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Research Article

A Hybrid Method for a Countable Family of Multivalued Maps, Equilibrium Problems, and Variational Inequality Problems

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We introduce a new monotone hybrid iterative scheme for finding a common element of the set of common fixed points of a countable family of nonexpansive multivalued maps, the set of solutions of variational inequality problem, and the set of the solutions of the equilibrium problem in a Hilbert space. Strong convergence theorems of the purposed iteration are established.

1. Introduction

Let D be a nonempty convex subset of a Banach spaces E. Let F be a bifunction from $D \times D$ to \mathbb{R} , where \mathbb{R} is the set of all real numbers. The equilibrium problem for F is to find $x \in D$ such that $F(x,y) \geq 0$ for all $y \in D$. The set of such solutions is denoted by EP(F). The set D is called *proximal* if for each $x \in E$, there exists an element $y \in D$ such that $\|x - y\| = d(x, D)$, where $d(x,D) = \inf\{\|x - z\| : z \in D\}$. Let CB(D), K(D), and P(D) denote the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of D, respectively. The *Hausdorff metric* on CB(D) is defined by

$$H(A,B) = \max \left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\}$$
(1.1)

for $A, B \in CB(D)$. A single-valued map $T: D \to D$ is called *nonexpansive* if $||Tx - Ty|| \le ||x - y||$ for all $x, y \in D$. A multivalued map $T: D \to CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \le CB(D)$

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 $\|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T: D \to D$ (resp., $T: D \to CB(D)$) if p = Tp (resp., $p \in Tp$). The set of fixed points of T is denoted by F(T). The mapping $T: D \to CB(D)$ is called *quasi-nonexpansive* [1] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in D$ and all $p \in F(T)$. It is clear that every nonexpansive multivalued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive; see [2].

The mapping $T:D\to CB(D)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in D such that $d(x_n,Tx_n)\to 0$ as $n\to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to p\in D$. We note that if D is compact, then every multivalued mapping $T:D\to CB(D)$ is *hemicompact*.

A mapping $T: D \to CB(D)$ is said to satisfy *Condition* (*I*) if there is a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with f(0) = 0, f(r) > 0 for $r \in (0, \infty)$ such that

$$d(x,Tx) \ge f(d(x,F(T))) \tag{1.2}$$

for all $x \in D$.

In 1953, Mann [3] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in [0,1].

However, we note that Mann's iteration process (1.3) has only weak convergence, in general; for instance, see [4–6].

In 2003, Nakajo and Takahashi [7] introduced the method which is the so-called CQ method to modify the process (1.3) so that strong convergence is guaranteed. They also proved a strong convergence theorem for a nonexpansive mapping in a Hilbert space.

Recently, Tada and Takahashi [8] proposed a new iteration for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping T in a Hilbert space H.

In 2005, Sastry and Babu [9] proved that the Mann and Ishikawa iteration schemes for multivalued map T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. More precisely, they proved the following result for nonexpansive multivalued map with compact domain.

In 2007, Panyanak [10] extended the above result of Sastry and Babu [9] to uniformly convex Banach spaces but the domain of *T* remains compact.

Later, Song and Wang [11] noted that there was a gap in the proofs of Theorem 3.1 [10] and Theorem 5 [9]. They further solved/revised the gap and also gave the affirmative answer to Panyanak [10] question using the following Ishikawa iteration scheme. In the main results, domain of T is still compact, which is a strong condition (see [11, Theorem 1]) and T satisfies condition (I) (see [11, Theorem 1]).

In 2009, Shahzad and Zegeye [2] extended and improved the results of Panyanak [10], Sastry and Babu [9], and Song and Wang [11] to quasi-nonexpansive multivalued maps. They also relaxed compactness of the domain of T and constructed an iteration scheme which removes the restriction of T, namely, $Tp = \{p\}$ for any $p \in F(T)$. The results provided an affirmative answer to Panyanak [10] question in a more general setting. In the main results,

T satisfies Condition (I) (see [2, Theorem 2.3]) and T is hemicompact and continuous (see [2, Theorem 2.5]).

A mapping $A:D\to H$ is called α -inverse-strongly monotone [12] if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \ge \alpha ||Ax - Ay||^2, \quad \forall x, y \in D.$$
 (1.4)

Remark 1.1. It is easy to see that if $A:D\to H$ is α -inverse-strongly monotone, then it is a $(1/\alpha)$ -Lipschitzian mapping.

Let $A: D \to H$ be a mapping. The classical variational inequality problem is to find a $u \in D$ such that

$$\langle Au, v - u \rangle \ge 0, \quad \forall v \in D.$$
 (1.5)

The set of solutions of variational inequality (3.9) is denoted by VI(D, A).

Question. How can we construct an iteration process for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of a variational inequality problem, and the set of common fixed points of nonexpansive multivalued maps?

In the recent years, the problem of finding a common element of the set of solutions of equilibrium problems and the set of fixed points of single-valued nonexpansive mappings in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors; for instance, see [8, 13–20] and the references cited theorems.

In this paper, we introduce a monotone hybrid iterative scheme for finding a common element of the set of a common fixed points of a countable family of nonexpansive multivalued maps, the set of variational inequality, and the set of solutions of an equilibrium problem in a Hilbert space.

2. Preliminaries

The following lemmas give some characterizations and a useful property of the metric projection P_D in a Hilbert space.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let D be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in D, denoted by $P_D x$, such that

$$||x - P_D x|| \le ||x - y||, \quad \forall y \in D.$$
 (2.1)

 P_D is called the *metric projection* of H onto D. We know that P_D is a nonexpansive mapping of H onto D.

Lemma 2.1 (see [21]). Let D be a closed and convex subset of a real Hilbert space H and let P_D be the metric projection from H onto D. Given $x \in H$ and $z \in D$, then $z = P_D x$ if and only if the following holds:

$$\langle x - z, y - z \rangle \le 0, \quad \forall y \in D.$$
 (2.2)

Lemma 2.2 (see [7]). Let D be a nonempty, closed and convex subset of a real Hilbert space H and $P_D: H \to D$ the metric projection from H onto D. Then the following inequality holds:

$$\|y - P_D x\|^2 + \|x - P_D x\|^2 \le \|x - y\|^2, \quad \forall x \in H, \ \forall y \in D.$$
 (2.3)

Lemma 2.3 (see [21]). Let H be a real Hilbert space. Then the following equations hold:

(i)
$$||x - y||^2 = ||x||^2 - ||y||^2 - 2\langle x - y, y \rangle$$
, for all $x, y \in H$;

(ii)
$$||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2$$
, for all $t \in [0,1]$ and $x, y \in H$.

Lemma 2.4 (see [22]). Let D be a nonempty, closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set

$$\left\{ v \in D : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v \rangle + a \right\}$$
 (2.4)

is convex and closed.

For solving the equilibrium problem, we assume that the bifunction $F: D \times D \to \mathbb{R}$ satisfies the following conditions:

- (A1) F(x,x) = 0 for all $x \in D$;
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \le 0$ for all $x, y \in D$;
- (A3) for each $x, y, z \in D$, $\limsup_{t \downarrow 0} F(tz + (1 t)x, y) \le F(x, y)$;
- (A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in D$.

Lemma 2.5 (see [13]). Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $D \times D$ to \mathbb{R} satisfying (A1)–(A4) and let r > 0 and $x \in H$. Then, there exists $z \in D$ such that

$$F(z,y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \quad \forall y \in D.$$
 (2.5)

Lemma 2.6 (see [18]). For r > 0, $x \in H$, defined a mapping $T_r : H \to D$ as follows:

$$T_r(x) = \left\{ z \in D : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \forall y \in D \right\}. \tag{2.6}$$

Then the following holds:

(1) T_r is a single value;

(2) T_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$||T_r x - T_r y||^2 \le \langle T_r x - T_r y, x - y \rangle; \tag{2.7}$$

- (3) $F(T_r) = EP(F)$;
- (4) EP(F) is closed and convex.

In the context of the variational inequality problem,

$$u \in VI(D, A) \iff u = P_D(u - \lambda A u), \quad \forall \lambda > 0.$$
 (2.8)

A set-valued mapping $T: H \to 2^H$ is said to be monotone if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply that $\langle f - g, x - y \rangle \geq 0$. A monotone mapping $T: H \to H$ is said to be maximal [23] if the graph G(T) of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping is maximal if and only if for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \geq 0$, $\forall (y, g) \in G(T)$ imply that $f \in Tx$. Let $A: D \to H$ be an inverse strongly monotone mapping and let $N_D v$ be the normal cone to D at $v \in D$, that is,

$$N_D v = \{ w \in H : \langle v - u, w \rangle \ge 0, \forall u \in D \}, \tag{2.9}$$

and define

$$Tv = \begin{cases} Av + N_D v, & v \in D, \\ \emptyset, & v \notin D. \end{cases}$$
 (2.10)

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(D, A)$ (see, e.g., [24]).

In general, the fixed point set of a nonexpansive multivalued map T is not necessary to be closed and convex (see [25, Example 3.2]). In the next Lemma, we show that F(T) is closed and convex under the assumption that $Tp = \{p\}$ for all $p \in F(T)$.

Lemma 2.7. Let D be a closed and convex subset of a real Hilbert space H. Let $T:D\to CB(D)$ be a nonexpansive multivalued map with $F(T)\neq\emptyset$ and $Tp=\{p\}$ for each $p\in F(T)$. Then F(T) is a closed and convex subset of D.

Proof. First, we will show that F(T) is closed. Let $\{x_n\}$ be a sequence in F(T) such that $x_n \to x$ as $n \to \infty$. We have

$$d(x,Tx) \le d(x,x_n) + d(x_n,Tx)$$

$$\le d(x,x_n) + H(Tx_n,Tx)$$

$$\le 2d(x,x_n).$$
(2.11)

It follows that d(x,Tx) = 0, so $x \in F(T)$. Next, we show that F(T) is convex. Let $p = tp_1 + (1-t)p_2$ where $p_1, p_2 \in F(T)$ and $t \in (0,1)$. Let $z \in Tp$; by Lemma 2.3, we have

$$||p-z||^{2} = ||t(z-p_{1}) + (1-t)(z-p_{2})||^{2}$$

$$= t||z-p_{1}||^{2} + (1-t)||z-p_{2}||^{2} - t(1-t)||p_{1}-p_{2}||^{2}$$

$$= td(z,Tp_{1})^{2} + (1-t)d(z,Tp_{2})^{2} - t(1-t)||p_{1}-p_{2}||^{2}$$

$$\leq tH(Tp,Tp_{1})^{2} + (1-t)H(Tp,Tp_{2})^{2} - t(1-t)||p_{1}-p_{2}||^{2}$$

$$\leq t||p-p_{1}||^{2} + (1-t)||p-p_{2}||^{2} - t(1-t)||p_{1}-p_{2}||^{2}$$

$$= t(1-t)^{2}||p_{1}-p_{2}||^{2} + (1-t)t^{2}||p_{1}-p_{2}||^{2} - t(1-t)||p_{1}-p_{2}||^{2}$$

$$= 0.$$
(2.12)

Hence p = z. Therefore, $p \in F(T)$.

3. Main Results

In the following theorem, we introduce a new monotone hybrid iterative scheme for finding a common element of the set of a common fixed points of a countable family of nonexpansive multivalued maps, the set of variational inequality, and the set of solutions of an equilibrium problem in a Hilbert space, and we prove strong convergence theorem without the condition (I).

Theorem 3.1. Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $D \times D$ to \mathbb{R} satisfying (A1)–(A4), let $A:D \to H$ be an α -inverse strongly monotone mapping, and let $T_i:D \to CB(D)$ be nonexpansive multivalued maps for all $i \in \mathbb{N}$ with $\Omega:=\bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \cap VI(D,A) \neq \emptyset$ and $T_ip=\{p\}, \forall p \in \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $\alpha_{i,n} \in [0,1)$ with $\limsup_{n\to\infty} \alpha_{i,n} < 1$ for all $i \in \mathbb{N}$, $\{r_n\} \subset [b,\infty)$ for some $b \in (0,\infty)$, and $\{\lambda_n\} \subset [c,d]$ for some $c,d \in (0,2\alpha)$. For an initial point $x_0 \in H$ with $C_1=D$ and $x_1=P_{C_1}x_0$, let $\{x_n\},\{y_n\},\{s_{i,n}\},$ and $\{u_n\}$ be sequences generated by

$$F(u_{n}, y) + \frac{1}{r_{n}} \langle y - u_{n}, u_{n} - x_{n} \rangle \geq 0, \quad \forall y \in D,$$

$$y_{n} = P_{D}(u_{n} - \lambda_{n} A u_{n}),$$

$$s_{i,n} = \alpha_{i,n} y_{n} + (1 - \alpha_{i,n}) z_{i,n},$$

$$C_{i,n+1} = \{ z \in C_{i,n} : ||s_{i,n} - z|| \leq ||y_{n} - z|| \leq ||x_{n} - z|| \},$$

$$C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1},$$

$$x_{n+1} = P_{C_{n+1}} x_{0}, \quad \forall n \in \mathbb{N},$$

$$(3.1)$$

where $z_{i,n} \in T_i y_n$. Then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $z_0 = P_{\Omega} x_0$.

Proof. We split the proof into six steps.

Step 1. Show that $P_{C_{n+1}}x_0$ is well defined for every $x_0 \in H$.

Since $0 < c \le \lambda_n \le d < 2\alpha$ for all $n \in \mathbb{N}$, we get that $P_C(I - \lambda_n A)$ is nonexpansive for all $n \in \mathbb{N}$. Hence, $\bigcap_{n=1}^{\infty} F(P_C(I - \lambda_n A)) = VI(D,A)$ is closed and convex. By Lemma 2.6(4), we know that EP(F) is closed and convex. By Lemma 2.7, we also know that $\bigcap_{i=1}^{\infty} F(T_i)$ is closed and convex. Hence, $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \cap VI(D,A)$ is a nonempty, closed and convex set. By Lemma 2.4, we see that $C_{i,n+1}$ is closed and convex for all $i,n \in \mathbb{N}$. This implies that C_{n+1} is also closed and convex. Therefore, $P_{C_{n+1}}x_0$ is well defined. Let $p \in \Omega$ and $i \in \mathbb{N}$. From $u_n = T_{r_n}x_n$, we have

$$||u_n - p|| = ||T_{r_n} x_n - T_{r_n} p|| \le ||x_n - p||$$
(3.2)

for every $n \ge 0$. From this, we have

$$||s_{i,n} - p|| = ||\alpha_{i,n}y_n + (1 - \alpha_{i,n})z_{i,n} - p||$$

$$\leq \alpha_{i,n}||y_n - p|| + (1 - \alpha_{i,n})||z_{i,n} - p||$$

$$\leq \alpha_{i,n}||y_n - p|| + (1 - \alpha_{i,n})d(z_{i,n}, T_i p)$$

$$\leq \alpha_{i,n}||y_n - p|| + (1 - \alpha_{i,n})H(T_i y_n, T_i p)$$

$$\leq ||y_n - p||$$

$$= ||P_D(u_n - \lambda_n A u_n) - P_D(p - \lambda_n A p)||$$

$$\leq ||u_n - p||$$

$$\leq ||x_n - p||.$$
(3.3)

So, we have $p \in C_{i,n+1}$, hence $\Omega \subset C_{i,n+1}$, $\forall i \in \mathbb{N}$. This shows that $\Omega \subset C_{n+1} \subset C_n$.

Step 2. Show that $\lim_{n\to\infty} ||x_n - x_0||$ exists.

Since Ω is a nonempty closed convex subset of H, there exists a unique $v \in \Omega$ such that

$$z_0 = P_{\Omega} x_0. \tag{3.4}$$

From $x_n = P_{C_n} x_0$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_n$, $\forall n \ge 0$, we get

$$||x_n - x_0|| \le ||x_{n+1} - x_0||, \quad \forall n \ge 0.$$
 (3.5)

On the other hand, as $\Omega \subset C_n$, we obtain

$$||x_n - x_0|| \le ||z_0 - x_0||, \quad \forall n \ge 0.$$
 (3.6)

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore, $\lim_{n\to\infty} ||x_n - x_0||$ exists.

Step 3. Show that $x_n \to q \in D$ as $n \to \infty$.

For m > n, by the definition of C_n , we see that $x_m = P_{C_m} x_0 \in C_m \subset C_n$. By Lemma 2.2, we get

$$||x_m - x_n||^2 \le ||x_m - x_0||^2 - ||x_n - x_0||^2.$$
(3.7)

From Step 2, we obtain that $\{x_n\}$ is Cauchy. Hence, there exists $q \in D$ such that $x_n \to q$ as $n \to \infty$.

Step 4. Show that $q \in F$.

From Step 3, we get

$$||x_{n+1} - x_n|| \longrightarrow 0 \tag{3.8}$$

as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$||s_{i,n} - x_n|| \le ||s_{i,n} - x_{n+1}|| + ||x_{n+1} - x_n|| \le 2||x_{n+1} - x_n|| \longrightarrow 0$$
(3.9)

as $n \to \infty$ for all $i \in \mathbb{N}$,

$$||y_n - x_n|| \le ||y_n - x_{n+1}|| + ||x_{n+1} - x_n|| \le 2||x_{n+1} - x_n|| \longrightarrow 0$$
(3.10)

as $n \to \infty$. Hence, $y_n \to q$ as $n \to \infty$. It follows from (3.9) and (3.10) that

$$||z_{i,n} - y_n|| = \frac{1}{1 - \alpha_{i,n}} ||s_{i,n} - y_n|| \longrightarrow 0$$
 (3.11)

as $n \to \infty$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, we have

$$d(q, T_{i}q) \leq ||q - y_{n}|| + ||y_{n} - z_{i,n}|| + d(z_{i,n}, T_{i}q)$$

$$\leq ||q - y_{n}|| + ||y_{n} - z_{i,n}|| + H(T_{i}y_{n}, T_{i}q)$$

$$\leq ||q - y_{n}|| + ||y_{n} - z_{i,n}|| + ||y_{n} - q||.$$
(3.12)

From (3.11), we obtain $d(q, T_i q) = 0$. Hence $q \in F$.

Step 5. Show that $q \in EP(F)$.

By the nonexpansiveness of P_D and the inverse strongly monotonicity of A, we obtain

$$||y_{n} - p||^{2} \leq ||u_{n} - \lambda_{n}Au_{n} - (p - \lambda_{n}Ap)||^{2}$$

$$\leq ||u_{n} - p||^{2} + \lambda_{n}(\lambda_{n} - 2\alpha)||Au_{n} - Ap||^{2}$$

$$= ||T_{r_{n}}x_{n} - T_{r_{n}}p||^{2} + \lambda_{n}(\lambda_{n} - 2\alpha)||Au_{n} - Ap||^{2}$$

$$\leq ||x_{n} - p||^{2} + c(d - 2\alpha)||Au_{n} - Ap||^{2}.$$
(3.13)

This implies

$$c(2\alpha - d)\|Au_n - Ap\|^2 \le \|x_n - p\|^2 - \|y_n - p\|^2$$

$$\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).$$
(3.14)

It follows from (3.10) that

$$\lim_{n \to \infty} ||Au_n - Ap|| = 0. {(3.15)}$$

Since P_D is firmly nonexpansive, we have

$$||y_{n} - p||^{2} = ||P_{D}(u_{n} - \lambda_{n}Au_{n}) - P_{D}(p - \lambda_{n}Ap)||^{2}$$

$$\leq \langle u_{n} - \lambda_{n}Au_{n}\rangle - (p - \lambda_{n}Ap), y_{n} - p\rangle$$

$$= \frac{1}{2} \Big(||(u_{n} - \lambda_{n}Au_{n}) - (p - \lambda_{n}Ap)||^{2}$$

$$+ ||y_{n} - p||^{2} - ||(u_{n} - \lambda_{n}Au_{n}) - (p - \lambda_{n}Ap) - (y_{n} - p)||^{2} \Big)$$

$$\leq \frac{1}{2} \Big(||u_{n} - p||^{2} + ||y_{n} - p||^{2} - ||(u_{n} - y_{n}) - \lambda_{n}(Au_{n} - Ap)||^{2} \Big)$$

$$\leq \frac{1}{2} \Big(||x_{n} - p||^{2} + ||y_{n} - p||^{2} - ||u_{n} - y_{n}||^{2} + 2\lambda_{n}\langle u_{n} - y_{n}, Au_{n} - Ap\rangle \Big)$$

$$\leq \frac{1}{2} \Big(||x_{n} - p||^{2} + ||y_{n} - p||^{2} - ||u_{n} - y_{n}||^{2} + 2\lambda_{n}\langle u_{n} - y_{n}, Au_{n} - Ap|| \Big).$$
(3.16)

This implies that

$$||y_n - p||^2 \le ||x_n - p||^2 - ||u_n - y_n||^2 + 2\lambda_n ||u_n - y_n|| ||Au_n - Ap||.$$
(3.17)

It follows that

$$||u_n - y_n||^2 \le ||x_n - y_n|| (||x_n - p|| + ||y_n - p||) + 2d||u_n - y_n|| ||Au_n - Ap||.$$
(3.18)

From (3.10) and (3.15), we get

$$\lim_{n \to \infty} ||u_n - y_n|| = 0. \tag{3.19}$$

It follows from (3.10) and (3.19) that

$$\lim_{n \to \infty} ||u_n - x_n|| = 0. {(3.20)}$$

Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in D.$$
 (3.21)

From the monotonicity of F, we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge F(y, u_n), \quad \forall y \in D, \tag{3.22}$$

hence

$$\left\langle y - u_n, \frac{u_n - x_n}{r_n} \right\rangle \ge F(y, u_n), \quad \forall y \in D.$$
 (3.23)

From (3.20) and condition (A4), we have

$$0 \ge F(y,q), \quad \forall y \in \mathcal{D}. \tag{3.24}$$

For t with $0 < t \le 1$ and $y \in D$, let $y_t = ty + (1 - t)q$. Since $y, q \in D$ and D is convex, then $y_t \in D$ and hence $F(y_t, q) \le 0$. So, we have

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1 - t)F(y_t, q) \le tF(y_t, y). \tag{3.25}$$

Dividing by t, we obtain

$$F(y_t, y) \ge 0, \quad \forall y \in D. \tag{3.26}$$

Letting $t \downarrow 0$ and from (A3), we get

$$F(q, y) \ge 0, \quad \forall y \in D. \tag{3.27}$$

Therefore, we obtain $q \in EP(F)$.

Step 6. Show that $q \in VI(D, A)$.

Since T is the maximal monotone mapping defined by (2.10),

$$Tx = \begin{cases} Ax + N_D x, & x \in D, \\ \emptyset, & x \notin D. \end{cases}$$
 (3.28)

For any given $(x, u) \in G(T)$, hence $u - Ax \in N_D x$. It follows that

$$\langle x - y_n, u - Ax \rangle \ge 0. \tag{3.29}$$

On the other hand, since $y_n = P_D(u_n - \lambda_n A u_n)$, we have

$$\langle x - y_n, y_n - (u_n - \lambda_n A u_n) \rangle \ge 0, \tag{3.30}$$

and so

$$\left\langle x - y_n, \frac{y_n - u_n}{\lambda_n} + Au_n \right\rangle \ge 0. \tag{3.31}$$

From (3.29), (3.31), and the α -inverse monotonicity of A, we have

$$\langle x - y_{n}, u \rangle \geq \langle x - y_{n}, Ax \rangle$$

$$\geq \langle x - y_{n}, Ax \rangle - \left\langle x - y_{n}, \frac{y_{n} - u_{n}}{\lambda_{n}} + Au_{n} \right\rangle$$

$$= \langle x - y_{n}, Ax - Ay_{n} \rangle + \langle x - y_{n}, Ay_{n} - Au_{n} \rangle - \left\langle x - y_{n}, \frac{y_{n} - u_{n}}{\lambda_{n}} \right\rangle$$

$$\geq \langle x - y_{n}, Ay_{n} - Au_{n} \rangle - \left\langle x - y_{n}, \frac{y_{n} - u_{n}}{\lambda_{n}} \right\rangle.$$
(3.32)

It follows that

$$\lim_{n \to \infty} \langle x - y_n, u \rangle = \langle x - q, u \rangle \ge 0. \tag{3.33}$$

Again since *T* is maximal monotone, hence $0 \in Tq$. This shows that $q \in VI(D, A)$.

Step 7. Show that $q = z_0 = P_{\Omega}x_0$.

Since $x_n = P_{C_n} x_0$ and $\Omega \subset C_n$, we obtain

$$\langle x_0 - x_n, x_n - p \rangle \ge 0 \quad \forall p \in \Omega.$$
 (3.34)

By taking the limit in (3.34), we obtain

$$\langle x_0 - q, q - p \rangle \ge 0 \quad \forall p \in \Omega.$$
 (3.35)

This shows that $q = P_{\Omega}x_0 = z_0$.

From Steps 3 to 5, we obtain that $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $z_0 = P_{\Omega}x_0$. This completes the proof.

Theorem 3.2. Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i: D \to CB(D)$ be nonexpansive multivalued maps for all $i \in \mathbb{N}$ with $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(D,A) \neq \emptyset$ and $T_ip = \{p\}$, for all $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $\alpha_{i,n} \in [0,1)$ with $\limsup_{n\to\infty} \alpha_{i,n} < 1$ and $\{\lambda_n\} \subset [c,d]$ for some $c,d \in (0,2\alpha)$. For an initial point $x_0 \in H$ with $C_1 = D$ and $x_1 = P_{C_1}x_0$, let $\{x_n\}$, $\{y_n\}$, and $\{s_{i,n}\}$ be sequences generated by

$$y_{n} = P_{D}(x_{n} - \lambda_{n}Ax_{n}),$$

$$s_{i,n} = \alpha_{i,n}y_{n} + (1 - \alpha_{i,n})z_{i,n},$$

$$C_{i,n+1} = \{z \in C_{i,n} : ||s_{i,n} - z|| \le ||y_{n} - z|| \le ||x_{n} - z||\},$$

$$C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1},$$

$$x_{n+1} = P_{C_{n+1}}x_{0}, \quad \forall n \in \mathbb{N},$$

$$(3.36)$$

where $z_{i,n} \in T_i y_n$. Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $z_0 = P_{\Omega} x_0$.

Proof. Putting F(x,y) = 0 for all $x,y \in D$ in Theorem 3.1, we obtain the desired result directly from Theorem 3.1.

Theorem 3.3. Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i: D \to CB(D)$ be nonexpansive multivalued maps for all $i \in \mathbb{N}$ with $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$, for all $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $\alpha_{i,n} \in [0,1)$ with $\limsup_{n\to\infty} \alpha_{i,n} < 1$. For an initial point $x_0 \in H$ with $C_1 = D$ and $x_1 = P_{C_1}x_0$, let $\{x_n\}$ and $\{s_{i,n}\}$ be sequences generated by

$$s_{i,n} = \alpha_{i,n} x_n + (1 - \alpha_{i,n}) z_{i,n},$$

$$C_{i,n+1} = \{ z \in C_{i,n} : ||s_n - z|| \le ||x_n - z|| \},$$

$$C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1},$$

$$x_{n+1} = P_{C_{n+1}} x_{0}, \quad \forall n \in \mathbb{N},$$

$$(3.37)$$

where $z_{i,n} \in T_i y_n$. Then, $\{x_n\}$ converge strongly to $z_0 = P_{\Omega} x_0$.

Proof. Putting A = 0 in Theorem 3.2, we obtain the desired result directly from Theorem 3.2.

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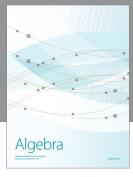
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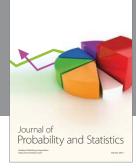
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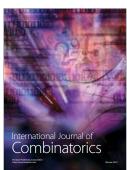








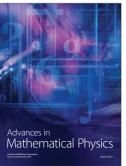


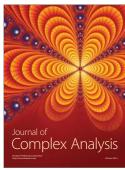


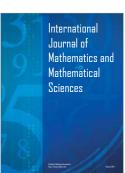


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