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A Hörmander-Type Spectral Multiplier Theorem for Operators Without Heat Kernel

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Abstract. Hörmander's famous Fourier multiplier theorem ensures the L_p-boundedness of $F(-\Delta_{\mathbb{R}}D)$ whenever $F \in \mathcal{H}(s)$ for some $s > \frac{D}{2}$, where we denote by $\mathcal{H}(s)$ the set of functions satisfying the Hörmander condition for s derivatives. Spectral multiplier theorems are extensions of this result to more general operators $A \geq 0$ and yield the L_p-boundedness of $F(A)$ provided $F \in \mathcal{H}(s)$ for some s sufficiently large. The harmonic oscillator $A = -\Delta_{\mathbb{R}} + x^2$ shows that in general $s > \frac{D}{2}$ is not sufficient even if A has a heat kernel satisfying Gaussian estimates. In this paper, we prove the L_p-boundedness of $F(A)$ whenever $F \in \mathcal{H}(s)$ for some $s > \frac{D+1}{2}$, provided A satisfies generalized Gaussian estimates. This assumption allows to treat even operators A without heat kernel (e.g. operators of higher order and operators with complex or unbounded coefficients) which was impossible for all known spectral multiplier results.

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0. – Introducion

In this paper, we present a new spectral multiplier result motivated by Hörmander's famous Fourier multiplier theorem. In terms of the functional calculus $F \mapsto F(-\Delta)$ of the Laplace operator Δ on $\mathbb{R}^D$, Hörmander's theorem says the following:

$$F \in \mathcal{H}(s) \quad \text{for some} \quad s > \frac{D}{2} \Longrightarrow F(-\Delta) \in \mathfrak{L}(L_p(\mathbb{R}^D)) \quad \text{for all} \quad p \in (1, \infty)\,.$$

Here we denote by $\mathcal{H}(s)$ the set of functions satisfying the Hörmander condition for s derivatives:

$$\mathcal{H}(s) := \{F : \mathbb{R}_+ \to \mathbb{C} \text{ bounded Borel funcion; } \sup_{t>0} \|\omega F(t\cdot)\|_{H^s(\mathbb{R}_+)} < \infty\}\,,$$

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where $\omega \in \mathbb{C}_c^\infty(\mathbb{R}_+)$ is a fixed 'partition of unity' function [i.e. $\sum_{l\in\mathbb{Z}} \omega(2^l t) = 1$ for all $t \in \mathbb{R}_+$]. Christ [C] and Mauceri and Meda [MM] generalized this result to homogeneous Laplacians Δ on Lie groups G of some homogeneous dimension D, i.e. $|B(x,r)| \sim r^D$ for all $x \in G$, $r > 0$. Indeed, they obtained independently

$$F \in \mathcal{H}(s) \quad \text{for some} \quad s > \frac{D}{2} \Longrightarrow F(-\Delta) \in \mathfrak{L}(L_p(G)) \quad \text{for all} \quad p \in (1,\infty)\,.$$

In order to treat more general elliptic operators and irregular domains, Duong, Ouhabaz and Sikora [DOS] extended this result to arbitrary non-negative self-adjoint operators A on (subsets of) metric measured spaces (Ω,μ,d) of some dimension D, i.e. $|B(x,\lambda r)| \le C\lambda^D |B(x,r)|$ for all $x \in \Omega$, $r > 0$, $\lambda \ge 1$. They showed

$$\text{(H)} \quad F \in \mathcal{H}(s) \quad \text{for some} \quad s > \frac{D}{2} \Longrightarrow F(A) \in \mathfrak{L}(L_p(\Omega)) \;\; \text{for all} \;\; p \in (1,\infty)\,,$$

provided A satisfies the so-called Plancherel estimate

$$\text{(P)} \quad \| F(tA) |B(\cdot, r_t)|^{1/2} \|_{1\to 2} \le C \|F\|_{L_2([0,1])} \;\; \text{for all} \;\; F \in L_\infty([0,1]),\ t > 0$$

and A satisfies Gaussian estimates, i.e. the e^{-tA} have integral kernels $k_t(x,y)$ for which one has a pointwise upper bound of the following type:

$$\text{(GE)} \qquad |k_t(x,y)| \le |B(x,r_t)|^{-1} g\left(\frac{d(x,y)}{r_t}\right) \quad \text{for all} \quad x, y \in \Omega,\ t > 0\,.$$

Here the r_t are suitable positive radii and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a suitable decay function. Note that (GE) without any additional assumption like (P) does not imply (H) since the harmonic oscillator $A = -\Delta + x^2$ on $\Omega = \mathbb{R}$ satisfies (GE) and has the following property [T]:

$$F \in \mathcal{H}(s) \quad \text{for some} \quad s \le \frac{D}{2} + \frac{1}{6} - \varepsilon \not\Rightarrow F(A) \in \mathfrak{L}(L_p(\mathbb{R})) \quad \text{for all} \quad p \in (1,\infty)\,.$$

Furthermore, note that an elliptic operator A of order m on $\mathbb{R}^D$ with bounded measurable coefficients satisfies (GE) if $m \ge D$ [AT], [D1] or $m = 2$ and the coefficients are real [A]. On the other hand, in general A does not satisfy (GE) if $m < D$ [D3], [ACT] or the coefficients are unbounded [LSV]. But in many of these cases A still satisfies so-called generalized Gaussian estimates [D1], [ScV]. By this we mean an estimate of the following type:

$$\text{(GGE)} \qquad \|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p_o \to p_o'} \le |B(x,r_t)|^{\frac{1}{p_o'} - \frac{1}{p_o}} g\left(\frac{d(x,y)}{r_t}\right)$$
$$\text{for all} \quad x, y \in \Omega,\ t > 0$$

and for some $p_o \in [1,2)$. Notice that (GGE) for $p_o = 1$ is equivalent to (GE) [BK1]. The main result of the present paper is that (GGE) without any additional assumption implies the following adaptation of (H):

$$(\tilde{\mathrm{H}}) \quad F \in \mathcal{H}(s) \text{ for some } s > \frac{D+1}{2} \Longrightarrow F(A) \in \mathfrak{L}(L_p(\Omega)) \text{ for all } p \in (p_o, p_o').$$

We want to mention that, for the class of operators A satisfying (GGE), the interval $[p_o, p_o']$ is optimal for the existence of the semigroup $(e^{-tA})_{t\in\mathbb{R}_+}$ on L_p [D3]; this shows the optimality of our spectral multiplier theorem $(\tilde{\mathrm{H}})$.

Our main tool for the proof of $(\tilde{\mathrm{H}})$ is the singular integral theory developped in [BK2] which generalizes the classical singular integral theory based on Hörmander's well-known weak type $(1,1)$ condition for integral operators (in a weakened version due to Duong and McIntosh [DM]). This new singular integral theory based on (GGE) allows to extend other L_2-properties of A (than the boundedness of $F(A)$ for $F \in \mathcal{H}(s)$ is considered in this paper) to L_p for $p \in (p_o, p_o')$. We mention the properties of having maximal regularity [BK1], an H^∞ functional calculus [BK2] or Riesz transforms [BK3], [HM].

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1. – Main result

We begin with some basic notation and assumptions. For the rest of this paper, (Ω, μ, d) is a metric measure space of dimension D, i.e.

$$|B(x,\lambda r)| \le C\lambda^D |B(x,r)| \quad \text{for all} \quad x \in \Omega,\ r>0,\ \lambda \ge 1\,.$$

Here we denote by $B(x,r)$ the ball of center x and radius r, and by $|B(x,r)|$ or $v_r(x)$ its volume; by $L_p^\omega(\Omega)$ we denote the weak $L_p(\Omega)$-spaces. Furthermore, we fix once and for all real numbers $p_o \in [1,2)$, $m \in [2,\infty)$ and the following notation:

$$r_t := t^{1/m} \quad \text{and} \quad g(t) := Ce^{-bt^{\frac{m}{m-1}}} \quad \text{for all} \quad t \in \mathbb{R}_+\,.$$

Here C and b are positive constants whose values are of no interest and might change from one appearance of the function g to the next without mentioning it. We denote by $\mathcal{H}(s)$ the set of functions satisfying the Hörmander condition for s derivatives:

$$\mathcal{H}(s) := \{F : \mathbb{R}_+ \to \mathbb{C} \text{ bounded Borel function;} \ \sup_{t>0} \|\omega F(t\cdot)\|_{H^s(\mathbb{R}_+)} < \infty\}\,,$$

where $\omega \in C_c^\infty(\mathbb{R}_+)$ is a fixed 'partition of unity' function [i.e. $\sum_{l\in\mathbb{Z}} \omega(2^l t) = 1$ for all $t \in \mathbb{R}_+$]. Now we can present the main result of this paper.

THEOREM 1.1. *Let* (Ω, μ, d) *be a space of dimension* D *and* A *a non-negative self-adjoint operator on* $L_2(\Omega)$ *such that* (GGE) *holds. Then, for all* $s > \frac{D+1}{2}$, *there exists* $C > 0$ *such that*

$$\|F(A)\|_{\mathfrak{L}(L_{p_o}(\Omega), L^{\omega}_{p_o}(\Omega))} \le C(\|F\|_{L_\infty(\mathbb{R}_+)} + \sup_{t>0} \|\omega F(t\cdot)\|_{H^s(\mathbb{R}_+)})$$

for all $F \in \mathcal{H}(s)$. *In particular, the following implication holds*:

$$F \in \mathcal{H}(s) \quad \textit{for some} \quad s > \frac{D+1}{2} \Longrightarrow F(A) \in \mathfrak{L}(L_p(\Omega)) \quad \textit{for all} \quad p \in (p_o, p_o').$$

REMARK.

(a) An important example are the Riesz means $R_\alpha(A)$, where $R_\alpha(x) := (1-x)^\alpha_+$. Observe that $R_\alpha \in \mathcal{H}(s)$ if and only if $s < \alpha + \frac{1}{2}$. Hence, in the situation of Theorem 1.1 one has

$$\|R_\alpha(tA)\|_{p\to p} \le C_{p,\alpha} \quad \text{for all} \quad t > 0, \, p \in (p_o, p_o'), \, \alpha > \frac{D}{2}.$$

(b) Another important example are the imaginary powers $A^{i\tau}$, $\tau \in \mathbb{R}$. If we denote $P_\tau(x) := x^{i\tau}$ then $\|\omega P_\tau(t\cdot)\|_{H^s(\mathbb{R}_+)} \le C_s(1+|\tau|)^s$ for all $\tau \in \mathbb{R}$, $s, t > 0$. Hence, in the situation of Theorem 1.1 one has

$$\|A^{i\tau}\|_{p\to p} \le C_{p,s}(1+|\tau|)^s \quad \text{for all} \quad \tau \in \mathbb{R}, \; p \in (p_o, p_o'), \; s > \frac{D+1}{2}.$$

(c) Theorem 1.1 is optimal with respect to p since, for the class of operators A satisfying (GGE), the interval $[p_o, p_o']$ is optimal for the existence for the semigroup $(e^{-tA})_{t\in\mathbb{R}_+}$ on L_p [D3].

(d) Concerning optimality with respect to s (the number of derivatives), we mention that our condition $s > \frac{D+1}{2}$ cannot be replaced by $s > \frac{D}{2} + \alpha$ with $\alpha < \frac{1}{6}$. Indeed, the Riesz means $R_\alpha(A)$ of the harmonic oscillator $A = -\Delta + x^2$ on $\mathbb{R}$ do not satisfy $R_\alpha(A) \in \mathfrak{L}(L_p(\mathbb{R}))$ for all $p \in (1, \infty)$ unless $\alpha \ge \frac{1}{6}$ [T, Theorem 2.1]. On the other hand, A satisfies (GGE) for $p_o = 1$, and $R_\alpha \in \mathcal{H}(s)$ for all $s < \alpha + \frac{1}{2}$.
Under the additional assumptions (P) from above and $p_o = 1$, our condition $s > \frac{D+1}{2}$ can be replaced by $s > \frac{D}{2}$ [DOS, Theorem 3.1].

(e) By standard methods [DM], [BK2], Theorem 1.1 can be extended to the case where Ω is only a subset of a space of dimension D. This allows to treat elliptic operators A on irregular domains $\Omega \subset \mathbb{R}^D$; see Section 2.1 below.

2. – Examples

In this section, we give some examples of elliptic operators A for which Theorem 1.1 applies, i.e. for which (GGE) holds.

2.1. – Higher order operators with bounded coefficients and Dirichlet boundary conditions on irregular domains

These operators A are given by forms $\mathfrak{a} : V \times V \to \mathbb{C}$ of the type

$$\mathfrak{a}(u, v) = \int_\Omega \sum_{|\alpha|=|\beta|=k} a_{\alpha\beta} \partial^\alpha u \, \overline{\partial^\beta v} \, dx \, ,$$

where $V := \overset{\circ}{H}{}^k(\Omega)$ for some arbitrary (irregular) domain $\Omega \subset \mathbb{R}^D$. We assume $a_{\alpha\beta} = \overline{a_{\beta\alpha}} \in L_\infty(\mathbb{R}^D)$ for all α, β and Garding's inequality

$$\mathfrak{a}(u, u) \geq \delta \|\nabla^k u\|_2^2 \quad \text{for all} \quad u \in V \, ,$$

for some $\delta > 0$ and $\|\nabla^k u\|_2^2 := \sum_{|\alpha|=k} \|\partial^\alpha u\|_2^2$. Then $\mathfrak{a}$ is a closed symmetric form, and the associated operator A on $L_2(\Omega)$ is given by $u \in D(A)$ and $Au = g$ if and only if $u \in V$ and $\langle g, v\rangle = \mathfrak{a}(u, v)$ for all $v \in V$.

In this situation, we have for $p_o := \frac{2D}{m+D} \vee 1$ and $m := 2k$ [D1], [AT, Section 1.7]:

$$\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p_o \to p_o'} \leq r_t^{D\left(\frac{1}{p_o'} - \frac{1}{p_o}\right)} g\left(\frac{d(x, y)}{r_t}\right) \quad \text{for all} \quad x, y \in \Omega, t > 0 \, .$$

Hence, by Remark (e) above, the conclusion of Theorem 1.1 holds:

$$F \in \mathcal{H}(s) \quad \text{for some} \quad s > \frac{D+1}{2} \Longrightarrow F(A) \in \mathfrak{L}(L_p(\Omega)) \quad \text{for all} \quad p \in (p_o, p_o') \, .$$

2.2. – Schrödinger operators with singular potentials on $\mathbb{R}^D$

Now we study Schrödinger operators $A = -\Delta + V$ on $\mathbb{R}^D$, $D \geq 3$, where $V = V_+ - V_-$, $V_\pm \geq 0$ are locally integrable, and V_+ is bounded for simplicity (for the general case, see e.g. [ScV]). We assume the following form bound:

$$\langle V_- u, u\rangle \leq \gamma(\|\nabla u\|_2^2 + \langle V_+ u, u\rangle) + c(\gamma)\|u\|_2^2 \quad \text{for all} \quad u \in H^1(\mathbb{R}^D)$$

and some $\gamma \in (0, 1)$. Then the form sum $A := -\Delta + V = (-\Delta + V_+) - V_-$ is defined and the associated form is closed and symmetric with form domain $H^1(\mathbb{R}^D)$. By standard arguments using ellipticity and Sobolev inequality, (GGE) holds for $p_o = \frac{2D}{D+2}$ and $m = 2$ [after replacing A by $A + c(\gamma)$]. Due to [LSV], $(e^{-tA})_{t \in \mathbb{R}_+}$ is bounded on $L_q(\mathbb{R}^D)$ for all $q \in (p_\gamma, p_\gamma')$ and

$p_\gamma := \frac{2D}{D(1+\sqrt{1-\gamma})+2(1-\sqrt{1-\gamma})} < \frac{2D}{D+2}$. Hence, by interpolation, one obtains (GGE) even for all $p_o \in (p_\gamma, 2)$. Thus, Theorem 1.1 yelds

$$F \in \mathcal{H}(s) \text{ for some } s > \frac{D+1}{2} \Longrightarrow F(A) \in \mathfrak{L}(L_p(\mathbb{R}^D)) \text{ for all } p \in (p_\gamma, p_\gamma') \,.$$

2.3. – Elliptic operators on Riemannian manifolds

Let $A = -\Delta$ be the Laplacian on a Riemannian manifold Ω. Let d be the geodesic distance and u the Riemannian measure. Assume that Ω satisfies the so-called volume doubling property and that the heat kernel $k_t(x, y)$ satisfies

$$k_t(x, x) \le C|B(x, \sqrt{t})|^{-1} \quad \text{for all} \quad x \in \Omega,\ t > 0\,.$$

Then $(e^{t\Delta})_{t\in\mathbb{R}_+}$ satisfies (GE) [G] or, equivalently, (GGE) for $p_o = 1$ and $m = 2$. On Riemannian manifolds satisfying a local higher order Sobolev inequality, (GGE) holds even for suitable higher order elliptic operators A [BC].

3. – Proof of the main result

The main tool for the proof of Theorem 1.1 is the following result [BK2, Theorem 1.1] which generalizes Hörmander's well-known weak type (1, 1) condition for integral operators (in a weakened version due Duong and McIntosh [DM]) and provides a weak type (p_o, p_o) condition for arbitrary operators.

THEOREM 3.1. *Let (Ω, d, μ) be a space of homogeneous type and A a non-negative selfadjoint operator on $L_2(\Omega)$ such that* (GGE) *holds. Let $T \in \mathfrak{L}(L_2(\Omega))$ satisfy*

$$N_{p_o',r_t/2}((TD^n e^{-tA})^* \chi_{B(y,4r_t)^c} f)(y) \le C(M_2 f)(x) \tag{1}$$

for all $t > 0$, $f \in L_{p_o'}(\Omega)$, $x \in \Omega$, $y \in B(x, r_t/2)$ and some $n \in \mathbb{N}$. Then we have $T \in \mathfrak{L}(L_{p_o}(\Omega), L^\omega_{p_o}(\Omega))$.

Here we used the following notation:

$$M_p f(x) := \sup_{r>0} N_{p,r} f(x) \quad [p\text{-maximal operator}]$$

$$N_{p,r} f(x) := |B(x,r)|^{-1/p} \|f\|_{L_p(B(x,r))},\ D^n f(t) := \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(kt)\,.$$

Hence $I - D^n e^{-tA}$ can be seen as an approximation of the identity of order n since we formally have $\frac{D^n f(t)}{t^n} \to (-1)^n f^{(n)}(0)$ for $t \to 0$.

Another central tool for the prof of Theorem 1.1 is the following result on the extension of generalized Gaussian estimates for real times $t \in \mathbb{R}_+$ to complex times $z \in \mathbb{C}_+$; its proof is given in [B, Theorem 2.1].

THEOREM 3.2. *Let* (Ω, μ, d) *be a space of dimension* D *and* $1 \le p \le 2 \le q \le \infty$. *Let* A *be a non-negative selfadjoint operator on* $L_2(\Omega)$ *such that*

$$\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p\to q} \le |B(x,r_t)|^{\frac{1}{q}-\frac{1}{p}} g\left(\frac{d(x,y)}{r_t}\right)$$

for all $t \in \mathbb{R}_+$, $x, y \in \Omega$. *Then we have*

$$\|\chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)}\|_{p\to q} \le |B(x,r_z)|^{\frac{1}{q}-\frac{1}{p}} g\left(\frac{|z|}{Rez}\right)^{D\left(\frac{1}{p}-\frac{1}{q}\right)} g\left(\frac{d(x,y)}{r_z}\right)$$

for all $z \in \mathbb{C}_+$, $x, y \in \Omega$ *and* $r_z := (Rez)^{\frac{1}{m}-1}|z|$.

Some rather technical features of generalized Gaussian estimates are summarized in the following lemma; see [BK4, Proposition 2.1] and [BK2, Lemma 3.3(a)] for the proofs. We will denote by $A(x,r,k)$ the annular region $A(x,r,k) := B(x,(k+1)r)\backslash B(x,r)$.

LEMMA 3.3. *Let* (Ω, μ, d) *be a space of dimension* D *and* $1 \le p \le q \le \infty$. *Let* R *be a linear operator and* $r > 0$.

(i) *The following are equivalent*:

(a) *We have for all* $x, y \in \Omega$:

$$\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{p\to q} \le v_r(x)^{\frac{1}{q}-\frac{1}{p}} g\left(\frac{d(x,y)}{r}\right).$$

(b) *We have for all* $x, y \in \Omega$ *and* $u \in [p,q]$:

$$\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{u\to q} \le v_r(x)^{\frac{1}{q}-\frac{1}{u}} g\left(\frac{d(x,y)}{r}\right).$$

(c) *We have for all* $x \in \Omega$ *and* $k \in \mathbb{N}$:

$$\|\chi_{B(x,r)} R \chi_{A(x,r,k)}\|_{p\to q} \le v_r(x)^{\frac{1}{q}-\frac{1}{p}} g(k).$$

(ii) *If* (a) *holds then we have for all* $s > 0$, $f \in L_q(\Omega)$, $x \in \Omega$, $y \in B(x,s)$:

$$N_{q,s}(RP_{B(y,5s)^c} f)(y) \le g(r^{-1}s)(1+s^{-1}r)^{D/q} M_p f(x).$$

In order to prove the assertion $F(A) \in \mathfrak{L}(L_{p_o}(\Omega), L^{\omega}_{p_o}(\Omega))$ of Theorem 1.1 by means of our weak type (p_o, p_o) criterion Theorem 3.1, we have to check line (1) for $T = F(A)$. The main step is the following.

PROPOSITION 3.4. *Let* (Ω, d, μ) *be a space of dimension* D *and* A *a non-negative selfadjoint operator on* $L_2(\Omega)$ *such that* (GGE) *holds. Then, for all* $s > \frac{D+1}{2}$, *there exist* $\varepsilon_1, \varepsilon_2, C > 0$ *such that we have for all* $F \in L_\infty(\mathbb{R}_+)$ *and* $\sigma > 0$:

$$N_{p_o',r_t/2}(F(\sigma A)^* \chi_{B(y,4r_t)^c} f)(y) \le C\left(\left(\frac{t}{\sigma}\right)^{-\varepsilon_1} \vee \left(\frac{t}{\sigma}\right)^{-\varepsilon_2}\right) \| F \cdot \exp \|_{H^s(\mathbb{R}_+)} (M_2 f)(x)$$

for all $t > 0$, $f \in L_{p_o'}(\Omega)$, $x \in \Omega$, $y \in B(x, r_t/2)$.

PROOF. By Lemma 3.3(i), the $L_{p_o} \to L_{p_o'}$ estimate (GGE) in the hypothesis implies the following $L_2 \to L_{p_o'}$ estimate:

$$\| \chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)} \|_{2 \to p_o'} \le |B(x, r_t)|^{\frac{1}{p_o'} - \frac{1}{2}} g\left(\frac{d(x, y)}{r_t}\right) .$$

By Theorem 3.2, the latter extends to complex times $z \in \mathbb{C}_+$ as follows, denoting $r_z = (Rez)^{\frac{1}{m}-1}|z|$ and $\alpha = D(\frac{1}{2} - \frac{1}{p_o'})$:

$$\| \chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)} \|_{2 \to p_o'} \le |B(x, r_z)|^{\frac{1}{p_o'} - \frac{1}{2}} \left(\frac{|z|}{Rez}\right)^{\alpha} g\left(\frac{d(x, y)}{r_z}\right) .$$

This implies by Lemma 3.3(ii) for $R = (\frac{|z|}{Rez})^{-\alpha} e^{-zA}$:

$$N_{p_o',r_t/2}(e^{-zA} \chi_{B(y,4r_t)^c} f)(y) \le \left(\frac{|z|}{Rez}\right)^{\alpha} \left(1 + \frac{r_z}{r_t}\right)^{D/p_o'} g\left(\frac{r_t}{r_z}\right) (M_2 f)(x)$$

for all $t > 0$, $f \in L_{p_o'}(\Omega)$, $x \in \Omega$, $y \in B(x, r_t/2)$. The latter for $z = (1 + i\tau)\sigma$ allows to estimate $N_{p_o',r_t/2}(F(\sigma A)^* \chi_{B(y,4r_t)^c} f)(y)$ by using the Fourier inversion formula for $G := F \cdot \exp$ (this approach is taken from [DOS, Lemma 4.3]):

$$F(\sigma A)^* = \int_{\mathbb{R}} e^{-(1+i\tau)\sigma A}\, \overline{\widehat{G}(\tau)} d\tau .$$

Indeed, since $r_{(1+i\tau)\sigma} = \sqrt{1 + \tau^2}\sigma^{1/m}$, we can estimate as follows:

$$\begin{aligned}
&N_{p_o',r_t/2}(F(\sigma A)^* \chi_{B(y,4r_t)^c} f)(y) \\
&\quad\le \int_{\mathbb{R}} N_{p_o',r_t/2}(e^{-(1+i\tau)\sigma A} \chi_{B(y,4r_t)^c} f)(y) |\widehat{G}(\tau)| d\tau \\
&\quad\le \int_{\mathbb{R}} \sqrt{1+\tau^2}^{\alpha} \left(1 + \sqrt{1+\tau^2}\, \frac{\sigma^{1/m}}{t^{1/m}}\right)^{D/p_o'} g\left(\sqrt{1+\tau^2}^{-1} \frac{t^{1/m}}{\sigma^{1/m}}\right) |\widehat{G}(\tau)| d\tau\, M_2 f(x) \\
&\quad\le \left(1 + \frac{\sigma}{t}\right)^{D/p_o' m} \int_{\mathbb{R}} \sqrt{1+\tau^2}^{D/2} g\left(\sqrt{1+\tau^2}^{-1} \frac{t^{1/m}}{\sigma^{1/m}}\right) |\widehat{G}(\tau)| d\tau\, M_2 f(x) \\
&\quad\le \left(1 + \frac{\sigma}{t}\right)^{D/p_o' m} \left(\int_{\mathbb{R}} (1+\tau^2)^{\frac{D}{2} - s} g\left(\sqrt{1+\tau^2}^{-1} \frac{t^{1/m}}{\sigma^{1/m}}\right)^2 d\tau\right)^{1/2} \|G\|_{H^s} M_2 f(x).
\end{aligned}$$

Hence the assertion is proved once we show for $\beta := s - \frac{D}{2} > \frac{1}{2}$:

$$\int_0^\infty (1+\tau^2)^{-\beta} g(\sqrt{1+\tau^2}^{-1} a)\, d\tau \le C a^{1-2\beta} \quad \text{for all} \quad a \ge 2\,.$$

First, the change of variables $u = \sqrt{1+\tau^2}^{-1}$ yields

$$\int_0^\infty (1+\tau^2)^{-\beta} g(\sqrt{1+\tau^2}^{-1} a)\, d\tau = a^{2(1-\beta)} \int_{a^{-1}}^\infty g(u^{-1}) u^{1-2\beta} (a^2u^2-1)^{-1/2} du\,.$$

Since $(a^2u^2-1)^{1/2} \ge \frac{\sqrt{3}}{2} au$ for all $u \in [2a^{-1}, \infty)$, we have

$$\int_{2a^{-1}}^\infty g(u^{-1}) u^{1-2\beta} (a^2u^2-1)^{-1/2} du \le \frac{2}{\sqrt{3}} a^{-1} \int_0^\infty g(u^{-1}) u^{-2\beta} du = C a^{-1}\,.$$

On the other hand, the remaining part of the integral can be estimated by

$$\begin{aligned}\int_{a^{-1}}^{2a^{-1}} g(u^{-1}) u^{1-2\beta} (a^2u^2-1)^{-1/2} du &\le g(a/2) a^{2\beta-1} \int_{a^{-1}}^{2a^{-1}} (au-1)^{-1/2} du \\ &= g(a/2) a^{2(\beta-1)} \int_0^1 v^{-1/2} dv\,. \qquad \square\end{aligned}$$

The last preparatory step for the proof of Theorem 1.1 is the following lemma.

LEMMA 3.5. *Let* $n \in \mathbb{N}$, $\varepsilon > 0$ *and* $E(u) := \sum_{k=0}^n \binom{n}{k} (-1)^k e^{-ku}$, $u \in \mathbb{R}_+$. *Then*

$$\|E(\sigma\cdot)\|_{C^n([\varepsilon,\varepsilon^{-1}])} \le C(1 \wedge \sigma^n) \quad \textit{for all} \quad \sigma > 0\,.$$

PROOF. Fix $m \in \{0, \dots, n\}$. First we treat the case of small σ. Since $\frac{E^{(m)}(t)}{t^{n-m}} \to \frac{n!}{(n-m)!}$ for $t \to 0$, we have $|E^{(m)}(t)| \le C_0 t^{n-m}$ for all $t \in [0,1]$. This implies for all $\sigma \in [0, \varepsilon]$ and $u \in [0, \varepsilon^{-1}]$:

$$|E(\sigma\cdot)^{(m)}(u)| = \sigma^m |E^{(m)}(\sigma u)| \le \sigma^m C_0 (\sigma u)^{n-m} \le C_0 \varepsilon^{n-m} \sigma^n\,.$$

Now we treat the case of large σ. Since $E^{(m)}(t) = \sum_{k=0}^n c_{k,m,n} e^{-kt}$ with $c_{0,m,n} = 0$ for $m > 0$, we deduce for all σ, $u \in [\varepsilon, \infty)$:

$$|E(\sigma\cdot)^{(m)}(u)| = \sigma^m |E^{(m)}(\sigma u)| \le \sigma^m \sum_{k=0}^n |c_{k,m,n}| e^{-k\sigma\varepsilon} \le C_1\,. \qquad \square$$

Finally, we come to the proof of Theorem 1.1. We use the symbol $\preceq$ to indicate domination up to constants independent of the relevant parameters.

PROOF OF THEOREM 1.1. We want to apply our weak type (p_o, p_o) criterion Theorem 3.1 for $T = F(A)$. Hence we have to show

$$N_{p_o',r_t/2}((F(A)D^n e^{-tA})^* \chi_{B(y,4r_t)^c} f)(y) \preceq \sup_{h>0} \|\omega F_h\|_{H^s} (M_2 f)(x) \tag{2}$$

for all $t > 0$, $f \in L_{p_o'}(\Omega)$, $x \in \Omega$, $y \in B(x, r_t/2)$ and some $n \in \mathbb{N}$. Choose $\varepsilon_1 \geq \varepsilon_2 > 0$ as in Proposition 3.4 and $n \in \mathbb{N}$ such that $n > \varepsilon_1 \vee s$. Denote $\delta := (n - \varepsilon_1) \wedge \varepsilon_2 > 0$ and $\varphi(u) := u^{-\varepsilon_1} \vee u^{-\varepsilon_2}$, $E(u) := \sum_{k=o}^{n} \binom{n}{k} (-1)^k e^{-ku}$ for all $u \in \mathbb{R}_+$. Furthermore, for $\sigma > 0$ we denote the dilations $F_\sigma := F(\sigma \cdot)$, $\omega_\sigma := \omega(\sigma \cdot)$ and $E_\sigma := E(\sigma \cdot)$. Observe that $E_\sigma(A) = D^n e^{-\sigma A}$ and by Lemma 3.5

$$\varphi(\sigma)\|E_\sigma\|_{C^n(\operatorname{supp}\omega)} \preceq \varphi(\sigma)(1 \wedge \sigma^n) \leq \sigma^{-\delta} \wedge \sigma^{\delta} \quad \text{for all} \quad \sigma > 0\,. \tag{3}$$

Now (2) follows from Proposition 3.4, applied for $\omega F_{2^{-l}} E_{t2^{-l}}$ instead of F and $\sigma = 2^l$, and then summation over $l \in \mathbb{Z}$:

$$\begin{aligned}
& N_{p_o',r_t/2}((F(A)D^n e^{-tA})^* \chi_{B(y,4r_t)^c} f)(y) \\
&= N_{p_o',r_t/2}\left(\sum_{l\in\mathbb{Z}} ((w_{2^l} F)(A) D^n e^{-tA})^* \chi_{B(y,4r_t)^c} f\right)(y) && \left[\textstyle\sum \omega_{2^l} = 1\right] \\
&\leq \sum_{l\in\mathbb{Z}} N_{p_o',r_t/2}((w F_{2^{-l}} E_{t2^{-l}})(2^l A)^* \chi_{B(y,4r_t)^c} f)(y) && [E_\sigma(A) = D^n e^{-\sigma A}] \\
&\preceq \sum_{l\in\mathbb{Z}} \varphi(t2^{-l}) \|w F_{2^{-l}} E_{t2^{-l}} \cdot \exp\|_{H^s} M_2 f(x) && [\text{Proposition 3.4}] \\
&\preceq \sup_{h>0} \|\omega F_h\|_{H^s} M_2 f(x) \sum_{l\in\mathbb{Z}} \varphi(t2^{-l}) \|E_{t2^{-l}} \cdot \exp\|_{C^n(\operatorname{supp}\omega)} && [n \geq s] \\
&\preceq \sup_{h>0} \|\omega F_h\|_{H^s} M_2 f(x) \sum_{l\in\mathbb{Z}} (t2^{-l})^{-\delta} \wedge (t2^{-l})^{\delta} && [\text{line (3)}] \\
&\preceq \sup_{h>0} \|\omega F_h\|_{H^s} M_2 f(x)\,.
\end{aligned}$$

□

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