A Hyperdense Semantic Domain for Hybrid Dynamic Systems to Model Different Classes of Discontinuities

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ABSTRACT

The physics of technical systems, such as embedded and cyber-physical systems, is frequently modeled using the notion of continuous time. The underlying continuous phenomena may, however, occur at a time scale much faster than the system behavior of interest. In such situations, it is desirable to approximate the detailed continuous-time behavior by discontinuous change. Two classes of discontinuous change can be identified: pinnacles and mythical modes. This work shows how pinnacles are well modeled using a hyperreal notion of time while a superdense notion of time applies well to mythical modes. Thus, the combination, called *hyperdense* time, is proposed to allow for the expression of the semantics of both pinnacles and mythical modes. Further, the hyperdense semantic domain is translated into a computational representation as a three-dimensional model of time. In particular, continuous-time behavior is mapped onto floating point numbers, while the mythical mode and pinnacle event iterations each map onto an integer dimension. A modified Newton's cradle is used as a case study and to illustrate the computational implementation.

1. INTRODUCTION

With the advent of ubiquitous embedded computation, the complexity of engineered systems has grown by leaps and bounds. Where much of the increase in complexity has been driven by the availability of higher level programming languages combined with powerful compilers, more recently engineers have been turning to yet a higher level of abstraction. This higher level of abstraction is often somewhat colloquially referred to as 'models', making the implicit assumption that the computer code is the system under study rather than the actual implementation for which requirements are formulated.

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Recently attention has been gained by the notion of *Cyber-Physical Systems* (CPS). These systems tightly integrate the physical world, the computation, and the information space in a technological sense. Model-Based Design [14, 30] is a prime exponent to develop such systems [40]. Given the open nature of CPS and the necessity to include abstract representations in their development process, the availability of quality models for system-level studies is fundamental to a successful design effort.

In other work [29], the use of models has indeed been explored as an approach to mitigate system integration issues. Because of the delicate interaction between the various component and subsystem behaviors in their implementation, the use of models in addressing system integration challenges is not straightforward, however. In particular, it becomes essential to create 'good' models of the physics, that is, models that embody correctly the pertinent physical effects while not giving rise to behaviors that have no physical manifestation.

A critical characteristic of physics models for system-level studies is the manner in which their dynamics are captured. Generally, physical systems are well described by continuous-time behavior, for example, based on the foundations of thermodynamics [6, 13]. However, the dynamics of some of the continuous behavior is much faster than the time scale of interest, and, therefore, it may be efficient to model the resulting effect by discontinuous changes instead. The corresponding models then combine continuous behavior with discontinuous effects, which can be represented as a mathematical system with hybrid dynamics.

Complications arise because, as recognized by previous work [10, 27], certain laws of physics do not hold when discontinuities are introduced in the model behavior. For example, *continuity of power* is violated much like *conservation of energy* because of *instantaneous dissipation*. Other laws, however, do hold across discontinuities (e.g., *conservation of charge*, *conservation of momentum*, etc.) and these laws help determine consistent transfer across discontinuities of the dynamic state in a physics model. It is the objective of the work presented here to further develop these domain-specific constraints while formalizing their behavior so as to support multiparadigm modeling [32] across the fields of signal processing, control engineering, computer science, and thermodynamics.

Because of a natural bias toward computational implementations, a first approach to formalizing discontinuities in models of physical systems is to employ a semantic domain that is continuous but interspersed with discontinuous changes. In such a computational implementation, it is tempting to model sequences of discontinuous changes as occurring at a single point in time, which naturally leads to an orthogonal integer dimension so as to attribute an order to consecutive discontinuous changes. However, the resulting *superdense time* [21], $\mathbb{R}\times\mathbb{N}$ is insufficient to precisely describe the intricate behavior of discontinuities in physical system models [31]. In particular, an ontology of hybrid dynamics in physical system models that was developed in previous work [34] includes a class of behaviors referred to as *mythical modes* [35] that can be well formalized in superdense time. The ontology also includes a class of behaviors called *pinnacles*, though, which requires a time advance during discontinuous change, and, therefore, cannot be formalized by the integer dimensions of superdense time.

As an alternative, related work [17] proposed another approach to formalizing discontinuities based on the *hyperreals* of nonstandard analysis [19]. More recently, this approach has been further developed in an effort to define the semantics of hybrid dynamic systems [2, 3]. Because hyperreals allow an infinitesimal time advance, they present a semantic domain onto which pinnacles map well. With mythical modes being well presented by an orthogonal integer dimension, this paper proposes a semantic domain that combines the two in order to be able to span the range of discontinuous behaviors that have been identified in previous work [23, 32].

The resulting *hyperdense* semantic domain enables modeling of physics from first principles based on the theory of thermodynamics. A formalization of the intricate behavior on a hyperdense domain is then mapped onto a computational implementation. The corresponding executable models generate consistent and physically meaningful behavior even in the face of complicated interactions between classes of discontinuities. The conceptual modeling from first principles reduces the conceptual investment that is generally required when introducing discontinuities in physics models [7], an investment that is significantly (if not prohibitively) exacerbated in the case of interacting discontinuities.

Section 2 discusses the notion of discontinuity in physical modeling. In particular, hybrid bond graphs are introduced as a means to represent the nuances of physical phenomena. Examples of models are provided to explain the discontinuities, such as pinnacles and mythical mode. Section 3 relates the discontinuous changes to the notion of time. Superdense and hyperreal time semantic domains are discussed to provide a basis for a combination thereof. The resulting hyperdense time notion is then related to mythical mode and pinnacle behavior. In Section 4, a computational representation for hyperdense time is provided. A case study of a variant of Newton's cradle illustrates the value of the approach and its computational implementation is discussed. Section 5 discusses related work on the topic. Section 6 completes the treatise.

2. DISCONTINUITIES IN PHYSICS MODELS

The Heaviside principle [15] forms the foundation of a continuity assumption that is well developed in the theory of thermodynamics (e.g., [9]). In previous work [18, 36], *bond graphs* have been developed as a formalism to model continuous-time behavior of physical systems. After a brief introduction to thermodynamics and bond graphs, this section reviews how the bond graph formalism may be extended with an ideal switching element to form *hybrid bond graphs* [25]. Note that the principles apply to physics modeling languages in general such as Modelica [12] and MathWorks® Simscape™ [22].

2.1 Bond graphs for physical modeling

Modeling physics for system-level studies is often based on the concept of *reticulation* [5]. This is the foundation of the *lumped parameter assumption*, that is, physical phenomena can be isolated and represented by well-defined parameter values. For example, a spring may exert a varying force along its longitudinal dimension but is often represented by a spring constant that captures the force between two points of attachment given a displacement between these two points.

Across physics domains (e.g., electrical, hydraulic, thermal, chemical, etc.) thermodynamics identifies two types of variables that are subject to dynamic behavior. These variables represent either: (i) *extensive* quantities or (ii) *intensive* quantities. The extensive quantities depend on the extent of a lumped phenomenon whereas the intensive quantities do not. For example, momentum is an extensive quantity because halving the mass that stores it also halves the momentum. In contrast, velocity is an intensive quantity as halving a mass with a given velocity does not change the velocity of the two halves.

The dynamics of extensities and intensities are related by *conduction*, that is, when there is a difference in intensities, a change in extensity follows. For example, a difference in velocities between two bodies results in a force acting between them that causes a change in momentum ($F = m\frac{dv}{dt} = \frac{dp}{dt}$). Because of the generality across physical domains of this phenomena, it is useful to define a generalized measure based on a notion of intensity difference, *effort*, and a notion of corresponding change in extensity *flow*. If this measure is taken to be the product, it corresponds to the well-known notion of *power* (change of energy). For example, $v \cdot F$ equates power much like in the electrical domain the product of the intensity difference (voltage, v) and change of extensity (current i) equates power.

The relations in thermodynamics models can then be summarized as illustrated in Fig. 1 as a 'triangle of state' (after Henry Paynter's 'tetrahedron of state'). A component represents a lumped parameter. This component may store a quantity as its extensity. Based on the constituent behavior of the component, this extensity relates to an intensity. Given the connection structure that the component is part of, the intensity results in a change of extensity, a flow. The flow adds to the stored extensity (by means of integration over time). Generally, the relation between intensity and extensity is one of a storage element whereas the relation between the intensity and flow is one of dissipation. Note that the role of a quantity depends on the physical domain

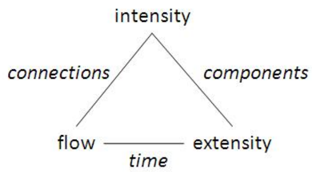


Figure 1: Triangle of state

(e.g., whereas a voltage is an intensity in the electrical domain, it is a flow in the magnetic domain).

In an energy-based representation, any change in dynamic variable values then is the result of two variables acting: an effort, e, and a flow, f. Moreover, in a model there are two basic energy-based phenomena: (i) storage of either effort (C) or flow (I) and (ii) dissipation (R) while the context of the model is defined by ideal sources of either effort (Se) or flow (Sf).

With the various idealized physical phenomena identified as I, C, R, Se, and Sf, the final step is to establish how they can be connected. Turning to reticulation again it is desirable to identify ideal connection behavior. Given the energy-based approach, this connection behavior should center around flow of energy, power, between physical phenomena. Moreover, the connection behavior should be ideal in that it neither dissipates nor stores energy (the latter condition implying continuity of power).

Given these conditions, a connection element between a number of physical phenomena relates the efforts and the flows of all of these interacting phenomena in such a way that the sum of their product equates 0 (otherwise, there would be either dissipation or storage), $\sum_i e_i \cdot f_i = 0$. The two orthogonal implementations of this are that either all efforts are the same while the flows sum to zero, or the other way around. In the electrical domain, this corresponds to either Kirchhoff's current law or Kirchhoff's voltage law. In bond graph terminology these connections are represented by *junctions*, the former by a 0 junction ($\forall_{i \neq j} e_i = e_j$ and $\sum_i f_i = 0$) and the latter by a 1 junction ($\forall_{i \neq j} f_i = f_j$ and $\sum_i e_i = 0$).

To introduce discontinuities into the bond graph modeling formalism again the principle of reticulation is followed. This requires an idealized form of discontinuous change in dynamic behavior, which is well represented by a change in the junction structure because this structure is ideal, void of dissipation and storage. Implementing discontinuities then becomes a matter of dynamically modifying the junction structure in an idealized manner. This idealized reconfiguration amounts to a junction between phenomena being active or not [41]. In other words, a 0 junction can be active ($\forall_{i \neq j} e_i = e_j$ and $\sum_i f_i = 0$) or not ($\forall_i Se_i = 0$) and a 1 junction exhibits the dual behavior when active ($\forall_{i \neq j} f_i = f_j$ and $\sum_i e_i = 0$) or not ($\forall_i Sf_i = 0$). Note that when a junction is not active, indeed no power flows across it. These junctions that can change their mode from active (*on*) to inactive (*off*) are called *controlled junctions*.

2.2 Discontinuities in physics models

With the ability for a junction to switch its mode it becomes necessary to specify when a mode switch should occur. Given the popularity of finite state machines (FSM) to capture discrete event logic, a controlled junction is equipped with a FSM that determines the *on* or *off* mode of the junction. This determination requires two aspects of the state logic to be captured: (i) how the state of the FSM maps onto the *on* and *off* mode of the junction and (ii) how the physical quantities map onto transition conditions of the FSM. Continuity of power leads to the notion that discontinuities in physical quantities result from a lack of detail in modeled phenomena, which come in two classes: (i) storage and (ii) dissipation. The discontinuous behavior that emerges in turn for each of these is discussed next.

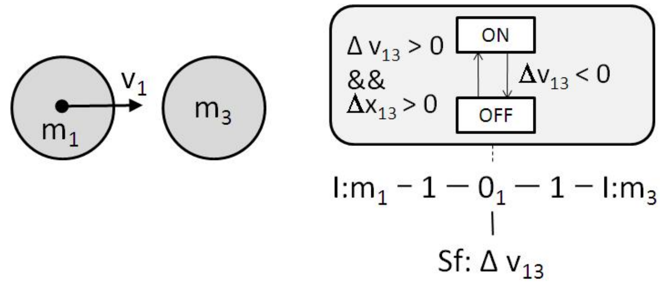


Figure 2: Ideally plastic collision

2.2.1 Pinnacles

Discontinuities in physics models are often studied based on multibody collisions, albeit with impact models that generally are subject to a large set of assumptions (e.g., perfectly simultaneous collisions, that are ideally plastic, along the normal direction, with only a translational component, and between bodies of equal mass). In a hybrid bond graph model, the impact of such a collision between two bodies, m_1 and m_3, can be modeled as depicted in Fig. 2. The two bodies are modeled as inertias, I, connected to a common velocity, 1, junction. These junctions represent the respective velocities, v_1 and v_3, which are connected via a common force, 0, junction. This 0 junction is controlled and when *off* it exerts force 0 on both bodies. Upon collision, the 0 junction turns *on* and it now enforces a velocity balance such that $v_1 - v_3 + \Delta v = 0$, where Δv_{13} is computed by an ideal flow source, Sf, as $\Delta v_{13} = v_1^- - v_3^-$, with the '-' superscript referring to signals immediately preceding the collision.

The FSM controlling the *on*/*off* mode of the 0 junction models the impact by switching from *off* to *on* when the bodies make contact ($\Delta x > 0$) and when they are moving toward one another ($\Delta v > 0$). Here the $\Delta v > 0$ is essential to model that there is impact because of a collision as opposed to the bodies only being in contact. As soon as the bodies move away from one another ($\Delta v < 0$), the 0 junction switches to *off*, irrespective of whether the bodies are touching.

During behavior generation, when $\Delta x > 0$ && $\Delta v > 0$ holds, a collision occurs and the flow source enforcing the velocity difference Δv^- because of impact becomes active. Based on this velocity difference and conservation of momentum ($\sum_i m_i v_i^- = \sum_i m_i v_i$), the velocities upon collision can be computed. The state of the velocity of the bodies is then reinitialized and this leads to the condition $\Delta v < 0$ being satisfied. Thus, a consecutive mode change occurs where the FSM moves to the *off* mode again. In the *off* mode the bodies behave as independent masses, and, therefore, no further changes in the physical state occur. Since the discrete mode

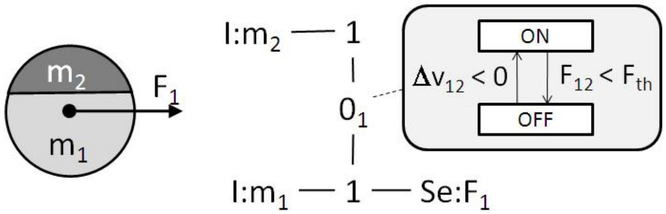


Figure 3: Two bodies with a breakaway force

changes have thus converged, the system proceeds to evolve in continuous time.

The end result is that the bodies m_1 and m_2 evolve according to a mode of continuous evolution. With a point in time at which two mode changes occur: (i) first, a collision mode occurs to model the impact that necessitates a reinitialization (discontinuous change) and (ii) second, the system changes back to a mode of continuous evolution. The collision mode that is active only to effect a reinitialization of physical state is referred to as a *pinnacle* [27]. Note that in more detail this collision mode would comprise a *storage* effect that captures the compression and expansion of the bodies.

2.2.2 Mythical mode changes

A different interaction effect between two bodies takes place based on friction effects. For example, consider two bodies m_1 and m_2 with m_2 at rest on top of m_1. When at a point in time a large enough external force is exerted on m_1, m_1 will start moving with a corresponding velocity. However, if the force is sufficiently large that the breakaway friction force $F_{breakaway}$ between m_1 and m_2 is exceeded, m_2 may remain at rest.

A hybrid bond graph model of such a system is depicted in Fig. 3. In this model, an ideal source of effort exerts an external force F_{in}. This force is applied to m_1 because it is connected to the common velocity 1 junction that represents the velocity of m_1. When *on*, a controlled 0 junction connects the 1 junction that represents the velocity of m_2, which forces m_1 and m_2 to move with the same velocity. The FSM for the controlled 0 junctions shows that the junction changes to its *off* mode when the force between m_1 and m_2 exceeds the breakaway force, $F > F_{breakaway}$. In the *off* mode, the 0 junction exerts 0 force on both m_1 and m_2, and so they move independently. The FSM also shows that if the velocity difference between m_1 and m_2 falls below a threshold velocity ($\Delta V < v_{th}$) the two bodies 'stick' to each other again. Note that this model does not account for viscous friction when the two bodies move relative to one another as the focus of this paper is on the discontinuous behavior.

During behavior generation, initially the 0 junction is in its *on* mode because the bodies are at rest on top of one another and the system evolves in continuous time. When the point in time occurs at which F_{in} changes discontinuously, new velocities for both m_1 and m_2 are computed. These velocities, however, may require a force to be exerted on m_2 that causes the condition $F > F_{breakaway}$ to be satisfied and the 0 junction changes to its *off* mode. Once in the *off* mode, if the velocity difference is sufficiently large, no further mode changes occur. Since the discrete mode changes have thus converged, the system proceeds to evolve in continuous time.

The end result is that the two bodies m_1 and m_2 evolve according to a mode of continuous evolution until a point in time at which a force is exerted. At this point in time, the corresponding velocities and forces are computed and based on the newly computed values the connection between the two bodies changes mode such that they are dynamically independent. In the mode of independence the system evolves in continuous time again. Since there is no effect of the external force on the velocity of m_2, in order to arrive at the proper values for reinitialization of v_1 and v_2, the mode where the external force becomes active while m_1 and m_2 are still connected is considered to have no effect on the physical state, which is referred to as a *mythical mode* [27]. Note that in more detail the mythical mode would comprise a very small nonlinear *dissipation* effect that captures the quick buildup of force before disconnecting.

3. MODELS OF TIME

In the following subsections, background information on superdense and hyperreal time models is provided. These time semantics are required to arrive at a combination in the form of a hyperdense time that allows for representing pinnacles and mythical modes in concert. It is shown how previous results can now be mapped onto a hyperdense semantic domain.

3.1 Introduction to superdense time

Time-event sequence is a semantic domain for describing event-based models. Intuitively, time-event sequences are instantaneous events separated by non-negative real numbers that describe time durations between the events. Events separated by zero duration are simultaneous, but have a well-defined causal ordering.

Superdense time was introduced to represent time-event sequences as functions of time [21]. Superdense time is a totally ordered subset of $\mathbb{R}_+ \times \mathbb{N}$, where the non-negative real number represents the real time and the natural number represents the causal ordering. Simultaneous events at time t are mapped to $(t, 0), (t, 1), \ldots$ superdense time instants such that the ordering of the events is preserved.

The (total) ordering of superdense time is given by the following definitions: $(t, n) = (t', n') \Leftrightarrow t = t' \wedge n = n'$, and $(t, n) < (t', n') \Leftrightarrow t < t' \vee (t = t' \wedge n < n')$. Therefore, superdense time is a time model that can be used to describe simultaneous events as functions of time, while retaining the causality of events.

Mythical modes emerge as an artifact of logical inference to determine a new mode in which physical state can change. As such, mythical modes do not affect the dynamic state of a physical system. Moreover, different logic formulations may traverse different mythical modes yet still arrive at the same resulting mode where physical state changes can occur. Consequently, the logical evaluation has no corresponding manifestation in the dynamic state of a physical system and occurs at a single point in time along a logical inferencing dimension. This behavior corresponds to the superdense semantic domain.

3.2 Introduction to hyperreal time

The system of hyperreals extends the theories with real numbers to a larger number system that includes infinitesimal and infinite numbers. An infinitesimal ϵ is any nonzero number, such that $|\epsilon| < \frac{1}{n}$ for any $n \in \mathbb{N}$. Intuitively, the idea behind hyperreals is to extend the dense field of $\mathbb{R}$ with infinitely many points around each real number such that any real sentence that holds for one or more real functions also holds for the hyperreal natural extensions of these functions [19] (transfer principle).

In the ultrapower construction [16], hyperreals are represented as infinite sequences of real numbers $\langle u_1, u_2, \ldots\rangle \in \mathbb{R}^{\mathbb{N}}$ with real numbers embedded as constant sequences (i.e., a real number r is the sequence of $\langle r, r, \ldots\rangle \in \mathbb{R}^{\mathbb{N}}$). These sequences, together with elementwise addition and multiplication operations, form a commutative ring but not a field (since the multiplication of two non-zero numbers could result in zero: $\langle 0, 1, 0, \ldots\rangle \times \langle 1, 0, 1, \ldots\rangle = \langle 0, 0, \ldots\rangle$). This issue is remedied by considering equivalence classes of $\mathbb{R}^{\mathbb{N}}$ defined by a free ultrafilter U of $\mathbb{N}$.

Let J be a nonempty set. An ultrafilter on J is a nonempty collection U of subsets of J having the following properties: $\emptyset \notin U$; $A \in U$ and $B \in U$ implies $A \cap B \in U$; $A \in U$ and $A \subseteq B \subseteq J$ implies $B \in U$; for all $A \subseteq J$, either $A \in U$ or $J \setminus A \in U$. The ultrafilter U is free if the intersection of all its members is the empty set: $\bigcap_{A \in U} A = \emptyset$.

Given a free ultrafilter U of $\mathbb{N}$, an equivalence relation $=_U$ can be defined over $\mathbb{R}^{\mathbb{N}}$: $u =_U v$ holds for sequences $u = \langle u_1, u_2, \ldots\rangle$ and $v = \langle v_1, v_2, \ldots\rangle$ iff $\{i \mid u_i = v_i\} \in U$. The hyperreals are then defined as the quotient of $\mathbb{R}^{\mathbb{N}}$ by U, ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/U$.

As a semantic domain, hyperreals have the advantage that around any real time instant there are hyperreal time instants that are closer to it than any other real time instant. Such extension of time greatly simplifies the semantic specification of discontinuities, in particular, the description of pinnacles that represent fast physical behaviors where the dynamic state changes discontinuously. As a result, a pinnacle corresponds to a distinct state of physical behavior. In physics, such a distinct state corresponds to a distinct point in time. Because the continuous behavior represented by a pinnacle is considered to occur infinitely fast, time is considered to advance by an infinitesimal amount for a pinnacle to implement the physical state change. This behavior corresponds to the hyperreal domain.

3.3 Hyperdense time

The models of time provide the ingredients for a semantic domain that is sufficiently rich to formalize the behavior at discontinuities in physical system models including mythical mode behavior and pinnacle behavior, as well as combinations of both. It is a straightforward extension to introduce a hyperdense time model as a “combination” of the superdense and hyperreal time models. We define the hyperdense time ${}^*\mathbb{R}_+ \times \mathbb{N}$ as the product of the non-negative hyperreals and natural numbers. Such a time model can be used for representing both infinitesimal time advancements as different hyperreals, as well as establishing a causal ordering among superdense time instances with the same hyperreal. The particular value of such a precise semantic description lies in the ability to develop consistent computational behavior generation algorithms.

As discussed before, mythical mode changes correspond to immediate (zero-time) changes during which the state variables are not updated, i.e., 0/0. Pinnacles, on the other hand, are found at $\epsilon/\mathbb{R}$ since they correspond to real jumps in an infinitesimally small time interval.

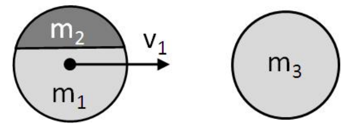


Figure 4: A variant of Newton’s cradle

4. COMPUTATIONAL REPRESENTATION OF HYPERDENSE SEMANTICS

Because of the discreteness of computational values and limited precision of a computational implementation, the semantic domain of computational models cannot represent superdense nor hyperreal domains. Therefore, (i) a modeling formalism must include a means for the modeler to distinguish between pinnacles and mythical modes and (ii) the behavior generation algorithms must include sophistication that addresses the differences between superdense and hyperreal semantic domains. The resulting semantic domain for the computational implementation maps continuous time behavior onto floating point numbers, $\mathbb{F}$, while the mythical mode and pinnacle event iteration each map onto an integer domain. The first principle hyperdense semantic domain ${}^*\mathbb{R}_+ \times \mathbb{N}$ then maps onto a three-dimensional model of time $\mathbb{F} \times \mathbb{N}^{-1} \times \mathbb{N}$ in a computational implementation, where the pinnacle domain comprises the natural numbers extended with -1 in order to capture the physical state immediately preceding the point in time at which discontinuities occur. This section first presents a paradigmatic example followed by a computational implementation. This implementation can now be formalized based on mapping the first principles hyperdense semantic domain onto a computational three-dimensional time semantic domain.

4.1 A paradigmatic example

In Fig. 4, a variant of Newton’s cradle is shown. One of the bodies, m_1, has another body, m_2, positioned on top of it. Stiction effects between m_1 and m_2 cause them to behave as one body with combined mass as long as the breakaway force between them, $F_{breakaway}$, is not exceeded. The body m_1 may collide with another body, m_3, according to a perfectly elastic collision, $\Delta v_{32} = -\Delta v_{32}^{-}$, where Δv is the difference in velocities $(v_3 - v_2)$ after the collision and Δv_{32}^{-} is the difference in velocities before the collision.

The bond graph model in Fig. 5 shows the three masses as inertias, I, each of them connected to a common velocity junction, 1, which have as velocity on all connected ports the velocity of the directly connected mass. The 1 junctions are connected by common force junctions, 0. The 0 junctions are *controlled junctions* in that a finite state machine determines whether their *on* or *off* state is active. The plastic collision is modeled as ideally plastic (i.e., a coefficient of restitution $\epsilon = 1$) by a modulated flow source *MSf*. If the controlled junction 0_1 is in its *on* state, this flow source enforces a difference in velocities of m_1 and m_3, possibly accounting for the rigidly connected mass m_2. If the controlled junction 0_1 is in its *off* state, a 0 force is exerted on both m_1 and m_3 (possibly accounting for m_2). The controlled junction 0_2 exerts a 0 force in its *off* state as well, which is when the force at the contact point between m_1 and m_2 exceeds the breakaway force. Note that for clarity no viscous friction in case m_1 and m_2 move independently in continuous time

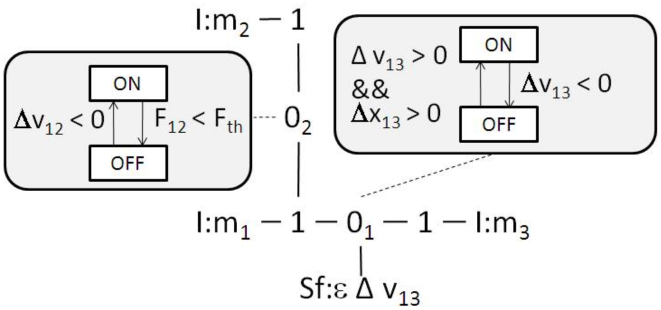


Figure 5: Model of the variant of Newton's cradle

is modeled. If the difference in velocities between m_1 and m_2 falls below a threshold level, the stiction effect becomes active, which is modeled by 0_2 changing to its *on* state. In the *on* state, a difference in velocities of m_2 and m_3 of 0 is enforced.

Upon collision of m_1 and m_3, if the difference in velocities v_1 and v_2, Δv_{12}, is less than the threshold velocity v_{th}, stiction is active and m_1 and m_2 behave as one body with mass $m_1 + m_2$. In case $m_1 + m_2 < m_3$, this would result in a return velocity of m_1 (i.e., m_1 would start moving in the opposite direction compared to the velocity before the collision). However, the momentum of m_1 may be such that an impulsive force [8] arises between m_1 and m_2 that triggers the $F_{breakaway}$ transition, causing the two bodies to move independently. In this case, if $m_1 = m_3$ and $\epsilon = 1$, there is no return velocity for m_1 but instead it acts as in the case of Newton's cradle where m_3 assumes all of the momentum of m_1 while m_1 comes to rest. In this case the velocity of m_2 is not affected by the collision.

The importance of a semantic domain that combines both superdense as well as hyperreals is prominently displayed by this example. While the condition for 0_2 to switch from *on* to *off* occurs in 0 time, the condition for 0_1 to switch from *on* to *off* occurs in infinitesimal, ϵ, time. A critical consequence of this phenomenon is that, although in reasoning about the system, 0_1 first changes its state to *on*, after which the change of state in 0_2 to *off* is determined, the change of state in 0_1 back to *off* is not effected until *after* the change of state in 0_2 to *off*.

4.2 A computational implementation

To generate behaviors for the system in Fig. 4, a computational model is designed based on the hybrid bond graph model in Fig. 5 by using the hybrid bond graph modeling and simulation tool HyBrSim [28]. Figure 6 shows the hybrid bond graph in Fig. 5 modeled in HyBrSim with the finite state machines of the controlled junctions shown in Fig. 7.

The HyBrSim model shows the power bonds in the hybrid bond graph as harpoons (i.e., one-sided arrow heads). The bond graph elements are shown as black rectangles with their type on the right-hand side of each rectangle. The left-hand side of each rectangle displays the name of the element as well as its parameter. Mapping the bond graph elements (0, 1, I, and Sf) onto the model in Fig. 5 is straightforward, where the HyBrSim models shows that the masses all are

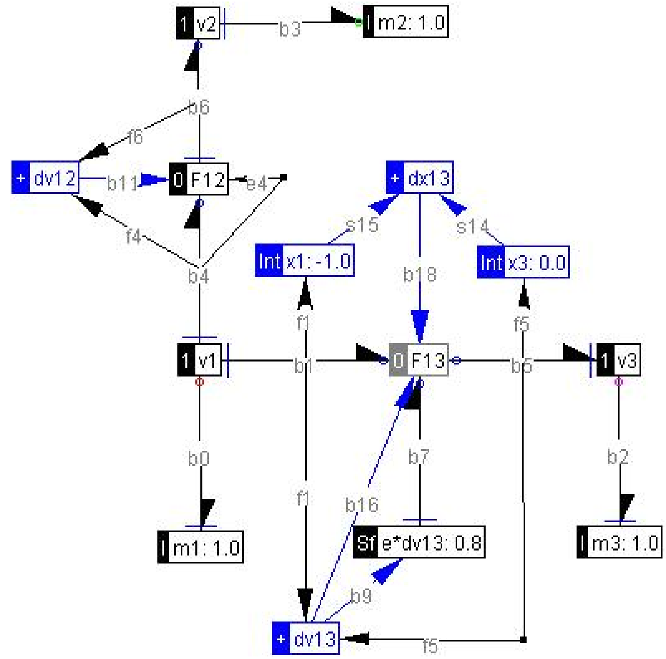


Figure 6: Hybrid bond graph of the modified Newton's cradle

chosen as 1.0 and the coefficient of restitution, ϵ, is chosen to be 0.8.

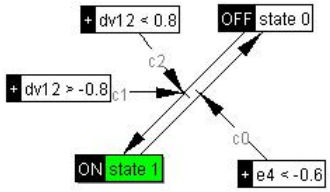
In addition to the bond graph elements, Fig. 6 shows signal elements of type + and *Int*. The latter integrate their input with respect to time, which in Fig. 6 is either $f1$ (the flow variable of bond $b1$ that represents the velocity of m_1) or $f5$ (the flow variable of bond $b5$ that represents the velocity of m_3). The result of the integration is the position of m_1 and m_3, respectively.

The type + elements add their input (with possible negative sign, which is a property of each particular element) and so the *dx13* element subtracts the position of m_3, represented by signal *s14*, from the position of m_1, represented by signal *s15*. The resulting output of *dx13* is input to the controlled 0-junction *F13* and so become a source in the state transition diagram in Fig. 7(b) that controls the mode of *F13*. Similarly, the + element *dv13* subtracts the velocity of m_3, represented by *f5* from the velocity of m_1 represented by *f1*. The resulting output is input to the controlled 0-junction *F13* as well and so also becomes a source (with two incarnations) in the state transition diagram Fig. 7(b).

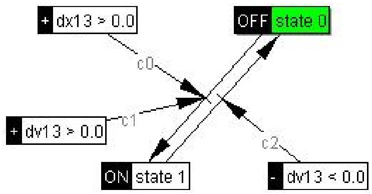
Finally, the + element *dv12* subtracts the velocity of m_2, represented by the flow *f6* on bond *b6* from the velocity of m_1 represented by the flow *f4* on bond *b4*. The resulting output is input to the controlled 0-junction *F12* and so becomes a source (with two incarnations) in the state transition diagram Fig. 7(a). In addition, the state transition diagram shows a source *e4*, which is shown in the HyBrSim model to be the effort on bond *b4* and this corresponds to the force exerted across the 0 junction *F12*.

4.3 Behavior generation

A key aspect of the computational model in HyBrSim is the language elements that allow a modeler to map the conceptual differences in discontinuities on a hyperdense seman-



(a) F12



(b) F13

Figure 7: Finite state machines for controlled junctions

tic domain onto a computational three-dimensional model of time. In HyBrSim this is facilitated by allowing the modeler to state whether a guard in the state transition diagrams is evaluated based on *a priori* or *a posteriori* variable values in the model. This is illustrated in the state transition diagrams of Fig. 7 by the guard elements being annotated on the left-hand side by either a '-' sign (e.g., $dv13 < 0.0$) or a '+' sign (e.g., $dv13 > 0.0$), respectively.

In the case of an *a priori* guard, the variable values that are employed are those of the most recent hyperreal advance. In other words, while the model iterates across mythical modes, the truth value of the *a priori* guard does not change. In contrast, in the case of an *a posteriori* guard, the most recent variable values (either of a pinnacle or a mythical mode) are employed to compute its truth value. In addition to the language elements for expressing the different types of discontinuous behaviors, the behavior generation of HyBrSim is based on the *mythical mode algorithm* [26] to enable evaluating *a posteriori* truth values in between hyperreal advances (corresponding to 'accepted' and *physically meaningful* states). Finally, it should be noted that HyBrSim implements conservation of physical extensities (charge, mass, flux, etc.) to hold across structural changes in a hybrid bond graph [28], which completes the necessary computational machinery for behavior generation.

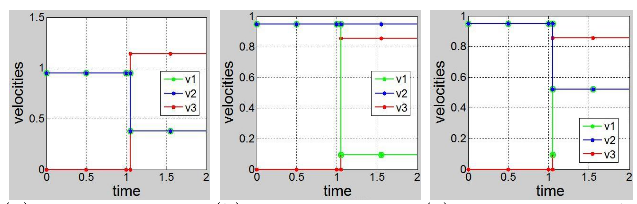
Parameter studies can now be conducted to analyze three qualitatively different behaviors: (i) a partially plastic collision between the combined $m_1 + m_2$ mass and m_3 may occur without further discontinuous effects; (ii) the collision may cause the breakaway force F_{th} to be exceeded so that m_1 and m_2 behave as independent masses, after which continuous behavior resumes; and (iii) before continuous behavior resumes, the difference in velocities of m_1 and m_2 is such that it falls below the threshold velocity and m_1 and m_2 act as a single mass again. The studies are presented based on computational behavior generation with the progression of behavior on both the hyperdense (as a 2 tuple $\langle time, mythical\ mode\rangle$ with time as hyperreals that progress by infinitesimals, ϵ, to represent pinnacles) and the three-dimensional (as a corresponding 3 tuple $\langle time, pinnacle, mythical\ mode\rangle$) semantic domains. This

time	0_{F12}	0_{F13}	p_{m1}	p_{m2}	p_{m3}
$\langle t_{collide} - \epsilon, 0\rangle$ $\langle t_{collide}, -1, 0\rangle$	*on*	*off*	0.95	0.95	0
$\langle t_{collide}, 0\rangle$ $\langle t_{collide}, 0, 0\rangle$	*on*	*on*	0.38	0.38	1.14
$\langle t_{collide} + \epsilon, 0\rangle$ $\langle t_{collide}, 1, 0\rangle$	*on*	*off*	0.38	0.38	1.14

Table 1: Sequence of mode changes in infinitesimal steps for $v_{th} = 0.8$ and $F_{th} = -0.6$

notation is used in the following tables while referring to the column labeled time.

Behavior generation for the first case is shown in Fig. 8(a) with the numerical values of the detailed discontinuous behavior upon collision shown in Table 1. With the chosen parameters ($m_1 = m_2 = m_3 = 1.0$ and $\epsilon = 0.8$) and initial conditions for the velocities and momenta ($v_{1,0} = p_{1,0} = v_{2,0} = p_{2,0} = 0.95$, $v_{3,0} = 0$), the table shows how upon collision, the mode of controlled 0 junction F_{13} changes from *off* to *on*. The resulting collision causes a change in momentum across all three bodies involved and time advances by an infinitesimal amount. Upon the advance the collision mode of F_{13} turns *off* and the bodies proceed to move in a continous fashion with the changed momentum.



(a) $F_{th} = -0.6$ and $v_{th} = 0.8$ (b) $F_{th} = -0.5$ and $v_{th} = 0.8$ (c) $F_{th} = -0.5$ and $v_{th} = 0.875$

Figure 8: Simulation results of the computational algorithm ($m_1 = m_2 = m_3 = 1.0$, $v_{1,0} = v_{2,0} = 0.95$, $v_{3,0} = 0$, $\epsilon = 0.8$)

Behavior generation for the second case is shown in Fig. 8(b) with the numerical values of the detailed discontinuous behavior upon collision shown in Table 2. With the $F_{th} = -0.5$ parameter, the change in momenta upon collision ($t = \langle t_{collide}, 0\rangle$) exceeds the breakaway force and before an infinitesimal advance, a consecutive mode change occurs at $t = \langle t_{collide}, 1\rangle$ that causes the controlled 0 junction between m_1 and m_2 to turn *off*. This new set of modes of the controlled junctions effects a further change in momenta after which no further instantaneous mode changes occur. Therefore, time advances by an infinitesimal step ($t = \langle t_{collide} + \epsilon, 0\rangle$), after which the system proceeds to evolve continuously.

Behavior generation for the third case is shown in Fig. 8(c) with the numerical values of the detailed discontinuous behavior upon collision shown in Table 3. Here, the previous scenario is followed up till $t = \langle t_{collide} + \epsilon, 0\rangle$. However, after the controlled 0 junction that models the collision effect changes its mode to *off*, the difference in velocities of m_1 and m_2 falls below the threshold velocity and the masses become rigidly connected again by the 0 junction F_{12} switching back *on* instantaneously (i.e., at $t = \langle t_{collide} + \epsilon, 1\rangle$). Once in this mode, no further mode changes occur and the system proceeds to evolve continuously.

time	0_{F12}	0_{F13}	p_{m1}	p_{m2}	p_{m3}
$\langle t_{collide} - \epsilon, 0\rangle$ $\langle t_{collide}, -1, 0\rangle$	*on*	*off*	0.95	0.95	0
$\langle t_{collide}, 0\rangle$ $\langle t_{collide}, 0, 0\rangle$	*on*	*on*	0.38	0.38	1.14
$\langle t_{collide}, 1\rangle$ $\langle t_{collide}, 0, 1\rangle$	*off*	*on*	0.095	0.95	0.855
$\langle t_{collide} + \epsilon, 0\rangle$ $\langle t_{collide}, 1, 0\rangle$	*off*	*off*	0.095	0.95	0.855

Table 2: Sequence of mode changes in 0 and infinitesimal steps for $v_{th} = 0.8$ and $F_{th} = -0.5$

time (t)	0_{F12}	0_{F13}	p_{m1}	p_{m2}	p_{m3}
$\langle t_{collide} - \epsilon, 0\rangle$ $\langle t_{collide}, -1, 0\rangle$	*on*	*off*	0.95	0.95	0
$\langle t_{collide}, 0\rangle$ $\langle t_{collide}, 0, 0\rangle$	*on*	*on*	0.38	0.38	1.14
$\langle t_{collide}, 1\rangle$ $\langle t_{collide}, 0, 1\rangle$	*off*	*on*	0.095	0.95	0.855
$\langle t_{collide} + \epsilon, 0\rangle$ $\langle t_{collide}, 1, 0\rangle$	*off*	*off*	0.095	0.95	0.855
$\langle t_{collide} + \epsilon, 1\rangle$ $\langle t_{collide}, 1, 1\rangle$	*on*	*off*	0.5225	0.5225	0.855
$\langle t_{collide} + 2\epsilon, 0\rangle$ $\langle t_{collide}, 2, 0\rangle$	*on*	*off*	0.5225	0.5225	0.855

Table 3: Sequence of mode changes in 0 and infinitesimal steps for $v_{th} = 0.875$ and $F_{th} = -0.5$

The sequences of mode changes is depicted in Table 3, which clearly shows the difference in effects as the model evolves in the computational representation of superdense and hyperreal time. The ability to differentiate between $t = \langle t_{collide}, 1\rangle$ and $t = \langle t_{collide} + \epsilon, 0\rangle$ enables making the distinction between the pinnacle effect of 0 junction F_{13} and the mythical mode effect of 0 junction F_{12}. Otherwise, 0 junction F_{13} would have switched back *off* simultaneously with 0 junction F_{12} switching *off*. This would either: (i) not allow modeling of inferencing (mythical) modes or (ii) have the collision effect (incorrectly) computed for m_1 and m_2 comprising a combined mass of $m_1 + m_2$.

Note that in evaluating the truth value of the guard with F_{th} as a variable requires the ability to evaluate impulses. Because of the discontinuous changes in velocities, the accelerations become impulsive, and, therefore, the forces that are compared against F_{th} become impulsive as well. Because of the infinite magnitude of an impulse, F_{th} would always be exceeded and the guard would always evaluate to true. Instead, HyBrSim allows the use of either a numerical approximation of the impulse (based on a small numerical delta in time) or the use of the impulsive area in the comparison to evaluate the truth value of the guard [24]. For example, in the case of the model in Fig. 6, the area of the impulse corresponds to the discontinuous change in momentum of m_1.

5. RELATED WORK

A number of research activities are presently being explored in the hybrid dynamic system community. Based on the superdense time notion of Maler, Manna, and Pnueli [21], Lee and Zheng have developed an operational semantics of hybrid systems [20]. Because of the focus on tool development, their work concentrates on computational models, in which case the notion of hyperreals can be reduced to an integer dimension. In contrast with the three-dimensional model of time developed for computational models in this paper, support for mythical mode iterations is not considered in their work. Moreover, employing superdense time for impact modeling requires a direct mapping of physics onto a computational representation that differs in the first principles semantic domain. In turn, this requires a high conceptual investment and relies on additional validation to compensate for the omission of a modeling stage based on a theory of physics (e.g., [38]).

In a different exploration, Bliudze and Krob [3] and Benveniste, Bourke, and Pouzet have pursued nonstandard analysis [2, 3] to formalize the semantics of hybrid automata models. Though the intent is less about developing a first principles modeling approach, by developing a mathematical representation that helps in formulating a computational implementation, the work provides an unbiased foundation that does relate well to an underlying physics theory. Still, compared to the work presented in this paper, their work currently does not support a semantics for mythical mode behavior. Generally, the choice of hybrid automata as a modeling formalism requires explicit formulation of much of the behavior of a system that is being modeled and so in the context of the work presented here it would require a conceptual investment to explicitly determine the mythical mode overall end effects.

Other work on formalizing hybrid automata [1] relies on a superdense semantic domain and so comparisons with a hyperdense semantic domain apply. Similar to hybrid automata, related work on dynamic logic [37] requires an explicit formulation of discontinuities with the corresponding conceptual investment that it necessitates.

Somewhat further removed but still related in the underlying intent are efforts by Bourke and Pouzet [4] to formalize the semantics of hybrid system modeling languages. In particular, formalizing the combination of synchronous behavior and continuous-time behavior generated by a variable-step numerical solver requires a precise understanding of discrete event behavior and its bearing on continuous evolution. Focus in their work is an eventual computational implementation and as such other work by Mosterman, Zander, Hamon, and Deckla [33] has taken a perspective in which continuous-time differential equations behavior is defined by a numerical solver that is applied in the computational approximation. In addition to the lack of support for mythical mode iteration, this work also does not specifically focus on a first principles theory for modeling physics.

While the challenges of mythical mode iteration are acknowledged in related work by Söderman and Strömberg [39], the classification of *transient*, *semi-transient*, and *nontransient* modes in the implementation by Edström, Strömberg, Söderman, and Top [11] does not provide support for mythical modes. In their work, a mode transition system (MTS) is derived from a declarative formulation of a *switched bond graph*. Depending on the execution semantics of the operational MTS, a parallel execution may handle mythical modes by compiling a static transition structure that accounts for these modes. Furthermore, though the use of bond graphs bases the work by Edström , Strömberg, Söderman, and Top [11] on a solid theory of physics, the MTS concentrates on a computational implementation (i.e., interpreted as integer semantics), and, therefore, the use of hyperreals to represent pinnacles (*semi-transient* modes in MTS) is not part of their work [10].

Previous work by Mosterman and Biswas [27] provided a theory of physics to support mythical mode iteration but did not elaborate to the same extent on pinnacle iteration. The lack of a hyperreals in the semantic domain lead to modeling a sequence of collisions as a sequence of isolated points on the reals. The resulting gaps in time between these points challenged the understanding of physics from a thermodynamics perspective, because it includes continuity principles. Hyperreals allow formulating such collision sequences with infinitesimal advances between the collisions and thus, the semantic pitfall of gaps in time are avoided.

6. CONCLUSION

A critical characteristic of physical system models is the manner in which the system dynamics are captured and abstracted. Generally, physical systems are well described by continuous-time behavior. However, if the dynamics of some of the continuous behavior is much faster than the time scale of interest it is sometimes more efficient to model the resulting effect by discontinuous change. The corresponding models then combine continuous behavior with discontinuous change and the system comprises hybrid dynamics.

In this paper, the discontinuous change is represented by pinnacles and mythical modes. These two classes of modes, in turn, are represented by different temporal semantics. Pinnacles require physical time to advance during discontinuous change. This change is modeled using a notion of time as hyperreals. Mythical modes do not allow changes in the physical state and so occur without time increment. As such, these modes are represented by a notion of superdense time.

Consequently, a combined notion of time that allows for the expression of both semantics is required in order to analyze pinnacles and mythical modes in concert. Moreover, so as to support behavior generation, a computational representation is necessary that captures the interaction between these two classes of discontinuities in a consistent and executable manner.

In the presented work, a *hyperdense* notion of time is introduced as a combination of superdense and hyperreal domains. The particular value of using such a precise semantic description lies in the ability to develop consistent computational behavior generation algorithms. To this end, the hyperdense notion of time is mapped onto a three-dimensional computational model of time. In particular, continuous-time behavior is mapped onto floating point numbers, while the mythical mode and pinnacle event iteration each map onto an integer domain. A modified Newton's cradle is used as a case study and to illustrate the implementation.

The benefits of having a hyperdense semantics include support for combining the expertise of physics modeling with computer science and control engineering. Such combined expertise becomes increasingly valuable as cyber-physical systems are rising to prominence in the realm of modern technical systems. Moreover, providing a computational semantics based on the hyperdense semantic domain allows for designing better interdisciplinary behavioral models derived from first principle foundations. Impact scenarios, for example, in multibody systems can be studied with a sound theoretical foundation about the physical interactions and nuances that are taking place before, during, and after the collision moment.

Moreover, the work provides support for implicit modeling that is ubiquitous in industry as well as a foundation for relating implicit models to explicit models that are often used in academia. For example, nonterminating behavior in the superdense sense (causing time to halt) can now be strictly separated from nonterminating behavior in the hyperreal sense (causing time to advance by infinitesimals).

7. REFERENCES

[1] R. Alur, C. Courcoubetis, T. A. Henzinger, and P.-H. Ho. Hybrid automata: An algorithmic approach to the specification and verification of hybrid systems. In R. Grossman, A. Nerode, A. Ravn, and H. Rischel, editors, *Lecture Notes in Computer Science*, volume 736, pages 209–229. Springer-Verlag, 1993.

[2] A. Benveniste, T. Bourke, B. Caillaud, and M. Pouzet. Non-standard semantics of hybrid systems modelers. *Journal of Computer and System Sciences*, 78(3):877–910, May 2012.

[3] S. Bliudze and D. Krob. Modelling of complex systems: Systems as dataflow machines. *Fundamenta Informaticae*, 91:1–24, 2009.

[4] T. Bourke and M. Pouzet. Zélus, a Synchronous Language with ODEs. In *International Conference on Hybrid Systems: Computation and Control (HSCC 2013)*, Philadelphia, USA, April 8–11 2013. ACM.

[5] P. Breedveld. Multibond graph elements in physical systems theory. *Journal of the Franklin Institute*, 319(1/2):1–36, January/February 1985.

[6] P. C. Breedveld. *Physical Systems Theory in Terms of Bond Graphs*. PhD dissertation, University of Twente, Enschede, Netherlands, 1984.

[7] P. C. Breedveld. The context-dependent trade-off between conceptual and computational complexity illustrated by the modeling and simulation of colliding objects. In *CESA '96 IMACS Multiconference*, Lille, France, July 1996. Ecole Centrale de Lille.

[8] B. Brogliato. *Nonsmooth Mechanics*. Springer-Verlag, London, 1999. ISBN 1-85233-143-7.

[9] H. Callen. *Thermodynamics*. John Wiley & Sons, Inc., New York/London, 1960.

[10] K. Edström. *Switched Bond Graphs: Simulation and Analysis*. PhD dissertation, Linköping University, Sweden, 1999.

[11] K. Edström, J.-E. Strömberg, U. Söderman, and J. Top. Modelling and simulation of a switched power converter. In F. E. Cellier and J. J. Granda, editors, *1997 International Conference on Bond Graph Modeling and Simulation (ICBGM '97)*, pages 195–200, Phoenix, AZ, Jan. 1997. Society for Computer Simulation.

[12] H. Elmqvist, B. Bachmann, F. Boudaud, J. Broenink, D. Brück, T. Ernst, R. Franke, P. Fritzson, A. Jeandel, P. Grozman, K. Juslin, D. Kågedahl, M. Klose, N. Loubere, S. E. Mattsson, P. Mosterman, H. Nilsson, M. Otter, P. Sahlin, A. Schneider, H. Tummescheit, and H. Vangheluwe. Modelica^{tm}–a unified object-oriented langauge for physical systems modeling: Language specification, Dec. 1999. version 1.3, http://www.modelica.org/.

[13] G. Falk and W. Ruppel. *Energie und Entropie: Eine Einführung in die Thermodynamik*. Springer-Verlag, Berlin, Heidelberg, New York, 1976. ISBN 3-540-07814-2.

[14] J. Friedman and J. Ghidella. Using model-based design for automotive systems engineering – requirements analysis of the power window example.

In *Proceedings of the SAE 2006 World Congress & Exhibition*, pages CD–ROM: 2006–01–1217, Detroit, MI, Apr. 2006.

[15] O. Heaviside. On the forces, stresses, and fluxes of energy in the electromagnetic field. *Proceedings of the Royal Society of London*, 50:126–129, 1891.

[16] A. Hurd and P. Loeb. *An Introduction to Nonstandard Real Analysis*. Pure and Applied Mathematics. Elsevier Science, 1985.

[17] Y. Iwasaki, A. Farquhar, V. Saraswat, D. Bobrow, and V. Gupta. Modeling time in hybrid systems: How fast is "instantaneous"? In *1995 International Conference on Qualitative Reasoning*, pages 94–103, Amsterdam, May 1995. University of Amsterdam.

[18] D. Karnopp, D. Margolis, and R. Rosenberg. *Systems Dynamics: A Unified Approach*. John Wiley and Sons, New York, 2 edition, 1990.

[19] H. J. Keisler. *Elementary Calculus: An Infinitesimal Approach*. Prindle, Weber and Schmidt, Dover, 3 edition, 2012.

[20] E. Lee and H. Zheng. Operational semantics of hybrid systems. In *International Conference on Hybrid Systems: Computation and Control (HSCC 2005)*, pages 25–53, Zürich, Switzerland, Mar. 2005.

[21] O. Maler, Z. Manna, and A. Pnueli. From timed to hybrid systems. In *Real-Time: Theory in Practice*, Lecture Notes in Computer Science, pages 447–484. Springer, 1992.

[22] MathWorks®. *MATLAB® and Simulink® product families*, Sept. 2012.

[23] P. J. Mosterman. *Hybrid Dynamic Systems: A hybrid bond graph modeling paradigm and its application in diagnosis*. PhD dissertation, Vanderbilt University, 1997.

[24] P. J. Mosterman. HyBrSim—a modeling and simulation environment for hybrid bond graphs. *Journal of Systems and Control Engineering*, 216(1):35–46, 2002.

[25] P. J. Mosterman and G. Biswas. Behavior generation using model switching a hybrid bond graph modeling technique. In F. E. Cellier and J. J. Granda, editors, *1995 International Conference on Bond Graph Modeling and Simulation (ICBGM '95)*, number 1 in Simulation, pages 177–182, Las Vegas, Jan. 1995. Society for Computer Simulation, Simulation Councils, Inc. Volume 27.

[26] P. J. Mosterman and G. Biswas. Modeling discontinuous behavior with hybrid bond graphs. In *1995 International Workshop on Qualitative Reasoning*, pages 139–147, Amsterdam, May 1995. University of Amsterdam.

[27] P. J. Mosterman and G. Biswas. A theory of discontinuities in dynamic physical systems. *Journal of the Franklin Institute*, 335B(3):401–439, Jan. 1998.

[28] P. J. Mosterman and G. Biswas. A java implementation of an environment for hybrid modeling and simulation of physical systems. In *Proceedings of the International Conference on Bond Graph Modeling*, pages 750–755, Mexico City, Mexico, Sept. 1999.

[29] P. J. Mosterman, J. Ghidella, and J. Friedman. Model-based design for system integration. In *Proceedings of The Second CDEN International Conference on Design Education, Innovation, and Practice*, pages CD–ROM: TB–3–1 through TB–3–10, Kananaskis, Alberta, Canada, July 2005.

[30] P. J. Mosterman, S. Prabhu, and T. Erkkinen. An industrial embedded control system design process. In *Proceedings of The Inaugural CDEN Design Conference (CDEN'04)*, Montreal, Canada, July 2004. CD-ROM: 02B6.

[31] P. J. Mosterman, G. Simko, and J. Zander. A hyperdense semantic domain for discontinuous behavior in physical system models. In *Proceedings of the 7th International Workshop on Multi-Paradigm Modeling at the ACM/IEEE 16th International Conference on Model Driven Engineering Languages and Systems (MoDELS) conference*, Miami, FL, Sept. 2013.

[32] P. J. Mosterman and H. Vangheluwe. Computer automated multi-paradigm modeling in control system design. In *Proceedings of the IEEE International Symposium on Computer-Aided Control System Design*, pages 65–70, Anchorage, Alaska, Sept. 2000.

[33] P. J. Mosterman, J. Zander, G. Hamon, and B. Denckla. A computational model of time for stiff hybrid systems applied to control synthesis. *Control Engineering Practice*, 20(1):2–13, 2012.

[34] P. J. Mosterman, F. Zhao, and G. Biswas. An ontology for transitions in physical dynamic systems. In *AAAI98*, pages 219–224, July 1998.

[35] T. Nishida and S. Doshita. Reasoning about discontinuous change. In *Proceedings AAAI-87*, pages 643–648, Seattle, Washington, 1987.

[36] H. M. Paynter. *Analysis and Design of Engineering Systems*. The M.I.T. Press, Cambridge, Massachusetts, 1961.

[37] A. Platzer. Differential dynamic logic for hybrid systems. *Journal of Automated Reasoning*, 41(2):143–189, 2008.

[38] D. E. Post and L. G. Votta. Computational science demands a new paradigm. *Physics Today*, 58(8):35–41, Jan. 2005.

[39] U. Söderman and J.-E. Strömberg. Switched bond graphs: Multiport switches, mathematical characterization and systematic composition of computational models. Technical Report LiTH-IDA-R-95-7, Department of Computer and Information Science, Linköping University, Linköping, Sweden, 1995.

[40] Steering Committee for Foundations in Innovation for Cyber-Physical Systems. Foundations for Innovation: Strategic Opportunities for the 21st Century Cyber-Physical Systems—Connecting computer and information systems with the physical world. Technical report, National Institute of Standards and Technology (NIST), Mar. 2013.

[41] J.-E. Strömberg, J. Top, and U. Söderman. Variable causality in bond graphs caused by discrete effects. In *Proceedings of the International Conference on Bond Graph Modeling*, pages 115–119, San Diego, California, 1993.