A hybrid boundary-element/finite-element method for the computation of design sensitivities in structural shape optimisation

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Abstract

This paper presents an accurate method of calculating design sensitivities of behaviour variables, such as displacements and stress, in FE structural shape optimisation problems. Its applicability to *unstructured* meshes is emphasised.

The semi-analytical method is applied to the determination of the design sensitivities of behaviour variables in structural shape optimisation. It is shown how the computation of the geometric sensitivities (i.e. the sensitivity of internal points in the problem domain to boundary perturbations) is a primary task to carry out the sensitivity analysis. A BEM based point-tracking method, which is particularly applicable to unstructured meshes, is developed and used, in conjunction with FD approximation, to determine these geometric sensitivities.

Three benchmark examples, with unstructured meshes, are used to show the application of the method and results from these show that the geometric sensitivity calculations are easily carried out using the BEM point tracking approach. Design sensitivities calculated compare favourably with analytical solutions.

Introduction

A primary task in Structural Shape Optimisation is the calculation of the design sensitivities. These are the sensitivity of the design variables, such as displacement and stress, to changes in the shape of the domain, usually described by a shape design vector. The design sensitivities are then used at the optimisation stage to find the direction towards the most optimal shape. The requirements of the sensitivity analysis are twofold. Firstly, it must provide a reasonable estimate of the sensitivities so that the solution will remain in the feasible region after the calculated changes are made to the shape variables. Secondly it must be as computationally efficient as possible.

Methods of sensitivity analysis fall into two basic categories, namely the semi-analytical and the finite difference, and a comparison of the accuracy of these two methods is found in Barthelemy and Haftka [1]. In the present paper a procedure for the implementation of the semi-analytical method, which utilises the hybrid BE/FE method developed, is considered and numerical studies are examined.

FE Design sensitivities by the semi-analytical method

The semi-analytical method is a specific variant of the implicit differentiation approach and has been described in other references [2,3,4]. It has been used successfully in structural shape optimisation systems for both static and dynamic problems: examples of these may be found in Hinton et al [5,6,7]. A brief outline of the method is provided here.

The FEM formulation for a linear elastic continuum domain results in the set of linear equations

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{1}$$

in which $\mathbf{K}$ is the stiffness matrix, $\mathbf{f}$ is the force vector and $\mathbf{u}$ is the unknown set of displacements. Defining b, a design variable that describes some aspect of the shape of the domain, we may differentiate (1) to get

$$\frac{\partial \mathbf{K}}{\partial b}\mathbf{u} + \mathbf{K}\frac{\partial \mathbf{u}}{\partial b} = \frac{\partial \mathbf{f}}{\partial b} \tag{2}$$

If we assume that $\mathbf{f}$ is independent of b and recast (2) using the prime notation to denote differentiation with respect to b, we get

$$\mathbf{K}\mathbf{u}' = -\mathbf{K}'\mathbf{u} \tag{3}$$

We may solve this equation for the displacement sensitivities $\mathbf{u}'$, and since for a given analysis the factorised $\mathbf{K}$ matrix is available, this is computationally inexpensive. If we consider the case for a plane isoparametric finite element then $\mathbf{K} = \sum \mathbf{K}_e$, where $\mathbf{K}_e$ is the element stiffness matrix, given by

$$\mathbf{K}_e = \int_{-1}^{+1}\int_{-1}^{+1} \mathbf{B}^T\mathbf{D}\mathbf{B}\,|\,\mathbf{J}\,|\,d\xi\, d\eta \tag{4}$$

$\mathbf{J}$, the Jacobian transformation matrix, is defined as

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum \frac{\partial N_i}{\partial \xi} x_i & \sum \frac{\partial N_i}{\partial \xi} y_i \\ \sum \frac{\partial N_i}{\partial \eta} x_i & \sum \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \tag{5}$$

where N_i are the shape functions for an isoparametric finite element. Obviously $\mathbf{K}' = \sum \mathbf{K}'_e$ which means that differentiating (4) results in

$$\mathbf{K}'_e = \int_{-1}^{+1}\int_{-1}^{+1} \mathbf{B}^T\mathbf{DB}\,|\,\mathbf{J}\,|'\,d\xi\,d\eta + \int_{-1}^{+1}\int_{-1}^{+1}\left[\mathbf{B}'^T\mathbf{DB} + \mathbf{B}^T\mathbf{DB}'\right]|\,\mathbf{J}\,|\,d\xi\,d\eta \tag{6}$$

in which the constitutive matrix, **D**, is considered independent of b.

Since the behaviour variable $\sigma = \mathbf{DBu}$ then

$$\sigma' = \mathbf{DB'u} + \mathbf{DBu'} \tag{7}$$

where $\mathbf{u}'$ are the displacement sensitivities obtained from (3). The above formulation requires the computation of the $\mathbf{B}'$ matrices and the $|\,\mathbf{J}\,|'$ term and it will be shown that they depend on the geometric sensitivities, $\mathbf{x}'$ and $\mathbf{y}'$. However the calculation of the geometric sensitivities is first considered.

Geometric sensitivities using the BEM

The geometric sensitivities may be calculated using the FD approximation

$$x' = \frac{x(b+\Delta b) - x(b)}{\Delta b} \tag{8}$$

in which Δb is a perturbation of the design variable and $x(b)$ is the current geometry. Robinson et al [8,9] have developed a BEM formulation using only prescribed kinematic Dirichlet boundary conditions that will calculate the movement of boundary and interior points subject to a change in shape Δb. The method only requires the boundary nodal positions and the change in their position due the perturbation. It applies the BEM and subsequently the change in position of any internal node may be calculated. Therefore the geometric location $x(b+\Delta b)$ may be computed as

$$x(b+\Delta b) = x(b) + u(\Delta b) \tag{9}$$

where $u(\Delta b)$ is the displacement of a node due to the applied boundary shape change Δb. This procedure may be visualised as an unloaded elastic sheet, having the same shape as the true structure, subject to prescribed displacements of every node on the boundary. The internal nodes will also move in relation to the shape changes and computing these displacements will be referred to as *point tracking*.

From the above, it is clear that the only information that the method requires is the boundary nodal positions and the associated displacements for the analysis. Once the boundary tractions have been determined, the displacements at a general interior point may be computed in the usual BE manner [10]. Hence the displacements of the interior mesh nodal points of any domain are easily calculated. It is noteworthy that this method is excellent for the calculation of the geometric sensitivities of *unstructured* meshes.

Jacobian derivatives

The derivatives of the jacobian, **J**, with respect to the shape variable, b, is

$$\frac{\partial \mathbf{J}}{\partial b} = \frac{\partial}{\partial b}\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum \frac{\partial N_i}{\partial \xi} x_i' & \sum \frac{\partial N_i}{\partial \xi} y_i \\ \sum \frac{\partial N_i}{\partial \eta} x_i' & \sum \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \tag{10}$$

and $$|\mathbf{J}|' = \frac{\partial}{\partial b}\left(\frac{\partial x}{\partial \xi}\right)\frac{\partial y}{\partial \eta} + \frac{\partial}{\partial b}\left(\frac{\partial y}{\partial \eta}\right)\frac{\partial x}{\partial \xi} - \frac{\partial}{\partial b}\left(\frac{\partial y}{\partial \xi}\right)\frac{\partial x}{\partial \eta} - \frac{\partial}{\partial b}\left(\frac{\partial x}{\partial \eta}\right)\frac{\partial y}{\partial \xi} \tag{11}$$

Therefore (10) and (11) may be readily calculated from the geometric sensitivities computed in (8). The $(\mathbf{J}^{-1})'$ may also be similarly defined. The $\mathbf{B}'$ sub-matrices in (6) and (7) are defined as

$$\frac{\partial \mathbf{B}_i}{\partial b} = \frac{\partial}{\partial b}\begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial b}\left(\frac{\partial N_i}{\partial x}\right) & 0 \\ 0 & \frac{\partial}{\partial b}\left(\frac{\partial N_i}{\partial y}\right) \\ \frac{\partial}{\partial b}\left(\frac{\partial N_i}{\partial y}\right) & \frac{\partial}{\partial b}\left(\frac{\partial N_i}{\partial x}\right) \end{bmatrix} \tag{12}$$

and since

$$\frac{\partial}{\partial b}\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = (\mathbf{J}^{-1})' \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} \tag{13}$$

these are also easily computed.

Examples

The following three examples show the power of the point tracking technique developed and it's application to the calculation of the design sensitivities. In particular, it's applicability to *unstructured* meshes is demonstrated as all the example meshes are generated using adaptive mesh refining. The calculated percentage error of each mesh considered is given in each of the tables.

All examples were subjected to a range of shape design perturbation factors: the applied perturbation being the perturbation factor multiplied by shape design variable.

Example 1 - *Two dimensional thin slab*

The plate has dimensions and loading as shown in Fig. *1(a)*. Young's Modulus is 10^7 psi and Poisson's ratio is 0.3. The shape design variable is taken as the depth of the section and a series of meshes were generated with increasing accuracy. Three of these meshes are shown in Fig. *1(b), (c)* and *(d)*. The sensitivity of the vertical deflection at point A $(\partial v/\partial b)$ was calculated and this is compared with an exact value from Özakça [11] in Table 1. It can be seen that it compares favourably for all meshes and perturbation factors.

Mesh error %	No. of elements	$\partial v/\partial b$ ($\times 10^{-6}$) $\Delta b/b$ 10^{-1}	10^{-3}	10^{-5}	10^{-7}	10^{-9}	Exact
12.47	51	-5.520	-5.520	-5.520	-5.520	-5.520	-5.426
3.63	209	-5.551	-5.551	-5.551	-5.551	-5.551	-5.426
2.34	354	-5.553	-5.553	-5.553	-5.553	-5.553	-5.426
2.18	388	-5.554	-5.554	-5.554	-5.554	-5.554	-5.426
1.96	411	-5.557	-5.557	-5.557	-5.557	-5.557	-5.426

Table 1: *The sensitivity of the vertical displacement at point A in Example 1*

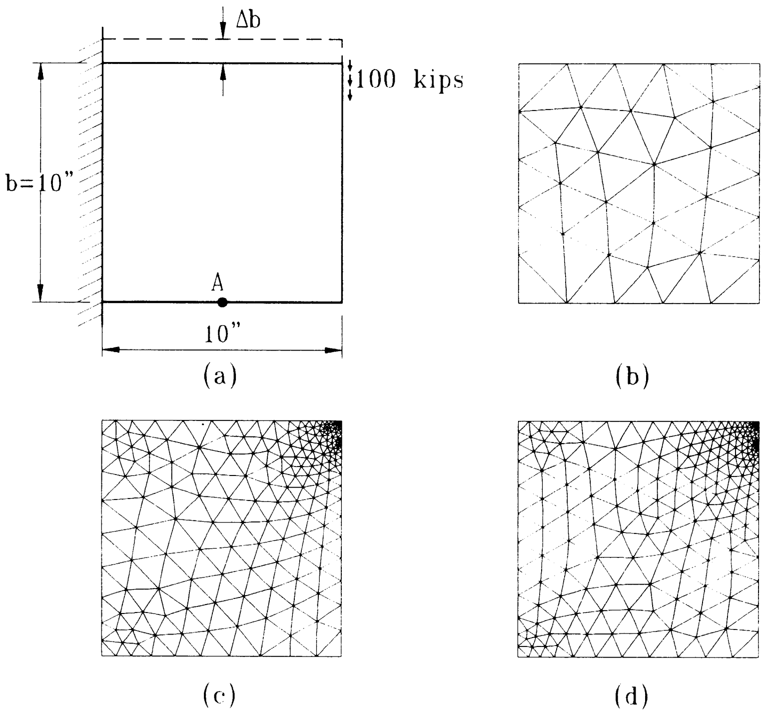

Figure 1: *(a) Plate dimensions and loading; (b) Mesh 1 – (51 elements); (c) Mesh 3 – (354 elements); (d) Mesh 5 – (411 elements)*

Example 2 *- Cantilever Beam*

The beam has dimensions and loading as shown in Fig. *2(a)*. Young's Modulus is 10^7 psi and Poisson's ratio is 0.3. The shape design variable is taken as the depth of the section and a series of meshes were generated with increasing accuracy. Three of these meshes are shown in Fig. *2(b)*, *(c)* and *(d)*. The sensitivities of the vertical deflection $(\partial v/\partial b)$ and the direct stress $(\partial \sigma_x/\partial b)$ at point A were calculated and these are compared with exact values derived from elasticity theory [12] in Tables 2 and 3. It can be seen that the $\partial v/\partial b$ terms compare well for all meshes and perturbation factors down to 10^{-7}. However the $\partial \sigma_x/\partial b$ compare favourably for all meshes and perturbation factors.

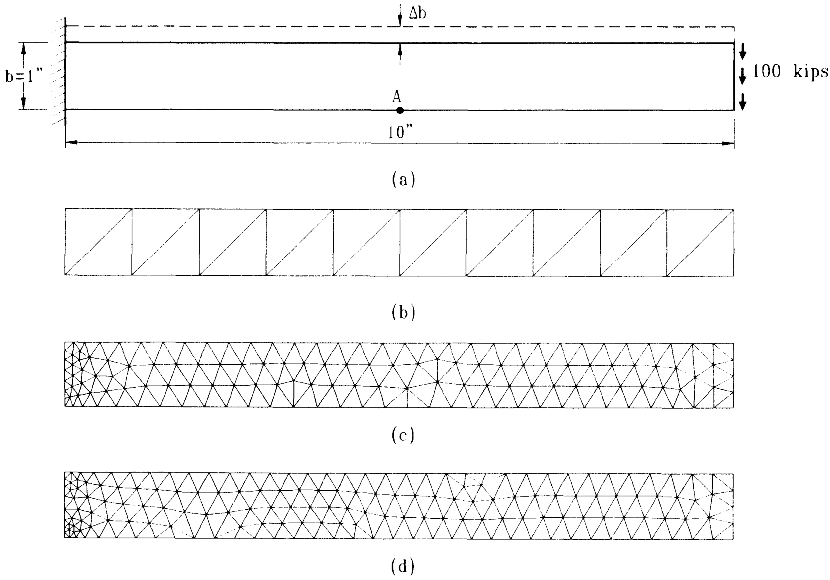

Figure 2: *(a) Beam dimensions and loading; (b) Mesh 1 – (20 elements); (c) Mesh 3 – (215 elements); (d) Mesh 5 – (267 elements)*

		$\partial v/\partial b$					
Mesh error %	**No. of elements**	**Δb/b**					
		10^{-1}	**10^{-3}**	**10^{-5}**	**10^{-7}**	**10^{-9}**	**Exact**
6.40	20	-0.0372	-0.0372	-0.0372	-0.0420	-0.5159	-0.0377
1.45	186	-0.0376	-0.0376	-0.0376	-0.0376	-0.0376	-0.0377
1.32	215	-0.0376	-0.0376	-0.0376	-0.0380	-0.0795	-0.0377
1.06	238	-0.0376	-0.0376	-0.0376	-0.0388	-0.1603	-0.0377
0.99	267	-0.0376	-0.0376	-0.0376	-0.0382	-0.0958	-0.0377

Table 2: *The sensitivity of the vertical displacement at point A in Example 2*

		$\partial \sigma_x/\partial b$					
Mesh error %	**No. of elements**	**Δb/b**					
		10^{-1}	**10^{-3}**	**10^{-5}**	**10^{-7}**	**10^{-9}**	**Exact**
6.40	20	-6015	-6015	-6015	-6015	-6011	-6000
1.45	186	-6044	-6044	-6044	-6044	-6044	-6000
1.32	215	-5952	-5952	-5952	-5952	-5939	-6000
1.06	238	-6046	-6046	-6046	-6052	-6053	-6000
0.99	267	-6043	-6043	-6044	-6043	-6042	-6000

Table 3: *The sensitivity of the direct stress at point A in Example 2*

Example 3 - *Thick Cylinder*

The cylinder has dimensions and loading as shown in Fig. *4(a)*. Young's Modulus is 200 kN/mm^2 and Poisson's ratio is 0.3. The shape design variable is taken as the outside radius of the section and a series of meshes were generated with increasing accuracy. Two of these meshes are shown in Fig. *3(b)* and *(c)*. The sensitivities of the horizontal deflection $(\partial u/\partial b)$ and the direct stress $(\partial \sigma_x/\partial b)$ at point A were calculated and these are compared with exact values derived from elasticity theory [12] in Tables 4 and 5. It can be seen that the $\partial u/\partial b$ terms compare well for all meshes and perturbation factors down to 10^{-5}. However the $\partial \sigma_x/\partial b$ compare favourably for all meshes but only for perturbation factors down to 10^{-3}.

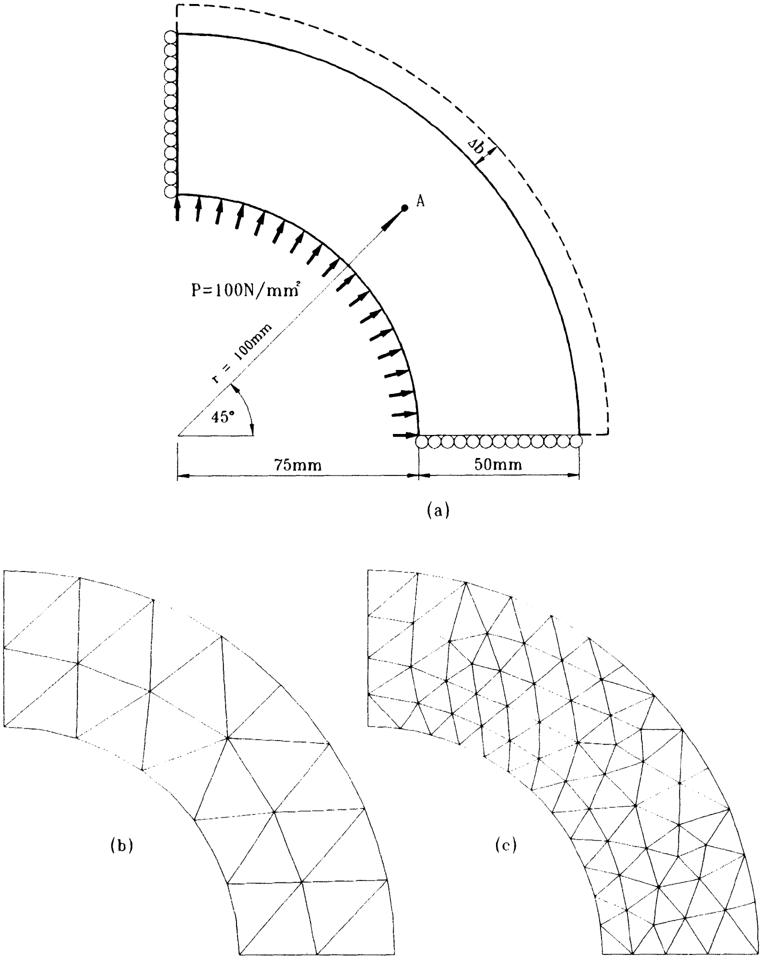

Figure 3: *(a) Cylinder dimensions and loading; (b) Mesh 1 – (25 elements); (c) Mesh 4 – (115 elements)*

		$\partial u/\partial b$					
Mesh error %	**No. of elements**	**Δb/b**					**Exact**
		10^{-1}	**10^{-3}**	**10^{-5}**	**10^{-7}**	**10^{-9}**	
3.58	25	-0.0171	-0.0171	-0.0171	-0.0137	0.3232	-0.0142
1.12	84	-0.0172	-0.0172	-0.0136	0.3425	35.9548	-0.0142
1.03	94	-0.0171	-0.0170	-0.0151	0.1783	19.5187	-0.0142
0.87	115	-0.0170	-0.0170	-0.0144	0.2475	26.4362	-0.0142

Table 4: *The sensitivity of the horizontal displacement at point A in Example 3*

		$\partial \sigma_x/\partial b$					
Mesh error %	**No. of elements**	**Δb/b**					**Exact**
		10^{-1}	**10^{-3}**	**10^{-5}**	**10^{-7}**	**10^{-9}**	
3.58	25	-1.463	-1.463	-1.460	-0.82	58.9	-1.406
1.12	84	-1.405	-1.411	-2.028	-64.17	-6150.7	-1.406
1.03	94	-1.413	-1.417	-1.757	-35.99	-3430.4	-1.406
0.87	115	-1.414	-1.418	-1.875	-47.71	-4571.2	-1.406

Table 5: *The sensitivity of the σ_x at point A in Example 3*

Closure

The BE *point tracking* method has been applied successfully to the calculation of design sensitivities using the semi-analytical method of sensitivity analysis. It has been shown to give both displacement and stress sensitivity results comparable to analytical solutions. It is also readily applicable to sensitivity analysis using *unstructured* FE meshes which is particularly useful in structural shape optimisation systems.

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