A HYPERBOLIC STEFAN PROBLEM*

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Abstract. Heat conduction is considered in a semi-infinite solid subjected to a high step change in surface heat flux, such that melting occurs. A time-dependent relaxation model for the energy flux is assumed, leading to a non-Fourier, non-linear equation for the thermal field, which is solved under suitable conditions on the interface displacement.

1. Introduction. Criticism to the Fourier model for heat conduction, which leads to a physically unacceptable infinite speed of propagation of the energy transfer, was put forward, in the past, by several authors. Such a criticism, initially based on purely speculative grounds, follows from a variety of approaches to the problem, from the first consideration in a work by Cattaneo [1], where a model for the heat conduction process was substantiated—in the case of gaseous media—by means of the kinetic theory, to the statistical mechanics of nonequilibrium irreversible processes [2].[1]

In any case, when the Fourier law is rebuted, a time-dependent relaxation model is proposed for the heat flux and the thermal field, which, in the Fourier case, is governed by a parabolic equation, obeys to a hyperbolic wave equation.

The temperature distribution evaluated by the latter model more significantly differs from the Fourier model predictions as the involved fluxes of heat and their time variations increase. Recent technological developments have drawn increasing attention to non-Fourier heat transfer models as situations where their effects can start playing a significant role become more frequent. Examples are provided by targets irradiated by high intensity electromagnetic radiation of nuclear origin or by high power Laser beams. When these powerful energy sources act on a solid material, very high temperatures can be achieved and, in most cases, a change of phase, like ablation or melting, or a change of state, like recrystallization, occur. All these circumstances are connected with the absorption of some

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[1] Due to the large number of papers on this problem and on the classic Stefan problem, the bibliography listed at the end of this article has been necessarily reduced to a minimum. Further references on these subjects can be found, for instance, in [2, 4, 5, 7, 8].

form of latent heat in correspondence to the displacement of an interface between the phases or two crystalline states of the same substance. These phenomena, when investigated within the framework of the Fourier heat conduction equation, originate the classic parabolic Stefan problem.

To the authors' knowledge no attention has been paid, up to now, to the non-Fourier heat conduction associated to a change of phase in a solid material. This work deals with a first, simple, hyperbolic problem. In particular, it will not yet be dealt with the possible further refinement of a relaxation model also for the melting process.

2. Basic equations. In the following, the one-dimensional heat conduction in a semi-infinite body, subjected to a step change in the surface heat flux, is considered when melting occurs (Fig. 1). The Cattaneo time-dependent relaxation model for the heat flux in a homogeneous material will be assumed

$$\tau\frac{\partial Q}{\partial T} + Q = -K\frac{\partial U}{\partial X} \tag{1}$$

where T and X are the time- and space-coordinates, respectively. Q is the heat flux, U is the temperature, K is the thermal conductivity and τ is the relaxation time. For the meaning and the evaluation of the relaxation time more information can be found, for instance, in the articles by Chester [3] and Brazel and Nolan [9]. The equation of energy conservation is

$$\rho C\frac{\partial U}{\partial T} + \frac{\partial Q}{\partial X} = 0 \tag{2}$$

with ρ and C the density and the specific heat, respectively. Density and physical properties are assumed to be constant.

When Eq. (1) and (2) are combined, the temperature field will be governed by the equation

$$\tau\frac{\partial^2 U}{\partial T^2} + \frac{\partial U}{\partial T} = k\frac{\partial^2 U}{\partial X^2} \tag{3}$$

where k is the thermal diffusivity. Equation (3) holds in the solid as well as in the liquid region, respectively (S) and (L) of Fig. 1. For the sake of simplicity, it will be assumed, in

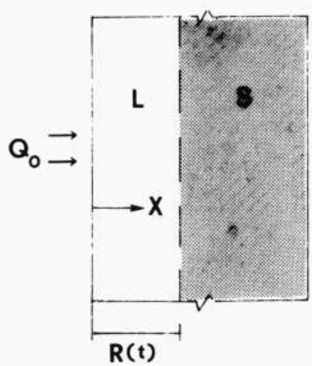

FIG. 1. Geometry of the problem.

the following, that the densities, the transport properties and the relaxation times are equal for both regions. After appending the indexes L and S to the quantities relative to the liquid and to the solid, respectively, the differential problem to be considered is:

Region (L)

$$\tau \frac{\partial^2 U_L}{\partial T^2} + \frac{\partial U_L}{\partial T} = k \frac{\partial^2 U_L}{\partial X^2}, \tag{4}$$

$$Q_L = -K \frac{\partial U_L}{\partial X} - \frac{\partial Q_L}{\partial T}, \tag{5}$$

$$U_L = U_i, \quad \frac{\partial U_L}{\partial T} = 0; \qquad X > 0, \quad T = 0, \tag{6a, b}$$

$$Q_L = Q_0 H(T); \qquad X = 0, \tag{7}$$

$$U_L = U_m, \quad Q_L - Q_S = Q_m \frac{dR}{dT}; \qquad X = R(T) \tag{8a, b}$$

where $X = R(T)$ is the equation for the interface between (L) and (S), Q_0 is a positive constant, Q_m is the latent heat of melting and $H(T)$ is the Heaviside step function. Furthermore, U_m and U_i are the temperature of melting and the initial temperature, respectively.

Region (S)

$$\tau \frac{\partial^2 U_S}{\partial T^2} + \frac{\partial U_S}{\partial T} = k \frac{\partial^2 U_S}{\partial X^2}, \tag{9}$$

$$Q_S = -K \frac{\partial U_S}{\partial X} - \tau \frac{\partial Q_S}{\partial T}, \tag{10}$$

$$U_S = U_i, \quad \frac{\partial U_S}{\partial T} = 0; \qquad X > 0, \quad T = 0, \tag{11a, b}$$

$$U_S = U_i; \qquad X \to \infty. \tag{12}$$

At the interface, the matching conditions are given by Eq. (8b) and

$$U_L = U_S; \qquad X = R(T). \tag{13}$$

When Eqs. (5) and (10) are substituted into Eq. (8b), that matching condition is expressed in the form:

$$\frac{K}{Q_m} \frac{\partial}{\partial X}(U_S - U_L) = \frac{dR}{dT} + \frac{d^2 R}{dT^2}. \tag{14}$$

The system (4)–(14) can be expressed in a non-dimensional form by assuming the following proper set of dimensionless variables:

$$t = T/\tau, \quad x = X/(k\tau)^{1/2}, \quad u_j = (U_j - U_i)/\Delta U, \quad j = L, S, m,$$

$$q = Q/Q_0, \quad v = (dR/dT)/v_0, \quad \Delta U = Q_0 (k\tau)^{1/2}/K,$$

where v_0 is a reference speed, of the order of magnitude of the speed of the interface plane, whose dimensionless equation is $x = r(t)$. The speed of propagation of the damped thermal wave $(k/\tau)^{1/2}$ is of course greater than v_0. With the assumptions reported above, one has for $j = L, S$

$$\frac{\partial^2 u_j}{\partial t^2} + \frac{\partial u_j}{\partial t} = \frac{\partial^2 u_j}{\partial x^2}, \tag{15}$$

$$u_j = 0, \quad \frac{\partial u_j}{\partial t} = 0; \qquad x > 0, \quad t = 0 \tag{16a, b}$$

and

$$\frac{\partial u_L}{\partial X} = -[1 + \delta(t)]; \qquad x = 0, \tag{17}$$

$$u_S = 0; \qquad t \geqslant 0, \quad x \to \infty, \tag{18}$$

$$u_j = u_m, \quad \frac{\partial}{\partial x}(u_S - u_L) = A(v + \dot{v}); \qquad x = r(t) \tag{19a, b}$$

where the dimensionless product A (which is inversely proportional to the Stefan number) is equal to $Q_m v_0 / Q_0$, and $\delta(t)$ is the Dirac delta distribution.

The problem expressed by Eqs. (15)–(19) is equivalently formulated by the following time-dependent heat conduction problem with a moving heat source for the entire domain $(L) + (S)$

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - A(v + \dot{v})\delta\{x - r(t)\} \tag{20}$$

subjected to the condition $u\{r(t)\} = u_m$ and to the conditions corresponding to Eqs. (16)–(18).

3. Solution for small A **and** t. Equation (20) is suitably expressed for solution by means of the Green function method. Taking into account the pertinent boundary and initial conditions, the solution can be formally written, in a way analogous to the parabolic case [5], in the form

$$u(t, x) = u_0(t, x) + Au_1(t, x) \tag{21}$$

where

$$u_0(t, x) = H(t - x)\Big\{\exp(-\tfrac{1}{2}t)I_0\big[\tfrac{1}{2}(t^2 - x^2)^{1/2}\big] + \int_0^t \exp(-\tfrac{1}{2}y)I_0\big[\tfrac{1}{2}(y - x)^{1/2}\big]H(y - x)\,dy\Big\}, \tag{22}$$

$$u_1(t, x) = -\tfrac{1}{2}\int_0^t \{v(t') + \dot{v}(t')\}H\{t - t' - |x - r(t')|\}F(t, x; t')\,dt' - \tfrac{1}{2}\int_0^t \{v(t') + \dot{v}(t')\}H\{t - t' - [x + r(t')]\}G(t, x; t')\,dt' \tag{23}$$

and where

$$F(t, x; t') = \exp\{-(t - t')/2\} I_0\left\{\tfrac{1}{2}\left[(t - t')^2 - [x - r(t')]^2\right]^{1/2}\right\},$$

$$G(t, x; t') = \exp\{-(t - t')/2\} I_0\left\{\tfrac{1}{2}\left[(t - t')^2 - [x + r(t')]^2\right]^{1/2}\right\}.$$

It should be noted that $u_0(t, x)$ corresponds to the solution in the case where no change of phase occurs, as obtained in [6]. Difficulties arise for the evaluation of the expression (21), which are conceptually analogous to those met in the parabolic Stefan problem for the semi-infinite slab [7–8]. In the hyperbolic case, in the presence of very intense heat fluxes at the surface, it seems appropriate to assume v_0 sufficiently small in comparison with $(k/\tau)^{1/2}$, so that it is reasonable to suppose that A is also small. As a consequence, the two integrals which appear in Eq. (23) can be evaluated by assuming for $v(t)$ and $\dot{v}(t)$, namely the speed and the rate of change of the speed of the interface, their values corresponding to the relation $x = r_0(t)$ which is obtained by putting

$$u_0 = u_m. \tag{24}$$

This approximate procedure has also been followed in some classic Stefan problem and, in the linearization process, the matching condition (19a) will only approximately be satisfied. Unfortunately, Eq. (24) does not provide an explicit relation for the interface displacement as a function of t. However, for small time values and, consequently as $t \geqslant x$, for small x, the integrand appearing in Eq. (22) can be expanded as a power series of t, and this leads to an explicit expression for $x = r_0(t)$.

As an example, when the power series expansion is carried out and only terms of the order of magnitude of t^2 are retained, then one has for the interface displacement

$$r_0(t) = \left\{8 - (a(t))^{1/2}\right\}/3 \tag{25}$$

where

$$a(t) = 64 - 48\left(1 - u_m + \tfrac{1}{2}t - t^2/16\right). \tag{26}$$

At this point an approximate expression for $u_1(t, x)$ can be evaluated from Eq. (23) which represents the perturbing effect of the latent heat of melting on the termal field.

4. Conclusion. As a conclusion, some remarks will be made and some numerical results will be provided.

Remark (i). A more accurate evaluation of the relation $x = r(t)$ can be obtained by assuming

$$r(t) = r_0(t) + A r_1(t).$$

In this case, the perturbation term $r_1(t)$ can be evaluated by the simple relation

$$r_1(t) = -u_1\{t, r_0(t)\} \Big/ \left(\frac{\partial u_0(t, x)}{\partial x}\right)_{x = r_0}.$$

Remark (ii). Since from Eq. (21), $\lim_{t \to 0} u_0(t, 0) = 1$, as already pointed out in [6], the particular value $u_m = 1$ corresponds to an instant rise of the surface temperature up to the melting value at $t = 0$ and, consequently, in this case, $r_0(0) = 0$. Of course, this situation does not occur in the parabolic problem.

In general the equation $r_0(t_c; u_m) = 0$ provides the characteristic value of time t_c, as a function of u_m, for which the temperature at the surface reaches the melting point. In the case $u_m > 1$, for $0 \leqslant t \leqslant t_c$ the solution will reduce exactly to u_0, since melting has not yet taken place. The case where $u_m < 1$ can not be discussed in the frame of a small perturbation theory, for which the basic solution is the one relative to the no melting case. In fact, u_m less than one corresponds to an initial temperature jump at $x = 0$ beyond the melting temperature.

Remark (iii). The thermal perturbation $u_1(t, x)$ is always less or at most equal to zero. This follows of course from considerations of energy conservation. More interesting is to observe that, in contrast with the classic Stefan problem, u_1 depends upon the rate of change of the speed of the interface.

Remark (iv). Within the limits of the hypotheses of Chapter 3, the speed of the interface is a decreasing function of the time t.

Fig. 2 shows the x-distribution at a given instant in a typical situation. The numerical evaluation of the solution has been carried out according to the simplified procedure

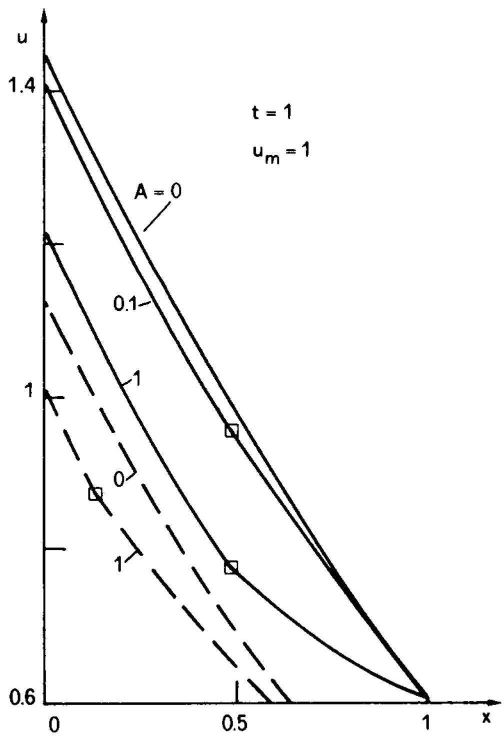

FIG. 2. Temperature distributions. Solid lines correspond to the hyperbolic case and dashed lines correspond to the parabolic case. The squares indicate the first approximation of the position of the interface.

exposed in the preceding chapter. It can be easily realized that, for increasing times, the perturbation of the temperature field due to the release of the latent heat of melting decreases at given A. In the same figure the solutions for $A = 0$ and $A = 1$ of a corresponding parabolic problem are also given. In this case, the melting of an initially isothermal semi-infinite solid was considered, under a constant flux of energy for $t > 0$ at $x = 0$, and with the proper Fourier matching conditions at the interface. The technique for the evaluation of $r(t)$ was the same as in the hyperbolic case. Finally, Table 1 shows some computed results. In particular, for significant values of A and u_m and at indicative times, the values of u_0, $u^{(1)}$ and $u^{(2)}$ at $x = 0$ are given, where $u^{(1)}$ and $u^{(2)}$ represent the solutions obtained by limiting the series expansion in Eq. (22) to the first power and to the second power of t, respectively. In the same table the values of the interface displacement, speed and rate of change of the speed are also reported, together with the results for the above mentioned parabolic case, where meaningful. The differences between the results in the Fourier model and in the non-Fourier model of heat conduction are still quite large at $t = 1$. They decrease at increasing times and tend to disappear as t tends to infinity, at each given position along the spatial coordinate.

TABLE 1

A	u_m	t	$u_0(x=0)$	$u^{(1)}(x=0)$	$u^{(2)}(x=0)$	r	v	$-\dot{v}$
	Hyperbolic problem							
1	1.0	0.1	1.05	1.03	1.02	0.05	0.50	0.03
1	1.0	1.0	1.45	1.28	1.22	0.48	0.46	0.06
1	1.2	1.0	1.45	1.27	1.18	0.25	0.41	0.07
	Parabolic problem							
1	1.0	1.0	1.13		1.01	0.13	0.61	

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