UNIVERSITY
OF TRENTO

DIPARTIMENTO DI INGEGNERIA E SCIENZA DELL'INFORMAZIONE

38123 Povo – Trento (Italy), Via Sommarive 14
http://www.disi.unitn.it

A HYBRID APPROACH BASED ON PSO AND HADAMARD DIFFERENCE SETS FOR THE SYNTHESIS OF SQUARE THINNED ARRAYS

M. Donelli, A. Martini, and A. Massa

January 2011

Technical Report # DISI-11-031

A Hybrid Approach Based on PSO and Hadamard Difference Sets for the Synthesis of Square Thinned Arrays

Massimo Donelli, Anna Martini, and Andrea Massa

ELEDIA Research Group - Department of Information and Communication Technology

University of Trento, Via Sommarive 14, I-38050 Trento - Italy

Tel. +39 0461 882057, Fax +39 0461 882093

E-mail: *andrea.massa@ing.unitn.it*,

{*massimo.donelli, anna.martini*}*@dit.unitn.it*

Web-site: *http://www.eledia.ing.unitn.it*

A Hybrid Approach Based on PSO and Hadamard Difference Sets for the Synthesis of Square Thinned Arrays

Massimo Donelli, Anna Martini, and Andrea Massa

Abstract

A hybrid approach for the synthesis of planar thinned antenna arrays is presented. The proposed solution exploits and combines the most attractive features of a particle swarm algorithm and those of a combinatorial method based on the noncyclic difference sets of Hadamard type. Numerical experiments validate the proposed solution, showing improvements with respect to previous results.

Key words:

Thinned arrays, particle swarm optimizer (PSO), sidelobe control, difference sets.

1 Introduction

This paper deals with arrays composed by elements placed on a uniform lattice with half-wavelength spacing between adjacent points. Active elements are fed with equal amplitude currents while the remaining ones are turned off (i.e., connected to a matched or dummy load). For a given lattice size, since the resolution mainly depends on the aperture dimension in wavelength, the thinning operation allows one to obtain an approximately unaltered main lobe width with reduced cost, weight, power consumption, and heat dissipation. On the other hand, for a given number of active elements, narrower beamwidth can be achieved since larger antenna sizes are possible. However, the thinning operation usually implies a reduced antenna gain and a non-negligible loss of the sidelobe level control.

In principle, an exhaustive search is possible to synthesize the thinned configuration with the minimum peak sidelobe level (PSL). As a matter of fact, unlike non-uniformly spaced arrays, which are determined by means of both algorithmic and random placement techniques [1]-[4], only a finite number of thinned configurations is allowed [i.e., $\begin{pmatrix} V \\ K \end{pmatrix}$, V and K being the number of array elements and of turned on elements, respectively]. Nevertheless, it is evident that an exhaustive sampling of the solution space is feasible only dealing with small arrays [2], while more sophisticated thinning techniques are needed when large arrays are at hand. In such a framework, the use of optimization approaches has led to significant advancements [5]-[7]. As a matter of fact, Simulated Annealing [8], Genetic Algorithms (GAs) [9], and Particle Swarm Optimizers (PSOs) [10] allow to manage very large configurations and unlike dynamic programming [11] they are not vulnerable to local minima thanks to their hill-climbing search. However, such algorithms are quite slow when compared with deterministic techniques due to their global nature. In particular, the convergence rate considerably reduces in the neighborhood of the optimal solution.

In order to speed up the optimization process, all the available a-priori information or initial solutions from available combinatorial tools must be taken into account. The hybrid approach proposed in [12] is based on such a philosophy. The efficiency of the GA-based search process has been improved by means of a suitable integration with the deterministic approach proposed by Leeper [13][14]. When an array of V elements is thinned according to a cyclic difference set

(CDS) [15], the array power pattern is forced to pass through V uniformly spaced, equal, and constant values that are less than $1/K$ times the main lobe peak. Unfortunately, the synthesized power pattern presents undesired ripples, the greatest one usually located near the main lobe. Combining combinatorial and stochastic methods has allowed to limit the ripple amplitudes, leading to improved results in terms of PSL with respect to both CDS- and GAs-optimized arrays [12]. CDSs can be applied to thin both linear and planar arrays [14]. As regards to planar arrays, the synthesis has been carried out starting from two-dimensional sequences of ones and zeros generated from linear CDSs by means of ad-hoc placement algorithms [16]. Nevertheless, in order to keep the CDSs' positive features, the two-dimensional sequences can be located only on rectangular grids with coprime side-lengths[(1)]. Consequently, the CDS-based approaches cannot be applied to square arrays.

Leaning upon the same philosophy of [12] (i.e., combining combinatorial and optimization-based methods to exploit their positive features and to compensate their drawbacks at the same time), this contribution is aimed at proposing a hybrid approach for the synthesis of square thinned arrays. The approach is based on the use of noncyclic two-dimensional difference sets of Hadamard type (HSs) [17] to initialize an optimization procedure. In principle, HSs allow to deal also with rectangular arrays [18]. Nevertheless, we focus on square configurations since in this case for each K value such that some HS exists and for $K \leq 300$ the totality of possible HS-based square thinned arrays has been analyzed and the HS allowing the lowest PSL has been provided [18].

As far as the optimizer is concerned, taking into account the advantages of PSOs when compared to GAs (i.e., easier implementation and calibration, and ability to control the convergence of the optimization as well as its stagnation), a customized and suitably integrated PSO is used as optimization algorithm.

The paper is organized as follows. Section 2 briefly resumes some useful array notations. In Section 3, the hybrid strategy is detailed pointing out the key features of the HSs and of the standard PSOs (SPSOs) methods. The numerical validation is reported in Section 4, where the hybrid approach is assessed through a comparative analysis. Finally, some concluding remarks are given in Section 5.

[(1)] Two integers are said to be coprime or relatively prime if they have no common factor other than 1 or, equivalently, if their greatest common divisor is 1. The integer side-lengths are coprime if the diagonal does not intersect any other lattice point.

2 Antenna Arrays Notation

Let us consider a planar array on the $x-y$ plane whose half-wavelength spaced elements are located on a uniform lattice of $V = V_x \times V_y$ locations numbered from (0,0) to $(V_x - 1, V_y - 1)$. The corresponding array factor is defined as[(2)]

$$f(u,v) \triangleq \sum_{m=0}^{V_x-1} \sum_{n=0}^{V_y-1} a_{m,n} e^{j\pi(mu+nv)} \tag{1}$$

where $a_{m,n} = 1$ if there is an element at the (m,n)-th location and $a_{m,n} = 0$, otherwise, $u = \sin\theta\cos\phi$ and $v = \sin\theta\sin\phi$. Moreover, the PSL [4] turns out to be

$$PSL \triangleq \frac{1}{K} \max_{(u,v)\in L} |f(u,v)|^2, \tag{2}$$

where L denotes the sidelobe region. Finally, the thinning percentage Γ is given by

$$\Gamma \triangleq \frac{K}{V}. \tag{3}$$

3 Hybrid Optimization Strategy

The objective of the synthesis is to find in a computationally-efficient way the array configuration that represents the best compromise in terms of PSL and thinning percentage. Towards this end, we propose a hybrid optimization method. In order to point out the arguments that justify an integrated strategy, the key features of the HS-based method and of the PSO-based synthesis technique are firstly discussed. Then, the integrated procedure is carefully described.

3.1 HS-based Synthesis Method

By definition, a $(V,\ K,\ \Lambda)$-HS [17] is a set of K points defined on a integer $V_x \times V_y$ grid of V elements

$$HS = \{(b_0, c_0),\ (b_1, c_1),\ \ldots,\ (b_{K-1}, c_{K-1})\} \tag{4}$$

[(2)] To simplify the array factor expression and without loss of generality, the steering angles have been set to zero.

with $0 \leq b_j \leq V_x - 1$ and $0 \leq c_j \leq V_y - 1$, such that: (*a*) for any grid point (m, n), $0 \leq m \leq V_x - 1$ and $0 \leq n \leq V_y - 1$, there exist exactly Λ pairs $\{(b_i, c_i), (b_j, c_j)\}$ that satisfy the equations $m \equiv (b_i - b_j) mod\, V_x$ and $n \equiv (c_i - c_j) mod\, V_y$, where "$mod\, V$" means that the difference has to be taken modulo V; (*b*) either $V = 4N^2,\ K = 2N^2 - N,\ \Lambda = N^2 - N$ or $V = 4N^2,\ K = 2N^2 + N,\ \Lambda = N^2 + N$, N being equal to 2^r or $3 \cdot 2^r$, $r \geq 0$ [18].

By placing the array elements at the HS locations, it is possible to synthesize a thinned array with a normalized power pattern that, in the sidelobe region, is forced to take the constant value $(K - \Lambda)/K^2$ in correspondence with the points of the lattice whose coordinates are multiple of $(2\pi/V_x, 2\pi/V_y)$ [18]. Therefore, there are some constraints on the sidelobe level, but undesired ripples still verify. Moreover, it is not possible to define a HS whatever $(V,\ K,\ \Lambda)$ value, but only if (*b*) holds true. Accordingly, there is a constraint on the thinning percentage Γ achievable through the HS-based synthesis method.

3.2 PSO-based Synthesis Method

PSOs are stochastic multiple-agents optimization algorithms extensively applied in the framework of antenna array optimization (see [19] and the references cited therein). By imitating the social behavior of groups of insect and animals in their food searching activities, they are based upon the cooperation among *particles*. The ensemble of the particles, referred to as *swarm*, explores the solution space to find out the best position (i.e., the optimum of a suitably defined cost function). During the optimization process, each particle updates its position on the basis of its own previous best position and of the swarm's previous best position.

In principle, applying PSO-based strategies to the synthesis of thinned arrays allows to effectively explore the whole solution space and to figure out arrays with any thinning percentage. However, if no *a-priori* information is available and/or exploited, a satisfactory solution can be found only after a non-negligible number of iterations. Clearly, the computational burden grows with the dimension of the solution space (e.g., when large arrays are at hand).

3.3 Hybrid Synthesis Method

The considerations outlined in the previous Sections suggest that an improvement could be achieved by integrating the HS-based method into the optimization process. More specifically,

we expect that the hybrid strategy outperforms the HS-based method in terms of both thinning percentage and PSL, and the PSO-based method in terms of convergence rate and computational costs.

Because of the discrete nature of the problem at hand, a binary PSO is used. The p-th trial solution is a two-dimensional array

$$X^p = \left\{ a^p_{m,n};\ m = 0, ..., V_x - 1;\ n = 0, ..., V_y - 1 \right\}, \quad a^p_{m,n} \in \{0, 1\} \tag{5}$$

$p = 0, ..., P - 1$ being the particle index and P the swarm dimension. Moreover, in order to ensure the best compromise in terms of both PSL and Γ, the cost function to be minimized is defined as follows

$$F = \alpha F_{PSL} + \beta F_{\Gamma} \tag{6}$$

where

$$F_{PSL} = \frac{\max_{(u,v)\in L} |f(u,v)|^2}{\max_{\forall(u,v)} |f(u,v)|^2} \tag{7}$$

$$F_{\Gamma} = \frac{1}{V} \sum_{m=0}^{V_x - 1} \sum_{n=0}^{V_y - 1} a_{m,n}. \tag{8}$$

The weighting coefficients α and β are chosen so that the final solution has both PSL and Γ lower, or equal, than those of the optimal HS-based array configuration $[H]^{(V,K,\Lambda)}_{OPT}$ provided in [18], i.e.,

$$\alpha = \begin{cases} 1, & PSL \leq PSL_{[H]^{(V,K,\Lambda)}_{OPT}} \\ \Upsilon \gg 1, & PSL > PSL_{[H]^{(V,K,\Lambda)}_{OPT}} \end{cases} \tag{9}$$

$$\beta = \begin{cases} 1, & \Gamma \leq \Gamma_{[H]^{(V,K,\Lambda)}_{OPT}} \\ \Upsilon \gg 1, & \Gamma > \Gamma_{[H]^{(V,K,\Lambda)}_{OPT}} \end{cases}. \tag{10}$$

In the following, the steps of the hybrid algorithm are summarized.

- **Initialization Step ($i = 0$).** The positions of the P particles of the swarm (i.e., the initial trial solutions) are generated according to the optimal HS-based array configuration

$[H]_{OPT}^{(V,K,\Lambda)}$, which is known and provided in [18]. More in detail:

$$X^{0,p} = \begin{cases} [H]_{OPT}^{(V,K,\Lambda)}, & p = 0 \\ \wp_p\left\{[H]_{OPT}^{(V,K,\Lambda)}\right\}, & 1 \leq p \leq \sqrt{V-1} \\ \mu_p\left\{[H]_{OPT}^{(V,K,\Lambda)}\right\}, & \sqrt{V-1}+1 \leq p \leq P-1 \end{cases} \tag{11}$$

$\wp_p\left\{[H]_{OPT}^{(V,K,\Lambda)}\right\}$ and $\mu_p\left\{[H]_{OPT}^{(V,K,\Lambda)}\right\}$ being a cyclic shift and a randomly mutated version of $[H]_{OPT}^{(V,K,\Lambda)}$, respectively. On the other hand, the *velocity* of each particle $S^{0,p} = \left\{s_{m,n}^{0,p};\ m = 0, ..., V_x - 1;\ n = 0, ..., V_y - 1\right\}$ is randomly generated:

$$s_{m,n}^{0,p} = \begin{cases} 1, & \rho_{m,n}^{0,p} \geq 0.5 \\ 0, & \rho_{m,n}^{0,p} < 0.5 \end{cases} \tag{12}$$

where $\rho_{m,n}^{0,p}$ is a random number with uniform distribution in the range $[0, 1]$.

- **Evaluation Step.** The optimality of each trial solution at the i-th iteration is evaluated (i.e., $F^{i,p} = F(X^{i,p})$, $p = 0, ..., P-1$) and the *personal best* $B^{i,p} = \left\{b_{m,n}^{i,p};\ m = 0, ..., V_x - 1;\ n = 0, ..., V_y - 1\right\} = \arg\left\{\min_{h=0,...,i}\left[F(X^{h,p})\right]\right\}$ as well as the *global best* $G^i = \left\{g_{m,n}^{i};\ m = 0, ..., V_x - 1;\ n = 0, ..., V_y - 1\right\} = \arg\left\{\min_{p=0,...,P-1}\left[F(B^{i,p})\right]\right\}$ positions are updated. The iterations index is increased ($i = i + 1$) and the termination criteria are checked. If the maximum number of iteration I is reached or the cost of the global best is smaller than a given threshold η, then the optimization process stops and the global best G^i is assumed as the problem solution.

- **Velocity Updating Step.** The velocity of each particle is updated according to the following relationship

$$s_{m,n}^{i,p} = \begin{cases} S_{MAX}, & t_{m,n}^{i,p} > S_{MAX} \\ -S_{MAX}, & t_{m,n}^{i,p} < -S_{MAX} \\ t_{m,n}^{i,p}, & \left|t_{m,n}^{i,p}\right| \leq S_{MAX} \end{cases} \tag{13}$$

where S_{MAX} is a constant clamping value [10] and

$$t_{m,n}^{i,p} = \omega s_{m,n}^{i-1,p} + c_1\rho_1(b_{m,n}^{i,p} - a_{m,n}^{i,p}) + c_2\rho_2(g_{m,n}^{i} - a_{m,n}^{i,p}) \tag{14}$$

ω, c_1, and c_2 being constant parameters called *inertial weight*, *cognition* and *social acceleration*, respectively. Moreover, ρ_1 and ρ_2 are random positive coefficients drawn from a uniform distribution with predefined upper limits [20].

- **Position Updating Step.** The position of each particle is updated as follows

$$x_{m,n}^{i,p} = \begin{cases} 1, & \rho_{m.n}^{i,p} < T(s_{m,n}^{i,p}) \\ 0, & \rho_{m.n}^{i,p} \geq T(s_{m,n}^{i,p}) \end{cases} \tag{15}$$

$T(s_{m,n}^{i,p})$ being the sigmoid function [21]

$$T(s_{m,n}^{i,p}) = \frac{1}{1 + e^{-s_{m,n}^{i,p}}}. \tag{16}$$

The optimization algorithm restarts from the **"Evaluation Step**".

4 Numerical Results

This section provides the numerical assessment of the proposed hybrid approach, denoted by the acronym HSPSO. Moreover, a comparative study with the HS-based strategy and the SPSO is discussed. From the exhaustive set of numerical experiments, selected representative results concerned with small arrays ($V = 36$), medium arrays ($V = 144$), and large arrays ($V = 576$) are presented. The initial trial solutions of the HSPSO have been generated by relying on the optimal HSs-based arrays determined by Kopilovich [18] when $K = 21$, $K = 78$, and $K = 300$, respectively. On the other hand, the SPSO has been randomly initialized. As regards to the parameters setup of both the HSPSO and the SPSO, the reference values are given in Table I. For each test case, both the HSPSO and the SPSO have been executed $T = 100$ times. Figures 1-3 show the best array configurations obtained by applying the HSs-based strategy, the SPSO, and the HSPSO. In the case of large arrays (i.e., $V = 576$) the corresponding power patterns are depicted as well (see Fig.3). The values of the PSL and of the thinning percentage Γ are given in Table II. Finally, Table III shows the CPU-time necessary for the initialization τ_0 , the CPU-time for iteration τ_i, and the iteration number i_η at which the convergence threshold η has been reached. All the reported values have been obtained by averaging on the T trials.

As it can be noticed, the HSPSO significantly outperforms the HSs-based approach in terms of both PSL and thinning percentage Γ. The improvements allowed by the PSO are particularly evident when dealing with large arrays as confirmed by the results concerned with the configuration having $V = 576$ elements (see Fig. 3 and right-hand side of Tab. II). The HSPSO power pattern has a lower PSL, $PSL_{HS} - PSL_{HSPSO} \cong 3.25dB$. Moreover, such a pattern is obtained with a massive thinning, $\Gamma_{HSPSO} \cong 0.44$, with a notable reduction of the active elements, $K_{HS} - K_{HSPSO} = 46$.

Now, let us compare the SPSO and the HSPSO in order to analyze how a suitable integration affects the optimization performances. We start by observing that in all the considered test cases the CPU-time necessary for the initialization through HSs is negligible with respect to the CPU-time for iteration and it is comparable with the CPU-time of a random initialization. When the small array (i.e., $V = 36$) is at hand, the HSPSO significantly outperforms the SPSO. Besides an expected increase of the convergence rate ($\frac{[i_\eta]_{SPSO}}{[i_\eta]_{HSPSO}} \approx 2.5$, Tab. III) as well as a reduction of the overall CPU-time for the synthesis ($\frac{[\tau_0+\tau_i \times i_\eta]_{SPSO}}{[\tau_0+\tau_i \times i_\eta]_{HSPSO}} \approx \frac{[\tau_i \times i_\eta]_{SPSO}}{[\tau_i \times i_\eta]_{HSPSO}} \approx 3$, Tab. III), a smart initialization leads also to a considerably improved solution. The HSPSO allows a significant thinning that cannot be attained with the SPSO, $\Gamma_{HSPSO} \cong 0.42$ vs. $\Gamma_{SPSO} = 0.5$, along with a lower PSL (Tab. II and Fig. 1). When dealing with larger arrays, the initialization through HSs still reduces the computational costs ($\frac{[i_\eta]_{SPSO}}{[i_\eta]_{HSPSO}} \approx \frac{[\tau_i \times i_\eta]_{SPSO}}{[\tau_i \times i_\eta]_{HSPSO}} \approx 1.9$ when $V = 144$; $\frac{[i_\eta]_{SPSO}}{[i_\eta]_{HSPSO}} \approx \frac{[\tau_i \times i_\eta]_{SPSO}}{[\tau_i \times i_\eta]_{HSPSO}} \approx 1.8$ when $V = 576$, Tab. III), but the SPSO and the HSPSO practically allow identical performances in terms of thinning percentage and PSL (Tab. II and Fig. 3). However, it must be taken into account that at present the collection of HSs is complete only for $V \leq 36$ [18]. Thus, HS-based arrays with lower PSL could be found when $V > 36$, leading to final solutions with improved PSLs and thinning percentages besides the advantages in terms of computational indexes already obtained without fully exploiting the potentialities of the method.

5 Conclusions

In this letter, a hybrid approach devoted to the synthesis of square thinned arrays has been presented. The proposed solution exploits and combines the positive features of the combinatorial techniques and of the optimization strategies to obtain in a computationally-effective way the

best compromise solution in terms of both peak side-lobe level and thinning percentage. An exhaustive set of numerical experiments has been performed showing that the hybrid approach outperforms both the SPSO and the HSs-based strategy. In particular, when compared to the HS-based synthesis method, the HSPSO allows a massive thinning by improving at the same time the obtained power pattern. Such an event highlights the capability of the integrated stochastic optimization to control the unwanted ripples that occur when the HSs-based method is applied. Moreover, when compared to the standard PSO-based optimization, the hybrid strategy allows a significant improvement in the convergence rate.

References

[1] Y. T. Lo, "A mathematical theory of antenna arrays with randomly spaced elements," *IEEE Trans. Antennas Propagat.*, vol. 12, no. 3, pp. 257-268, 1964.

[2] Y. T. Lo and S. W. Lee, "A study of space-tapered arrays," *IEEE Trans. Antennas Propagat.*, vol. 14, no. 1, pp. 22-30, 1966.

[3] B. D. Steinberg, "Comparison between the peak sidelobe of the random array and algorithmically designed aperiodic arrays," *IEEE Trans. Antennas Propagat.*, vol. 21, no. 3, pp. 366-370, 1973.

[4] B. D. Steinberg, *Principles of Aperture and Arrays System Design*. New York: Wiley, 1976.

[5] C. S. Ruf, "Numerical annealing of low-redundancy linear arrays," *IEEE Trans. Antennas Propagat.*, vol. 41, no. 1, pp. 85-90, 1993.

[6] D. J. O'Neill, "Element placement in thinned arrays using genetic algorithms," *Proc. Oceans '94*, vol. 2, pp. 301-306, 1994.

[7] R. L. Haupt, "Thinned arrays using genetic algorithms," *IEEE Trans. Antennas Propagat.*, vol. 42, no. 7, pp. 993-999, 1994.

[8] P. J. M. van Laarhoven and E. H. L. Aarts, *Simulated Annealing: Theory and Applications*. Holland: D. Reidel, 1987.

[9] D. E. Goldberg, *Genetic Algorithms in Search, Optimization and Machine Learning*. Reading, MA: Addison-Wesley, 1989.

[10] J. Kennedy, R. C. Eberhart, and Y. Shi, *Swarm Intelligence*. San Francisco: Morgan Kaufmann, 2001.

[11] T. C. Cheston and J. Frank, "Phased array radar antennas," in *Radar Handbook*, M. Skolnik, Ed. New York: Mc Graw-Hill, 1990.

[12] S. Caorsi, A. Lommi, A. Massa, and M. Pastorino, "Peak sidelobe level reduction with a hybrid approach based on GAs and difference sets," *IEEE Trans. Antennas Propagat.*, vol. 52, no. 4, pp. 1116-1121, 2004.

[13] D. G. Leeper, "Thinned aperiodic antenna arrays with improved peak sidelobe level control," January 31, 1978. U. S. Patent.

[14] D. G. Leeper, "Isophoric arrays - Massively thinned phased arrays with well-controlled sidelobes," *IEEE Trans. Antennas Propagat.*, vol. 47, no. 12, pp. 1825-1835, 1999.

[15] L. D. Baumert, *Cyclic Difference Sets*. New York: Springer-Verlag, 1971.

[16] N. J. A. Sloane and F. J. MacWilliams, "Pseudo-random sequences and arrays," *Proc. IEEE*, vol. 64, no. 12, pp. 1715-1729, 1976.

[17] R. J. Turyn, "Character sums and difference sets," *Pacific J. Math.*, vol. 15, no. 1, pp. 319-346, 1965.

[18] L. E. Kopilovich, "Square array antennas based on Hadamard difference sets," *IEEE Trans. Antennas Propagat.*, vol. 56, no. 1, pp. 263-266, 2008.

[19] N. Jin and Y. Rahmat-Samii , "Advances in particle swarm optimization for antenna designs: real-number, binary, single-objective and multiobjective implementations," *IEEE Trans. Antennas Propagat.*, vol. 55, no. 3, pp. 556-567, 2007.

[20] R. C. Eberhart and Y. Shi, "Particle swarm optimization: developments, applications and resources," *Proc. Congress on Evolutionary Computation*, Seul, Corea, 2001, pp. 81-86.

[21] J. Kennedy and R. Eberhart, “A discrete binary version of the particle swarm algorithm,” *Proc. IEEE Int. Conf. Systems, Man, and Cybernetics*, 1997, pp. 4104-4108.

Figure Captions

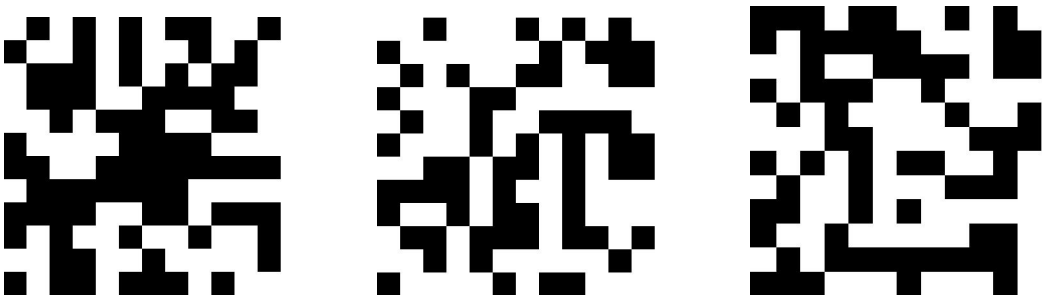
- **Figure 1.** Small arrays ($V = 36$). Elements grid when (*a*) the HS-based method, (*b*) the SPSO, and (*c*) the HSPSO are applied.
- **Figure 2.** Medium arrays ($V = 144$). Elements grid when (*a*) the HS-based method, (*b*) the SPSO, and (*c*) the HSPSO are applied.
- **Figure 3.** Large arrays ($V = 576$). Synthesized power patterns and elements grid when (*a*) the HS-based method, (*b*) the SPSO, and (*c*) the HSPSO are applied.

Table Captions

- **Table I.** PSO setup.
- **Table II.** Performance indexes.
- **Table III.** Computational indexes.

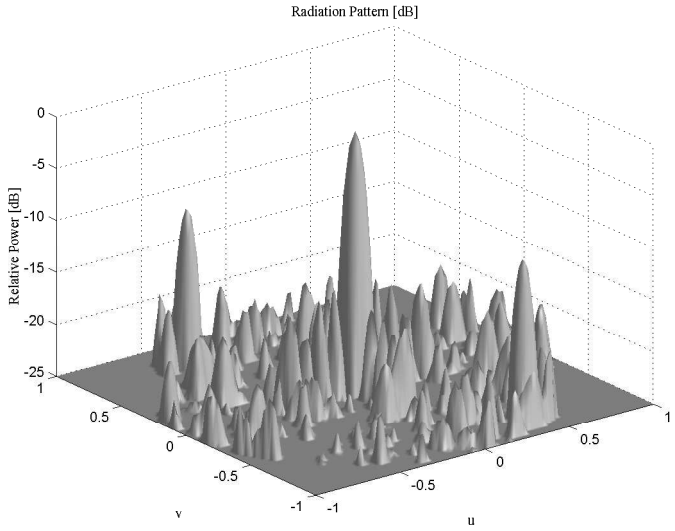
(*a*) (*b*) (*c*)

Fig. 1 - M. Donelli *et al.,* ”A Hybrid Approach ...”

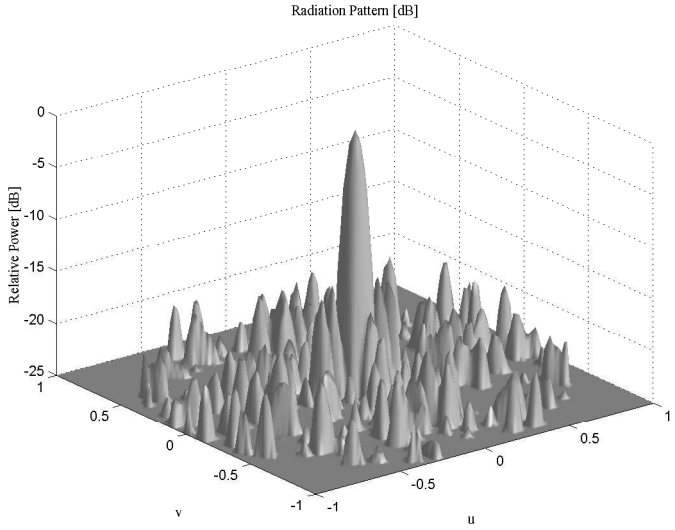


(*a*) (*b*) (*c*)

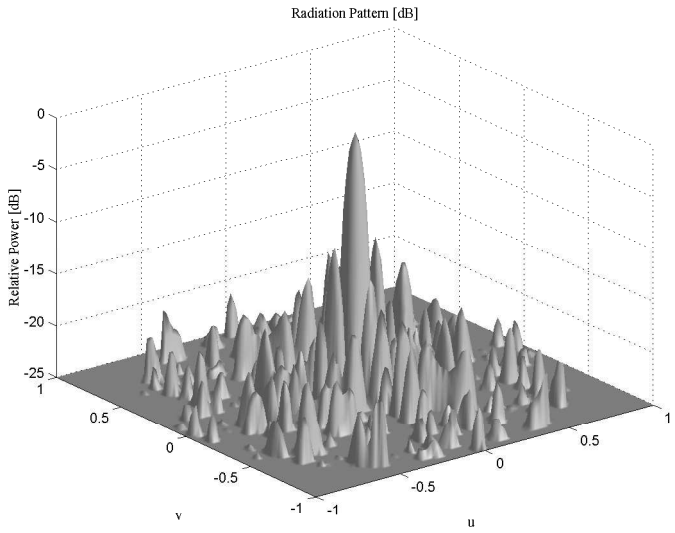
Fig. 2 - M. Donelli *et al.,* ”A Hybrid Approach ...”

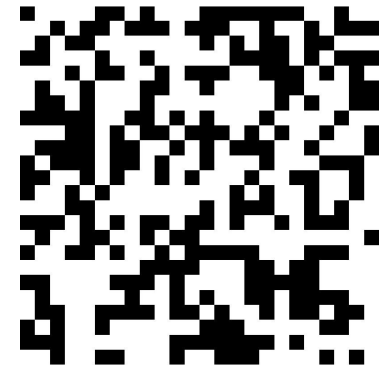


(*a*)



(*b*)





(*c*)

Fig. 3 - M. Donelli *et al.,* ”A Hybrid Approach ...”

P	I	η	ω	c_1	c_2	$\max \rho_1$	$\max \rho_2$	V_{MAX}
$\frac{V}{10}$	200	10^{-2}	0.4	2.0	2.0	1.0	1.0	1

Tab. I - M. Donelli *et al.,* "A Hybrid Approach ..."

	$V = 36$		$V = 144$		$V = 576$	
Method	**PSL** $[dB]$	Γ	**PSL** $[dB]$	Γ	**PSL** $[dB]$	Γ
HS	-12.55	0.58	-15.47	0.54	-15.72	0.52
SPSO	-12.72	0.50	-15.59	0.44	-18.53	0.43
HSPSO	-13.06	0.42	-16.74	0.48	-18.97	0.44

	$V = 36$			$V = 144$			$V = 576$		
Method	$\tau_0\ [s]$	$\tau_i\ [s]$	i_η	$\tau_0\ [s]$	$\tau_i\ [s]$	i_η	$\tau_0\ [s]$	$\tau_i\ [s]$	i_η
SPSO	1×10^{-3}	0.97	129	3×10^{-3}	14.89	145	0.31	51.12	189
HSPSO	2×10^{-3}	0.82	51	5×10^{-3}	15.01	75	0.20	50.21	108