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A Hybrid Fuzzy Fractional Order PID Sliding-Mode Controller design using PSO algorithm for interconnected Nonlinear Systems

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Abstract: The aim of this paper is to develop a hybrid fuzzy fractional order sliding mode controller (FFOSMC) for a class of interconnected nonlinear systems. Firstly a $PI^{\alpha}D^{\alpha}$ sliding surface is proposed, on which the control law is designed. Mathematical proof for the stability condition and convergence of the system is presented, taking into account the theory of the fractional order calculus. In order to reduce the chattering phenomenon in sliding mode control (SMC), a Takagi-Sugeno fuzzy logic controller is used to replace the discontinuity in the signum function, and to ensure optimal performance in the closed loop system, the PSO algorithm is used.
Finally the effectiveness of the proposed approach of FFOSMC-based PSO algorithm compared with the FFOSMC using PD^{α} sliding surface and FSMC using the conventional PID sliding surface is demonstrated by simulation results for a coupled double pendulum system.

Keywords: Fractional calculus, SMC, FOSMC, FFOSMC, PSO, Nonlinear systems.

1. INTRODUCTION

With the technological progress of the numerical tools of calculation, the stabilization of interconnected nonlinear systems constitutes at the present time a research axis and development very privileged; the nonlinearity and the coupling in systems makes its control very delicate and complex to implement; to resolve this problem, several approaches were developed in the literature.

The SMC for example was largely proved its efficiency through the reported theoretical studies (Slotine et al. (1991); Emel'yanov (1967)). The first step of SMC design is to select a sliding surface that models the desired closed-loop performance in state variable space. The second step is to design the equivalent and a hitting control law such as the system state trajectories forced toward the sliding surface and slides along it to the desired attitude.

In the literature, several methods for selecting sliding surface have been reported. The approach in (Shi-Yuan et al. (2002); Diantong et al. (2005); Chih-Min et al. (2006); Lon-Chen et al. (2007)) uses a proportional-derivative type sliding surface, where the order of derivation is an integer. In (Ahcene et al. (2009); Djamel et al. (2003)) the sliding surface based on nonlinear continuous function is adopted in order to obtain static feedback. Due to the fact that the fractional order calculus plays an important role in various domains (Chunna et al. (2005); Saptarshi et al. (2012); Elham Amini et al. (2012); Khoichi et al. (1993)); a PD^{α} sliding surface is proposed in (Delavari et al. (2010); BiTao et al. (2012)), and a novel sliding surface in (Chian-Song (2012)) is developed by introducing sign and fractional integral terminal sliding modes. also authors in (Mohammad Pourmahmood (2012)) have proposed a novel fractional-order integral type sliding surface.

Motivated by the above discussion this paper designs a sliding surface based on the fractional order proportional-integral-derivative ($PI^{\alpha}D^{\alpha}$), the best choice of the proposed sliding surface gains can accelerates the movement of the process towards setpoint and eliminates the residual steady-state error that occurs with a pure proportional-derivative action and improves settling time and stability of the system.

Then, to make the developed surface globally attractive and invariant, the control law is designed.

An advantage of these methods of control (SMC) is their robustness to parameter variations and bounded external disturbances. The robustness is attributed to the discontinuous term in the control input. However, this discontinuous term also causes an undesirable effect called chattering, especially, when take the sliding surface without integrator. Sometimes this discontinuous control action can even cause the system performance to be unstable. To reach a better compromise between small chattering and good tracking precision, various compensation strategies have been proposed. For example, integral sliding control (Jung-Hoon et al. (1992); Chern et al. (1993); Baik et al. (1996)), A fuzzy sliding mode control strategy (Her-Terng et al. (2006)). Though introducing a fuzzy logic controller and taking off the *sgn* function in the hitting control law of SMC may reduce the chatter amplitude.

The selection of suitable parameters of fuzzy fractional order sliding mode controller (FFOSMC) is a significant problem, that it can be solved either by manually changing the values or to use some optimization methods, in this paper we are interested by the particle swarm optimization algorithm (PSO).

Finally, a coupled double pendulum system is used to test the performance and effectiveness of the proposed control method using a simulation approach.

The rest of this article is organized as follows. Basic definitions of fractional calculus are described in Section 2.The Fuzzy fractional order sliding mode controller design, in Section 3. After that the PSO approach is described in Section 4. The optimization of FFOSMC with PSO in Section 5. And finally the simulation results and conclusion are given in Sections 6 and 7, respectively.

2. BASIC DEFINITIONS OF FRACTIONAL CALCULUS

The fractional differ-integral operators denoted by ${}_aD_t^\alpha$ (fractional calculus) are a generalization of integration and differentiation of the operators of a non integer order. In the literature we find different definitions of fractional differ-integral, but the commonly used are:

The Riemann-Liouville (RL) definition:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dt}\right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}}d\tau \quad (1) $$

The Caputo's definition:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^m(\tau)}{(t-\tau)^{1-(m-\alpha)}}d\tau \quad (2) $$

where $m-1 < \alpha < m$ and $\Gamma(.)$ is the well known Euler's gamma function, and its definition is:

$$ \Gamma(x) = \int_0^\infty e^{-t}t^{(x-1)}dt, \; x>0 \quad (3) $$

on the other hand, Grunwald-Letnikov (GL) reformulated the definition of the fractional order differ-integral as follows:

$$ {}_aD_t^\alpha f(t) = {}^{lim}_{h\longrightarrow 0} \frac{1}{h^\alpha}\sum_{k=0}^{(t-\alpha)/h}(-1)^k \binom{\alpha}{k} f(t-kh) \quad (4) $$

Because the numerical simulation of a fractional differential equation is not simple as that of an ordinary differential equation, so the Laplace transform method is often used as being a tool for the resolution of the problems arising in engineering (Oldham et al. (1974); Kenneth et al. (1993)).

In the following section, we give the Laplace transforms of the fractional order derivative given previously.

The Laplace transform of (RL) definition is as follow (Oldham et al. (1974); Podlubny (1999)):

$$ L\left\{{}_0D_t^\alpha f(t);s\right\} = s^\alpha F(s) - \sum_{k=0}^{(m-1)} s^k \left[{}_0D_t^{(\alpha-k-1)}f(t)\right]_{t=0} \quad (5) $$

Where the Laplace transform of Caputo's definition is given by (Podlubny (1999)):

$$ L\left\{{}_0D_t^\alpha f(t);s\right\} = s^\alpha F(s) - \sum_{k=0}^{(m-1)} s^{\alpha-k-1}f^k(0) \quad (6) $$

where $s = jw$ denotes the Laplace operator. For the zero initial conditions, the Laplace transforms of fractional derivative of Riemann-Liouville, Caputo and Grunwald-Letnikov are reduced to (7) (Manuel et al. (2009); Podlubny (1999)).

$$ L({}_0D_t^\alpha f(t)) = s^\alpha F(s) \quad (7) $$

In this paper the fractional order element s^α is approximated by Oustaloup's filter. In which, this filter (Alain et al. (2000)) is based on the approximation of a function of the form:

$$ G(s) = s^\alpha, \alpha \in R^+ \quad (8) $$

By a rational function:

$$ \hat{G}(s) = K' \prod_{k=-N'}^{N'} \frac{s+w_k'}{s+w_k} \quad (9) $$

Where the parameters of this function (zeros, poles, and gain) can be determined by the following formulas:

$$ \begin{aligned} w_k' &= w_b(w_h/w_b)^{(k+N'+0.5(1-\alpha))/(2N'+1)} \\ w_k &= w_b(w_h/w_b)^{(k+N'+0.5(1+\alpha))/(2N'+1)} \\ K' &= w_h^\alpha \end{aligned} \quad (10) $$

$(2N'+1)$ is the order of the filter, w_b and w_h are respectively the low and high transient-frequencies. In this paper we consider the 5^{th} order Oustaloup's rational approximation for the FO operator within the frequency range $w \in (10^{-2}, 10^2)$

3. FUZZY FRACTIONAL ORDER SLIDING MODE CONTROLLER DESIGN

We consider a large-scale nonlinear system comprised of n interconnected subsystems defined by:

$$ \begin{aligned} \dot{x}_{2j-1} &= x_{2j}, \; j=1,2,...,n \\ \dot{x}_{2j} &= f_j(x) + b_j(x)u_j \\ x_0 &= x(t_0) \end{aligned} \quad (11) $$

Where $x = [x_1, x_2, ..., x_{2n}]^T \in R^{2n}$ is the state vector, $f_j(x)$ and $b_j(x)$ are nonlinear functions, u_j is the control input of the j^{th} subsystem designed to track a command $x_{(2j-1)d}$ closely. Without losing generality, assume $b_j(x) > 0$ for all x. For this kind of system, we find several control methods, such as, fuzzy control, PID control, sliding mode control,...etc.

3.1 Fractional Order Sliding Mode Controller (FOSMC)

Because the use of centralized control generally results in complex control laws, which can not easily implemented; the overall system (11) will be decomposed into n subsystems, each of them is controlled independently. Interconnections are considered as perturbations for each subsystem.

For the j^{th} subsystem of (11), firstly we propose the following $PI^{\alpha}D^{\alpha}$ $(0 \prec \alpha \prec 1)$ sliding surface using Caputo's definition as:

$$S_j = k_{pj}e_{2j-1} + k_{ij}D_t^{-\alpha_j}(e_{2j-1}) + D_t^{\alpha_j}(e_{2j-1}) \quad (12)$$

Remark: It is clear that selecting $\alpha_j = 1$, a classical PID sliding surfaces can be recovered.

The fractional derivatives caputo right hand definition (RHD) (Shantanu (2011)) of function $f(t)$ gives, $D_t^{\alpha}(f(t)) = D_t^{(\alpha-m)}\frac{d^m}{dt^m}(f(t))$ where m is an integer greater than α. From this we can write the sliding surface S_j as follows:

$$S_j = k_{pj}e_{2j-1} + k_{ij}D_t^{-\alpha_j}(e_{2j-1}) + D_t^{(\alpha_j-1)}(\dot{e}_{2j-1}) \quad (13)$$

Where $e_{2j-1} = x_{2j-1} - x_{(2j-1)d}$, and k_{pj}, k_{ij} are positive constants.

It is obvious from (12) that keeping system states on the sliding surface S_j, $\forall t > 0$ will guarantee that the tracking error vector asymptotically approach to zero. The corresponding sliding condition is:

$$\frac{1}{2}\frac{d}{dt}(S_j^2) = S_j\dot{S}_j \leq 0 \quad (14)$$

The general control structure that satisfies the stability condition of the sliding motion can be written as:

$$\begin{aligned} u_j &= u_{eqj} + \frac{1}{b_j(x)}D_t^{(1-\alpha_j)}(u_{hj}) \\ &= u_{eqj} + \frac{1}{b_j(x)}D_t^{(1-\alpha_j)}(-K_{sj}sgn(S_j)) \end{aligned} \quad (15)$$

Where u_{eqj} is called the equivalent control law that is derived by setting $\dot{S}_j = 0$; and K_{sj} is a positive constant.

we refer to (Podlubny (1999)) for more details. Differentiating both sides of Eq (13) to the order unity yields the equality in (16):

$$\begin{aligned} \dot{S}_j &= k_{pj}\dot{e}_{2j-1} + k_{ij}D_t^{-\alpha_j}(\dot{e}_{2j-1}) + D_t^{(\alpha_j-1)}(\ddot{e}_{2j-1}) \\ &= k_{pj}\dot{e}_{2j-1} + k_{ij}D_t^{-\alpha_j}(\dot{e}_{2j-1}) \\ &+ D_t^{(\alpha_j-1)}(\ddot{x}_{2j-1} - \ddot{x}_{(2j-1)d}) \end{aligned} \quad (16)$$

From Eq(16) one can conclude that:

$$\begin{aligned} D_t^{(1-\alpha_j)}(\dot{S}_j) &= k_{pj}D_t^{(1-\alpha_j)}(\dot{e}_{2j-1}) + k_{ij}D_t^{(1-2\alpha_j)}(\dot{e}_{2j-1}) \\ &+ (\ddot{x}_{2j-1} - \ddot{x}_{(2j-1)d}) \end{aligned} \quad (17)$$

as discussed above u_{eqj} is obtained by setting $\dot{S}_j = 0$; and we have the fractional order derivative of 0 is 0; for this by setting $D_t^{(1-\alpha_j)}(\dot{S}_j) = 0$, the equivalent control is obtained, and it has the flowing formula:

$$\begin{aligned} u_{eqj} &= \frac{-1}{b_j(x)}(f_j(x) - \ddot{x}_{(2j-1)d} + k_{pj}D_t^{(1-\alpha_j)}(\dot{e}_{2j-1}) \\ &+ k_{ij}D_t^{(1-2\alpha_j)}(\dot{e}_{2j-1})) \end{aligned} \quad (18)$$

To verify the stability condition, substituting Eq(15) with the given u_{eqj} into Eq(11) yields:

$$\begin{aligned} \dot{x}_{2j} = \ddot{x}_{2j-1} = \ddot{x}_{(2j-1)d} &- k_{pj}D_t^{(1-\alpha_j)}(\dot{e}_{2j-1}) \\ &-k_{ij}D_t^{(1-2\alpha_j)}(\dot{e}_{2j-1}) \\ &-K_{sj}D_t^{(1-\alpha_j)}(sgn(S_j)) \end{aligned} \quad (19)$$

Eq(19) becomes

$$\begin{aligned} &\ddot{x}_{2j-1} - \ddot{x}_{(2j-1)d} + k_{pj}D_t^{(1-\alpha_j)}(\dot{e}_{2j-1}) \\ &+k_{ij}D_t^{(1-2\alpha_j)}(\dot{e}_{2j-1}) = -K_{sj}D_t^{(1-\alpha_j)}(sgn(S_j)) \end{aligned} \quad (20)$$

By using Eq (17), the Eq (20) can be rewritten as follows:

$$D_t^{(1-\alpha_j)}(\dot{S}_j) = -K_{sj}D_t^{(1-\alpha_j)}(sgn(S_j)) \quad (21)$$

Differentiate (21) to the order $(\alpha_j - 1)$. Since $(\alpha_j - 1) \prec 0$ this indeed corresponds to fractional order integration, corresponding to negative valued α in Caputo's definition in (2); and taking into account the property of Caputo's derivative ${}_aD_t^{-\alpha}({}_aD_t^{\alpha}f(t)) = f(t) - \sum_{i=0}^{m-1}\frac{f^i(a)}{i!}(t-a)^i$.

$$\dot{S}_j + \dot{S}_j(0) = -K_{sj}(sgn(S_j)) + K_{sj}(sgn(S_j(0))) \quad (22)$$

considering $\dot{S}_j(0) = 0, (sgn(S_j(0))) = 0$ for $m = 1$; this lets us have $\dot{S}_j = -K_{sj}.(sgn(S_j))$.

Thus by using (14) we can obtain:

$$\begin{aligned} S_j\dot{S}_j &= S_j(-K_{sj}(sgn(S_j))) \\ &= -K_{sj}\left|S_j\right| \leq 0 \end{aligned} \quad (23)$$

Lemma 1. (Denis (1996)) Consider the following autonomous linear fractional-order system:

$${}_0D_t^{\alpha}x(t) = Ax(t),\ x(0) = x_0 \quad (24)$$

where $x \in R^n, A = (a_{ij}) \in R^{n\times n}, 0 \prec \alpha \prec 1$, is asymptotically stable if and only if (see figure 1):

$$|arg(eig(A))| > \alpha\frac{\pi}{2} \quad (25)$$

In this case, the components of the state decay towards 0 like $t^{-\alpha}$.

Proof. When the sliding mode occurs, system (12) can be represented as follows:

$$k_{pj}(e_{2j-1}) + k_{ij}D_t^{-\alpha_j}(e_{2j-1}) + D_t^{\alpha_j}(e_{2j-1}) = 0 \quad (26)$$

Taking the fractional derivative of order α_j of Eq.(26), with respect to $D_t^{\alpha}(f(t)) = D_t^{\alpha_1}D_t^{\alpha_2}...D_t^{\alpha_n}(f(t))$, $(\alpha =$

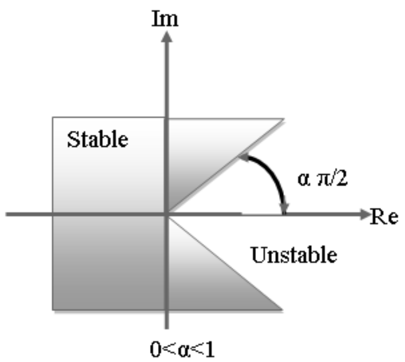

Fig. 1. Stable domain of fractional order system in s^{α} plane

$\alpha_1+\alpha_2+...+\alpha_n, \alpha_i \prec 1$) (Kenneth et al. (1993); Shantanu (2011)) yields:

$$k_{ij}(e_{2j-1}) + k_{pj} D_t^{\alpha_j}(e_{2j-1}) + D_t^{2\alpha_j}(e_{2j-1}) = 0 \quad (27)$$

The derivative operator can be the Caputo's definition.

Therefore, the sliding mode dynamics is obtained by the following equations:

$$\begin{aligned} D_t^{\alpha_j}(e_{2j-1}) &= e_{2j} \\ D_t^{\alpha_j}(e_{2j}) &= -k_{ij}e_{2j-1} - k_{pj}e_{2j} \end{aligned} \quad (28)$$

or in a matrix equation form as:

$$\begin{bmatrix} D_t^{\alpha}(e_{2j-1}) \\ D_t^{\alpha}(e_{2j}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_{ij} & -k_{pj} \end{bmatrix} \begin{bmatrix} e_{2j-1} \\ e_{2j} \end{bmatrix} \quad (29)$$

The sliding surface parameters k_{pj} and k_{ij} are selected to be positive such that the eigenvalues of matrix A satisfy the stability condition of Lemma 1.

In summary the proposed $PI^{\alpha}D^{\alpha}$ sliding surface can guarantee the stability, in the sense of Lemma 1 and Lyapunov theorem. However, a large control gain K_{sj} often causes the chattering effect. In order to tackle this problem, a continuous fuzzy logic control term Δu_j is used to approximate u_{hj}

3.2 Fuzzy Fractional Order Sliding Mode Controller

Takagi-Sugeno (Takagi and Sugeno, 1985) fuzzy models are used by many authors to design controllers for nonlinear systems (Zsofia et al. (2013); Mohamed Laid et al. (2011)), because it has the ability to approximate any nonlinear behavior.

in this paper the Fuzzy Fractional Order Sliding Mode Controller (FFOSMC) is considered as a hybrid controller, it can be regarded as a T-S fuzzy controller that controls the fractional order sliding surface S_j approach to zero.

the structure of a fuzzy controller design consists of: 1) the definition of input-output fuzzy variables; 2) decision-making related to fuzzy control rules; 3) fuzzy inference logic; and 4) defuzzification.

For the proposed FFOSMC, we used the sliding surface (S_j) as input at the fuzzy controller, and Δu_j is the fuzzy controller output. The structure is shown in figure 2:

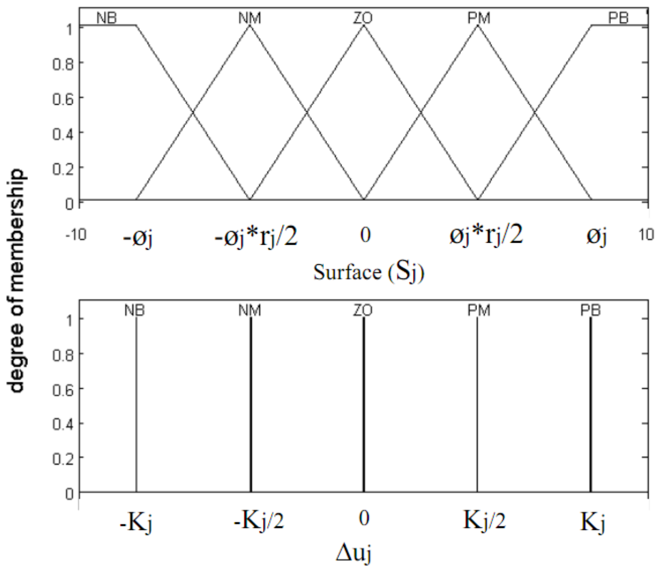

Fig. 3. Membership functions of input variable (S_j) and FLC output (Δu_j) for the FFOSMC of the j^{th} subsystem

Where:

$$\begin{aligned} u_j &= u_{eqj} + u_{fj} \\ &= u_{eqj} + \frac{1}{b_j(x)} D_t^{(1-\alpha_j)}(\Delta u_j) \end{aligned} \quad (30)$$

Assuming that the input and output of the j^{th} fuzzy controller has five level language variables, its membership function is shown in figure 3. ϕ_j and K_j are used to expand or shrink the divisions of the membership functions along the universes of discourse, r_j is a coefficient to be used to adjust the width of the input membership function of the linguistic variable Zero (Ahcene et al. (2008)).

Such linguistic expressions can be used to form the fuzzy control rules as below:

Rule 1: IF S_j is NB, THEN Δu_j is PB.

Rule 2: IF S_j is NM, THEN Δu_j is PM.

Rule 3: IF S_j is ZO, THEN Δu_j is ZO.

Rule 4: IF S_j is PM, THEN Δu_j is NM.

Rule 5: IF S_j is PB, THEN Δu_j is NB.

Where **NB** denotes "Negative Big", **NM** denotes "Negative Mid", **ZO** denotes "Zero", **PB** denotes "Positive Big", and **PM** denotes "Positive Mid".

The FLC output (Δu) is determined using the weighted average method (Ahcene et al. (2008)).

4. PARTICLE SWARM OPTIMIZATION

Particle swarm optimization (PSO) is an evolutionary computation technique developed by Kennedy and Eberhart in 1995 (Russell et al. (1995)). The inspiration underlying the development of this algorithm was the social behaviour of animals, such as the flocking of birds and the schooling of fish, and the swarm theory. It has been proven to be efficient in solving optimization problem especially for nonlinearity and non differentiability, multiple opti-

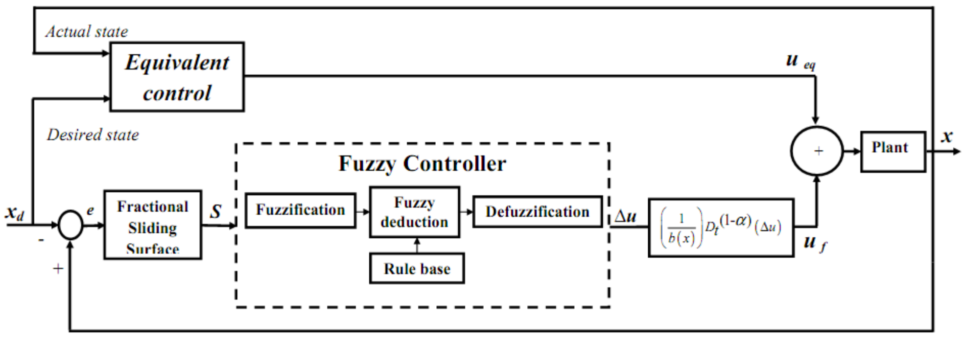

Fig. 2. Structure of the proposed FFOSMC for each susbsystem

mum, high dimensionality (Wei-Der et al. (2010); Juing-Shian et al. (2009)) and parameters identification (Tlili et al. (2013)).

In PSO, the velocity of each particle is modified iteratively by its individual best position (*pbest*), and the global best position (*gbest*) found by particles in its neighborhood. As a result, each particle searches around a region defined by its individual best position (*pbest*) and the global best position (*gbest*) from its neighborhood. Henceforth we use V_i to denote the velocity of the i^{th} particle in the swarm, p_i denote its position. At each step (or iteration) J, by using the individual best position, (*pbest*), and global best position, (*gbest*), the velocity and position of each particle are updated by the following two equations:

$$V_i(J) = W[V_i(J-1) + c_1 rand_1(pbest_i - p_i(J-1)) + c_2 rand_2(gbest - p_i(J-1))] \tag{31}$$

$$p_i(J) = p_i(J-1) + V_i(J) \tag{32}$$

Where $rand_1$ and $rand_2$ are random numbers between 0 and 1; c_1 and c_2 are positive constant learning rates; W is called the constriction factor (Maurice (1999)) and is defined by (33):

$$W = \frac{2}{\left|2 - c - \sqrt{c^2 - 4.c}\right|}, c = c_1 + c_2, \ c \succ 4 \tag{33}$$

In each step J the position is confined within the range of $[p_{min}, p_{max}]$. If the position violates these limits, it is forced to its proper values (Wei-Der et al. (2010)).

$$p_i = \begin{cases} p_{min} & if \quad p_i < p_{min} \\ p_i & if \ p_{min} < p_i < p_{max} \\ p_{max} & if \quad p_i > p_{max} \end{cases} \tag{34}$$

Changing position by this way enables the i^{th} particle to search around its individual best position *pbest*, and global best position, *gbest*.

The following shows the design step for implementing the PSO algorithm (Wei-Der et al. (2010)).

Step 1. Initialize particles with random position and velocity on dimension in the problem space.

Step 2.If a prescribed number of iterations (generations) is achieved, and then stop the algorithm.

Step 3.For each particle, evaluate the desired optimization fitness function, and record each particle's best previous position (*pbest*), and global best position (*gbest*).

Step 4.Change the velocity and position according to equations (31) and (32) respectively, for each particle

Step 5. Check each particle's position using (34).

Step 6. Go back to Step 2.

5. OPTIMIZATION OF FFOSMC WITH PSO

The design problem is defined as finding the optimum values of the fuzzy fractional order sliding mode controller parameters in the closed-loop system. The parameters vector composed by the positions of the membership functions (when the conclusions are fixed), the gains k_{pj}, k_{ij}, and the fractional orders α_j.

Let $p_i = [\phi_j, r_j, K_j, k_{pj}, k_{ij}, \alpha_j]$ the vector of selective parameters of FFOSMC, where the regions of these selective parameters are mentioned as follows.

$$0.1 < \phi_j < 10 \ , \quad 0.1 < r_j < 1 \ , \quad 0.1 < K_j < 20 \ , \\ 0.01 < k_{pj}, k_{ij} < 20 \ , \quad 0.1 < \alpha_j < 0.98$$

To converge toward the optimal solution, the PSO algorithm must be guided by the cost function. Hence, it should be properly defined before the PSO algorithm is executed. In the present study, the used cost function (F) is defined by the following formula:

$$F = \sum_{i=1}^{N} (\sum_{j=1}^{n} (\gamma_{2j-1} |e_{2j-1}(i)| + \gamma_{2j} |u_j(i)|)) \tag{35}$$

Where $e_{2j-1}(i)$ is the trajectory error of i^{th} sample, $u_j(i)$ is the control signal of i^{th} sample and N is the number of sample. The weighting factors γ_{2j-1} and γ_{2j} are used to give more flexibility to the designer depending on the nature of application and relative importance of low error and control signal. In this present study the weighting factors γ_{2j-1} and γ_{2j} $\forall j = 1, 2, ..., n$ are set to 3 and 0.1 respectively. Figure 4 illustrates the block structure of the FFOSMC optimizing process using PSO algorithm.

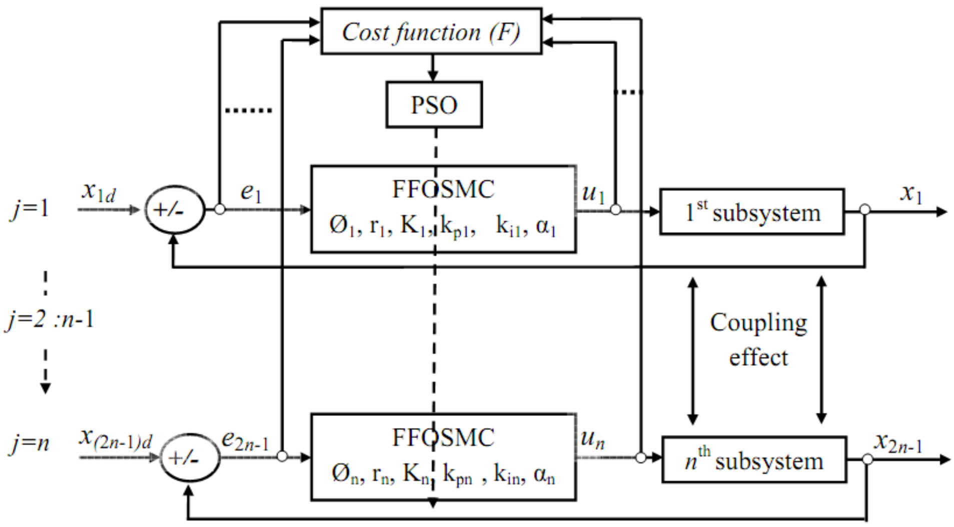

Fig. 4. Tuning process of FFOSMC parameters with PSO

6. SIMULATION RESULTS AND DISCUSSION

In this section, we shall demonstrate that the FFOSMC tuned with PSO is applicable to the problem of trajectory tracking control of a coupled double pendulum system, which is illustrated on figure 5. Tuning process by PSO is also applied to the FFOSMC using PD^{α} sliding surface and FSMC using the conventional PID sliding surface.

The simulation is carried out using the "Matlab/Simulink" tools within 0.01 sample time. The population size of PSO algorithm is set to 20 particles. The parameters c_1, c_2 and W are set to 2.05, 2.05 and 0.7298 respectively, and the maximum number of iteration J is set to 50 iterations.

The dynamic of the coupled double pendulum system is described below.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= (\frac{m_1 g r}{J_1})sin(x_1) + (\frac{kr^2}{4J_1})(sin(x_3) - sin(x_1))\\ &\quad + \frac{kr}{2J_1}(l-b) + \frac{u_1}{J_1}\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= (\frac{m_2 g r}{J_2})sin(x_3) + (\frac{kr^2}{4J_1})(sin(x_2) - sin(x_3))\\ &\quad - \frac{kr}{2J_2}(l-b) + \frac{u_2}{J_2}\end{aligned} \tag{36}$$

Where $x_1 = \theta_1$, and $x_3 = \theta_2$ are the angular displacements of the pendulums with respect to the vertical axis, u_1 and u_2 are the control input torques:

In this simulation example, the following parameters are used;

$k = 100N/M, g = 9.81m/s^2, m_1 = 2kg, m_2 = 2.5kg, J_1 = 0.5kg, J_2 = 0.625kg, l = 0.5m, r = 0.5m, b = 0.4m.$

by using the theoretical development given previously in section 3.1, firstly the fractional order PID sliding surfaces S_1 and S_2 are given by:

$$\begin{aligned}S_1 &= k_{p1}e_1 + k_{i1}D_t^{-\alpha_1}(e_1) + D_t^{\alpha_1}(e_1)\\ S_2 &= k_{p2}e_3 + k_{i2}D_t^{-\alpha_2}(e_3) + D_t^{\alpha_2}(e_3)\\ e_1 &= x_1 - x_{1d}\\ e_3 &= x_3 - x_{3d}\end{aligned} \tag{37}$$

And the control signals by the following:

$$\begin{aligned}u_1 &= \tfrac{-1}{b_1(x)}(f_1(x) - \ddot{x}_{1d} + k_{p1}D_t^{(1-\alpha_1)}(\dot{e}_1)\\ &\quad + k_{i1}D_t^{(1-2\alpha_1)}(\dot{e}_1) - D_t^{(1-\alpha_1)}(\Delta u_1))\\ u_2 &= \tfrac{-1}{b_2(x)}(f_2(x) - \ddot{x}_{3d} + k_{p2}D_t^{(1-\alpha_2)}(\dot{e}_3)\\ &\quad + k_{i2}D_t^{(1-2\alpha_2)}(\dot{e}_3) - D_t^{(1-\alpha_2)}(\Delta u_2))\end{aligned} \tag{38}$$

The reference trajectories used in the simulations are given by:

$$\begin{aligned}x_{1d} &= 0.12(cos(\pi t) + sin(\tfrac{2\pi t}{3}))\quad rad\\ x_{3d} &= 0.12(sin(\pi t) + cos(\tfrac{\pi t}{3}))\quad rad\end{aligned} \tag{39}$$

Figure 6 shows the cost function evolution during the optimization process, after 50 iteration the PSO algorithm converges to the optimal parameters on table 1. The angles position (x_1, x_3) and control signals (u_1, u_2) are illustrated on figure 7.

From table 1, we observe that the Lemma 1 is verified, then the system is stable; and to prove this: we have for the first subsystem; $|arg(eig(\lambda_1))| = |arg(eig(\lambda_2))| = 1.9765 > 0.4770\frac{\pi}{2}$; where (λ_1, λ_2) are the eigenvalues of matrix A in Eq(28) with the substitution of k_p and k_i. For the second subsystem we have also; $|arg(eig(\lambda_1))| = |arg(eig(\lambda_2))| = 1.7677 > 0.6317\frac{\pi}{2}$;

For the robustness evaluation of the controllers tuned by PSO, the parameters m_1, m_2 and r are changed to the values $3.5kg, 4kg$ and $0.7m$ respectively, where the parameters of the different controllers are kept unchanged; the simulation results are shown in figure 8.

in order to compare the performance of the proposed controller with the FFOSMC using PD^{α} sliding surface

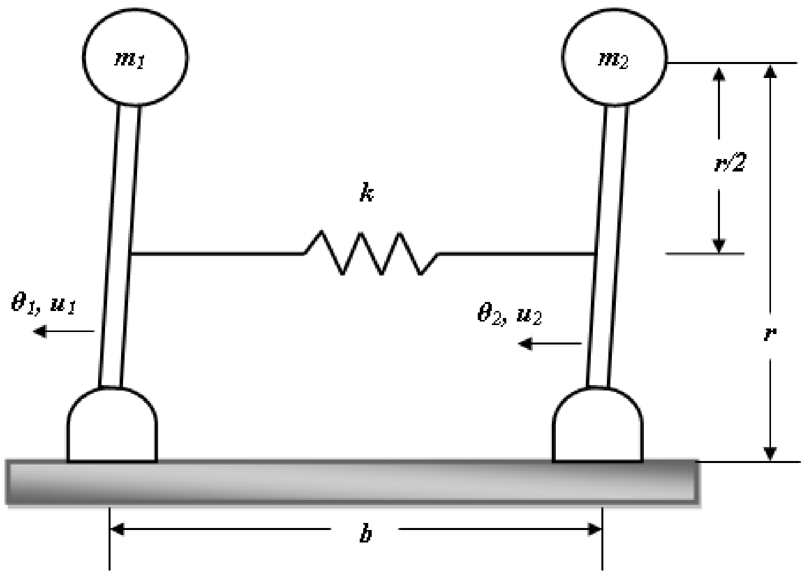

Fig. 5. coupled double pendulum system

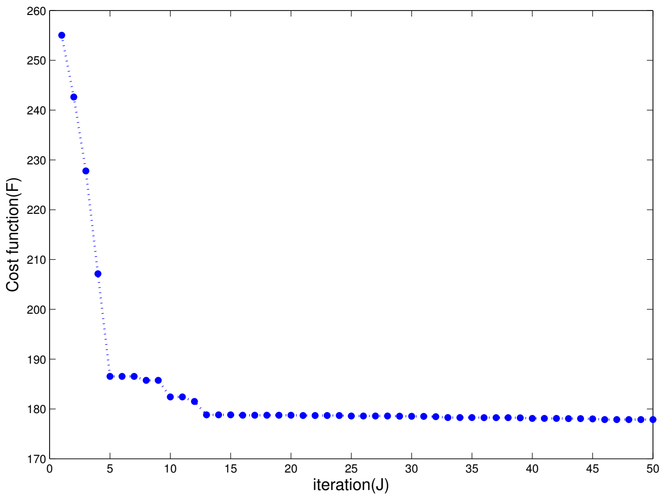

Fig. 6. Evolution of the cost function (F) vs iteration (J)

Table 1. Optimal parameters for FFOSMC

controller parameters	1st subsystem	2nd subsystem
ϕ	0.1131	0.1000
r	0.1451	0.1091
K	1.0165	0.8927
k_p	0.9444	0.2877
k_i	1.4313	0.5406
α	0.4770	0.6317

and FSMC using the conventional PID sliding surface, we define two cost functions F_1 and F_2, such as:

$$F_1 = \sum_{i=1}^{N}(\sum_{j=1}^{n}(e_{2j-1}^2(i))) \tag{40}$$

and

$$F_2 = \sum_{i=1}^{N}(\sum_{j=1}^{n}(u_j^2(i))) \tag{41}$$

Table 2. Comparative study for the first subsystem, a) without any disturbance, b) with parameters variation

		$PI^\alpha D^\alpha$	PD^α	PID
a)	F_1	0.0962	0.1422	0.2278
	F_2	1.7990×10^3	1.5881×10^3	980.7940
b)	F_1	0.0962	0.1422	0.2278
	F_2	4.3012×10^3	3.9391×10^3	3.0448×10^3

Table 3. Comparative study for the second subsystem, a) without any disturbance, b) with parameters variation

		$PI^\alpha D^\alpha$	PD^α	PID
a)	F_1	0.1451	0.1767	0.2600
	F_2	3.2281×10^3	2.8596×10^3	2.4612×10^3
b)	F_1	0.1451	0.1767	0.2600
	F_2	8.2032×10^3	7.8582×10^3	7.2046×10^3

Table 4. Comparative study for the coupled system, a) without any disturbance, b) with parameters variation

		$PI^\alpha D^\alpha$	PD^α	PID
a)	F_1	0.2413	0.3188	0.4878
	F_2	5.0271×10^3	4.4450×10^3	3.4420×10^3
b)	F_1	0.2413	0.3188	0.4878
	F_2	1.2504×10^4	1.1797×10^4	1.0249×10^4

The simulation results for each performance index are given in tables 2, 3 and 4.

From the simulation results, it can be seen that the proposed FFOSMC using $PI^\alpha D^\alpha$ sliding surface performs better control specification such as fast response and trajectory tracking task compared with FFOSMC using PD^α sliding surface and conventional sliding mode controller using PID sliding surface. However when compared with respect to small magnitude of control signal, the conventional sliding mode controller gives better results compared with the other using the theory of fractional calculus.

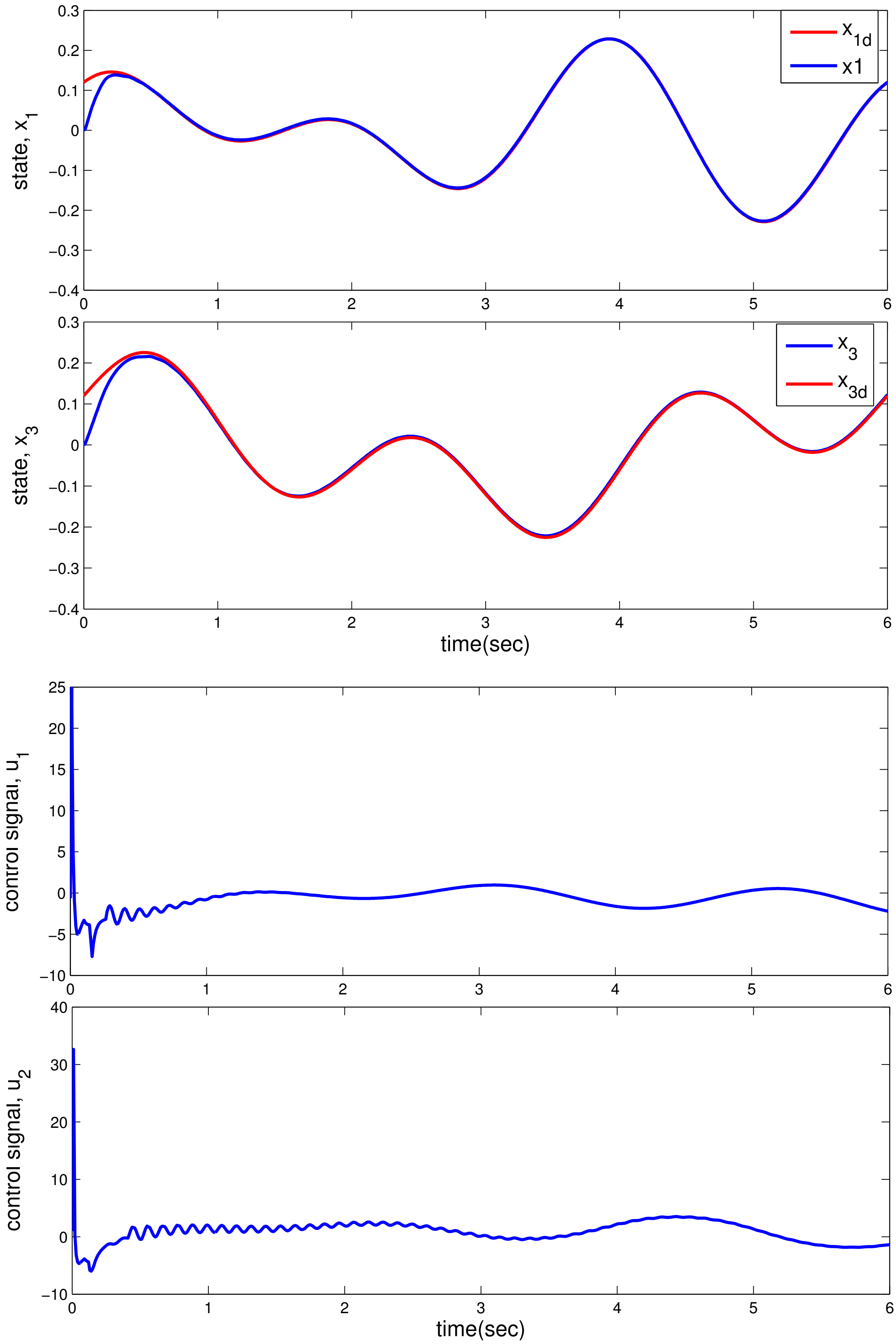

Fig. 7. Simulation results of the coupled double pendulum system using optimized fuzzy fractional order sliding mode controller

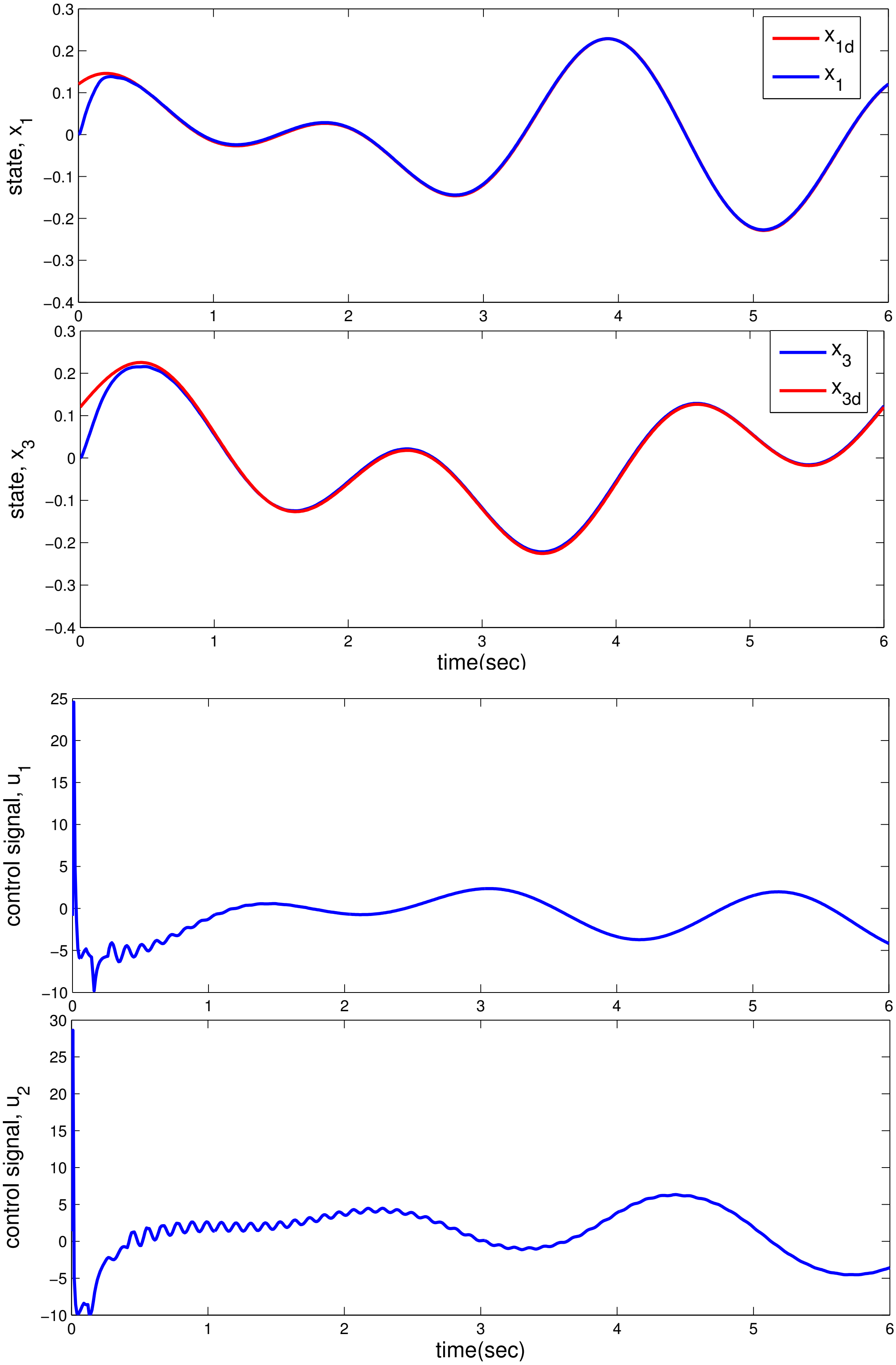

Fig. 8. Simulation results of the coupled double pendulum system using optimized fuzzy fractional order sliding mode controller under parameters variation

also with parameters variation we can see that the different controllers tuned with PSO give better results; and the system remains stable.

7. CONCLUSION

In this paper a Fuzzy Fractional Order Sliding Mode Controller that combines the advantages in term of robustness of the fractional calculus, fuzzy logic for its ability to express the amount of ambiguity in human reasoning and sliding mod controller in term of robustness to parameters variation and external disturbances, is investigated for the coupled double pendulum system.

Firstly, $PI^{\alpha}D^{\alpha}$ surface sliding mode controller is used. The design yields an equivalent control term with an addition of fuzzy logic control to approximate the discontinuous control term and to alleviate the chattering phenomenon. Then the application of the PSO method can perform an efficient search for the optimal parameters of both FSMC and FFOSMC, and achieve good accuracy.

Finally, the simulation results show the effectiveness of the proposed controller algorithm for interconnected nonlinear systems.

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