r_i-r_{i-1}}{r_i}} + [\ell_{i-1}e_{i-1}]^{\frac{2r_n-r_i}{r_{i-1}}-1}\right|. \end{aligned}$$

By using Young's inequality to each terms in above relation, one can prove

$$\begin{aligned} -\ell_{i-1}e_i\left(\hat{z}_i^{\frac{2r_n-r_i}{r_i}} - (\eta_i + \ell_{i-1}z_{i-1})^{\frac{2r_n-r_i}{r_{i-1}}}\right) &\leq e_i^{2r_n/r_r} \\ + \frac{1}{16}\left(\xi_i^{2r_n/r_i} + \xi_{i-1}^{2r_n/r_{i-1}}\right) &+ h_i(l_{i-1})e_{i-1}^{2r_n/r_i} \end{aligned}$$

for a polynomial function $h_i(\ell_{i-1})$. ∎

Proof of Proposition 4.4: Since v is C^1, we expand the function as

$$v(\hat{z}) - v^*(z) = \sum_{i=2}^{n} e_i \int_0^1 \left.\frac{\partial v(X)}{\partial X_i}\right|_{X=z-\lambda e} d\lambda. \quad \text{(A.8)}$$

By the homogeneity of $v^*(z)$ whose degree is $r_n + \tau$, $\frac{\partial v(X)}{\partial X_i}$ is homogeneous of degree $r_n+\tau-r_i$. This, together with (A.6), yields

$$\frac{\partial v(X)}{\partial X_i}|_{X=z-\lambda e} \leq c\|\xi\|_{\Delta_z}^{r_n+\tau-r_i} + c\|e\|_{\Delta_z}^{r_n+\tau-r_i}.$$

Therefore, by Young's inequality

$$\xi_n^{\frac{2r_n-\tau}{r_n}-1}\left(v(\hat{z}) - v^*(z)\right) \leq c|\xi_n^{(2r_n-\tau)/r_n-1}|\sum_{i=2}^{n}|e_i| \times\left(\|\xi\|_{\Delta_z}^{r_n+\tau-r_i} + \|e\|_{\Delta_z}^{r_n+\tau-r_i}\right) \leq \sum_{i=1}^{n}\frac{\xi_i^{\frac{2r_n}{r_i}}}{4} + \tilde{\alpha}\sum_{i=2}^{n}e_i^{\frac{2r_n}{r_i}}$$

for a constant $\tilde{\alpha} \geq 0$. ∎

REFERENCES

[1] A. Bacciotti and L. Roiser. *Liapunov Fucntions and Stability in Control Theory*, volume 267 of *Lecture Notes in Control and Information Sciences*. Springer, 2001.

[2] G. Besancon. State affine systems and obsever based control. *NOLCOS*, 2:399–404, 1998.

[3] D. Bestle and M. Zeitz. Canonical form observer design for non-linear time-variable systems. *International Journal of Control*, 38(2):419–431, August 1983.

[4] Z. Chen and J. Huang. Global output feedback stabilization for uncertain nonlinear systems with output dependent incremental rate. In *Proceeding of 2004 American Control Conference*, 2004.

[5] W. P. Dayawansa, C. F. Martin, and G. Knowles. Asymptotic stabilization of a class of smooth two-dimensional systems. *SIAM J. Control Optim.*, 28(6):1321–1349, 1990.

[6] W. P. Dayawnasa. Recent advances in the stabilization problem for low dimensional systems. *NOLCOS*, pages 1–8, 1992.

[7] J.-P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems applications to bioreactors. *IEEE Trans. Automat. Control*, 37(6):875–880, 1992.

[8] W. Hahn. *Stability of motion*. Springer-Verlag, 1967.

[9] H. Hermes. Homogeneous coordinates and continuous asymptotically stabilizing feedback controls. In *Differential equations (Colorado Springs, CO, 1989)*, volume 127 of *Lecture Notes in Pure and Appl. Math.*, pages 249–260. Dekker, New York, 1991.

[10] M. Kawski. Stabilization of nonlinear systems in the plane. *Systems Control Lett.*, 12(2):169–175, 1989.

[11] M. Kawski. Homogeneous stabilizing feedback laws. *Control Theory Adv. Tech.*, 6(4):497–516, 1990.

[12] H. K. Khalil and A. Saberi. Adaptive stabilization of a class of nonlinear systems using high-gain feedback. *IEEE Trans. Automat. Control*, 32(11):1031–1035, 1987.

[13] A. J. Krener and A. Isidori. Linearization by output injection and nonlinear observers. *Systems Control Lett.*, 3(1):47–52, 1983.

[14] A. J. Krener and W. Respondek. Nonlinear observers with linearizable error dynamics. *SIAM J. Control Optim.*, 23(2):197–216, 1985.

[15] A. J. Krener and M. Xiao. Nonlinear observer design in the Siegel domain. *SIAM J. Control Optim.*, 41(3):932–953, 2002.

[16] R. Marino and P. Tomei. Dynamic output feedback linearization and global stabilization. *Systems Control Lett.*, 17(2):115–121, 1991.

[17] F. Mazenc, L. Praly, and W. P. Dayawansa. Global stabilization by output feedback: examples and counterexamples. *Systems Control Lett.*, 23(2):119–125, 1994.

[18] L. Praly and Z. Jiang. On global output feedback stabilization of uncertain nonlinear systems. In *Proc. of the 42nd IEEE CDC*, 2003.

[19] C. Qian and W. Lin. A continuous feedback approach to global strong stabilization of nonlinear systems. *IEEE Trans. Automat. Control*, 46(7):1061–1079, 2001.

[20] C. Qian and W. Lin. Output feedback control of a class of nonlinear systems: a nonseparation principle paradigm. *IEEE Trans. Automat. Control*, 47(10):1710–1715, 2002.

[21] C. Qian and W. Lin. New results on nonsmooth output feedback stabilization of nonlinear systems. In W. D. A. Lindquist and Y. Zhou, editors, *New Directions in Control Theory and Applications*, Lecture Notes in Control and Inform. Sci. Springer, Berlin, 2004.

[22] C. Qian and W. Lin. Recursive observer design and nonsmooth output feedback stabilization of inherently nonlinear systems. In *Proc. of 43rd IEEE CDC*, Atlantis, Bahamas, pp. 4927-4932, 2004.

[23] J. Tsinias. A theorem on global stabilization of nonlinear systems by linear feedback. *Systems Control Lett.*, 17(5):357–362, 1991.

[24] B. Yang and W. Lin. Homogeneous observers, iterative design, and global stabilization of high-order nonlinear systems by smooth output feedback. *IEEE Trans. Automat. Control*, 49(7):1069– 1080, 2004.