*A Hypergraph Blow-Up Lemma**

Peter Keevash
School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK; e-mail: p.keevash@qmul.ac.uk.

Received 1 October 2007; accepted 5 October 2010; received in final form 19 October 2010
Published online 10 March 2011 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/rsa.20362

ABSTRACT: We obtain a hypergraph generalisation of the graph blow-up lemma proved by Komlós, Sarközy and Szemerédi, showing that hypergraphs with sufficient regularity and no atypical vertices behave as if they were complete for the purpose of embedding bounded degree hypergraphs. © 2011 Wiley Periodicals, Inc. Random Struct. Alg., 39, 275–376, 2011

Keywords: hypergraphs; hypergraph regularity; hypergraph embeddings

1. INTRODUCTION

Szemerédi's regularity lemma [44] has impressive applications in many areas of modern graph theory, including extremal graph theory, Ramsey theory and property testing. Roughly speaking, it says that any graph can be approximated by an average with respect to a partition of its vertex set into a bounded number of classes, the number of classes depending only on the accuracy of the desired approximation, and not on the number of vertices in the graph. A key property of this approximation is that it leads to a 'counting lemma', allowing an accurate prediction of the number of copies of any small fixed graph spanned by some specified classes of the partition. We refer the reader to [28] for a survey of the regularity lemma and its applications. An analogous theory for hypergraphs has only been developed very recently, with independent and rather different approaches given by Rödl et al. (e.g. [37, 41, 43]) and Gowers [13], subsequently reformulated and developed in [1, 7, 16, 38, 45, 46]. In a very short space of time the power of this hypergraph theory has already been amply demonstrated, e.g. by a multidimensional generalisation of Szemerédi's theorem on arithmetic progressions [13] and a linear bound for the Ramsey number of hypergraphs with bounded maximum degree [4, 35].

The blow-up lemma is a powerful tool developed by Komlós, Sarközy and Szemerédi [24] for using the regularity lemma to embed spanning subgraphs of bounded degree. An

---

Correspondence to: P. Keevash
*Supported by ERC grant (239696); EPSRC grant (EP/G056730/1)

informal statement is that graphs with sufficient regularity and no atypical vertices behave as if they were complete for the purpose of embedding bounded degree graphs. In [25, 26] they used it to prove Seymour's conjecture on the minimum degree needed to embed the kth power of a Hamilton cycle, and the Alon-Yuster conjecture on the minimum degree needed for a graph to have an H-factor, for some fixed graph H (a question finally resolved in [32]). There are many other applications to embedding spanning subgraphs, see the survey [33]. There are also several results on embedding spanning subhypergraphs, such as perfect matchings or (various definitions of) Hamilton cycles, see the survey [39]. For the most part, the proofs of the known hypergraph results have not needed any analogue of the blow-up lemma. An exception is an embedding lemma for some special spanning subhypergraphs proved in [30] for loose Hamilton cycles in 3-graphs, although the 'absorbing' method of Rödl, Ruciński and Szemerédi was subsequently shown to be a simpler method for Hamilton cycles [15, 21, 40]. Another partial hypergraph blow-up lemma is to embedding bounded degree subgraphs of linear size, obtained independently in [4] and in [35] (for 3-graphs). The application to linear Ramsey numbers of bounded degree hypergraphs was also subsequently proved by simpler means in [3]. However, one would not necessarily expect that embedding lemmas can always be avoided by using alternative methods, so a hypergraph blowup lemma would be a valuable tool.

In this paper we prove such a result that gives conditions for embedding any spanning hypergraph of bounded degree. We will not attempt a formal statement of our result in this introduction, as it will take us a considerable amount of work to set up the necessary notation and terminology, particularly for the key notion of super-regularity, which has some additional subtleties that do not appear in the graph case. The proof will be by means of a randomised greedy embedding algorithm, which is very similar that used in [24]. However the analysis is more involved, due both to the additional complications of hypergraph regularity theory, and the need to work with an approximating hypergraph rather than the true hypergraph (see Section 3 for further explanation). There are many potential applications of our theorem to hypergraph generalisations of results for graphs that were obtained with the graph blow-up lemma. We will illustrate the method by proving a hypergraph generalisation of a result of Kühn and Osthus [31, Theorem 2] on packing bipartite graphs.

The rest of this paper is organised as follows. In the next section we prove the blow-up lemma of Komlós, Sarközy and Szemerédi [24]. This is mostly for expository purposes, although there are some small differences in our proof, and it will be useful to refer back to the basic argument when discussing additional complications that arise for hypergraphs. We will prove our main result at first in the special case of 3-graphs (with some additional simplifications); this case is already sufficiently complex to illustrate the main ideas of our proofs. In section 3 we discuss hypergraph regularity theory (following the approach of Rödl et al.) and motivate and define super-regularity for 3-graphs. We prove the 3-graph blow-up lemma in section 4. In section 5 we develop some additional theory that is needed for applications of the 3-graph blow-up lemma, based on the Regular Approximation Lemma of Rödl and Schacht [41]. We also give a 'black box' reformulation of the blow-up lemma that will be more easily accessible for future applications. We illustrate this by generalising a result in [31] to packing tripartite 3-graphs. The final section concerns the general hypergraph blow-up lemma. As well as generalising from 3-graphs to k-graphs, we allow additional generalisations that will be needed in future applications, including restricted positions and complex-indexed complexes (defined in that section). The proof is mostly similar to that for 3-graphs, so we only give full details for those aspects that are different. We conclude with some remarks on potential developments and applications of the blow-up lemma.

Notation. We will introduce a substantial amount of terminology and notation throughout the paper, which is summarised in the index. Before starting our discussion we establish the following basic notation. We write $[n] = \{1, \dots, n\}$. If X is a set and k is a number then $\binom{X}{k} = \{Y \subseteq X : |Y| = k\}$, $\binom{X}{\le k} = \cup_{i \le k} \binom{X}{i}$ and $\binom{X}{<k} = \cup_{i<k} \binom{X}{i}$. $a \pm b$ denotes an unspecified real number in the interval $[a - b, a + b]$. It is convenient to regard a finite set X as being equipped with the uniform probability measure $\mathbb{P}(\{x\}) = 1/|X|$, so that we can express the average of a function f defined on X as $\mathbb{E}_{x \in X} f(x)$. A *k-graph* H consists of a *vertex set* $V(H)$ and an *edge set* $E(H)$, each edge being some k-tuple of vertices. We often identify H with $E(H)$, so that $|H|$ is the number of edges in H. A *k-complex* H consists of a vertex set $V(H)$ and an edge set $E(H)$, where each edge is a subset of the vertex set of size at most k, that is a *simplicial complex*, i.e. if $S \in E(H)$ and $T \subset S$ then $T \in E(H)$. For $S \subset V(H)$ the *neighbourhood* $(k - |S|)$-graph or $(k - |S|)$-complex is $H(S) = \{A \subset V(H) \setminus S : A \cup S \in E(H)\}$, and $|H(S)|$ is the *degree* of S. We also write $H^S = \{A \subseteq V(H) : S \subseteq A \in E(H)\}$. A *walk* in H is a sequence of vertices for which each consecutive pair are contained in some edge of H, and the distance between two vertices is the length of the shortest walk connecting them. The *vertex neighbourhood* $VN_H(x)$ is the set of vertices at distance exactly 1 from x (so x itself is not included). We will often have to consider hierarchies involving many real parameters, and it will be useful to use the notation $0 < \alpha \ll \beta$ to mean that there is an increasing function $f(x)$ so that the following argument is valid for $0 < \alpha < f(\beta)$. The parameter n will always be sufficiently large compared to all other parameters, and we use the phrase *with high probability* to refer to an event that has probability $1 - o_n(1)$, i.e. the probability tends to 1 as n tends to infinity.

2. THE GRAPH BLOW-UP LEMMA

This section is mostly expository. We introduce the basic notions of regularity and super-regularity for graphs and prove the blow-up lemma of Komlós, Sarközy and Szemerédi. This will serve as a warm-up to the hypergraph blow-up lemma, as our proof even in the graph case differs slightly from the original in a few details (although the general approach is the same). It will also be helpful to establish our notation in this simplified setting, and to refer back to the basic argument when explaining why certain extra complications arise for hypergraphs. To streamline the proof we focus on a slightly simplified setting, which still contains all the ideas needed for the general case. We hope that the general reader will find this section to be an accessible account of a proof that has a reputation for difficulty!

We start with a brief summary of the key notions in graph regularity, referring the reader to [28] for more details. Consider an r-partite graph G with vertex set V partitioned as $V = V_1 \cup \cdots \cup V_r$. Let G_{ij} be the bipartite subgraph of G with parts V_i and V_j, for $1 \le i \ne j \le r$. The *density* of G_{ij} is $d(G_{ij}) = \frac{|G_{ij}|}{|V_i||V_j|}$. Given $\epsilon > 0$, we say that G_{ij} is *ϵ-regular* if for all subsets $V_i' \subseteq V_i$ and $V_j' \subseteq V_j$ with $|V_i'| > \epsilon |V_i|$ and $|V_j'| > \epsilon |V_j|$, writing G_{ij}' for the bipartite subgraph of G with parts V_i' and V_j', we have $|d(G_{ij}') - d(G_{ij})| < \epsilon$. Then we say that G is ϵ-regular if each G_{ij} is ϵ-regular. Informally, we may say that each G_{ij} behaves like a random bipartite graph, up to accuracy ϵ. This statement is justified by the counting lemma, which allows one to estimate the number of copies of any fixed graph F, up to accuracy $O(\epsilon)$, using a suitable product of densities. For now we just give an example: if we write $T_{123}(G)$ for the set of triangles formed by the graphs G_{12}, G_{13}, G_{23}, then

$$d(T_{123}(G)) := \frac{|T_{123}(G)|}{|V_1||V_2||V_3|} = d(G_{12})d(G_{13})d(G_{23}) \pm 8\epsilon. \tag{1}$$

Remarkably, this powerful property can be applied in any graph G, via Szemerédi's Regularity Lemma, which can be informally stated as saying that we can decompose the vertex set of any graph on n vertices into $m(\epsilon)$ parts, such that all but at most ϵn^2 edges belong to bipartite subgraphs that are ϵ-regular.

The blow-up lemma arises from the desire to embed *spanning* graphs in G, meaning that they use every vertex in V. Suppose that $|V_i| = n$ for $1 \le i \le r$. The argument used to prove the counting lemma can be generalised to embed any bounded degree graph H provided that all components of H have size $o(n)$, and $\Omega(n)$ vertices of G are allowed to remain uncovered. However, one cannot guarantee an embedding of a spanning graph: the definition of ϵ-regularity does not prevent the existence of isolated vertices, so we may not even be able to find a perfect matching. This observation naturally leads us to the stronger notion of super-regularity. We say that G_{ij} is (ϵ, d_{ij})-*super-regular* if it is ϵ-regular and every vertex has degree at least $(d_{ij} - \epsilon)n$. It is well-known that one can delete a small number of vertices from a regular pair to make it super-regular (see Lemma 5.3). We say that G is (ϵ, d)-*super-regular* if each G_{ij} is either empty or (ϵ, d_{ij})-super-regular for some $d_{ij} \ge d$. Now we can state the graph blow-up lemma.

Theorem 2.1 (Graph blow-up lemma). *Suppose H is an r-partite graph on $X = X_1 \cup \cdots \cup X_r$ and G is an r-partite graph on $V = V_1 \cup \cdots \cup V_r$, where $|V_i| = |X_i| = n$ for $1 \le i \le r$ and H_{ij} is only non-empty when G_{ij} is non-empty. If H has maximum degree at most D and G is (ϵ, d)-super-regular, where $0 \ll 1/n \ll \epsilon \ll d, 1/r, 1/D$, then G contains a copy of H, in which for each $1 \le i \le r$ the vertices of V_i correspond to the vertices of X_i.*

Informally speaking, Theorem 2.1 embeds any bounded degree graph into any super-regular graph. Note that arbitrary part sizes are allowed in [24], but for simplicity we start by considering the case when they are all equal. The proof is via a random greedy algorithm for embedding H in G, which considers the vertices of X in some order and embeds them to V one at a time. We start by giving an informal description of the algorithm.

Initialisation. List the vertices of H in a certain order, as follows. Some vertices at mutual distance at least 4 are identified as buffer vertices and put at the end of the list. The neighbours of the buffer vertices are put at the start of the list. (The rationale for this order is that we hope to embed these neighbours in a nice manner while there is still plenty of room in the early stages of the algorithm, and then the buffer vertices still have many suitable places at the conclusion.) During the algorithm a queue of priority vertices may arise: it is initially empty.

Iteration. Choose the next vertex x to be embedded, either from the queue if this is non-empty, or otherwise from the list. The image $\phi(x)$ of x is chosen randomly in $V(G)$ among those free spots that do not unduly restrict the free spots for those unembedded neighbours of x. If some unembedded vertex has too few free spots it is added to the queue. Stop when all non-buffer vertices have been embedded. If the number of vertices that have ever been in the queue becomes too large before this point then abort the algorithm as a failure (this is an unlikely event).

Conclusion. Choose a system of distinct representatives among the free slots for the unembedded vertices to complete the embedding. (This will be possible with high probability.)

Now we will formally describe the random greedy algorithm to construct an embedding $\phi : V(H) \to V(G)$ such that $\phi(e) \in E(G)$ for every $e \in E(H)$. First we introduce more parameters with the hierarchy

$$0 \ll 1/n \ll \epsilon \ll \epsilon' \ll \epsilon_* \ll p_0 \ll \gamma \ll \delta_Q \ll p \ll d_u \ll \delta'_Q \ll \delta_B \ll d, 1/r, 1/D.$$

To assist the reader we list here the role of each parameter for easy reference. Parameters ϵ, ϵ' and ϵ_* are used to measure graph regularity. Parameter γ plays the role of κ in [24]: it is used to distinguish various cases at the conclusion of the algorithm when selecting the system of distinct representatives. Parameter d_u plays the role of γ in [24]: it is a universal lower bound on the proportion of vertices in a class of G free to embed any given vertex of H. The *queue threshold* parameter δ_Q corresponds to δ''' in [24]: the maximum proportional size for the queue before we will abort with failure. The *buffer* parameter δ_B corresponds to δ' in [24]: the proportional size of the buffer. We also introduce two probability parameters that are not explicitly named in [24], although they are key to the proof. Parameter p_0 appears in Lemma 2.6 in the upper bound for the probability that any given set A will be significantly under-represented in the free images for a vertex. Parameter p appears in Lemma 2.5 as a lower bound for the probability that a given unused vertex will be free as an image for a given buffer vertex at the conclusion of the algorithm. Note that the *queue admission* parameter δ'_Q is similar to but slightly different from the corresponding parameter δ'' in [24]: for any vertex z and time t we will compare $F_z(t)$ to an earlier free set $F_z(t_z)$, where t_z is the most recent time at which we embedded a neighbour of z.

Initialisation and notation. We choose a buffer set $B \subset X$ of vertices at mutual distance at least 4 in H so that $|B \cap X_i| = \delta_B n$ for $1 \le i \le r$. The maximum degree property of H implies that we can construct B simply by selecting vertices one-by-one greedily. For any given vertex in H there are fewer than D^4 vertices within distance 4, so at any point in the construction of B we have excluded at most $D^4 r\delta_B n$ vertices from any given X_i. Thus we can construct B if we choose $\delta_B < 1/(rD^4)$.

Let $N = \cup_{x\in B} N_H(x)$ be the vertices with a neighbour in the buffer. Since H has maximum degree D we have $|N \cap X_i| \le Dr\delta_B n < \sqrt{\delta_B} n$ for $1 \le i \le r$, if we choose $\delta_B < 1/(Dr)^2$.

We use t to denote time during the algorithm, by which we mean the number of vertices of H that have been embedded. At time t we denote the queue by $q(t)$ and write $Q(t) = \cup_{u\le t} q(u)$ for the vertices that have ever been in the queue by time t. Initially we set $q(0) = Q(0) = \emptyset$.

We order the vertices in a list $L = L(0)$ that starts with N and ends with B. Within N, we arrange that $N_H(x)$ is consecutive for each $x \in B$. This is possible by the mutual distance property in B, which implies that the neighbourhoods $N_H(x)$, $x \in B$ are mutually disjoint. We denote the vertex of H selected for embedding at time t by $s(t)$. This will be the first vertex of $L(t-1)$, unless the queue is non-empty, when this takes priority.

We write $F_x(t)$ for the vertices that are *free* to embed a given vertex x of H. Initially we set $F_x(0) = V_x$, where we write V_x for that part V_i of G corresponding to the part X_i of H that contains x. We also write $X_i(t) = X_i \setminus \{s(u) : u \le t\}$ for the unembedded vertices of X_i and $V_i(t) = V_i \setminus \{\phi(s(u)) : u \leq t\}$ for the available positions in V_i. We let $X(t) = \cup_{i=1}^r X_i(t)$ and $V(t) = \cup_{i=1}^r V_i(t)$. Initially we set $X_i(0) = X_i$ and $V_i(0) = V_i$.

Iteration. At time t, while there are still some unembedded non-buffer vertices, we select a vertex to embed $x = s(t)$ according to the following *selection rule*. If the queue $q(t)$ is non-empty then we let x be any member of the queue; otherwise we let x be the first vertex of the list $L(t-1)$. (A First In First Out rule for the queue is used in [24], but this is not essential to the proof.) We choose the image $\phi(x)$ of x uniformly at random among all elements $y \in F_x(t-1)$ that are 'good', a property that can be informally stated as saying that if we set $\phi(x) = y$ then the free sets at time t for the unembedded neighbours of x will have roughly their 'expected' size.

To define this formally, we first need to describe the update rule for the free sets when we embed x to some vertex y. First we set $F_x(t) = \{y\}$. We will have $F_x(t') = \{y\}$ at all subsequent times $t' \geq t$. Then for any unembedded z that is not a neighbour of x we set $F_z(t) = F_z(t-1) \setminus \{y\}$. Thus the size of $F_z(t)$ decreases by 1 if z belongs to the same part of X as x, but otherwise is unchanged. Finally, for any unembedded $z \in N_H(x)$ we set $F_z(t) = F_z(t-1) \cap N_G(y)$. Now we say that $y \in F_x(t-1)$ belongs to the *good* set $OK_x(t-1)$ if for every unembedded $z \in N_H(x)$ we have $|F_z(t)| = (1 \pm 2\epsilon')d_{xz}|F_z(t-1)|$. Here d_{xz} denotes that density $d(G_{ij})$ for which $x \in V_i$ and $z \in V_j$.

Having chosen the image $\phi(x)$ of x as a random good element y, we conclude the iteration by updating the list $L(t-1)$ and the queue $q(t-1)$. First we remove x from whichever of these sets it was taken. Then we add to the queue any unembedded vertex z for which $F_z(t)$ has become 'too small'. To make this precise, suppose $z \in L(t-1) \setminus \{x\}$, and let t_z be the most recent time at which we embedded a vertex in $N_H(z)$, or 0 if there is no such time. (Note that if $z \in N_H(x)$ then $t_z = t$.) We add z to $q(t)$ if $|F_z(t)| < \delta'_Q|F_z(t_z)|$. This defines $L(t)$ and $q(t)$.

Repeat this iteration until the only unembedded vertices are buffer vertices, but abort with failure if at any time we have $|Q(t) \cap X_i| > \delta_Q|X_i|$ for some $1 \leq i \leq r$. Let T denote the time at which the iterative phase terminates (whether with success or failure).

Conclusion. When all non-buffer vertices have been embedded, we choose a system of distinct representatives among the free slots $F_x(T)$ for the unembedded vertices $x \in X(T)$ to complete the embedding, ending with success if this is possible, otherwise aborting with failure.

Now we analyse the algorithm described above and show that it is successful with high probability. We start by recording two standard facts concerning graph regularity. The first fact states that most vertices in a regular pair have 'typical' degree, and the second that regularity is preserved by restriction to induced subgraphs. We maintain our notation that G is an r-partite graph on $V = V_1 \cup \cdots \cup V_r$. We fix any pair (i,j), write G_{ij} for the bipartite subgraph spanned by V_i and V_j and denote its density by d_{ij}. We give the short proofs of these facts here, both for completeness and as preparation for similar hypergraph arguments later.

Lemma 2.2 (Typical degrees). *Suppose G_{ij} is ϵ-regular. Then all but at most $2\epsilon|V_i|$ vertices in V_i have degree $(d_{ij} \pm \epsilon)|V_j|$ in V_j.*

Proof. We claim that there is no set $X \subseteq V_i$ of size $|X| > \epsilon|V_i|$ such that every $x \in X$ has degree less than $(d_{ij} - \epsilon)|V_j|$ in V_j. For the pair (X, V_j) would then induce a subgraph of density less than $d_{ij} - \epsilon$, contradicting the definition of ϵ-regularity. Similarly, there is no set $X \subseteq V_i$ of size $|X| > \epsilon|V_i|$ such that every $x \in X$ has degree greater than $(d_{ij} + \epsilon)|V_j|$ in V_j. ▪

Lemma 2.3 (Regular restriction). *Suppose G_{ij} is ϵ-regular, and we have sets $V_i' \subseteq V_i$ and $V_j' \subseteq V_j$ with $|V_i'| \geq \sqrt{\epsilon}|V_i|$ and $|V_j'| \geq \sqrt{\epsilon}|V_j|$. Then the bipartite subgraph G_{ij}' of G_{ij} induced by V_i' and V_j' is $\sqrt{\epsilon}$-regular of density $d_{ij}' = d_{ij} \pm \epsilon$.*

Proof. Since G_{ij} is ϵ-regular we have $d_{ij}' = d_{ij} \pm \epsilon$. Now consider any sets $V_i'' \subseteq V_i'$ and $V_j'' \subseteq V_j'$ with $|V_i''| \geq \sqrt{\epsilon}|V_i'|$ and $|V_j''| \geq \sqrt{\epsilon}|V_j'|$. Then $|V_i''| \geq \epsilon|V_i|$ and $|V_j''| \geq \epsilon|V_j|$, so V_i'' and V_j'' induce a bipartite subgraph of G_{ij} with density $d_{ij} \pm \epsilon \subset d_{ij}' \pm \sqrt{\epsilon}$. Therefore G_{ij}' is $\sqrt{\epsilon}$-regular. ▪

Our next lemma shows that the definition of good vertices for the algorithm is sensible, in that most free vertices are good. Before giving the lemma, we observe that the number of free vertices in any given class does not become too small at any point during the algorithm. We can quantify this as $|V_i(t)| \geq |B \cap V_i| - |Q(t) \cap V_i| \geq (\delta_B - \delta_Q)n \geq \delta_B n/2$, for any $1 \leq i \leq r$ and time t. To see this, note that we stop the iterative procedure when the only unembedded vertices are buffer vertices, and during the procedure a buffer vertex is only embedded if it joins the queue.

Lemma 2.4 (Good vertices). *Suppose we embed a vertex $x = s(t)$ of H at time t.*

Then $|OK_x(t-1)| \geq (1-\epsilon_*)|F_x(t-1)|$, and for every unembedded vertex z we have $|F_z(t)| \geq d_u n$.

Proof. We argue by induction on t. At time $t = 0$ the first statement is vacuous, as we do not embed any vertex at time 0, and the second statement follows from the fact that $F_z(0) = V_z$ has size n for all z. Now suppose $t \geq 1$. By induction we have $|F_z(t-1)| \geq d_u n$ for every unembedded vertex z. Then by Lemma 2.3, for any unembedded $z \in N_H(x)$, the bipartite subgraph of G induced by $F_x(t-1)$ and $F_z(t-1)$ is ϵ'-regular of density $(1 \pm \epsilon)d_{xz}$. Applying Lemma 2.2, we see that there are at most $2\epsilon'|F_x(t-1)|$ vertices $y \in F_x(t-1)$ that do not satisfy $|N_G(y) \cap F_z(t-1)| = (1 \pm 2\epsilon')d_{xz}|F_z(t-1)|$. Summing over at most D neighbours of x and applying the definition of good vertices in the algorithm we obtain the first statement that $|OK_x(t-1)| \geq (1 - 2D\epsilon')|F_x(t-1)| \geq (1-\epsilon_*)|F_x(t-1)|$.

Next we prove the second statement. Consider any unembedded vertex z. Let t_z be the most recent time at which we embedded a neighbour of z, or 0 if there is no such time. If $t_z > 0$ then we embedded some neighbour $w = s(t_z)$ of z at time t_z. Since we chose the image $\phi(w)$ of w to be a good vertex, by definition we have $|F_z(t_z)| = (1 \pm 2\epsilon')d_{wz}|F_z(t_z - 1)| > \frac{1}{2}d|F_z(t_z - 1)|$. If z is not in the queue then the rule for updating the queue in the algorithm gives $|F_z(t)| \geq \delta_Q'|F_z(t_z)|$. On the other hand, suppose z is in the queue, and that it joined the queue at some time $t' < t$. Since z did not join the queue at time $t' - 1$ we have $|F_z(t'-1)| \geq \delta_Q'|F_z(t_z)|$. Also, between times t' and t we only embed vertices that are in the queue: the queue cannot become empty during this time, as then we would have embedded z before x. During this time we embed at most $\delta_Q n$ vertices in V_z, as we abort the algorithm if the number of vertices in X_z that have ever been queued exceeds this.

Thus we have catalogued all possible ways in which the number of vertices free for z can decrease. It may decrease by a factor no worse than $d/2$ when a neighbour of z is embedded, and by a factor no worse than δ_Q' before the next neighbour of z is embedded, unless z joins the queue. Also, if z joins the queue we may subtract at most $\delta_Q n$ from the number of vertices free for z. Define i to be the number of neighbours of z that are embedded before z joins the queue if it does, or let $i = d(z)$ be the degree of z if z does not join the queue. Now z has at most D neighbours, and $|F_z(0)| = |V_z| = n$, so $|F_z(t)| \geq (\delta'_Q d/2)^{D-i}((\delta'_Q d/2)^i \delta'_Q n - \delta_Q n) \geq (\delta'_Q d/2)^D \delta'_Q n - \delta_Q n > d_u n$. ∎

Next we turn our attention to the time period during which we are embedding N, which we will refer to as the *initial phase* of the algorithm. We start by observing that the queue remains empty during the initial phase, and so N is embedded consecutively in the order given by the list L. To see this, we use a similar argument to that used for the second statement in Lemma 2.4. Consider any unembedded vertex z and suppose that the queue has remained empty up to the current time t. Then we have embedded at most $|N \cap X_z| < \sqrt{\delta_B} n$ vertices in V_z. Also, if we embed a neighbour w of z the algorithm chooses a good image for it, so by definition of good, the number of free images for z decreases by a factor no worse than $(1 - 2\epsilon')d_{wz} > d/2$ when we embed w. Since z has at most D neighbours we get $|F_z(t)| \geq (d/2)^D n - \sqrt{\delta_B} n > \delta'_Q n$. This shows that no unembedded vertex z is added to the queue during the initial phase.

Now we want to show that for any buffer vertex $x \in B$ there will be many free positions for x at the end of the algorithm. This is the point in the argument where super-regularity is essential. A vertex $v \in V_x$ will be free for x if it is not used for another vertex and we embed $N_H(x)$ in $N_G(v)$ during the initial phase. Our next lemma gives a lower bound on this probability, conditional on any embedding of the previous vertices not using v. We fix some $x \in B$ and write $N_H(x) = \{z_1, \ldots, z_g\}$, with vertices listed in the order that they are embedded. We let T_j be the time at which z_j is embedded. Since N is embedded consecutively we have $T_{j+1} = T_j + 1$ for $1 \leq j \leq g - 1$. We also define $T_0 = T_1 - 1$. Since vertices in B are at mutual distance at least 4 in H, at time T_0, when we have only embedded vertices from N, no vertices within distance 2 of x have been embedded. (This is the only place at which mutual distance 4 is important.)

Lemma 2.5 (Initial phase). *For any $v \in V_x$, conditional on any embedding of the vertices $\{s(u) : u < T_1\}$ that does not use v, with probability at least p we have $\phi(N_H(x)) \subseteq N_G(v)$.*

Proof. We estimate the probability that $\phi(N_H(x)) \subseteq N_G(v)$ using arguments similar to those we are using to embed H in G. We want to track the free positions within $N_G(v)$ for each unembedded vertex in $N_H(x)$, so we hope to not only embed each vertex of $N_H(x)$ in $N_G(v)$ but also to do so in a 'good' way, a property that can be informally stated as saying that the free positions within $N_G(v)$ will have roughly their expected size. To define this formally, suppose $1 \leq j \leq g$ and we are considering the embedding of z_j. We interpret quantities at time T_j with the embedding $\phi(z_j) = y$, for some unspecified $y \in F_{z_j}(T_j - 1)$. We say that $y \in F_{z_j}(T_j - 1) \cap N_G(v)$ is *good for* v, and write $y \in OK^v_{z_j}(T_j - 1)$, if for every unembedded $z \in N_H(x)$ we have $|F_z(T_j) \cap N_G(v)| = (1 \pm 2\epsilon')d_{zz_j}|F_z(T_j - 1) \cap N_G(v)|$. We let A_j denote the event that $y = \phi(z_j)$ is chosen in $OK^v_{z_j}(T_j - 1)$.

We claim that conditional on the events $A_{j'}$ for $j' < j$ and the embedding up to time $T_j - 1$, the probability that A_j holds is at least $d_u/2$. To see this we argue as in Lemma 2.4. First we show that $|F_z(T_j - 1) \cap N_G(v)| \geq d_u n$ for every unembedded neighbour z of x. Note that initially $F_z(0) \cap N_G(v) = V_z \cap N_G(v)$ has size at least $(d - \epsilon)n$ by super-regularity of G. Up to time T_0 we embed at most $|N \cap X_z| \leq \sqrt{\delta_B} n$ vertices in X_z, and do not embed any neighbours of z by the distance property mentioned before the lemma. At time $T_{j'}$ with $j' < j$, we are conditioning on the event that the algorithm chooses an image for $z_{j'}$ that is good for v, so the number of free images for z within $N_G(v)$ decreases by a factor no worse than $(1-2\epsilon')d_{zz_{j'}} > d/2$. Thus we indeed have $|F_z(T_j-1)\cap N_G(v)| \geq (d/2)^D|V_z\cap N_G(v)| - |N\cap X_z| \geq (d/2)^D(d-\epsilon)n - \sqrt{\delta_B}n \geq d_u n$.

Next, by Lemma 2.3, for any unembedded $z\in N_H(x)$, the bipartite subgraph of G induced by $F_{z_j}(T_j-1)\cap N_G(v)$ and $F_z(T_j-1)\cap N_G(v)$ is ϵ'-regular of density $(1\pm\epsilon)d_{zz_j}$. Applying Lemma 2.2, we see that there are at most $2\epsilon'|F_{z_j}(T_j-1)\cap N_G(v)|$ vertices $y\in F_{z_j}(T_j-1)\cap N_G(v)$ that do not satisfy $|F_z(T_j)\cap N_G(v)| = |N_G(y)\cap F_z(T_j-1)\cap N_G(v)| = (1\pm 2\epsilon')d_{zz_j}|F_z(T_j-1)\cap N_G(v)|$. Summing over at most D neighbours of z_j we see that $|OK^v_{z_j}(T_j-1)| \geq (1-2D\epsilon')|F_{z_j}(T_j-1)\cap N_G(v)|$. Also, by Lemma 2.4 we have $|OK_{z_j}(T_j-1)| \geq (1-\epsilon_*)|F_{z_j}(T_j-1)|$, so

$$\left|OK^v_{z_j}(T_j-1)\cap OK_{z_j}(T_j-1)\right| \geq |F_{z_j}(T_j-1)\cap N_G(v)| - 2\epsilon_* n \geq d_u n/2.$$

Since $\phi(z_j)=y$ is chosen uniformly at random from $OK_{z_j}(T_j-1)$, we see that A_j holds with probability at least $d_u/2$, as claimed.

To finish the proof, note that if all the events A_j hold then we have $\phi(N_H(x))\subseteq N_G(v)$. Multiplying the conditional probabilities, this holds with probability at least $(d_u/2)^D > p$. ▪

Our next lemma is similar to the 'main lemma' of [24]. An informal statement is as follows. Consider any subset Y of a given class X_i, and any 'not too small' subset A of vertices of V_i that could potentially be used for Y. Then it is very unlikely that no vertices in A will be used and yet only a small fraction of the free positions for every vertex in Y will belong to A.

Lemma 2.6 (Main lemma). *Suppose $1\leq i\leq r$, $Y\subseteq X_i$ and $A\subseteq V_i$ with $|A|>\epsilon_* n$. Let $E_{A,Y}$ be the event that (i) no vertices are embedded in A before the conclusion of the algorithm, and (ii) for every $z\in Y$ there is some time t_z such that $|A\cap F_z(t_z)|/|F_z(t_z)| < 2^{-D}|A|/n$. Then $\mathbb{P}(E_{A,Y}) < p_0^{|Y|}$.*

Proof. We start by choosing $Y'\subseteq Y$ with $|Y'|>|Y|/D^2$ so that vertices in Y' are mutually at distance at least 3 (this can be done greedily, using the fact that H has maximum degree D). It suffices to bound the probability of $E_{A,Y'}$. Note that initially we have $|A\cap F_z(0)|/|F_z(0)| = |A|/n$ for any $z\in X_i$. Also, if no vertices are embedded in A, then $|A\cap F_z(t)|/|F_z(t)|$ can only be less than $|A\cap F_z(t-1)|/|F_z(t-1)|$ for some z and t if we embed a neighbour of z at time t. It follows that if $E_{A,Y'}$ occurs, then for every $z\in Y'$ there is a first time t_z when we embed a neighbour w of z and have $|A\cap F_z(t_z)|/|F_z(t_z)| < |A\cap F_z(t_z-1)|/2|F_z(t_z-1)|$.

By Lemma 2.4 we have $|F_w(t_z-1)|\geq d_u n$, and by choice of t_z we have $|A\cap F_z(t_z-1)|/|F_z(t_z-1)| \geq 2^{-D}|A|/n$, so $|A\cap F_z(t_z-1)| \geq 2^{-D}\epsilon_* d_u n \geq \epsilon_*^2 n$. Then by Lemma 2.3, the bipartite subgraph of G induced by $A\cap F_z(t_z-1)$ and $F_w(t_z-1)$ is ϵ'-regular of density $(1\pm\epsilon)d_{zw}$. Applying Lemma 2.2, we see that there are at most $2\epsilon'|F_w(t_z-1)|$ 'exceptional' vertices $y\in F_w(t_z-1)$ that do not satisfy $|A\cap F_z(t_z)| = |N_G(y)\cap A\cap F_z(t_z-1)| = (1\pm 2\epsilon')d_{zw}|A\cap F_z(t_z-1)|$. On the other hand, the algorithm chooses $\phi(w)=y$ to be good, in that $|F_z(t_z)| = (1\pm 2\epsilon')d_{zw}|F_z(t_z-1)|$, so we can only have $|A\cap F_z(t_z)|/|F_z(t_z)| < |A\cap F_z(t_z-1)|/2|F_z(t_z-1)|$ by choosing an exceptional vertex y. But y is chosen uniformly at random from $|OK_w(t_z-1)| \geq (1-\epsilon_*)|F_w(t_z-1)|$ possibilities (by Lemma 2.4). It follows that, conditional on the prior embedding, the probability of choosing an exceptional vertex for y is at most $2\epsilon'|F_w(t_z-1)|/|OK_w(t_z-1)| < 3\epsilon'$.

Since vertices of Y' have disjoint neighbourhoods, we can multiply the conditional probabilities over $z\in Y'$ to obtain an upper bound of $(3\epsilon')^{|Y'|}$. Recall that this bound is for a subset of $E_{A,Y'}$ in which we have specified a certain neighbour w for every vertex $z \in Y'$. Taking a union bound over at most $D^{|Y'|}$ choices for these neighbours gives $\mathbb{P}(E_{A,Y}) \le \mathbb{P}(E_{A,Y'}) \le (3\epsilon' D)^{|Y'|} < p_0^{|Y|}$. ∎

Now we can prove the following theorem, which implies Theorem 2.1.

Theorem 2.7. *With high probability the algorithm embeds H in G.*

Proof. First we estimate the probability of the iteration phase aborting with failure, which happens when the number of vertices that have ever been queued is too large. We can take a union bound over all $1 \le i \le r$ and $Y \subseteq X_i$ with $|Y| = \delta_Q |X_i|$ of $\mathbb{P}(Y \subseteq Q(T))$. Suppose that the event $Y \subseteq Q(T)$ occurs. Then $A = V_i(T)$ is a large set of unused vertices, yet it makes up very little of what is free to any vertex in Y. To formalise this, note that by definition, for every $z \in Y$ there is some time t such that $|F_z(t)| < \delta'_Q |F_z(t_z)|$, where $t_z < t$ is the most recent time at which we embedded a neighbour of z. Since A is unused we have $A \cap F_z(t) = A \cap F_z(t_z)$, so $|A \cap F_z(t_z)|/|F_z(t_z)| = |A \cap F_z(t)|/|F_z(t_z)| \le |F_z(t)|/|F_z(t_z)| < \delta'_Q$. However, we noted before Lemma 2.4 that $|A| \ge \delta_B n/2$, so since $\delta'_Q \ll \delta_B$ we have $|A \cap F_z(t_z)|/|F_z(t_z)| < 2^{-D}|A|/n$. Taking a union bound over all possibilities for i, Y and A, Lemma 2.6 implies that the failure probability is at most $r \cdot 4^n \cdot p_0^{\delta_Q n} < o(1)$, since $p_0 \ll \delta_Q$.

Now we estimate the probability of the conclusion of the algorithm aborting with failure. Recall that buffer vertices have disjoint neighbourhoods, the iterative phase finishes at time T, and $|X_i(T)| \ge \delta_B n/2$. By Hall's criterion for finding a system of distinct representatives, the conclusion fails if and only if there is some $1 \le i \le r$ and $S \subseteq X_i(T)$ such that $|\cup_{z \in S} F_z(T)| < |S|$. We divide into cases according to the size of S.

$0 \le |S|/|X_i(T)| \le \gamma$. By Lemma 2.4 we have $|F_z(T)| \ge d_u n > \gamma n$ for every unembedded z, so this case cannot occur.

$\gamma \le |S|/|X_i(T)| \le 1 - \gamma$. In this case we use the fact that $A := V_i(T) \setminus \cup_{z \in S} F_z(T)$ is a large set of unused vertices which is completely unavailable to any vertex z in S: we have $|A| \ge |V_i(T)| - |S| \ge \gamma |X_i(T)| \ge \gamma \delta_B n/2 \ge \gamma^2 n$, yet $A \cap F_z(T) = \emptyset$, so $|A \cap F_z(T)|/|F_z(T)| = 0 < 2^{-D}|A|/n$. As above, taking a union bound over all possibilities for i, S and A, Lemma 2.6 implies that the failure probability is at most $r \cdot 4^n \cdot p_0^{\gamma^2 n} < o(1)$, since $p_0 \ll \gamma$.

$1 - \gamma \le |S|/|X_i(T)| \le 1$. In this case we claim that with high probability $\cup_{z \in S} F_z(T) = V_i(T)$, so in fact Hall's criterion holds. It suffices to consider sets $S \subseteq X_i(T)$ of size exactly $(1-\gamma)|X_i(T)|$. The claim fails if there is some $v \in V_i(T)$ such that $v \notin F_z(T)$ for every $z \in S$. Since v is unused, it must be that we failed to embed $N_H(z)$ in $N_G(v)$ for each $z \in S$. By Lemma 2.5, these events have probability at most $1 - p$ conditional on the prior embedding. Multiplying the conditional probabilities and taking a union bound over all $1 \le i \le r$, $v \in V_i$ and $S \subseteq X_i(T)$ of size $(1-\gamma)|X_i(T)|$, the failure probability is at most $rn\binom{n}{(1-\gamma)n}(1-p)^{(1-\gamma)|X_i(T)|} < o(1)$. This estimate uses the bounds $\binom{n}{(1-\gamma)n} \le 2^{\sqrt{\gamma} n}$, $(1-p)^{(1-\gamma)|X_i(T)|} < e^{-p\delta_B n/4} < 2^{-p^2 n}$ and $\gamma \ll p$.

In all cases we see that the failure probability is $o(1)$. ∎

3. REGULARITY AND SUPER-REGULARITY OF 3-GRAPHS

When considering how to generalise regularity to 3-graphs, a natural first attempt is to mirror the definitions used for graphs. Consider an r-partite 3-graph H with vertex set V partitioned as $V = V_1 \cup \cdots \cup V_r$. Let H_{ijk} be the tripartite sub-3-graph of H with parts V_i, V_j and V_k, for any i, j, k. The *density* of H_{ijk} is $d(H_{ijk}) = \frac{|H_{ijk}|}{|V_i||V_j||V_k|}$. Given $\epsilon > 0$, we say that H_{ijk} is ϵ-*vertex-regular* if for all subsets $V_i' \subseteq V_i$, $V_j' \subseteq V_j$ and $V_k' \subseteq V_k$ with $|V_i'| > \epsilon|V_i|$, $|V_j'| > \epsilon|V_j|$ and $|V_k'| > \epsilon|V_k|$, writing H_{ijk}' for the tripartite sub-3-graph of H with parts V_i', V_j' and V_k', we have $|d(H_{ijk}') - d(H_{ijk})| < \epsilon$. There is a decomposition theorem for this definition analogous to the Szemerédi Regularity Lemma. This is often known as the 'weak hypergraph regularity lemma', as although it does have some applications, the property of vertex-regularity is not strong enough to prove a counting lemma (see e.g. [12] for further discussion).

To obtain a counting lemma one needs to take account of densities of triples of H with respect to sets of pairs of vertices, as well as densities of pairs with respect to sets of vertices. Thus we are led to define regularity for simplicial complexes. We make the following definitions.

Definition 3.1. *Suppose X is a set partitioned as $X = X_1 \cup \cdots \cup X_r$. We say $S \subseteq X$ is* r-partite *if $|S \cap X_i| \leq 1$ for $1 \leq i \leq r$. Write $K(X)$ for the* complete *collection of all r-partite subsets of X. We say that H is an* r-partite 3-complex *on $X = X_1 \cup \cdots \cup X_r$ if H consists of r-partite sets of size at most 3 and is a simplicial complex, i.e. if $T \subseteq S \in H$ then $T \in H$. The* index *of an r-partite set S is $i(S) = \{1 \leq i \leq r : S \cap X_i \neq \emptyset\}$. For $I \subseteq [r]$ we write H_I for the set of $S \in H$ with index $i(S) = I$, when this set is non-empty. If there are no sets $S \in H$ with $i(S) = I$ we can choose to either set $H_I = \emptyset$ or let H_I be* undefined, *provided that if H_I is defined then H_J is defined for all $J \subseteq I$. When not explicitly stated, the default is that H_I is undefined when there are no sets of index I. For any $S \in H$ we write $H_S = H_{i(S)}$ for the naturally defined $|S|$-partite $|S|$-graph in H containing S.*

To put this definition in concrete terms, whenever the following sets are defined, $H_{\{i\}}$ is a subset of X_i, $H_{\{i,j\}}$ is a bipartite graph with parts $H_{\{i\}}$ and $H_{\{j\}}$, and $H_{\{i,j,k\}}$ is a 3-graph contained in the set of triangles spanned by $H_{\{i,j\}}$, $H_{\{i,k\}}$ and $H_{\{j,k\}}$. Of course, the interesting part of this structure is the 3-graph together with its underlying graphs. We also have $H_\emptyset$, which is usually equal to $\{\emptyset\}$, i.e. the set of size 1 whose element is the empty set, although it could be empty if all other parts are empty. It is most natural to take $H_{\{i\}} = X_i$ for $1 \leq i \leq r$. We often allow $H_{\{i\}}$ to be a strict subset of X_i, but note that if desired we can make such a complex 'spanning' by changing the ground set to $X' = X_1' \cup \cdots \cup X_r'$, where $X_i' = H_{\{i\}}$, $1 \leq i \leq r$. When unspecified, our default notation is that H is an r-partite 3-complex on $X = X_1 \cup \cdots \cup X_r$. We will see later in Definitions 3.5 and 4.4 why we have been so careful to distinguish the cases H_I empty and H_I undefined in Definition 3.1.

Definition 3.2. *To avoid clumsy notation we will henceforth frequently identify a set with a sequence of its vertices, e.g. writing $H_i = H_{\{i\}}$ and $H_{ijk} = H_{\{i,j,k\}}$. We also use concatenation for set union, e.g. if $S = ij = \{i,j\}$ then $Sk = ijk = \{i,j,k\}$. For any $I \subseteq [r]$ we write $H_{I^{\leq}} = \cup_{I' \subseteq I} H_{I'}$ for the subcomplex of H consisting of all defined $H_{I'}$ with $I' \subseteq I$. We also write $H_{I^<} = H_{I^{\leq}} \setminus H_I = \cup_{I' \subsetneq I} H_{I'}$. Similarly, for any $S \subseteq X$ we write $H_{S^{\leq}} = \cup_{S' \subseteq S} H_{S'}$ and $H_{S^<} = \cup_{S' \subsetneq S} H_{S'}$. For any set system A we let $A^{\leq}$ be the complex generated by A, which consists of all subsets of all sets in A.*

It is clear that intersections and unions of complexes are complexes. We clarify exactly what these constructions are with the following definition.

Definition 3.3. *Suppose H and H' are r-partite 3-complexes on $X = X_1 \cup \cdots \cup X_r$. The* union $H \cup H'$ *is the r-partite 3-complex where $(H \cup H')_S = H_S \cup H'_S$ is defined whenever H_S or H'_S is defined. The* intersection $H \cap H'$ *is the r-partite 3-complex where $(H \cap H')_S = H_S \cap H'_S$ is defined whenever H_S and H'_S are defined.*

We often specify complexes as a sequence of sets, e.g. $(G_{12}, (G_i : 1 \le i \le 4), \{\emptyset\})$ has parts G_{12}, G_i for $1 \le i \le 4$, $\{\emptyset\}$ and is otherwise undefined. Now we come to a key definition.

Definition 3.4. *We let $T_{ijk}(H)$ be the set of triangles formed by H_{ij}, H_{jk} and H_{ik}. We say that a defined triple H_{ijk} is ϵ-*regular *if for every subgraph G of H with $|T_{ijk}(G)| > \epsilon |T_{ijk}(H)|$ we have*

$$\frac{|H \cap T_{ijk}(G)|}{|T_{ijk}(G)|} = \frac{|H_{ijk}|}{|T_{ijk}(H)|} \pm \epsilon.$$

*We also say that the entire 3-complex H is ϵ-*regular *if every defined triple H_{ijk} is ϵ-regular and every defined graph H_{ij} is ϵ-regular.*

Thus H_{ijk} is ϵ-regular if for any subgraph G with 'not too few' triangles of index ijk, the proportion of these triangles in G of index ijk that are triples in H_{ijk} is approximately equal to the proportion of triangles in H of index ijk that are triples in H_{ijk}. Note that we never divide by zero in Definition 3.4, as $\emptyset \neq T_{ijk}(G) \subseteq T_{ijk}(H)$. The definition applies even if every H_{ijk} is undefined, in which case we can think of H as a 2-complex with every graph H_{ij} being ϵ-regular. It also applies even if every H_{ij} is undefined, in which case we can think of H as a 1-complex (with no regularity restriction).[1]

This concept of regularity in 3-complexes is more powerful than vertex regularity, in that it does admit a counting lemma. In order to apply it we also need an analogue of the Szemerédi Regularity Lemma, stating that a general 3-complex can be decomposed into a bounded number of pieces, most of which are regular. Such a result does hold, but there is an important technical proviso that one cannot use the same parameter ϵ to measure regularity for both graphs and triples in the complex. In general, one needs to allow the densities of the graphs H_{ij} to be much smaller than the parameter used to measure the regularity of triples. This is known as a *sparse* setting, as contrasted with a situation when all densities are much larger than ϵ, which is known as a *dense* setting.

In the sparse setting, a counting lemma does hold, but we couldn't find any way to generalise the proof of the blow-up lemma. To circumvent this difficulty we will instead apply the Regular Approximation Lemma of Rödl and Schacht. Informally stated, this allows us to closely approximate any 3-graph G^0 by another 3-graph G, so that the 3-complex $G^{\le}$ generated by G can be decomposed (in a certain sense) into ϵ-regular complexes. We will

[1]Technically one should say that H_{ijk} is ϵ-regular *in the complex H*, as the definition depends on the graphs H_{ij}, H_{jk} and H_{ik}. For the sake of brevity we will omit this qualification, as it is not hard to see that when sufficiently regular these graphs are 'almost' determined by H_{ijk}: if say H_{ij} had many pairs not contained in triples of H_{ijk} then T_{ijk} would have many triangles none of which are triples of H_{ijk}, contradicting the definition of ϵ-regularity.

come to the formal statement later in the paper, but we mention it here to motivate the form of the blow-up lemma that we will prove. We will allow ourselves to work in the dense setting of ϵ-regular complexes, but we have to take account of the approximation of G^0 by G by 'marking' the edges $M = G \setminus G^0$ as forbidden. If we succeed in embedding a given 3-graph H in G without using M then we have succeeded in embedding H in G^0. (A similar set-up is used for the embedding lemma in [35].) We will refer to the pair $(G^{\leq}, M)$ as a *marked complex*.

In the remainder of this section we first motivate and then explain the definition of super-regularity for 3-graphs. It turns out that this needs to be significantly more complicated than for ordinary graphs. It is not sufficient to just forbid vertices of small degree. To see this, consider as an example a 4-partite 3-complex G on $X = X_1 \cup X_2 \cup X_3 \cup X_4$, with $|X_i| = n$, $1 \leq i \leq 4$ that is almost complete: say there are complete bipartite graphs on every pair of classes and complete 3-graphs on every triple of classes, except for one vertex x in X_4 for which the neighbourhood $G(x)$ is triangle-free. We can easily choose each $G(x)_{ij}$ to have size $n^2/4$ by dividing each class X_i, $1 \leq i \leq 3$ into two parts. Then H is $O(1/n)$-regular with densities $1 - O(1/n)$ and has minimum degrees at least $n^2/4$ in each triple, but x is not contained in any tetrahedron K_4^3, so we cannot embed a perfect K_4^3-packing.

Another complication is that the definition of super-regularity for 3-graphs is not 'local', in the sense that super-regularity of a graph G was defined by a condition for each of its bipartite subgraphs G_{ij}. Instead, we need to define super-regularity for the entire structure (G, M), where G is an r-partite 3-complex and M is a set of marked edges. To explain this point we need to look ahead to the analysis of the algorithm that we will use to prove the blow-up lemma. First we need an important definition that generalises the process of restricting a graph to a subset of its vertex set: we may also consider restricting a complex to a subcomplex in the following manner.

Definition 3.5. *Suppose H and G are r-partite 3-complexes on $X = X_1 \cup \cdots \cup X_r$ and G is a subcomplex of H. The* restriction *$H[G]$ is the r-partite 3-complex on $X = X_1 \cup \cdots \cup X_r$, where $H[G]_I$ is defined if and only if H_I is defined, and $H[G]_I$ consists of all $S \in H_I$ such that $A \in G$ for every $A \subseteq S$ such that G_A is defined.*

To put this in words, a given set S in H belongs to the restriction $H[G]$ if every subset A of S belongs to G, provided that the part of G corresponding to A is defined. At this point we will give some examples to illustrate Definition 3.5 and clarify the distinction between parts being empty or undefined in Definition 3.1. Consider a 3-partite 2-complex H on $X = X_1 \cup X_2 \cup X_3$ such that H_{12}, H_{13} and H_{23} are non-empty graphs. Suppose G is a 3-partite 1-complex on $X = X_1 \cup X_2 \cup X_3$. If G_1, G_2 and G_3 are defined then $H[G]$ is the 3-partite 2-complex on $X = X_1 \cup X_2 \cup X_3$ with $H[G]_i = G_i$ and $H[G]_{ij}$ equal to the bipartite subgraph of H_{ij} spanned by G_i and G_j. This corresponds to the usual notion of restriction for graphs. Note that all of the sets $H[G]_i$ and graphs $H[G]_{ij}$ are defined and some may be empty. However, if any of the G_i is undefined then it behaves as if it were equal to H_i. For example, if G_1 is undefined and G_2 is defined then $H[G]_1 = H_1$ and $H[G]_{12}$ is the bipartite subgraph of H_{12} spanned by H_1 and G_2. This highlights the importance of distinguishing between G_1 being empty or G_1 being undefined. Note that Definition 3.5 is monotone, in the sense that adding sets to any given defined part G_I of G does not remove any sets from any given part $H[G]_J$ of the restriction $H[G]$. We record the following obvious property for future reference:

$$H[G]_I = G_I \text{ when } G_I \text{ is defined.} \tag{2}$$

Another obvious property used later concerns the *empty complex* $(\{\emptyset\})$, which satisfies $H[(\{\emptyset\})] = H$ for any complex H. Next we will reformulate the definition of regularity for 3-complexes using the restriction notation. First we make the following definitions.

Definition 3.6. *Suppose H is an r-partite 3-complex on $X = X_1 \cup \cdots \cup X_r$. For any $I \subseteq [r]$ we let H^*_I denote the set of $S \in K(X)_I$ such that any strict subset $T \subsetneq S$ belongs to H_T when defined. When H_I is defined and $H^*_I \neq \emptyset$, we define the* relative density (at I) *of H as $d_I(H) = |H_I|/|H^*_I|$. We also call $d_I(H)$ the I-*density *of H. We define the* absolute density *of H_I as $d(H_I) = |H_I|/\prod_{i\in I}|X_i|$.*

To illustrate Definition 3.6, note that $H^*_i = X_i$ and $d_i(H) = d(H_i) = |H_i|/|X_i|$ when defined. If H_{ij} is defined then $H^*_{ij} = H_i \times H_j$ and $d_{ij}(H) = |H_{ij}|/|H^*_{ij}|$ is the density of the bipartite graph H_{ij} with parts H_i and H_j. We have $d(H_{ij}) = d_{ij}(H)d_i(H)d_j(H)$, so $d(H_{ij}) = d_{ij}(H)$ in the case when $H_i = X_i$ and $H_j = X_j$. Also, if H_{ij}, H_{ik} and H_{jk} are defined then $H^*_{ijk} = T_{ijk}(H)$ is the set of triangles in H of index ijk. We also note that if any of the H_{ij} is undefined it behaves as if it were equal to H^*_{ij}, e.g. if H_{ij} is undefined and H_{ik} and H_{jk} are defined then H^*_{ijk} is the set of triangles in $(H^*_{ij}, H_{ik}, H_{jk})$. As an illustration of Definition 3.5, we note that $H^*_I = K(X)[H_{I^<}]_I$, recalling that $K(X)$ is the complex of r-partite sets and $H_{I^<} = \cup_{I' \subsetneq I} H_{I'}$.

Now suppose H is a 3-partite 3-complex on $X = X_1 \cup X_2 \cup X_3$ such that H_{123} is defined and $H^*_{123} \neq \emptyset$. Suppose $G \subseteq H$ is a 2-complex such that G_{12}, G_{13} and G_{23} are defined. Then $G^*_{123} = T_{123}(G)$ and $H^*_{123} = T_{123}(H)$. By Definition 3.5 we have $H[G]_{123} = H_{123} \cap G^*_{123}$ and $H[G]^*_{123} = G^*_{123}$, so by Definition 3.6 $d_{123}(H[G]) = |H \cap G^*_{123}|/|G^*_{123}|$. Therefore H_{123} is ϵ-regular if whenever $|G^*_{123}| > \epsilon |H^*_{123}|$ we have $d_{123}(H[G]) = d_{123}(H) \pm \epsilon$.

For the remainder of this section we let H be an r-partite 3-complex on $X = X_1 \cup \cdots \cup X_r$, and G be an r-partite 3-complex on $V = V_1 \cup \cdots \cup V_r$, with $|V_i| = |X_i|$ for $1 \le i \le r$. We want to find an embedding ϕ of H in G. Our algorithm will consider the vertices of X in some order and embed them one at a time. At some time t in the algorithm, for each $S \in H$ there will be some $|S|$-graph $F_S(t) \subseteq G_S$ consisting of those sets $P \in G_S$ that are *free* for S, in that mapping S to P is *locally consistent* with the embedding so far. These free sets will be *mutually consistent*, in that

$$F_{S^\le}(t) = \cup_{S' \subseteq S} F_{S'}(t) \tag{3}$$

is a complex for every $S \in H$. We use the convention that $F_S(t)$ is undefined for any $S \notin H$. Note that (3) applies even for $S \notin H$.

Initially we define $F_S(0) = G_S$ for all $S \in H$. Now suppose we have defined $F_S(t-1)$ for all $S \in H$ and then at time t we embed some vertex $x \in X$ to some vertex $y \in F_x(t-1)$. *We will use this notation consistently throughout the paper*. Then for any $S \in H$ containing x we can only allow sets in $F_S(t)$ that correspond to mapping x to y, i.e. $F_S(t) = F_S(t-1)^y$, which is our notation for $\{P \in F_S(t-1) : y \in P\}$. Also, for any S in the neighbourhood complex $H(x)$, i.e. a set S not containing x such that $Sx = S \cup \{x\} \in H$, in order for $F_S(t)$ to be mutually consistent with $F_{Sx}(t) = \{P \in F_{Sx}(t-1) : y \in P\}$, we can only allow sets in $F_S(t)$ that are in the neighbourhood of y, i.e. $F_S(t) = F_{Sx}(t-1)(y)$, which is our notation for $\{P : P \cup \{y\} \in F_{S\cup\{x\}}(t-1)\}$. Finally, we need to consider the effect that embedding x has for sets S that do not contain x and are not even in the neighbourhood complex $H(x)$. Such a set S may contain a set S' in $H(x)$, so that $F_{S'}(t)$ is affected by the embedding of x. Then mutual consistency requires for that any set $P \in F_S(t)$, the subset of P corresponding to S'

must belong to $F_{S'}(t)$. We need to include these restrictions for all subsets S' of S. Also, as we are using the vertex y to embed x we have to remove it from any future free sets. Thus we are led to the following definition. (Lemma 3.10 will show that it is well-defined.)

Definition 3.7 (Update rule). *Suppose x is embedded to y at time t and $S \in H$. If $x \in S$ we define $F_S(t) = F_S(t-1)^y = \{P \in F_S(t-1) : y \in P\}$. If $x \notin S$ we define*

$$S.x = (S \setminus X_x) \cup \{x\}, \quad C_{S^\le}(t) = F_{S.x^\le}(t-1)(y) \quad \textit{and} \quad F_S(t) = F_{S^\le}(t-1)[C_{S^\le}(t)]_S \setminus y.$$

Note that $S.x$ is either $Sx = S \cup \{x\}$ if $i(x) \notin i(S)$ or obtained from Sx by deleting the element in S of index $i(x)$ if $i(x) \in i(S)$. Thus $S.x \subseteq X$ is r-partite. Also, the notation '$\setminus y$' means that we delete all sets containing y; this can have an effect only when $i(x) \in i(S)$. We will show below in Lemma 3.10 that Definition 3.7 makes sense, but first we will give an example to illustrate how it works.

Example 3.8. Suppose that H and G are 4-partite 3-complexes, and that we have 4 vertices $x_i \in X_i$, $1 \le i \le 4$ that span a tetrahedron K_4^3 in H, i.e. H contains every subset of $\{x_1, x_2, x_3, x_4\}$. Suppose also that we have the edges $x'_1x'_2x_3$ and $x'_1x'_3x'_4$ and all their subsets for some other 4 vertices $x'_i \in X_i$, $1 \le i \le 4$, and that there are no other edges of H containing any x_i or x'_i, $1 \le i \le 4$. Initially we have $F_S(0) = G_S$ for every $S \in H$. Suppose we start the embedding by mapping x_1 to some $v_1 \in V_1$ at time 1. Applying Definition 3.7 to sets S containing x_1 gives $F_{x_1}(1) = \{v_1\}$, $F_{x_1x_i}(1) = \{P \in G_{1i} : v_1 \in P\}$ for $2 \le i \le 4$, and $F_{x_1x_ix_j}(1) = \{P \in G_{1ij} : v_1 \in P\}$ for $2 \le i < j \le 4$.

Next we consider some examples of Definition 3.7 for sets not containing x_1. We have $C_{x_i^\le}(1) = (F_{x_1x_i}(0)(v_1), \{\emptyset\}) = (G(v_1)_i, \{\emptyset\})$ for $2 \le i \le 4$. Then $F_{x_i}(1) = F_{x_i^\le}(0)[C_{x_i^\le}(1)]_i \setminus v_1 = G(v_1)_i$. Similarly, we have $F_{x_ix_j}(1) = G(v_1)_{ij}$ for $2 \le i < j \le 4$. For $x_2x_3x_4$ we have

$$C_{x_2x_3x_4^\le}(1) = \bigcup_{S \subseteq x_2x_3x_4} F_{Sx_1}(0)(v_1) = G(v_1)_{23^\le} \cup G(v_1)_{24^\le} \cup G(v_1)_{34^\le}.$$

Therefore $F_{x_2x_3x_4}(1) = F_{x_2x_3x_4^\le}(0)[C_{x_2x_3x_4^\le}(1)]_{234} \setminus v_1$ consists of all triples in G_{234} that also form a triangle in the neighbourhood of v_1, and so complete v_1 to a tetrahedron in G.

For x'_2x_3, we have $C_{x'_2x_3}(1) = G(v_1)_{3^\le}$, so $F_{x'_2x_3}(1) = F_{x'_2x_3^\le}(0)[C_{x'_2x_3^\le}(1)]_{23} \setminus v_1$ consists of all pairs in G_{23} that contain a G_{13}-neighbour of v_1. For x'_2, $C_{x'_2{}^\le}(1) = (\{\emptyset\})$ is the empty complex, so $F_{x'_2}(1) = F_{x'_2{}^\le}(0)[C_{x'_2{}^\le}(1)]_2 \setminus v_1 = F_{x'_2}(0)$ is unaffected. (Recall that $J[(\{\emptyset\})] = J$ for any complex J.) Finally we give two examples in which the deletion of v_1 has some effect. For $x'_1x'_2x_3$, we have $x'_1x'_2x_3.x_1 = x_1x'_2x_3$, $F_{x_1x'_2x_3^\le}(0) = F_{x_1x_3^\le}(0) \cup F_{x'_2x_3^\le}(0) = G_{13^\le} \cup G_{23^\le}$, and $C_{x'_1x'_2x_3^\le}(1) = F_{x_1x'_2x_3^\le}(0)(v_1) = (G(v_1)_3, \{\emptyset\}) = G(v_1)_{3^\le}$. Then $F_{x'_1x'_2x_3^\le}(1) = G_{123^\le}[G(v_1)_{3^\le}] \setminus v_1$, so $F_{x'_1x'_2x_3}(1)$ consists of all triples T in G_{123} not containing v_1 such that $T \cap V_3$ is a neighbour of v_1. For $x'_1x'_3$, $C_{x'_1x'_3{}^\le}(1)$ is the empty complex, so $F_{x'_1x'_3}(1) = F_{x'_1x'_3}(0) \setminus v_1$ consists of all pairs in G_{13} that do not contain v_1.

Now we prove a lemma which justifies Definition 3.7 and establishes the 'mutual consistency' mentioned above, i.e. that $F_{S^\le}(t)$ is a complex. First we need a definition.

Definition 3.9. *Suppose $S \subseteq X$ is r-partite and $I \subseteq i(S)$. We write $S_I = S \cap \cup_{i \in I} X_i$. We also write $S_T = S_{i(T)}$ for any r-partite set T with $i(T) \subseteq i(S)$.*

Lemma 3.10. *Suppose $S \subseteq X$ is r-partite and $t \geq 1$. If $x \notin S$ then $C_{S^{\leq}}(t)$ is a subcomplex of $F_{S^{\leq}}(t-1)$ and $F_{S^{\leq}}(t) = F_{S^{\leq}}(t-1)[C_{S^{\leq}}(t)] \setminus y$ is a complex. If $x \in S$ then $F_S(t) = F_S(t-1)^y$, $F_{S\setminus x}(t) = F_S(t-1)(y)$ and $F_{S^{\leq}}(t) = F_{S^{\leq}}(t-1)^y \cup F_{S^{\leq}}(t-1)(y)$ is a complex.*

Proof. Note that $F_{S^{\leq}}(0) = G_{S^{\leq}}$ is a complex. We prove the statement of the lemma by induction on t. The argument uses the simple observation that if J is any complex and v is a vertex of J then $J(v)$ and $J^v \cup J(v)$ are subcomplexes of J.

First suppose that $x \notin S$. Since $F_{S^{\leq}}(t-1)$ is a complex by induction hypothesis, $C_{S^{\leq}}(t) = F_{S.x^{\leq}}(t-1)(y)$ is a subcomplex of $F_{S.x^{\leq}}(t-1)$, and so of $F_{S^{\leq}}(t-1)$. For any $S' \subseteq S$ write $J(S') = F_{S'^{\leq}}(t-1)[C_{S'^{\leq}}(t)] \setminus y$. Then $J(S')$ is a complex, since restriction to a complex gives a complex, and removing all sets containing y preserves the property of being a complex. Furthermore, $J(S')_{S'} = F_{S'}(t)$ by Definition 3.7. We also have $J(S)_{S'} = J(S')_{S'}$, since a set A' of index S' belongs to $F_{S'^{\leq}}(t-1)$ if and only if it belongs to $F_{S^{\leq}}(t-1)$, and any $B \subseteq A'$ belongs to $C_{S'^{\leq}}(t)$ if and only if it belongs to $C_{S^{\leq}}(t)$. Therefore $F_{S^{\leq}}(t) = J(S) = F_{S^{\leq}}(t-1)[C_{S^{\leq}}(t)] \setminus y$ is a complex.

Now suppose that $x \in S$. Then $F_S(t) = F_S(t-1)^y$ by definition. Next, note that $C_{S\setminus x^{\leq}}(t) = F_{S^{\leq}}(t-1)(y)$ is a complex, and $F_{S\setminus x^{\leq}}(t-1)[C_{S\setminus x^{\leq}}(t)]_{S\setminus x} = F_S(t-1)(y)$ by equation (2). Since S is r-partite, deleting y has no effect, and we also have $F_{S\setminus x}(t) = F_S(t-1)(y)$. Therefore $F_{S^{\leq}}(t) = \cup_{x \in S' \subseteq S}(F_{S'}(t-1)^y \cup F_{S'}(t-1)(y)) = F_{S^{\leq}}(t-1)^y \cup F_{S^{\leq}}(t-1)(y)$ is a complex. ∎

We will also use the following lemma to construct $F_S(t)$ iteratively from $\{F_{S'}(t) : S' \subsetneq S\}$.

Lemma 3.11. *Suppose that $S \in H$, $|S| \geq 2$, $x \notin S$ and $S \notin H(x)$. Write $F_{S^<}(t) = \cup_{S' \subsetneq S} F_{S'}(t)$. Then $F_{S^{\leq}}(t) = F_{S^{\leq}}(t-1)[F_{S^<}(t)]$.*

Proof. First note that for any A with $|A| < |S|$ we have $A \in F_{S^{\leq}}(t) \Leftrightarrow A \in F_{S^<}(t) \Leftrightarrow A \in F_{S^{\leq}}(t-1)[F_{S^<}(t)]$. Now suppose $A \in F_S(t)$. Then $A \in F_{S^{\leq}}(t-1)$ and $A' \in F_{S^<}(t)$ for every $A' \subsetneq A$, so $A \in F_{S^{\leq}}(t-1)[F_{S^<}(t)]$. Conversely, suppose that $A \in F_{S^{\leq}}(t-1)[F_{S^<}(t)]_S$. For any $S' \subsetneq S$ with $S' \in H(x)$ we have $F_{S'}(t) = F_{S'x}(t-1)(y)$, so $A_{S'} \in F_{S'x}(t-1)(y) \subseteq C_{S^{\leq}}(t)$. Also $C_{S^{\leq}}(t)_{S'}$ is undefined for $S' \notin H(x)$; in particular, our assumption that $S \notin H(x)$ means that $C_{S^{\leq}}(t)_S$ is undefined. Therefore $A_{S'} \in C_{S^{\leq}}(t)$ for every $S' \subseteq S$ such that $C_{S^{\leq}}(t)_{S'}$ is defined, i.e. $A \in F_{S^{\leq}}(t-1)[C_{S^{\leq}}(t)]_S$. Also, if $i(x) \in i(S)$, then writing $z = S_x = S \cap X_x \in S^<$, we have $A \cap V_x \in F_z(t) = F_z(t-1) \setminus y$, so $y \notin A$. Therefore $A \in F_{S^{\leq}}(t-1)[C_{S^{\leq}}(t)]_S \setminus y = F_S(t)$. ∎

We referred to 'local consistency' when describing the update rule because it only incorporates the effect that embedding x has on sets containing at least one neighbour of x. To illustrate this, recall that in Example 3.8 above we have $F_{x_2'}(1) = F_{x_2'}(0) = G_2$. Now H contains x_1x_3 and x_1 is embedded to v_1, so x_3 must be embedded to a vertex in $G(v_1)_3$. Also, H contains $x_2'x_3$, so any image of x_2' must have a neighbour in $G(v_1)_3$. This may not be the case for every vertex in G_2, so there is some non-local information regarding the embedding that has not yet been incorporated into the free sets at time 1. In light of this, we should admit that our description of sets in $F_S(t)$ as 'free' is a slight misnomer, as there may be a small number of sets in $F_S(t)$ that cannot be images of S under the embedding. This was the case even for the graph blow-up lemma, in which we described vertices in $F_x(t)$ as 'free' images for x but then only allowed the use of $OK_x(t) \subseteq F_x(t)$. On the other hand, our definition of free sets is relatively simple, and does contain enough information for the embedding. To see this, note that by definition of restriction $F_{S^{\leq}}(t)$ is a subcomplex of $G_{S^{\leq}}$ at every time

t, and when all vertices of S are embedded by ϕ we have $F_S(t) = \{\phi(S)\}$ with $\phi(S) \in G_S$. Furthermore, by removing all sets that contain y in the definition of $F_S(t)$ we ensure that no vertex is used more than once by ϕ. Therefore it does suffice to only consider local consistency in constructing the embedding, provided that the sets $F_S(t)$ remain non-empty throughout. The advantage is that we have the following simple update rule for sets S that are not local to x.

Lemma 3.12. *If S does not contain any vertex in $\{x\} \cup VN_H(x)$ then $F_S(t) = F_S(t-1) \setminus y$.*

Proof. Note that $C_{S^{\leq}}(t) = F_{Sx^{\leq}}(t-1)(y) = (\{\emptyset\})$ is the empty complex. ∎

We will also need to keep track of the marked triples M during the embedding algorithm. Initially, we just have some triples in G that are marked as forbidden for any triple of H. Then, as the algorithm proceeds, each pair of H is forbidden certain pairs of G, and each vertex of H is forbidden certain vertices of G. We adopt the following notation.

Definition 3.13. *For any triple $E \in H$ we write E^t for the subset of E that is unembedded at time t. We define the marked subset of $F_{E^t}(t)$ corresponding to E as*

$$M_{E^t,E}(t) = \{P \in F_{E^t}(t) : P \cup \phi(E \setminus E^t) \in M_E\}.$$

In words, $M_{E^t,E}(t)$ consists of all sets P in $F_{E^t}(t)$ that cannot be used as images for E^t in the embedding, because when we add the images of the embedded part $E \setminus E^t$ of E we obtain a marked triple. To illustrate this, suppose that in Example 3.8 we have some marked triples M. At time $t = 1$ we map x_1 to v_1, and then the free set for x_2x_3 is $F_{x_2x_3}(1) = G(v_1)_{23}$. Since the edges M_{123} are marked as forbidden, we will mark $M(v_1)_{23} \subseteq G(v_1)_{23}$ as forbidden by defining $M_{x_2x_3,x_1x_2x_3}(1) = M(v_1)_{23}$. As another illustration, recall that $F_{x_2x_3x_4}(1)$ consists of all triples in G_{234} that also form a triangle in the neighbourhood of v_1. Then $M_{x_2x_3x_4,x_2x_3x_4}(1) = M \cap F_{x_2x_3x_4}(1)$ consists of all triples in M_{234} that also form a triangle in the neighbourhood of v_1.

For any triple E containing x such that $E^{t-1} = x$ we will choose $y = \phi(x) \notin M_{x,E}(t-1)$. This will ensure that $\phi(E) \notin M$. The following lemma will enable us to track marked subsets. The proof is obvious, so we omit it.

Lemma 3.14.

(i) *If $x \in E$ then $E^t = E^{t-1} \setminus x$ and $M_{E^t,E}(t) = M_{E^{t-1},E}(t-1)(y)$.*
(ii) *If $x \notin E$ then $E^t = E^{t-1}$ and $M_{E^t,E}(t) = M_{E^{t-1},E}(t-1) \cap F_{E^t}(t)$.*

We need one more definition before we can define super-regularity. It provides some alternative notation for describing the update rule, but it has the advantage of not referring explicitly to any embedding or to another complex H.

Definition 3.15. *Suppose G is an r-partite 3-complex on $V = V_1 \cup \cdots \cup V_r$, $1 \leq i \leq r$, $v \in G_i$ and I is a subcomplex of $\binom{[r]}{\leq 3}$. We define $G^{I_v} = G[\cup_{S \in I} G(v)_S]$.*

We will now explain the meaning of Definition 3.15 and illustrate it using our running Example 3.8. To put it in words, G^{I_v} is the restriction of G obtained by only taking those sets $A \in G$ such that any subset of A indexed by a set S in I belongs to the neighbourhood $G(v)$, provided that the corresponding part $G(v)_S$ is defined. In Example 3.8 we have $C_{x_2^{\leq}}(1) = (G(v_1)_2, \{\emptyset\})$ and $F_{x_2}(1) = G_{2^{\leq}}[C_{x_2^{\leq}}(1)]_2 = G(v_1)_2$. Choosing $I = 2^{\leq} = (\{2\}, \{\emptyset\})$ we have $C_{x_2^{\leq}}(1) = \cup_{S\in I} G(v_1)_S$ and so $F_{x_2}(1) = G_2^{I_{v_1}}$. Similarly, choosing $I = 23^{\leq}$ we have $C_{x_2x_3^{\leq}}(1) = \cup_{S\in I} G(v_1)_S$ and $F_{x_2x_3}(1) = G_{23^{\leq}}[C_{x_2^{\leq}}(1)]_{23} = G(v_1)_{23} = G_{23}^{I_{v_1}}$. Also, choosing $I = 23^{\leq} \cup 24^{\leq} \cup 34^{\leq}$ we see that $F_{x_2x_3x_4}(1) = G_{234}^{I_{v_1}}$ consists of all triples in G_{234} that also form a triangle in the neighbourhood of v_1. In general, we can use this notation to describe the update rule for any complex H if we embed some vertex x of H to some vertex v of G at time 1. If $x \in S \in H$ we have $F_S(1) = G^v$ as before. If $x \notin S$ we let $I = \{i(S') : S' \subseteq S, S'x \in H\}$ and then $F_S(1) = G_S^{I_v} \setminus v$.

Finally, we can give the definition of super-regularity.

Definition 3.16 (Super-regularity). *Suppose G is an r-partite 3-complex on $V = V_1 \cup \cdots \cup V_r$ and $M \subseteq G_= := \{S \in G : |S| = 3\}$. We say that the marked complex (G, M) is $(\epsilon, \epsilon', d_2, \theta, d_3)$-*super-regular *if*

(i) *G is ϵ-regular, and $d_S(G) \geq d_{|S|}$ if $|S| = 2, 3$ and G_S is defined,*
(ii) *for every $1 \leq i \leq r$, $v \in G_i$ and S such that G_{Si} is defined, $|M(v)_S| \leq \theta |G(v)_S|$ if $|S| = 2$ and $G(v)$ is an ϵ'-regular 2-complex with $d_S(G(v)) = (1 \pm \epsilon') d_S(G) d_{Si}(G)$ for $S \neq \emptyset$,*
(iii) *for every vertex v and subcomplex I of $\binom{[r]}{\leq 3}$, $|(M \cap G^{I_v})_S| \leq \theta |G_S^{I_v}|$ if $|S| = 3$, and G^{I_v} is an ϵ'-regular 3-complex with densities (when defined)*

$$d_S(G^{I_v}) = \begin{cases} (1 \pm \epsilon') d_S(G) & \text{if } S \notin I, \\ (1 \pm \epsilon') d_S(G) d_{Si}(G) & \text{if } \emptyset \neq S \in I \text{ and } G_{Si} \text{ is defined.} \end{cases}$$

Just as one can delete a small number of vertices from an ϵ-regular graph to make it super-regular, we will see later (Lemma 5.9) that one can delete a small number of vertices from an ϵ-regular marked 3-complex to make it super-regular. For now we will just explain the meaning of Definition 3.16 with reference to our running example. First we remark that the parameters in the definition are listed according to their order in the hierarchy, in that $\epsilon \ll \epsilon' \ll d_2 \ll \theta \ll d_3$. Thus we consider a dense setting, in which the regularity parameters ϵ and ϵ' are much smaller than the density parameters d_2 and d_3. However, one should note that the marking parameter θ has to be larger than the density parameter d_2, which is the source of some technical difficulties in our arguments. We will bound the marked sets by a increasing sequence of parameters that remain small throughout the embedding. For now we just see what the definition of super-regularity tells us about the first step.

Condition (i) just says that G is a regular complex and gives lower bounds for the relative densities of its parts. Condition (ii) is analogous to the minimum degree condition in the definition of super-regularity for graphs. The second part of the condition says that the neighbourhood is a regular complex, and that its relative densities are approximately what one would expect (we will explain the formulae later). The first part says that the proportion of marked edges through any vertex is not too great. We need this to control the proportion of free sets that we have to mark as forbidden during the embedding algorithm. To illustrate this, suppose that in Example 3.8 we have some marked triples M. At time $t = 1$ we map x_1 to v_1, and then the free set for x_2x_3 is $F_{x_2x_3}(1) = G(v_1)_{23}$. Since the edges M_{123} are marked as forbidden, we will mark $M_{x_2x_3,x_1x_2x_3}(1) = M(v_1)_{23} \subseteq G(v_1)_{23}$ as forbidden.

Condition (ii) ensures that not too great a proportion is forbidden. Note that the density of the neighbourhood complex $G(v_1)$ will be much smaller than the marking parameter θ, so this does not follow if we only make the global assumption that M is a small proportion of G.

Condition (iii) is the analogue to condition (ii) for the restrictions that embedding some vertex can place on the embeddings of sets not containing that vertex. A very simple illustration is the case $I = (\{\emptyset\})$, which gives $|M_S| \leq \theta|G_S|$ when defined. (This could also be obtained from condition (ii) by summing over vertices v.) For a more substantial illustration, consider Example 3.8 and the subcomplex $I = 23^{\leq} \cup 24^{\leq} \cup 34^{\leq}$. We noted before the definition that $F_{x_2x_3x_4}(1) = G^{I_{v_1}}_{234}$ consists of all triples in G_{234} that also form a triangle in the neighbourhood of v_1. The second part of condition (iii) ensures that $F_{x_2x_3x_4^{\leq}}(1) = G^{I_{v_1}}_{234^{\leq}}$ is a regular complex, and that its relative densities are approximately what one would expect (again, we will explain the formulae later). The first part of condition (iii) again is needed to control the proportion of free sets that are marked. We will mark $M_{x_2x_3x_4,x_2x_3x_4}(1) = M \cap F_{x_2x_3x_4}(1) = (M \cap G^{I_{v_1}})_{234}$ as forbidden in $F_{x_2x_3x_4}(1)$, and the condition says that this is a small proportion. Again, since the neighbourhood of v_1 is sparse relative to G, this does not follow only from a global assumption that M is a small proportion of G.

As another illustration of condition (iii), suppose that we modify Example 3.8 by deleting the edge $x_1x_2x_3$ from H. Then $C_{x_2x_3x_4^{\leq}}(1) = G(v_1)_{24^{\leq}} \cup G(v_1)_{34^{\leq}}$ and $F_{x_2x_3x_4}(1) = F_{x_2x_3x_4^{\leq}}(0)[C_{x_2x_3x_4^{\leq}}(1)]_{234} \setminus v_1$ consists of all triples $S \in G_{234}$ such that S_{24} and S_{34} are edges in the neighbourhood of v_1. Taking $I = 24^{\leq} \cup 34^{\leq}$ we have $F_{x_2x_3x_4}(1) = G^{I_{v_1}}_{234}$. So condition (iii) tells us that also with this modified H, after embedding x_1 to v_1 the complex $F_{x_2x_3x_4^{\leq}}(1)$ is regular and we do not mark too much of $F_{x_2x_3x_4}(1)$ as forbidden.

4. THE 3-GRAPH BLOW-UP LEMMA

In this section we prove the following blow-up lemma for 3-graphs.

Theorem 4.1 (3-graph blow-up lemma). *Suppose H is an r-partite 3-complex on $X = X_1 \cup \cdots \cup X_r$ of maximum degree at most D, (G, M) is an $(\epsilon, \epsilon', d_2, \theta, d_3)$-super-regular r-partite marked 3-complex on $V = V_1 \cup \cdots \cup V_r$, where $n \leq |X_i| = |V_i| = |G_i| \leq Cn$ for $1 \leq i \leq r$, G_S is defined whenever H_S is defined, and $0 \ll 1/n \ll \epsilon \ll \epsilon' \ll d_2 \ll \theta \ll d_3, 1/r, 1/D, 1/C$. Then $G \setminus M$ contains a copy of H, in which for each $1 \leq i \leq r$ the vertices of V_i correspond to the vertices of X_i.*

Theorem 4.1 is similar in spirit to Theorem 2.1: informally speaking, we can embed any bounded degree 3-graph in any super-regular marked 3-complex. (The parallel would perhaps be stronger if we had also introduced marked edges in the graph statement; this can be done, but there is no need for it, so we preferred the simpler form.) We remark that we used the assumption $|V_i| = |X_i| = n$ in Theorem 2.1 for simplicity, but the assumption $n \leq |V_i| = |X_i| \leq Cn$ works with essentially the same proof, and is more useful in applications. (Arbitrary part sizes are permitted in [24], but this adds complications to the proof, and it is not clear why one would need them, so we will not pursue this option here.) There are various other generalisations that are useful in applications, but we will postpone discussion of them until we give the general hypergraph blow-up lemma. Theorem 4.1 is already sufficiently complex to illustrate the main ideas of our approach, so we prove it first so as not to distract the reader with additional complications.

The section contains six subsections, organised as follows. In the first subsection we present the algorithm that we will use to prove Theorem 4.1, and also establish some basic properties of the algorithm. Over the next two subsections we develop some theory: the second subsection contains some useful properties of restriction (Definition 3.5); the third contains some properties of regularity for 3-graphs, which are similar to but subtly different from known results in the literature. Then we start on the analysis of the algorithm, following the template established in the proof of Theorem 2.1. The fourth subsection concerns good vertices, and is analogous to Lemma 2.4. The fifth subsection concerning the initial phase is the most technical, containing three lemmas that play the role of Lemma 2.5 for 3-graphs. The final subsection concerns the conclusion of the algorithm, and is analogous to Lemma 2.6 and Theorem 2.7.

4.1. The Embedding Algorithm

As for the graph blow-up lemma, we will prove Theorem 4.1 via a random greedy algorithm to construct an embedding $\phi : V(H) \to V(G)$ such that $\phi(e) \in G \setminus M$ for every edge e of H. In outline, it is quite similar to the algorithm used for graphs, but when it comes to details the marked edges create significant complications. We introduce more parameters with the hierarchy

$$0 \le 1/n \ll \epsilon \ll \epsilon' \ll \epsilon_{0,0} \ll \cdots \ll \epsilon_{12D,3} \ll \epsilon_* \ll p_0 \ll \gamma \ll \delta_Q \ll p \ll d_u \ll d_2 \\ \ll \theta \ll \theta_0 \ll \theta_0' \ll \cdots \ll \theta_{12D} \ll \theta_{12D}' \ll \theta_* \ll \delta_Q' \ll \delta_B \ll d_3, 1/r, 1/D.$$

Most of these parameters do not require any further comment, as we explained their role in the graph blow-up lemma, and they will play the same role here. We need many more 'annotated ϵ' parameters to measure regularity here, but this is merely a technical inconvenience. The parameters $\epsilon_{i,j}$ with $0 \le i \le 12D$ and $0 \le j \le 3$ satisfy $\epsilon_{i,j} \ll \epsilon_{i',j'}$ when $i < i'$ or $i = i'$ and $j < j'$. Because of the marked edges, we also have new 'annotated θ' parameters, which are used to bound the proportion of free sets that are marked. It is important to note that the buffer parameter δ_B and queue admission parameter δ_Q' are larger than the marking parameter θ, which in turn is larger than the density parameter d_2. The result is that the queue may become non-empty during the initial phase, and then the set N of neighbours of the buffer B will not be embedded consecutively in the order given by the original list L. To cope with this, we need a modified selection rule that allows vertices in N to jump the queue.

Initialisation and notation. We choose a buffer set $B \subset X$ of vertices at mutual distance at least 9 in H so that $|B \cap X_i| = \delta_B |X_i|$ for $1 \le i \le r$. Since $n \le |X_i| \le Cn$ for $1 \le i \le r$ and H has maximum degree D we can construct B simply by selecting vertices one-by-one greedily. For any given vertex there are at most $(2D)^8$ vertices at distance less than 9, so at any point in the construction we have excluded at most $(2D)^8 r\delta_B Cn$ vertices from any given X_i. Thus we can construct B provided that $(2D)^8 r\delta_B C < 1$.

Let $N = \cup_{x \in B} VN_H(x)$ be the set of vertices with a neighbour in the buffer. Then $|N \cap X_i| \le 2Dr\delta_B Cn < \sqrt{\delta_B} n$ for $1 \le i \le r$.

We order the vertices in a list $L = L(0)$ that starts with N and ends with B. Within N, we arrange that $VN_H(x)$ is consecutive for each $x \in B$. We denote the vertex of H selected for embedding at time t by $s(t)$. This will usually be the first vertex of $L(t-1)$, but we will describe some exceptions to this principle in the selection rule below.

We denote the queue by $q(t)$ and write $Q(t) = \cup_{u \le t} q(u)$. We denote the vertices jumping the queue by $j(t)$ and write $J(t) = \cup_{u \le t} j(u)$. Initially we set $q(0) = Q(0) = j(0) = J(0) = \emptyset$.

We write $F_S(t)$ for the sets of G_S that are *free* to embed a given set S of H. We also use the convention that $F_S(t)$ is undefined if $S \notin H$. Initially we set $F_S(0) = G_S$ for $S \in H$. We also write $X_i(t) = X_i \setminus \{s(u) : u \le t\}$ for the unembedded vertices of X_i and $V_i(t) = V_i \setminus \{\phi(s(u)) : u \le t\}$ for the free positions in V_i. We let $X(t) = \cup_{i=1}^r X_i(t)$ and $V(t) = \cup_{i=1}^r V_i(t)$.

Iteration. At time t, while there are still some unembedded non-buffer vertices, we select a vertex to embed $x = s(t)$ according to the following *selection rule*. Informally, the rule is that our top priority is to embed any vertex neighbourhood $VN_H(x)$ with $x \in B$ as a consecutive sequence before embedding x itself or any other vertex with distance at most 4 from x, and our second priority is to embed vertices in the queue. Formally, the rule is:

- If $j(t-1) \neq \emptyset$ we select $x = s(t)$ to be any element of $j(t-1)$,
- If $j(t-1) = \emptyset$ and $q(t-1) \neq \emptyset$ we consider any element x' of $q(t-1)$.
  - If the distance from x' to all vertices in the buffer B is at least 5 then we select $x = x' = s(t)$.
  - Otherwise, there is a vertex $x'' \in B$ at distance at most 4 from x', and x'' is unique by the mutual distance property of B. If there are any unembedded elements of $VN_H(x'')$, we choose one of them to be $x = s(t)$, choosing x' itself if $x' \in VN_H(x'')$, and put all other unembedded vertices of $VN_H(x'')$ in $j(t)$. If all of $VN_H(x'')$ has been embedded we choose $x = x' = s(t)$.
- If $j(t-1) = q(t-1) = \emptyset$ we let $x = s(t)$ be the first vertex of $L(t-1)$.

We choose the image $\phi(x)$ of x uniformly at random among all elements $y \in F_x(t-1)$ that are 'good', a property that can be informally stated as saying that if we set $\phi(x) = y$ then the free sets $F_S(t)$ will be regular, have the correct density, and not create too much danger of using an edge marked as forbidden. Now we will describe the formal definition. Note that all expressions at time t are to be understood with the embedding $\phi(x) = y$, for some unspecified vertex y.

Definitions.

1. For a vertex x we write $\nu_x(t)$ for the number of elements in $VN_H(x)$ that have been embedded at time t. For a set S we write $\nu_S(t) = \sum_{y \in S} \nu_y(t)$. We also define $\nu'_S(t)$ as follows. When $|S| = 3$ we let $\nu'_S(t) = \nu_S(t)$. When $|S| = 1, 2$ we let $\nu'_S(t) = \nu_S(t) + K$, where K is the maximum value of $\nu'_{Sx'}(t')$ over vertices x' embedded at time $t' \le t$ with $S \in H(x')$; if there is no such vertex x' we let $\nu'_S(t) = \nu_S(t)$.
2. As in Definition 3.7, for any r-partite set S we define $F_S(t) = F_S(t-1)^y$ if $x \in S$ or $F_S(t) = F_{S\le}(t-1)[F_{S,x\le}(t-1)(y)]_S \setminus y$ if $x \notin S$. For any sets $S' \subseteq S$ we write $d_{S'}(F(t)) = d_{S'}(F_{S\le}(t))$; there is no ambiguity, as the density is the same for any S containing S'.
3. We say that S is *unembedded* if every vertex of S is unembedded, i.e. $s(u) \notin S$ for $u \le t$. We define an *exceptional* set $E_x(t-1) \subseteq F_x(t-1)$ by saying y is in $F_x(t-1) \setminus E_x(t-1)$ if and only if for every unembedded $\emptyset \neq S \in H(x)$,

$$\left.\begin{array}{l} d_S(F(t)) = (1 \pm \epsilon_{\nu'_S(t),0}) d_S(F(t-1)) d_{Sx}(F(t-1)) \\ \text{and } F_S(t) \text{ is } \epsilon_{\nu'_S(t),0}\text{-regular when } |S| = 2. \end{array}\right\} \qquad (*_{4.1})$$

Lemma 4.13 will imply that $E_x(t-1)$ is small compared to $F_x(t-1)$.

4. As in Definition 3.13, for any triple $E \in H$ we write E^t for the subset of E that is unembedded at time t and $M_{E^t,E}(t) = \{P \in F_{E^t}(t) : P \cup \phi(E \setminus E^t) \in M_E\}$. We define

$$D_{x,E}(t-1) = \{y \in F_x(t-1) : |M_{E^t,E}(t)| > \theta_{\nu'_{E^t}(t)}|F_{E^t}(t)|\}.$$

Intuitively, these sets consist of vertices y to which it is *dangerous* to embed x. Lemma 4.15 will show that only a small proportion of free vertices are dangerous.
5. Let $U(x)$ be the set of all triples $E \in H$ with $E^{t-1} \cap (VN_H(x) \cup x) \neq \emptyset$. We obtain the set of *good* elements $OK_x(t-1)$ from $F_x(t-1)$ by deleting $E_x(t-1)$ and $D_{x,E}(t-1)$ for every $E \in U(x)$.

We embed x as $\phi(x) = y$ where y is chosen uniformly at random from the good elements of $F_x(t-1)$. We conclude the iteration by updating $L(t-1)$, $q(t-1)$ and $j(t-1)$. First we remove x from whichever of these sets it was taken. Then we add to the queue any unembedded vertex z for which $F_z(t)$ has become 'too small'. To make this precise, suppose $z \in L(t-1) \setminus \{x\}$, and let t_z be the most recent time at which we embedded a vertex in $VN_H(z)$, or 0 if there is no such time. (Note that if $z \in VN_H(x)$ then $t_z = t$.) We add z to $q(t)$ if $|F_z(t)| < \delta'_Q|F_z(t_z)|$. This defines $L(t)$, $q(t)$ and $j(t)$.

Repeat this iteration until the only unembedded vertices are buffer vertices, but abort with failure if at any time we have $|Q(t) \cap X_i| > \delta_Q|X_i|$ for some $1 \leq i \leq r$. Let T denote the time at which the iterative phase terminates (whether with success or failure).

Conclusion. Suppose $x \in X(T)$ is unembedded at time T and we embed the last vertex of $VN_H(x)$ at time t_x^N. We define the following *available* sets for x. We let A_x be obtained from $F_x(t_x^N)$ by removing all sets $M_{x,E}(t_x^N)$ for triples E containing x. We let $A'_x = A_x \cap V_x(T)$. We choose a system of distinct representatives for $\{A'_x : x \in X(T)\}$ to complete the embedding, either ending with success if this is possible, or aborting with failure if it is not possible.

To justify this algorithm, we need to show that if it does not abort with failure then it does embed H in $G \setminus M$. We explained in the previous section why the 'local consistency' of the update rule implies that it embeds H in G, so we just need to show that no marked edge is used. This follows from the following lemma.

Lemma 4.2. *Suppose x is the last unembedded vertex of some triple E at time $t-1$. Then $D_{x,E}(t-1) = M_{x,E}(t-1)$. If $\phi(x) \notin M_{x,E}(t-1)$ then $\phi(E) \in G \setminus M$.*

Proof. Note that $E^{t-1} = x$, $E^t = \emptyset$ and $F_\emptyset(t) = \{\emptyset\}$ is a set of size 1. If we were to choose $y \in M_{x,E}(t-1)$ then we would get $M_{\emptyset,E}(t) = \{\emptyset\}$ and so $|M_{\emptyset,E}(t)| = 1 > \theta_{\nu'_{E^t}(t)} = \theta_{\nu'_{E^t}(t)}|F_\emptyset(t)|$. On the other hand, if we choose $y \notin M_{x,E}(t-1)$ then we get $M_{\emptyset,E}(t) = \emptyset$ and so $|M_{\emptyset,E}(t)| = 0 < \theta_{\nu'_{E^t}(t)} = \theta_{\nu'_{E^t}(t)}|F_\emptyset(t)|$. Therefore $D_{x,E}(t-1) = M_{x,E}(t-1)$. The second statement is now clear. ∎

It will often be helpful to use the following terminology pertaining to increments of $\nu_x(t)$. We think of time as being divided into *x-regimes*, defined by the property that vertices of $VN_H(x)$ are embedded at the beginning and end of x-regimes, but not during x-regimes. Thus the condition for adding a vertex z to the queue is that the free set for z has shrunk by a factor of δ'_Q during the current z-regime. Note that each vertex x defines its own regimes, and regimes for different vertices can intersect in a complicated manner.

Note that any vertex neighbourhood contains at most $2D$ vertices. Thus in the selection rule, any element of the queue can cause at most $2D$ vertices to jump the queue. Note also that when a vertex neighbourhood jumps the queue, its vertices are immediately embedded at consecutive times before any other vertices are embedded.

We collect here a few more simple observations on the algorithm.

Lemma 4.3.

(i) *For any* $1 \le i \le r$ *and time* t *we have* $|V_i(t)| \ge \delta_B n/2$.
(ii) *For any* t *we have* $|J(t)| \le 2D|Q(t)| \le \sqrt{\delta_Q}n$.
(iii) *We have* $\nu_x(t) \le 2D$ *for any vertex,* $\nu'_S(t) \le 6D$ *when* $|S| = 3$, $\nu'_S(t) \le 10D$ *when* $|S| = 2$, *and* $\nu'_S(t) \le 12D$ *when* $|S| = 1$. *Thus the* ϵ*-subscripts are always defined in* $(*_{4.1})$.
(iv) *For any* $z \in VN_H(x)$ *we have* $\nu_z(t) = \nu_z(t-1) + 1$, *so for any* $S \in H$ *that intersects* $VN_H(x)$ *we have* $\nu_S(t) > \nu_S(t-1)$.
(v) *If* $\nu_S(t) > \nu_S(t-1)$ *then* $\nu'_S(t) > \nu'_S(t-1)$.
(vi) *If* z *is embedded at time* $t' \le t$ *and* $S \in H(z)$ *then* $\nu'_S(t) \ge \nu'_{S_z}(t) > \nu'_{S_z}(t'-1)$.

Proof. As in the graph blow-up lemma, we stop the iterative procedure when the only unembedded vertices are buffer vertices, and during the procedure a buffer vertex is only embedded if it joins the queue. Therefore $|V_i(t)| \ge |B \cap V_i| - |Q(t) \cap V_i| \ge \delta_B n - \delta_Q Cn \ge \delta_B n/2$, so (i) holds. The fact that an element of the queue can cause at most $2D$ vertices to jump the queue gives (ii). Statements (iii) and (iv) are clear from the definitions. Statement (v) follows because $\nu'_S(t)$ and $\nu'_S(t-1)$ are obtained from $\nu_S(t)$ and $\nu_S(t-1)$ by adding constants that are maxima of certain sets, and the set at time t includes the set at time $t-1$. For (vi) note that $\nu'_S(t) = \nu_S(t) + K$, with $K \ge \nu'_{S_z}(t)$ and $\nu'_{S_z}(t) > \nu'_{S_z}(t'-1)$ by (iv) and (v). ∎

4.2. Restrictions of Complexes

Before analysing the algorithm in the previous subsection, we need to develop some more theory. In this subsection we prove a lemma that justifies various manipulations involving restrictions (Definition 3.5). We often consider situations when several restrictions are placed on a complex, and then it is useful to rearrange them. We define *composition* of complexes as follows.

Definition 4.4. *We write* $x \in^* S$ *to mean that* $x \in S$ *or* S *is undefined. Suppose* G *and* G' *are* r*-partite 3-complexes on* $X = X_1 \cup \cdots \cup X_r$. *We define* $(G * G')_S$ *if* G_S *or* G'_S *is defined and say that* $S \in (G * G')_S$ *if* $A \in^* G_A$ *and* $A \in^* G'_A$ *for any* $A \subseteq S$. *We say that* G, G' *are* separate *if there is no* $S \ne \emptyset$ *with* G_S *and* G'_S *both defined. Given complexes* $G^1, \ldots, G^t$ *we write* $\bigodot_{i=1}^t J_i = J_1 * \cdots * J_t$. *Similarly we write* $\bigodot_{i \in I} J_i$ *for the composition of a collection of complexes* $\{J^i : i \in I\}$.

Note that $G \cap G' \subseteq G * G' \subseteq G \cup G'$. To illustrate the relation $\in^*$, we note that if G is a subcomplex of H then $H[G]$ is the set of all $S \in H$ such that $A \in^* G_A$ for all $A \subseteq S$. It follows that $H * G = H[G]$, see (vi) in the following lemma. More generally, the composition $G * G'$

describes 'mutual restrictions', i.e. restrictions that G places on G' and restrictions that G' places on G. We record some basic properties of Definition 4.4 in the following lemma. Since the statement and proof are heavy in notation, we first make a few remarks to indicate that the properties are intuitive. Property (i) says that mutual restrictions can be calculated in any order. Property (ii) says that a neighbourhood in a mutual restriction is given by the mutual restriction of the neighbourhoods and the original complexes. Property (iii) says that separate restrictions act independently. The remaining properties give rules for rearranging repeated restrictions. The most useful cases are (v) and the second statement in (vi), which convert a repeated restriction into a single restriction (the other cases are also used, but their statements are perhaps less intuitive). One should note that the distinction made earlier between 'empty' and 'undefined' is important here; e.g. $(G * G')_S$ undefined implies that G_S and G'_S are undefined, but this is not true with 'undefined' replaced by 'empty'.

Lemma 4.5. *Suppose H is an r-partite 3-complex and G, G', G'' are subcomplexes of H.*

(i) *$*$ is a commutative and associative operation on complexes,*
(ii) $(G * G')_{S^\leq}(v) = G_{S\setminus v^\leq} * G'_{S\setminus v^\leq} * G_{S^\leq}(v) * G'_{S^\leq}(v)$ *for any $v \in S \in G \cup G'$,*
(iii) *If G, G' are separate then $G * G' = G \cup G'$ and $H[G][G'] = H[G'][G] = H[G \cup G']$,*
(iv) $H[G][G * G'] = H[G'][G * G']$,
(v) *If G' is a subcomplex of $H[G]$ then $H[G][G'] = H[G * G']$.*
(vi) *If G' is a subcomplex of G then $G * G' = G[G']$ and $H[G][G[G']] = H[G'][G[G']] = H[G][G']$. If also G'_S is defined whenever G_S is defined then $H[G][G'] = H[G']$.*

Proof. (i) By symmetry of the definition we have commutativity $G*G' = G'*G$. Next we show that $G*G'$ is a complex. Suppose $A \subseteq S' \subseteq S \in G*G'$. Since $S' \subseteq S \in G \cup G'$ we have $S' \in G \cup G'$. Since $A \subseteq S \in G*G'$ we have $A \in^* G_A$ and $A \in^* G'_A$. Therefore $S' \in G*G'$, so $G*G'$ is a complex. Now we show associativity, i.e. $(G*G')*G'' = G*(G'*G'')$. Suppose $S \in (G*G')*G''$. We claim that for any $A \subseteq S$ we have $A \in^* G_A$, $A \in^* G'_A$ and $A \in^* G''_A$. To see this, we apply the definition of $(G*G')*G''$ to get $A \in^* (G*G')_A$ and $A \in^* G''_A$. If $A \in (G*G')_A$ we have $A \in^* G_A$ and $A \in^* G'_A$. Otherwise, $(G*G')_A$ is undefined, so G_A and G'_A are undefined. This proves the claim. Now $S \in (G*G')*G''$ implies that $S \in G \cup G' \cup G''$, so $S \in G$ or $S \in G' \cup G''$. If $S \in G' \cup G''$ then by the claim we have $A \in^* G'_A$ and $A \in^* G''_A$ for $A \subseteq S$, so $S \in G'*G''$. Therefore $S \in G \cup (G'*G'')$. Also, if $A \subseteq S$ with $(G'*G'')_A$ defined then, for any $A' \subseteq A$, since $A' \subseteq S$, the claim gives $A' \in^* G'_{A'}$ and $A' \in^* G''_{A'}$. Therefore $A \in (G'*G'')_A$. This shows that $S \in G*(G'*G'')$, so $(G*G')*G'' \subseteq G*(G'*G'')$. Also, $G*(G'*G'') = (G''*G')*G \subseteq G''*(G'*G) = (G*G')*G''$, so $(G*G')*G'' = G*(G'*G'')$.

(ii) Suppose $v \in S \in G*G'$. Then $A \in^* G_A$ and $A \in^* G'_A$ for any $A \subseteq S$. Applying this to $A = A'v$ for any $A' \subseteq S \setminus v$ gives $A' \in^* G_A(v)$ and $A' \in^* G'_A(v)$. Thus $S \setminus v \in G_{S\setminus v^\leq} * G'_{S\setminus v^\leq} * G_{S^\leq}(v) * G'_{S^\leq}(v)$. Conversely, suppose that $v \in S \in G \cup G'$ and $S \setminus v \in G_{S\setminus v^\leq} * G'_{S\setminus v^\leq} * G_{S^\leq}(v) * G'_{S^\leq}(v)$. Then $A \in^* G_A$, $A \in^* G'_A$, $Av \in^* G_{Av}$ and $Av \in^* G'_{Av}$ for any $A \subseteq S \setminus v$, so $S \in G*G'$.

(iii) Suppose $\emptyset \neq A \subseteq S \in G$. Then $A \in G_A$ and G'_A is undefined, since G, G' are separate. Therefore $S \in G*G'$, so $G \subseteq G*G'$. Similarly $G' \subseteq G*G'$, so $G*G' = G \cup G'$. Next we note that $G' \subseteq H[G]$: if $S \in G'$ then $S \in H$ and G_A is undefined for any $\emptyset \neq A \subseteq S$. Similarly $G \subseteq H[G']$, so $H[G][G']$ and $H[G'][G]$ are well-defined. Now suppose $S \in H[G][G']$. Since $S \in H[G]$, for any $A \subseteq S$ we have $A \in^* G_A$. Since $S \in H[G][G']$, for any $A \subseteq S$ we have $A \in^* G'_A$. This shows that $S \in H[G'][G]$ and $S \in H[G' \cup G]$. Applying the same argument to any $S \in H[G'][G]$ we deduce that $H[G][G'] = H[G'][G] \subseteq H[G \cup G']$. Conversely, if $S \in H[G \cup G']$ then for any $A \subseteq S$ we have $A \in^* G_A \cup G'_A$. Since G, G' are separate, we have $A \in^* G_A$ and $A \in^* G'_A$. Therefore $S \in H[G][G'] = H[G'][G]$. It follows that $H[G][G'] = H[G'][G] = H[G \cup G']$.

(iv) We first show that $H[G][G * G'] \subseteq H[G'][G * G']$. Suppose $S \in H[G][G * G']$. Then $S \in H[G] \subseteq H$. Since $S \in H[G][G * G']$, for any $A \subseteq S$ we have $A \in^* (G * G')_A$. Consider any $A \subseteq S$ such that G'_A is defined. Then $(G * G')_A$ is defined, so $A \in (G * G')_A \subseteq G'_A$. Therefore $S \in H[G']$. Now consider any $A \subseteq S$ such that $(G * G')_A$ is defined. Since $S \in H[G][G * G']$ we have $A \in (G * G')_A$. Therefore $S \in H[G'][G * G']$. Similarly, $H[G'][G * G'] \subseteq H[G][G * G']$, so equality holds.

(v) First we show that $H[G * G'] \subseteq H[G][G']$. Suppose $S \in H[G * G']$. Then $S \in H$. Also, for any $A \subseteq S$ we have $A \in^* (G * G')_A$, and so $A \in^* G_A$ and $A \in^* G'_A$. This implies that $S \in H[G]$, and then that $S \in H[G][G']$. Now we show that $H[G][G'] \subseteq H[G * G']$. Suppose $S \in H[G][G']$. Then $S \in H[G] \subseteq H$. Since $S \in H[G]$, for any $A \subseteq S$ we have $A \in^* G_A$. Since $S \in H[G][G']$, for any $A \subseteq S$ we have $A \in^* G'_A$. Thus for any $A' \subseteq A \subseteq S$ we have $A' \in^* G_{A'}$ and $A' \in^* G'_{A'}$, so $A \in^* (G * G')_A$. Therefore $S \in H[G * G']$.

(vi) We first note that $G * G' = G[G']$ is immediate from Definitions 4.4 and 3.5. Then $H[G][G[G']] = H[G'][G[G']]$ follows from (iv). Now we show that $H[G][G[G']] = H[G][G']$. Suppose that $S \in H[G][G[G']]$. Then $S \in H[G]$. Consider any $A \subseteq S$ such that G'_A is defined. Since $S \in H[G][G[G']]$ we have $A \in G[G']_A$, and so $A \in G'_A$. Therefore $S \in H[G][G']$. Conversely, suppose that $S \in H[G][G']$. Consider any $A \subseteq S$ such that $G[G']_A$ is defined. Then G_A is defined, so $A \in G_A$, since $S \in H[G]$. Also, for any $A' \subseteq A$ with $G'_{A'}$ defined we have $A' \in G'_{A'}$, since $S \in H[G][G']$. Therefore $A \in G[G']_A$. This shows that $S \in H[G][G[G']]$. The second statement is immediate. ▪

4.3. Hypergraph Regularity Properties

In this subsection we record some useful properties of hypergraph regularity, analogous to the standard facts we mentioned earlier for graph regularity. Similar results can be found e.g. in [6, 9, 18], but with stronger assumptions on the hierarchy of parameters. However, with the same proof, we obtain Lemma 4.6 under weaker assumptions on the parameters, which will be crucial to the proof of Lemma 4.13. We start with two analogues to Lemma 2.2, the first concerning graphs that are neighbourhoods of a vertex, and the second sets that are neighbourhood of a pair of vertices.

Lemma 4.6 (Vertex neighbourhoods). *Suppose $0 < \epsilon \ll d$ and $0 < \eta \ll \eta' \ll d$ and G is a 3-partite 3-complex on $V = V_1 \cup V_2 \cup V_3$ with all densities $d_S(G) > d$. Suppose also that G_{13}, G_{12} are ϵ-regular, and G_{23}, G_{123} are η-regular. Then for all but at most $(4\epsilon + 2\eta')|G_1|$ vertices $v \in G_1$ we have $|G(v)_j| = (1 \pm \epsilon)d_{1j}(G)|G_i|$ for $j = 2, 3$ and $G(v)_{23}$ is an η'-regular graph of relative density $d_{23}(G(v)) = (1 \pm \eta')d_{123}(G)d_{23}(G)$.*

Proof. First we apply Lemma 2.2 to see that all but at most $4\epsilon|G_1|$ vertices in G_1 have degree $(d_{1j}(G) \pm \epsilon)|G_j|$ in G_{1j}, for $j = 2, 3$. Let G'_1 be the set of such vertices. It suffices to show the claim that all but at most $2\eta'|G_1|$ vertices $v \in G'_1$ have the following property: if $A^v_2 \subseteq G(v)_2$ and $A^v_3 \subseteq G(v)_3$ are sets with $|A^v_2| > \eta'|G(v)_2|$ and $|A^v_3| > \eta'|G(v)_3|$, then the bipartite subgraph $A^v_{23} \subseteq G_{23}$ spanned by A^v_2 and A^v_3 has $|A^v_{23}| = (d_{23}(G) \pm \eta)|A^v_2||A^v_3|$ edges, and the bipartite subgraph $A(v)_{23} \subseteq G(v)_{23}$ spanned by A^v_2 and A^v_3 has $|A(v)_{23}| = (1 \pm \eta'/2)d_{123}(G)|A^v_{23}|$ edges.

Suppose for a contradiction that this claim is false. Note that for any $v \in G_1'$ and sets $A_2^v \subseteq G(v)_2, A_3^v \subseteq G(v)_3$ with $|A_2^v| > \eta'|G(v)_2|, |A_3^v| > \eta'|G(v)_3|$ we have $|A_j^v| > \eta'(d_{1j}(G) - \epsilon)|G_j| > \eta|G_j|$ for $j = 2, 3$ so $|A_{23}^v| = (d_{23}(G) \pm \eta)|A_2^v||A_3^v|$ since G_{23} is η-regular. Then without loss of generality, we can assume that we have vertices $v_1, \ldots, v_t \in G_1'$ with $t > \eta'|G_1|$, and sets $A_2^{v_i} \subseteq G(v_i)_2$, $A_3^{v_i} \subseteq G(v_i)_3$ with $|A_2^{v_i}| > \eta'|G(v_i)_2|$ and $|A_3^{v_i}| > \eta'|G(v_i)_3|$, such that $|A(v_i)_{23}| < (1 - \eta'/2)d_{123}(G)|A_{23}^{v_i}|$ for $1 \le i \le t$. Define tripartite graphs $A^i = A_{23}^{v_i} \cup \{v_i a : a \in A_2^{v_i} \cup A_3^{v_i}\}$ and $A = \cup_{i=1}^t A_i$.

We can count the number of triangles in these graphs as $T_{123}(A) = \sum_{i=1}^t T_{123}(A^i) = \sum_{i=1}^t |A_{23}^{v_i}|$. Now $t > \eta'|G_1|$, $|A_{23}^{v_i}| > (d_{23}(G) - \eta)|A_2^{v_i}||A_3^{v_i}|$, $d_{23}(G) > d$, $|A_j^{v_i}| > \eta'|G(v_i)_j|$ and $|G(v_i)_j| > (d - \epsilon)|G_j|$ for $1 \le i \le t, j = 2, 3$, so

$$T_{123}(A) > \eta'|G_1| \cdot (d - \eta) \cdot \eta'(d - \epsilon)|G_2| \cdot \eta'(d - \epsilon)|G_3| > \eta|G_1||G_2||G_3| \ge \eta T_{123}(G).$$

Since G_{123} is η-regular we have $\frac{|G \cap T_{123}(A)|}{|T_{123}(A)|} = d_{123}(G) \pm \eta$. Therefore

$$|G \cap T_{123}(A)| > (d_{123}(G) - \eta)|T_{123}(A)| = (d_{123}(G) - \eta)\sum_{i=1}^t |A_{23}^{v_i}|.$$

But we also have

$$|G \cap T_{123}(A)| = \sum_{i=1}^t |A(v_i)_{23}| < \sum_{i=1}^t (1 - \eta'/2)d_{123}(G)\left|A_{23}^{v_i}\right| < (d_{123}(G) - \eta)\sum_{i=1}^t \left|A_{23}^{v_i}\right|,$$

contradiction. This proves the required claim. ∎

Lemma 4.7 (Pair neighbourhoods). *Suppose $0 < \epsilon \ll \epsilon' \ll d$ and G is an ϵ-regular 3-partite 3-complex on $V = V_1 \cup V_2 \cup V_3$ with all densities $d_S(G) > d$. Then for all but at most $\epsilon'|G_{12}|$ pairs $uv \in G_{12}$ we have $|G(uv)_3| = (1 \pm \epsilon')d_{123}(G)d_{13}(G)d_{23}(G)|G_3|$.*

Proof. Introduce another parameter η with $\epsilon \ll \eta \ll \epsilon'$. By Lemma 4.6, for all but at most $6\epsilon|G_1|$ vertices $v \in G_1$ we have $|G(v)_i| = (1 \pm \epsilon)d_{1i}(G)|G_i|$ for $i = 2, 3$ and $G(v)_{23}$ is an η-regular graph of relative density $d_{23}(G(v)) = (1 \pm \eta)d_{123}(G)d_{23}(G)$. Let G_1' be the set of such vertices $v \in G_1$. Then for any $v \in G_1'$, applying Lemma 2.2 to $G(v)_{23}$, we see that for all but at most $2\eta|G(v)_2| \le 2\eta|G_2|$ vertices in $u \in G(v)_2$, the degree of u in $G(v)_{23}$ satisfies

$$\begin{aligned}|G(uv)_3| &= (d_{23}(G(v)) \pm \eta)|G(v)_3| = ((1 \pm \eta)d_{123}(G)d_{23}(G) \pm \eta)(d_{13}(G) \pm \epsilon)|G_3| \\ &= (1 \pm \epsilon')d_{123}(G)d_{13}(G)d_{23}(G)|G_3|.\end{aligned}$$

Since $|G_{12}| = d_{12}(G)|G_1||G_2| > d|G_1||G_2|$, this estimate holds for all pairs $uv \in G_{12}$ except for at most $6\epsilon|G_1| \cdot |G_2| + |G_1| \cdot 2\eta|G_2| < \epsilon'|G_{12}|$. ∎

Next we give an analogue of Lemma 2.3, showing that regularity is preserved by restriction.

Lemma 4.8 (Regular restriction). *Suppose $0 < \epsilon \ll d$, G is a 3-partite 3-complex on $V = V_1 \cup V_2 \cup V_3$ with all densities $d_S(G) > d$, G_{123} is ϵ-regular, and $J \subseteq G$ is a 2-complex with $|J_{123}^*| > \sqrt{\epsilon}|G_{123}^*|$. Then $G[J]_{123}$ is $\sqrt{\epsilon}$-regular and $d_{123}(G[J]) = (1 \pm \sqrt{\epsilon})d_{123}(G)$.*

Proof. Since G_{123} is ϵ-regular, $|G[J]_{123}| = |G \cap J^*_{123}| = (d_{123}(G) \pm \epsilon)|J^*_{123}|$ and $d_{123}(G[J]) = |G[J]_{123}|/|G[J]^*_{123}| = |G[J]_{123}|/|J^*_{123}| = d_{123}(G) \pm \epsilon$. Now consider any subcomplex A of J with $|A^*_{123}| > \sqrt{\epsilon}|J^*_{123}|$. Then $|A^*_{123}| > \epsilon|G^*_{123}|$, so since G_{123} is ϵ-regular, $|G[J] \cap A^*_{123}| = |G \cap A^*_{123}| = (d_{123}(G) \pm \epsilon)|A^*_{123}| = (d_{123}(J[G]) \pm \sqrt{\epsilon})|A^*_{123}|$, i.e. $G[J]_{123}$ is $\sqrt{\epsilon}$-regular. ▪

It is worth noting the special case of Lemma 4.8 when J is a 1-complex. Then $G[J]$ is obtained from G by discarding some vertices, i.e. a restriction according to the traditional definition. In particular, we see that regularity implies vertex regularity (the weak property mentioned at the beginning of Section 3). We also record the following consequence of Lemma 4.8.

Corollary 4.9. *Suppose $0 < \epsilon \ll d$, G is a 3-partite 3-complex on $V = V_1 \cup V_2 \cup V_3$ with all densities $d_S(G) > d$ and G_{123} is ϵ-regular. Suppose also $0 < \eta \ll d$, $J \subseteq G$ is a 2-complex with all densities $d_S(J) > d$ and J_{12}, J_{13}, J_{23} are η-regular.*

Then $G[J]_{123}$ is $\sqrt{\epsilon}$-regular and $d_{123}(G[J]) = (1 \pm \sqrt{\epsilon})d_{123}(G)$.

Proof. We have $|J^*_{123}| = |T_{123}(J)| = (1 \pm 8\eta)d_{12}(J)d_{13}(J)d_{23}(J)|J_1||J_2||J_3| > \frac{1}{2}d^6|V_1||V_2||V_3| > \sqrt{\epsilon}|G^*_{123}|$ by the triangle counting lemma (1). The result now follows from Lemma 4.8. ▪

Next we note a simple relationship between relative and absolute densities.

Lemma 4.10. *Suppose $0 < \epsilon \ll d$, G is a 3-partite 3-complex on $V = V_1 \cup V_2 \cup V_3$ with all densities $d_S(G) > d$ and G is ϵ-regular. Then $d(G_{123}) = (1 \pm 8\epsilon)\prod_{S \subseteq 123} d_S(G)$.*

Proof. $d(G_{123}) = \frac{|G_{123}|}{|V_1||V_2||V_3|} = \frac{|G_{123}|}{|T_{123}(G)|} \cdot \frac{|T_{123}(G)|}{|V_1||V_2||V_3|} = d_{123}(G) \cdot (1 \pm 8\epsilon)\prod_{S \subsetneq 123} d_S(G)$ by (1). ▪

The following more technical lemma will be useful in the proof of Lemma 4.15. Later we will give a more general proof that is slicker, but conceptually more difficult, as it uses the 'plus complex' of Definition 6.8. For the convenience of the reader, in the 3-graph case we will use a proof that is somewhat pedestrian, but perhaps easier to follow.

Lemma 4.11. *Suppose $0 < \epsilon \ll \epsilon' \ll d$, G is a 4-partite 3-complex on $V = V_1 \cup V_2 \cup V_3 \cup V_4$ with all densities $d_S(G) > d$ and G is ϵ-regular.*

(i) *For any $P \in G_{123}$ and subcomplex I of $123^<$, let $G_{P,I}$ be the set of vertices $v \in G_4$ such that $P_S \cup v \in G_{S \cup 4}$ for all $S \in I$. Let B_I be the set of $P \in G_{123}$ such that we do not have $|G_{P,I}| = (1 \pm \epsilon')|V_4|\prod_{\emptyset \neq S \in I} d_{S \cup 4}(G)$. Then $|B_I| < \epsilon'|G_{123}|$.*
(ii) *For any $P' \in G_{12}$ and subcomplex I' of $12^<$ let $G'_{P',I'}$ be the set of vertices $v \in G_4$ such that $P'_{S'} \cup v \in G_{S' \cup 4}$ for all $S' \in I'$. Let $B'_{I'}$ be the set of $P' \in G_{12}$ such that we do not have $|G'_{P',I'}| = (1 \pm \epsilon')|V_4|\prod_{\emptyset \neq S' \in I'} d_{S \cup 4}(G)$. Then $|B'_{I'}| < \epsilon'|G_{12}|$.*

Proof. Introduce auxiliary constants with a hierarchy $\epsilon \ll \epsilon_1 \ll \epsilon_2 \ll \epsilon_3 \ll \epsilon'$. We consider selecting the vertices P_1, P_2, P_3 of P in turn, at each step identifying some exceptional sets P for which the stated estimate on $G_{P,I}$ might fail. First we choose P_1 so that $|G(P_1)_i| = (1 \pm \epsilon)d_{1i}(G)|G_i|$ and $G(P_1)_{ij}$ is an ϵ_1-regular graph of relative density $d_{ij}(G(P_1)) = (1 \pm \epsilon_1)d_{1ij}(G)d_{ij}(G)$ for distinct i,j in $\{2, 3, 4\}$. Applying

Lemma 4.6 with $\eta = \epsilon$ and $\eta' = \epsilon_1/4$, we see that this holds for all but at most $\epsilon_1|G_1|$ vertices $P_1 \in G_1$. Then the number of exceptional sets P at this stage is at most $\epsilon_1|G_1||V_2||V_3| = \epsilon_1 d_1(G)d(G_{123})^{-1}|G_{123}| < \sqrt{\epsilon_1}|G_{123}|$.

Now let $J^1 \subseteq G_{234<}$ be the 2-complex defined as follows. We define the singleton parts by J^1_4 equals $G(P_1)_4$ if $1 \in I$ or G_4 if $1 \notin I$, J^1_2 equals $G(P_1)_2$ if $12 \in I$ or G_2 if $12 \notin I$, and J^1_3 equals $G(P_1)_3$ if $13 \in I$ or G_3 if $13 \notin I$. We define the graphs by restriction to the singleton parts of the following: $G(P_1)_{24}$ if $12 \in I$ or G_{24} if $12 \notin I$, $G(P_1)_{34}$ if $13 \in I$ or G_{34} if $13 \notin I$, $G(P_1)_{23}$ if $123 \in I$ or G_{23} if $123 \notin I$. Then J^1 is $\sqrt{\epsilon_1}$-regular by Lemma 2.3. The graph densities $d_{ij}(J^1)$ are either $(1 \pm \epsilon_1)d_{1ij}(G)d_{ij}(G)$ or $(1 \pm \epsilon_1)d_{ij}(G)$, according as we restrict $G(P_1)_{ij}$ or G_{ij}.

Let $G^1 = G_{234\le}[J^1]$. Then G^1_{234} is ϵ_1-regular with $d_{234}(G^1) = (1 \pm \epsilon_1)d_{234}(G)$ by Corollary 4.9. Next we choose P_2 so that $|G^1(P_2)_i| = (1 \pm \epsilon_2)d_{2i}(G^1)|G^1_i|$ for $i = 3, 4$ and $G^1(P_2)_{34}$ is an ϵ_2-regular graph of relative density $d_{34}(G^1(P_2)) = (1 \pm \epsilon_2)d_{234}(G^1)d_{34}(G^1)$. By Lemma 4.6 this holds for all but at most $\epsilon_2|G^1_2|$ vertices $P_2 \in G^1_2$, so similarly to above, the number of exceptional sets P at this stage is at most $\sqrt{\epsilon_2}|G_{123}|$. Let $J^2 \subseteq G^1_{34\le}$ be the 2-complex in which J^2_4 is $G^1(P_2)_4$ if $2 \in I$ or G^1_4 if $2 \notin I$, J^2_3 is $G^1(P_2)_3$ if $23 \in I$ or G^1_3 if $23 \notin I$, and J^2_{34} is $G^1(P_2)_{34}$ if $23 \in I$ or G^1_{34} if $23 \notin I$. Then J^2_{34} is $\sqrt{\epsilon_2}$-regular by Lemma 2.3, with $d_{34}(J^2)$ either $(1 \pm \epsilon_2)d_{234}(G^1)d_{34}(G^1)$ or $(1 \pm \epsilon_2)d_{34}(G^1)$, according as we restrict $G^1(P_2)_{34}$ or G^1_{34}.

Now we choose P_3 so that $|J^2(P_3)_4| = (1 \pm \epsilon_3)d_{34}(J^2)|J^2_4|$. By Lemma 2.2 this holds for all but at most $\epsilon_3|J^2_3|$ vertices $P_3 \in J^2_3$, giving at most $\sqrt{\epsilon_3}|G_{123}|$ exceptional sets P here. In total, the number of exceptional sets at any stage is fewer than $\epsilon'|G_{123}|$. By construction, $G_{P,I}$ equals $J^2(P_3)_4$ if $3 \in I$ or J^2_4 if $3 \notin I$. If P is not exceptional then we can estimate $|G_{P,I}|$ by tracing back through the stages. At stage 3 we multiply $|J^2_4|$ by $(1 \pm \epsilon_3)d_{34}(J^2)$ if $3 \in I$, where $d_{34}(J^2)$ is $(1 \pm \epsilon_2)d_{234}(G^1)d_{34}(G^1)$ if $23 \in I$ or $(1 \pm \epsilon_2)d_{34}(G^1)$ if $23 \notin I$, where $d_{234}(G^1) = (1 \pm \epsilon_1)d_{234}(G)$ and $d_{34}(G^1)$ is $(1 \pm \epsilon_1)d_{134}(G)d_{34}(G)$ if $13 \in I$ or $(1 \pm \epsilon_1)d_{34}(G)$ if $13 \notin I$. Thus we obtain a factor of $d_{S4}(G)$ whenever $3 \in S \in I$. At stage 2, we obtain $|J^2_4|$ from $|G^1_4|$ by multiplying by $(1 \pm \epsilon_2)d_{24}(G^1)$ if $2 \in I$, where $d_{24}(G^1)$ is $(1 \pm \epsilon_1)d_{124}(G)d_{24}(G)$ if $12 \in I$ or $(1 \pm \epsilon_1)d_{24}(G)$ if $12 \notin I$. Thus we obtain a factor of $d_{S4}(G)$ whenever $2 \in S \in I$, $3 \notin S$. Finally, at stage 1, we obtain $|G^1_4|$ from $|G_4|$ by multiplying by $(1 \pm \epsilon_1)d_{14}(G^1)$ if $1 \in I$. Combining all factors we obtain $|G_{P,I}| = (1 \pm \epsilon')|V_4| \prod_{\emptyset \neq S \in I} d_{S4}(G)$.

This proves (i). The proof of (ii) is similar and much simpler (alternatively it could be deduced from (i)). We consider selecting the vertices P'_1 and P'_2 of P' in turn. We choose P'_1 so that $|G(P'_1)_4| = (1 \pm \epsilon)d_{14}(G)|G_4|$. We let G'_4 be G_4 if $1 \notin I$ or $G(P'_1)_4$ if $1 \in I$, and G'_{24} be the restriction of G_{24} to G_2 and G'_4. Then G'_{24} is ϵ_1-regular with $d_{24}(G') = (1 \pm \epsilon)d_{24}(G)$. We choose P'_2 so that $|G'_{24}(P'_2)| = (1 \pm \epsilon_1)d_{24}(G')|G'_4|$. Then $G'_{P',I'}$ is $G'_{24}(P'_2)$ if $2 \in I$ or G'_4 if $2 \notin I$. Now $|G'_{P',I'}|$ is obtained from $|G_4|$ by multiplying by $(1 \pm \epsilon)d_{14}(G)$ if $1 \in I$ and $(1 \pm \epsilon_1)(1 \pm \epsilon)d_{24}(G)$ if $2 \in I$, so $|G'_{P',I'}| = (1 \pm \epsilon')|V_4| \prod_{\emptyset \neq S' \in I'} d_{S \cup 4}(G)$. It is clear that there are at most $\epsilon'|G_{12}|$ exceptional sets P'. ∎

Finally we give another formulation of the neighbourhood Lemmas 4.6 and 4.7, showing that most vertices and pairs are close to 'average'.

Lemma 4.12 (Averaging). *Suppose $0 < \epsilon \ll \epsilon' \ll d$, G is a 3-partite 3-complex on $V = V_1 \cup V_2 \cup V_3$ with all densities $d_S(G) > d$ and G is ϵ-regular. Then*

(i) *for all but at most $\epsilon'|G_1|$ vertices $v \in G_1$ we have $|G(v)_{23}| = (1 \pm \epsilon')|G_{123}|/|G_1|$,*
(ii) *for all but at most $\epsilon'|G_{12}|$ pairs $uv \in G_{12}$ we have $|G(uv)_3| = (1 \pm \epsilon')|G_{123}|/|G_{12}|$.*

Proof. By Lemma 4.6 with $\eta = \epsilon$ and $\eta' = \epsilon'/4$, for all but at most $\epsilon'|G_1|$ vertices $v \in G_1$, $|G(v)_i| = (1 \pm \epsilon)d_{1i}(G)|G_i| = (1 \pm \epsilon)d_{1i}(G)d_i(G)|V_i|$ for $i = 2, 3$ and $d_{23}(G(v)) = (1 \pm \epsilon'/4)d_{123}(G)d_{23}(G)$. For such v we have $|G(v)_{23}| = d_{23}(G(v))|G(v)_2||G(v)_3| = (1 \pm \epsilon'/3)|V_2||V_3| \prod_{S\subseteq 123, S\neq 1} d_S(G)$. Also, $|G_{123}| = (1 \pm 8\epsilon)|V_1||V_2||V_3| \prod_{S\subseteq 123} d_S(G)$ by Lemma 4.10, so $|G(v)_{23}| = (1 \pm \epsilon')|G_{123}|/|G_1|$, giving (i). For (ii), Lemma 4.10 gives $|G_{123}|/|G_{12}| = (1 \pm 8\epsilon)|V_3| \prod_{3\in S\subseteq 123} d_S(G)$. Then by Lemma 4.7, replacing ϵ' with $\epsilon'/2$, for all but at most $\epsilon'|G_{12}|$ pairs $uv \in G_{12}$, $|G(uv)_3| = (1 \pm \epsilon'/2)d_{123}(G)d_{13}(G)d_{23}(G)|G_3| = (1 \pm \epsilon'/2)|V_3| \prod_{3\in S\subseteq 123} d_S(G) = (1 \pm \epsilon')|G_{123}|/|G_{12}|$. ∎

4.4. Good Vertices

We start the analysis of the algorithm by showing that most free vertices are good. Our first lemma handles the definitions for regularity and density in the algorithm.

Lemma 4.13. *The exceptional set $E_x(t-1)$ defined by $(*_{4.1})$ satisfies $|E_x(t-1)| < \epsilon_*|F_x(t-1)|$, and $F_S(t)$ is $\epsilon_{\nu'_S(t),1}$-regular with $d_S(F(t)) \geq d_u$ for every $S \in H$.*

Proof. We argue by induction on t. At time $t = 0$ the first statement is vacuous. The second statement at time 0 follows from the fact that $F_S(0) = G_S$ and our assumption that (G, M) is $(\epsilon, \epsilon', d_2, \theta, d_3)$-super-regular: condition (i) in Definition 3.16 tells us that G_S is ϵ-regular, with $d_S(G) \geq d_{|S|}$ if $|S| = 2, 3$. Also $d_S(G) = 1$ if $|S| = 0, 1$, as we assumed that $G_i = V_i$ in the hypotheses of Theorem 4.1. Now suppose $t \geq 1$ and $\emptyset \neq S \in H$ is unembedded, so $x \notin S$. We consider various cases for S to establish the bound on the exceptional set and the regularity property, postponing the density bound until later in the proof.

We start with the case when $S \in H(x)$. Suppose first that $S = vw$ with $xvw \in H$. By induction $F_{S'}(t-1)$ is $\epsilon_{\nu'_{S'}(t-1),1}$-regular and $d_{S'}(F(t-1)) \geq d_u$ for every $S' \subseteq xvw$. Write $\nu = \max\{\nu'_{xv}(t-1), \nu'_{xw}(t-1)\}$ and $\nu^* = \max\{\nu'_{vw}(t-1), \nu'_{xvw}(t-1)\}$. We claim that $\nu'_{vw}(t) > \nu^*$. This holds by Lemma 4.3: (iv) gives $\nu_{vw}(t) > \nu_{vw}(t-1)$, (v) gives $\nu'_{vw}(t) > \nu'_{vw}(t-1)$, and (vi) gives $\nu'_{vw}(t) > \nu'_{xvw}(t-1)$. Now applying Lemma 4.6, for all but at most $(4\epsilon_{\nu,1}+2\epsilon_{\nu^*,2})|F_x(t-1)|$ vertices $y \in F_x(t-1)$ we have $|F_v(t)| = |F_{xv}(t-1)(y)| = (1\pm\epsilon_{\nu,1})d_{xv}(F(t-1))|F_v(t-1)|$, $|F_w(t)| = |F_{xw}(t-1)(y)| = (1 \pm \epsilon_{\nu,1})d_{xw}(F(t-1))|F_w(t-1)|$, and $F_{vw}(t) = F_{xvw}(t-1)(y)$ is an $\epsilon_{\nu^*,2}$-regular graph with $d_{vw}(F(t)) = (1 \pm \epsilon_{\nu^*,2})d_{xvw}(F(t-1))d_{vw}(F(t-1))$. Since $\nu'_{vw}(t) > \nu^*$ we have $(*_{4.1})$ when $S = vw$ for such y. Note that it is important for this argument that Lemma 4.6 makes no assumption of any relationship between ν and ν^*. For future reference we also note that the density bounds at time $t-1$ imply that $d_{vw}(F(t)) > d_u^2/2$; we will show a lower bound of d_u later, but this interim bound will be useful before then.

The argument when $S = \{v\} \in H(x)$ has size 1 is similar and more straightforward. By Lemma 2.2, for all but at most $2\epsilon_{\nu'_{xv}(t-1),1}|F_x(t-1)|$ vertices $y \in F_x(t-1)$ we have $|F_v(t)| = |F_{xv}(t-1)(y)| = (1\pm\epsilon_{\nu'_{xv}(t-1),1})d_{xv}(F(t-1))|F_v(t-1)|$. Also, we have $\nu'_v(t) > \nu'_{xv}(t-1)$ by Lemma 4.3(vi), so $(*_{4.1})$ holds when $S = \{v\}$ for such y. We also note for future reference that $d_v(F(t)) > d_u^2/2$. In the argument so far we have excluded at most $6\epsilon_{12D,2}|F_x(t-1)|$ vertices $y \in F_x(t-1)$ for each of at most $3D$ sets $S \in H(x)$ with $|S| = 1$ or $|S| = 2$; this gives the required bound on $E_x(t-1)$. We also have the required regularity property of $F_S(t)$, but for now we postpone showing the density bounds.

Next we consider the case when $S \in H$ and $S \notin H(x)$. If $S = \{v\}$ has size 1 then $|F_v(t)| = |F_v(t-1)\backslash y| \geq |F_v(t-1)|-1$, so $d_v(F(t)) \geq d_v(F(t-1))-1/n > d_u/2$, say. Next suppose that $S = vw$ has size 2. Recall that Lemma 3.11 gives $F_S(t) = F_{S^\leq}(t-1)[F_{S^<}(t)]_S$. Then $F_{vw}(t)$ is the bipartite subgraph of $F_{vw}(t-1)$ induced by $F_v(t)$ and $F_w(t)$. We have

$F_v(t) = F_{xv}(t-1)(y)$ if $xv \in H$ or $F_v(t-1) \setminus y$ if $xv \notin H$. Similarly, $F_w(t) = F_{xw}(t-1)(y)$ if $xw \in H$ or $F_w(t-1) \setminus y$ if $xw \notin H$. Since we choose $y \notin E_x(t-1)$, if $xv \in H$ then $|F_v(t)| = (1 \pm \epsilon_{\nu'_v(t),1}) d_v(F(t-1)) d_{xv}(F(t-1)) > \frac{1}{2} d_u |F_v(t-1)|$, and if $xv \notin H$ then $|F_v(t)| = |F_v(t-1) \setminus y| \geq |F_v(t)| - 1$. Similarly, if $xw \in H$ then $|F_w(t)| > \frac{1}{2} d_u |F_w(t-1)|$, and if $xw \notin H$ then $|F_w(t)| \geq |F_w(t)| - 1$. Now $F_{vw}(t-1)$ is $\epsilon_{\nu'_{vw}(t-1),1}$-regular, so by Lemma 2.3, $F_{vw}(t)$ is $\epsilon_{\nu'_{vw}(t-1),2}$-regular and $d_{vw}(F(t)) = (1 \pm \epsilon_{\nu'_{vw}(t-1),2}) d_{vw}(F(t-1))$. This gives the required regularity property for $F_{vw}(t)$ in the case that vw intersects $VN_H(x)$, when we have $\nu'_{vw}(t) > \nu'_{vw}(t-1)$ by Lemma 4.3(v). In fact, we are only required to show that $F_{vw}(t)$ is $\epsilon_{\nu'_{vw}(t),1}$-regular, but this stronger regularity property will be useful for the case when vw and $VN_H(x)$ are disjoint. Now consider the case that vw and $VN_H(x)$ are disjoint. Let t' be the most recent time at which we embedded a vertex x' with a neighbour in vw. Then by Lemma 3.12, $F_{vw^\leq}(t)$ is obtained from $F_{vw^\leq}(t')$ just by deleting all sets containing any vertices that are embedded between time $t'+1$ and t. Thus $F_{vw}(t)$ is the bipartite subgraph of $F_{vw}(t')$ spanned by $F_v(t)$ and $F_w(t)$. Now $F_{vw}(t')$ is $\epsilon_{\nu'_{vw}(t'-1),2}$-regular, by the stronger regularity property just mentioned above. Since $\nu'_{vw}(t) \geq \nu'_{vw}(t') > \nu'_{vw}(t'-1)$, Lemma 2.3 gives the required regularity property for $F_{vw}(t)$. For future reference, we note that in either case the bound $d_{vw}(F(t-1)) > d_u$ implies that $d_{vw}(F(t)) > d_u/2$.

Continuing with the case when $S \in H$ and $S \notin H(x)$, we now suppose that $|S| = 3$. Again we use $F_S(t) = F_{S^\leq}(t-1)[F_{S^<}(t)]_S$. If $S' \subsetneq S$, whether $S' \in H(x)$ or $S' \notin H(x)$, we have shown above that $d_{S'}(F(t)) > d_u^2/2$, and if $|S'| = 2$ that $F_{S'}(t)$ is $\epsilon_{\nu'_{S'}(t),1}$-regular. Since $F_S(t-1)$ is $\epsilon_{\nu'_S(t-1),1}$-regular, Corollary 4.9 implies that $F_S(t)$ is $\epsilon_{\nu'_S(t-1),2}$-regular and $d_S(F(t)) = (1 \pm \epsilon_{\nu'_S(t-1),2}) d_S(F(t-1))$. This gives the required regularity property for $F_S(t)$ in the case that S intersects $VN_H(x)$, when we have $\nu'_S(t) > \nu'_S(t-1)$. As above, although we are only required to show that $F_S(t)$ is $\epsilon_{\nu'_S(t),1}$-regular, this stronger regularity property will be useful for the case when S and $VN_H(x)$ are disjoint. Suppose S and $VN_H(x)$ are disjoint. Let t' be the most recent time at which we embedded a vertex x' with a neighbour in S. Then by Lemma 3.12, $F_{S^\leq}(t)$ is obtained from $F_{S^\leq}(t')$ just by deleting all sets containing any vertices that are embedded between time $t'+1$ and t. Equivalently, $F_S(t) = F_S(t')[((F_v(t) : v \in S), \{\emptyset\})]$. Now $F_S(t')$ is $\epsilon_{\nu'_S(t'-1),2}$-regular, by the stronger regularity property just mentioned above. Since $\nu'_S(t) \geq \nu'_S(t') > \nu'_S(t'-1)$, Corollary 4.9 gives the required regularity property for $F_S(t)$.

Now we have established the bound on $E_x(t-1)$ and the regularity properties, so it remains to show the density bounds. First we consider any unembedded $S \in H$ with $|S| = 3$. We claim that

$$F_S(t) = G_{S^\leq}[F_{S^<}(t)]_S. \tag{4}$$

To see this we use induction. In the base case $t = 0$ we have $F_S(0) = G_S$ and $G_{S^\leq}[F_{S^<}(0)] = G_{S^\leq}[G_{S^<}] = G_{S^\leq}$, so $G_{S^\leq}[F_{S^<}(0)] = G_S$. For $t > 0$, Lemma 3.11 gives $F_S(t) = F_{S^\leq}(t-1)[F_{S^<}(t)]_S$, i.e. $F_S(t)$ consists of all triples in $F_S(t-1)$ that form triangles in $F_{S^<}(t)$. The induction hypothesis gives $F_S(t-1) = G_{S^\leq}[F_{S^<}(t-1)]_S$, and so we can write $F_S(t) = G_{S^\leq}[F_{S^<}(t-1)][F_{S^<}(t)]_S$. Now $F_{S^<}(t) \subseteq F_{S^<}(t-1)$, so Lemma 4.5(vi) gives $G_{S^\leq}[F_{S^<}(t-1)][F_{S^<}(t)] = G_{S^\leq}[F_{S^<}(t-1)[F_{S^<}(t)]] = G_{S^\leq}[F_{S^<}(t)]$. This proves (4). Also, we showed above that for every $S' \subsetneq S$, we have $d_{S'}(F(t)) > d_u^2/2$, and if $|S'| = 2$ then $F_{S'}(t)$ is $\epsilon_{\nu'_{S'}(t),1}$-regular. Since G_S is ϵ-regular, Corollary 4.9 shows that $F_S(t)$ is $\sqrt{\epsilon}$-regular and $d_S(F(t)) = (1 \pm \sqrt{\epsilon}) d_S(G) > d_3/2$. This gives the required bound $d_S(F(t)) > d_u$, although we will also use the stronger bound of $d_3/2$ below.

Next consider any unembedded pair $vw \in H$. Let $t' \leq t$ be the most recent time at which we embedded a vertex x' with $x'vw \in H$, or let $t' = 0$ if there is no such vertex x'. Note that we have $t' = t$ if $xvw \in H$. For $t^* \leq t$, let $J(t^*)$ be the 1-complex $(F_v(t^*), F_w(t^*), \{\emptyset\})$. We claim that $F_{vw}(t^*) = F_{vw\leq}(t')[J(t^*)]_{vw}$ for $t' \leq t^* \leq t$. This follows by induction, similarly to the argument when $|S| = 3$. When $t = t'$ the claim follows from $F_{vw\leq}(t')[J(t')] = F_{vw\leq}(t')$. For $t^* > t'$, we have $vw \notin H(x)$, so Lemma 3.11 gives $F_{vw}(t^*) = F_{vw\leq}(t^* - 1)[J(t^*)]_{vw}$. Since $F_{vw}(t^* - 1) = F_{vw\leq}(t')[J(t^* - 1)]_{vw}$ by induction, Lemma 4.5(vi) gives $F_{vw}(t^*) = F_{vw\leq}(t')[J(t^*-1)][J(t^*)]_{vw} = F_{vw\leq}(t')[J(t^*)]_{vw}$, as claimed. Now we claim that $d_{vw}(F(t)) > (d_3/4)^{i_t} d_2/2$, where we temporarily use i_t to denote the number of embedded vertices x' at time t with $x'vw \in H$. To see this, we argue by induction, noting that initially $d_{vw}(F(0)) = d_{vw}(G) > d_2$. Also, if $i_t = 0$ then $F_{vw}(t) = G_{vw\leq}[J(t)]_{vw}$, so $d_{vw}(F(t)) > d_2/2$ by Lemma 2.3. Now suppose that $i_t > 0$, so that t' and x' are defined above. By induction we have $d_{vw}(F(t' - 1)) > (d_3/4)^{i_t - 1} d_2/2$. Also $d_{x'vw}(F(t' - 1)) > d_3/2$ by the lower bound just proved for relative densities of triples, so $(*_{4.1})$ gives $d_{vw}(F(t')) = (1 \pm \epsilon_{v'_{vw}(t'),0}) d_{vw}(F(t' - 1)) d_{x'vw}(F(t' - 1)) > (d_3/4)^{i_t - 1} d_2/2 \cdot d_3/2$. Since $F_{vw}(t) = F_{vw\leq}(t')[J(t)]_{vw}$, Lemma 2.3 gives $d_{vw}(F(t)) > d_{vw}(F(t'))/2 > (d_3/4)^{i_t} d_2/2$, as claimed. Since $i_t \leq D$ we have $d_{vw}(F(t)) > 2d_3^{2D} d_2$, say. In particular we have the required bound of $d_{vw}(F(t)) > d_u$.

Finally we consider any unembedded vertex z. Let $t_z \leq t$ be the most recent time at which we embedded a neighbour $w = s(t_z)$ of z, or $t_z = 0$ if there is no such time. If $t_z > 0$ then by $(*_{4.1})$ and the above bound for pair densities we have $d_z(F(t_z)) > d_{wz}(F(t_z - 1)) d_z(F(t_z - 1))/2 > d_3^{2D} d_2 d_z(F(t_z - 1)))$. Now we consider cases according to whether z is in the list $L(t - 1)$, the queue $q(t - 1)$ or the queue jumpers $j(t - 1)$. Suppose first that $z \in L(t-1)$. Then the rule for updating the queue in the algorithm gives $|F_z(t)| \geq \delta'_Q |F_z(t_z)|$. Next suppose that $z \in j(t - 1) \cup q(t - 1)$, and z first joined $j(t') \cup q(t')$ at some time $t' < t$. Since z did not join the queue at time $t' - 1$ we have $|F_z(t' - 1)| \geq \delta'_Q |F_z(t_z)|$. Also, between times t' and t we only embed vertices that are in the queue or jumping the queue, or otherwise we would have embedded z before x. Now $|Q(t) \cap X_z| \leq \delta_Q C n$, otherwise we abort the algorithm, and $|J(t) \cap X_z| \leq \sqrt{\delta_Q} n$ by Lemma 4.3(ii), so we embed at most $2\sqrt{\delta_Q} n$ vertices in V_z between times t' and t. Thus we have catalogued all possible ways in which the number of vertices free for z can decrease. It may decrease by a factor of $d_3^{2D} d_2$ when a new z-regime starts, and by a factor δ'_Q during a z-regime before z joins the queue. Also, if z joins the queue or jumps the queue it may decrease by at most $2\sqrt{\delta_Q} n$ in absolute size. Since z has at most $2D$ neighbours, and $|F_z(0)| = |V_z| > n$, we have $|F_z(t)| \geq (\delta'_Q d_3^{2D} d_2)^{2D} \delta'_Q |V_z| - 2\sqrt{\delta_Q} n > d_u |V_z|$. ∎

In the preceding proof we needed to track the ϵ-subscripts in great detail to be sure that they always fall in the range allowed by our hierarchy. From now on it will often suffice and be more convenient to use a crude upper bound of ϵ_* for any epsilon parameter. We summarise some useful estimates in the following lemma.

Lemma 4.14.

(i) *If* $\emptyset \neq S \in H(x)$ *then* $d_S(F(t)) = (1 \pm \epsilon_*) d_S(F(t - 1)) d_{Sx}(F(t - 1))$ *and* $|F_S(t)| = (1 \pm \epsilon_*) |F_{Sx}(t - 1)| / |F_x(t - 1)|$.

(ii) *If* $S \notin H(x)$ *then* $d_S(F(t)) = (1 \pm \epsilon_*) d_S(F(t - 1))$.

(iii) *If* $S \in H$ *then* $d(F_S(t)) = (1 \pm \epsilon_*) \prod_{T \subseteq S} d_T(F(t))$.

(iv) *If* $S' \subseteq S \in H$ *then*

$$\frac{|F_S(t)|}{|F_{S'}(t)||F_{S \setminus S'}(t)|} = \frac{d(F_S(t))}{d(F_{S'}(t)) d(F_{S \setminus S'}(t))} = (1 \pm 4\epsilon_*) \prod_{T : T \subseteq S, T \nsubseteq S', T \nsubseteq S \setminus S'} d_T(F(t)).$$

(v) *If $S' \subseteq S \in H$ then $|F_S(t)(P)| = (1 \pm \epsilon_*)|F_S(t)|/|F_{S'}(t)|$ for all but at most $\epsilon_*|F_{S'}(t)|$ sets $P \in F_{S'}(t)$.*

(vi) *Statements (iii-v) hold replacing $F_{S\le}(t)$ by $F_{S\le}(t)[\Gamma]$ for any $\epsilon_{12D,3}$-regular subcomplex Γ of $F_{S\le}(t)$, such that $d_T(\Gamma) \ge \epsilon_*^2$ when defined.*

Proof. The first formula in (i) is a weaker form of $(*_{4.1})$. For the second formula, suppose first that $S = v$ has size 1. Then $|F_v(t)| = d_v(F(t))|V_v| = (1 \pm \epsilon_*)d_{xv}(F(t-1))d_v(F(t-1))|V_v| = (1 \pm \epsilon_*)d_{xv}(F(t-1))|F_v(t-1)| = (1 \pm \epsilon_*)|F_{xv}(t-1)|/|F_x(t-1)|$. In the case when $S = uv$ has size 2 we have $d_{S'}(F(t)) = (1 \pm \epsilon_{12D,1})d_{S'}(F(t-1))d_{S'x}(F(t-1))$ for S' equal to u, v or uv. Since $\epsilon_{12D,1} \ll \epsilon_*$, the formula follows from the same calculations as in Lemma 4.12(i). This proves (i). For (ii), note that if $|S| = 1$ then $F_S(t) = F_S(t) \setminus y$ has size $|F_S(t-1)|$ or $|F_S(t-1)| - 1$. Also, if $|S| = 2, 3$ we have $F_S(t) = F_{S\le}(t-1)[F_{S<}(t)]_S$ by Lemma 3.11. Statement (ii) then follows from Lemma 2.3 if $|S| = 2$ or Lemma 4.8 if $|S| = 3$. For (iii) we apply Lemma 4.10 when $|S| = 3$ or the identity $d(F_S(t)) = \prod_{T \subseteq S} d_T(F(t))$ when $|S| \le 2$. Statement (iv) follows by definition of absolute density and (iii). For (v) we apply Lemma 4.12. For (vi) we apply regular restriction to see that $F_{S\le}(t)[\Gamma]$ is $\epsilon_{12D,3}$-regular and then the same proofs. ∎

Our next lemma concerns the definitions for marked edges in the algorithm.

Lemma 4.15.

(i) *For every triple $E \in H$ we have $|M_{E^t,E}(t)| < \theta'_{\nu'_{E^t}(t)}|F_{E^t}(t)|$, and in fact $|M_{E^t,E}(t)| \le \theta_{\nu'_{E^t}(t)}|F_{E^t}(t)|$ for $E \in U(x)$.*

(ii) *For every x and triple $E \in U(x)$ we have $|D_{x,E}(t-1)| < \theta_{\nu'_{E^t}(t)}|F_x(t-1)|$.*

Proof. Throughout we use the notation $\overline{E} = E^{t-1}$, $\nu = \nu'_{\overline{E}}(t-1)$, $\nu^* = \nu'_{E^t}(t)$.

(i) To verify the bound for $t = 0$ we use our assumption that (G, M) is $(\epsilon, \epsilon', d_2, \theta, d_3)$-super-regular. We take $I = (\{\emptyset\})$, when for any ν we have $G^{I_\nu} = G$ by Definition 3.15. Then condition (iii) in Definition 3.16 gives $|M_E| \le \theta|G_E|$. Since $E^0 = E$, $M_{E,E}(0) = M_E$ and $F_E(0) = G_E$ we have the required bound. Now suppose $t > 0$. When $E \in U(x)$ we have $|M_{E^t,E}(t)| \le \theta_{\nu^*}|F_{E^t}(t)|$ by definition, since the algorithm chooses $y = \phi(x) \notin D_{x,E}(t-1)$. Now suppose $E \notin U(x)$, and let $t' < t$ be the most recent time at which we embedded a vertex x' with $E \in U(x')$. Then $E^{t'} = E^t$, $\nu'_{E^t}(t') = \nu^*$, and $|M_{E^t,E}(t')| \le \theta_{\nu'_{E^t}(t')}|F_{E^t}(t')|$ by the previous case. For any $z \in E^t$, we can bound $|F_z(t)|$ using the same argument as that used at the end of the proof of Lemma 4.13. We do not embed any neighbour of z between time $t'+1$ and t, so the size of the free set for z can only decrease by a factor of δ'_Q and an absolute term of $2\sqrt{\delta_Q}n$. Since $d_z(F(t')) \ge d_u \gg \delta_Q$ we have $|F_z(t)| \ge \delta'_Q|F_z(t')| - 2\sqrt{\delta_Q}n \ge \frac{1}{2}\delta'_Q|F_z(t')|$. By Lemma 3.11, for every $\emptyset \ne S \subseteq E^t$, $F_S(t)$ is obtained from $F_S(t')$ by restricting to the 1-complex $((F_z(t) : z \in S), \{\emptyset\})$. If $|S| = 2, 3$ then regular restriction (Lemmas 2.3 and 4.8) gives $d_S(F(t)) = (1 \pm \epsilon_*)d_S(F(t'))$. Now $d(F_{E^t}(t)) = (1 \pm \epsilon_*)\prod_{S \subseteq E^t} d_S(F(t))$, by Lemma 4.14(iii), and similarly $d(F_{E^t}(t')) = (1 \pm \epsilon_*)\prod_{S \subseteq E^t} d_S(F(t'))$. This gives

$$\frac{|F_{E^t}(t)|}{|F_{E^t}(t')|} = (1 \pm 3\epsilon_*)\prod_{S \subseteq E^t} \frac{d_S(F(t))}{d_S(F(t'))} = (1 \pm 10\epsilon_*)\prod_{z \in E^t} \frac{d_z(F(t))}{d_z(F(t'))} > (\delta'_Q/2)^3/2.$$

Therefore $|M_{E^t,E}(t)| \le |M_{E^t,E}(t')| \le \theta_{\nu^*}|F_{E^t}(t')| < 2(\delta'_Q/2)^{-3}\theta_{\nu^*}|F_{E^t}(t)| < \theta'_{\nu^*}|F_{E^t}(t)|$.

(ii) First we consider the case $x \in E$. Then $E^t = \overline{E} \setminus x$ and $\nu < \nu^*$ by Lemma 4.3(vi). Also $F_{E^t}(t) = F_{\overline{E}}(t-1)(y)$ and $M_{E^t,E}(t) = M_{\overline{E},E}(t-1)(y)$ (see Lemma 3.14), so

$$D_{x,E}(t-1) = \{y \in F_x(t-1) : |M_{\overline{E},E}(t-1)(y)| > \theta_{\nu^*}|F_{\overline{E}}(t-1)(y)|\}.$$

If $\overline{E} = \{x\}$ has size 1 then $D_{x,E}(t-1) = M_{x,E}(t-1)$ by Lemma 4.2, so $|D_{x,E}(t-1)| < \theta'_\nu|F_x(t-1)| < \theta_{\nu^*}|F_x(t-1)|$ by (i). If $|\overline{E}| \geq 2$ then $|F_{E^t}(t)| = |F_{\overline{E}}(t-1)(y)| = (1 \pm \epsilon_*)|F_{\overline{E}}(t-1)|/|F_x(t-1)|$ by Lemma 4.14 when $y \notin E_x(t-1)$. Now

$$\sum_{y \in D_{x,E}(t-1)} |M_{\overline{E},E}(t-1)(y)| > \theta_{\nu^*} \sum_{y \in D_{x,E}(t-1)\setminus E_x(t-1)} |F_{\overline{E}}(t-1)(y)|$$
$$> (1-\epsilon_*)\theta_{\nu^*}(|D_{x,E}(t-1)| - \epsilon_*|F_x(t-1)|)|F_{\overline{E}}(t-1)|/|F_x(t-1)|.$$

We also have an upper bound

$$\sum_{y \in D_{x,E}(t-1)} |M_{\overline{E},E}(t-1)(y)| \leq \sum_{y \in F_x(t-1)} |M_{\overline{E},E}(t-1)(y)| = |M_{\overline{E},E}(t-1)| < \theta'_\nu|F_{\overline{E}}(t-1)|,$$

where the last inequality holds by (i). Therefore

$$\frac{|D_{x,E}(t-1)|}{|F_x(t-1)|} < \frac{\theta'_\nu}{(1-\epsilon_*)\theta_{\nu^*}} + \epsilon_* < \theta_{\nu^*}.$$

Now we consider the case when $x \notin E$. Then $E^t = E^{t-1} = \overline{E}$. Note that when $E \in U(x)$ we have $\overline{E} \cap VN_H(x) \neq \emptyset$, so $\nu^* > \nu$. Suppose first that $\overline{E} \in H(x)$. Then $F_{\overline{E}}(t) = F_{\overline{E}x}(t-1)(y)$ and $M_{\overline{E},E}(t) = M_{\overline{E},E}(t-1) \cap F_{\overline{E}}(t)$ (see Lemma 3.14), so

$$D_{x,E}(t-1) = \{y \in F_x(t-1) : |M_{\overline{E},E}(t-1) \cap F_{\overline{E}x}(t-1)(y)| > \theta_{\nu^*}|F_{\overline{E}x}(t-1)(y)|\}.$$

Similarly to the previous case, by Lemma 4.14 we have

$$\Sigma := \sum_{y \in D_{x,E}(t-1)} |M_{\overline{E},E}(t-1) \cap F_{\overline{E}x}(t-1)(y)| > \theta_{\nu^*} \sum_{y \in D_{x,E}(t-1)\setminus E_x(t-1)} |F_{\overline{E}x}(t-1)(y)|$$
$$> (1-\epsilon_*)\theta_{\nu^*}(|D_{x,E}(t-1)| - \epsilon_*|F_x(t-1)|)|F_{\overline{E}x}(t-1)|/|F_x(t-1)|.$$

We also have $\Sigma \leq \sum_{y \in F_x(t-1)} |M_{\overline{E},E}(t-1) \cap F_{\overline{E}x}(t-1)(y)|$. This sum counts all pairs (y, P) with $P \in M_{\overline{E},E}(t-1)$, $y \in F_x(t-1)$ and $Py \in F_{\overline{E}x}(t-1)$, so we can rewrite it as

$$\Sigma \leq \sum_{P \in M_{\overline{E},E}(t-1)} |F_{\overline{E}x}(t-1)(P)|.$$

By Lemma 4.14(v) we have $|F_{\overline{E}x}(t-1)(P)| = (1 \pm \epsilon_*)\frac{|F_{\overline{E}x}(t-1)|}{|F_{\overline{E}}(t-1)|}$ for all but at most $\epsilon_*|F_{\overline{E}}(t-1)|$ sets $P \in F_{\overline{E}}(t-1)$. Therefore

$$\Sigma \leq |M_{\overline{E},E}(t-1)|(1+\epsilon_*)\frac{|F_{\overline{E}x}(t-1)|}{|F_{\overline{E}}(t-1)|} + \epsilon_*|F_{\overline{E}}(t-1)||F_x(t-1)|.$$

Combining this with the lower bound on Σ we obtain

$$(1-\epsilon_*)\theta_{\nu^*}\left(\frac{|D_{x,E}(t-1)|}{|F_x(t-1)|} - \epsilon_*\right) < (1+\epsilon_*)\frac{|M_{\overline{E},E}(t-1)|}{|F_{\overline{E}}(t-1)|} + \epsilon_*\frac{|F_{\overline{E}}(t-1)||F_x(t-1)|}{|F_{\overline{E}x}(t-1)|}.$$

Now $|M_{\overline{E},E}(t-1)| < \theta'_v |F_{\overline{E}}(t-1)|$ by (i), and $\frac{|F_{\overline{E}}(t-1)||F_x(t-1)|}{|F_{\overline{E}x}(t-1)|} \leq 2d_u^{-1} \ll \epsilon_*^{-1}$ by Lemma 4.14(iv), so

$$\frac{|D_{x,E}(t-1)|}{|F_x(t-1)|} < \frac{(1+\epsilon_*)\theta'_v + \sqrt{\epsilon_*}}{(1-\epsilon_*)\theta_{v^*}} + \epsilon_* < \theta_{v^*}.$$

It remains to consider the case when $x \notin E$ and $\overline{E} = E' \notin H(x)$. Since $\overline{E} \cap VN_H(x) \neq \emptyset$ we have $|\overline{E}| \geq 2$. Then $F_{\overline{E}^{\leq}}(t) = F_{\overline{E}^{\leq}}(t-1)[F_{\overline{E}^{<}}(t)]$ by Lemma 3.11 and $M_{\overline{E},E}(t) = M_{\overline{E},E}(t-1) \cap F_{\overline{E}}(t)$, so

$$D_{x,E}(t-1) = \left\{ y \in F_x(t-1) : \frac{|M_{\overline{E},E}(t-1) \cap F_{\overline{E}^{\leq}}(t-1)[F_{\overline{E}^{<}}(t)]|}{|F_{\overline{E}^{\leq}}(t-1)[F_{\overline{E}^{<}}(t)]|} > \theta_{v^*} \right\}.$$

Let I be the subcomplex of $\overline{E}^{<}$ consisting of all $S \subseteq \overline{E}$ with $S \in H(x)$. Then $P \in F_{\overline{E}^{\leq}}(t-1)[F_{\overline{E}^{<}}(t)]$ if and only if $P \in F_{\overline{E}^{\leq}}(t-1)$ and $P_S y \in F_{Sx}(t-1)$ for all $S \in I$. When we choose $y \notin E_x(t-1)$, Lemma 4.14 gives $d_S(F(t)) = (1 \pm \epsilon_*) d_S(F(t-1)) d_{Sx}(F(t-1))$ for $\emptyset \neq S \in I$ by (i), $d_S(F(t)) = (1 \pm \epsilon_*) d_S(F(t-1))$ for $S \subseteq \overline{E}$ with $S \notin I$ by (ii), and

$$d(F_{\overline{E}^{\leq}}(t-1)[F_{\overline{E}^{<}}(t)]) = (1 \pm \epsilon_*) \prod_{S \subseteq \overline{E}} d_S(F(t-1)) \prod_{\emptyset \neq S \in I} d_{Sx}(F(t-1))$$

by (vi), so

$$|F_{\overline{E}^{\leq}}(t-1)[F_{\overline{E}^{<}}(t)]| = (1 \pm 20\epsilon_*) |F_{\overline{E}^{\leq}}(t-1)| \prod_{\emptyset \neq S \in I} d_{Sx}(F(t-1)).$$

As in the previous cases we have

$$\begin{aligned}
\Sigma &:= \sum_{y \in D_{x,E}(t-1)} |M_{\overline{E},E}(t-1) \cap F_{\overline{E}^{\leq}}(t-1)[F_{\overline{E}^{<}}(t)]| \\
&> \theta_{v^*} \sum_{y \in D_{x,E}(t-1) \setminus E_x(t-1)} |F_{\overline{E}^{\leq}}(t-1)[F_{\overline{E}^{<}}(t)]| \\
&> (1 - 20\epsilon_*)\theta_{v^*} (|D_{x,E}(t-1)| - \epsilon_* |F_x(t-1)|)\, |F_{\overline{E}^{\leq}}(t-1)| \prod_{\emptyset \neq S \in I} d_{Sx}(F(t-1)).
\end{aligned}$$

For any $P \in F_{\overline{E}}(t-1)$, let $F_{P,I}$ be the set of $y \in F_x(t-1)$ such that $P_S y \in F_{Sx}(t-1)$ for all $S \in I$. Let B_I be the set of $P \in F_{\overline{E}}(t-1)$ such that we do not have

$$|F_{P,I}| = (1 \pm \epsilon_*) |F_x(t-1)| \prod_{\emptyset \neq S \in I} d_{Sx}(F(t-1)).$$

Then $|B_I| \leq \epsilon_* |F_{\overline{E}}(t-1)|$ by Lemma 4.11. Now $\Sigma \leq \sum_{y \in F_x(t-1)} |M_{\overline{E},E}(t-1) \cap F_{\overline{E}^{\leq}}(t-1)[F_{\overline{E}^{<}}(t)]|$, which counts all pairs (y, P) with $P \in M_{\overline{E},E}(t-1)$ and $y \in F_{P,I}$, so

$$\Sigma \leq |M_{\overline{E},E}(t-1)| (1 \pm \epsilon_*) |F_x(t-1)| \prod_{\emptyset \neq S \in I} d_{Sx}(F(t-1)) + \epsilon_* |F_{\overline{E}}(t-1)||F_x(t-1)|.$$

Combining this with the lower bound on Σ we obtain

$$(1 - 20\epsilon_*)\theta_{v^*} \left(\frac{|D_{x,E}(t-1)|}{|F_x(t-1)|} - \epsilon_* \right) < (1 + \epsilon_*) \frac{|M_{\overline{E},E}(t-1)|}{|F_{\overline{E}}(t-1)|} + \epsilon_* \prod_{\emptyset \neq S \in I} d_{Sx}(F(t-1))^{-1}.$$

Now $|M_{\overline{E},E}(t-1)| < \theta'_v|F_{\overline{E}}(t-1)|$ by (i) and all densities are at least $d_u \gg \epsilon_*$, so again we have

$$\frac{|D_{x,E}(t-1)|}{|F_x(t-1)|} < \frac{(1+\epsilon_*)\theta'_v + \sqrt{\epsilon_*}}{(1-20\epsilon_*)\theta_{v^*}} + \epsilon_* < \theta_{v^*}.$$

■

The following corollary is now immediate from Lemmas 4.13 and 4.15. Recall that $OK_x(t-1)$ is obtained from $F_x(t-1)$ by deleting $E_x(t-1)$ and $D_{x,E}(t-1)$ for $E \in U(x)$, and note that since H has maximum degree at most D we have $|U(x)| \le 2D^2$.

Corollary 4.16. $|OK_x(t-1)| > (1-\theta_*)|F_x(t-1)|$.

4.5. The Initial Phase

This subsection concerns the initial phase of the algorithm, during which we embed the neighbourhood N of the buffer B. There are several issues that make the analysis of this phase significantly more complicated than that of the graph blow-up lemma (which was given in Lemma 2.5). As we mentioned earlier, the buffer B is larger than before, so we embed many more vertices during the initial phase, and the queue may open. One potential problem is that there may be some vertex v and class V_i so that $G(v) \cap V_i$ is used excessively by the embedding – this is a concern, since $d_u \ll \delta_B$. The first lemma in this subsection shows that with high probability this does not happen.

Our goal is to show that for any vertex $x \in B$ there will be many *available* vertices $v \in V_x$ such that we embed $H(x)$ in $(G \setminus M)(v)$ during the initial phase. Then if v is not used before the conclusion of the algorithm we will be able to embed x as $\phi(x) = v$. We need to ensure that for every neighbour z of x and for every triple E of H containing a neighbour of z, the choice of image for z is good, in that the subcomplex of the free sets for $E^{\le}$ that is consistent with mapping x to v is suitably regular and does not have too many marked edges. As in Lemma 2.5, we aim to give a lower bound on the probability of this event, conditional on the previous embedding. The third lemma in this subsection achieves this.

The marked edges also add a complication to the conclusion of the algorithm, in which we need to verify Hall's condition for a system of distinct representatives of the available images for the unembedded buffer vertices. We need to show that for that any $W \subseteq V_x$ that is not too small, the probability that W does not contain a vertex available for x is quite small. This is achieved by the second lemma in this subsection. We present it before the lemma on mapping x to v because its proof is similar in spirit and somewhat simpler.

First we recall the key properties of the selection rule during the initial phase. Although the queue may become non-empty, jumping ensures that we embed all vertex neighbourhoods $VN_H(x)$, $x \in B$ at consecutive times, and before x or any other vertices at distance at most 4 from x. Then Lemma 3.12 implies that if we start embedding $VN_H(x)$ just after some time T_0 then $F_z(T_0) = V_z(T_0)$ consists of all vertices in V_z that have not yet been used by the embedding, for every z at distance at most 3 from x. We also recall that $|B \cap V_z| = \delta_B|V_z|$, $|N \cap V_z| < \sqrt{\delta_B}|V_z|$, $|Q(T_0) \cap V_z| \le \delta_Q|V_z|$ and $|J(T_0) \cap V_z| \le \sqrt{\delta_Q}|V_z|$ by Lemma 4.3(ii). Thus for any z at distance at most 3 from x we have

$$|F_z(T_0)| = |V_z(T_0)| > (1 - 2\sqrt{\delta_B})|V_z|. \tag{5}$$

We need the following supermartingale formulation of the Azuma-Hoeffding inequality, which can be easily derived from the martingale formulation quoted later as Theorem 5.16.

Theorem 4.17. *Suppose $Z_0, \dots, Z_n$ is a supermartingale, i.e. a sequence of random variables satisfying $\mathbb{E}(Z_{i+1}|Z_0, \dots, Z_i) \le Z_i$, and that $|Z_i - Z_{i-1}| \le c_i$, $1 \le i \le n$, for some constants c_i. Then for any $t \ge 0$,*

$$\mathbb{P}(Z_n - Z_0 \ge t) \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right).$$

Let T_I be the time at which we embed the last vertex of N, ending the initial phase. Then $T_I \le \sum_{i=1}^r 2\sqrt{\delta_B}|V_i| < \delta_B^{1/3} n$ by (5). Our first lemma shows that there are many available vertices in all neighbourhoods at time T_I. Note that by super-regularity the assumption $|G(v)_j| \ge d_u|V_j|$ in the lemma holds for every j such that $G_{i(v)j}$ is defined.

Lemma 4.18. *With high probability, for every vertex $v \in G$ and $1 \le j \le r$ with $|G(v)_j| \ge d_u|V_j|$ we have*

$$|G(v)_j \cap V_j(T_I)| > \left(1 - \delta_B^{1/3}\right)|G(v)_j|.$$

Proof. Suppose $|G(v)_j| \ge d_u|V_j|$. The ratio $|G(v)_j \cap V_j(t)|/|G(v)_j|$ only decreases when we embed a vertex in $G(v)_j$. We separate the analysis according to two effects. One effect is that we embed a vertex $z \in N \cap X_j$ to an image $\phi(z) \in G(v)_j$, where $G(v)_j$ is not too large a fraction of the free images for z. The other effect is that we we embed a vertex $z \in N \cap X_j$ to an image $\phi(z) \in G(v)_j$, and the embedding of some neighbour w of z previously caused the fraction of $G(v)_j$ in the free images for z to increase significantly. To analyse these effects we write T_z for the time at which a vertex z is embedded and define the following sets:

- Let $\Lambda_j(t)$ be the set of embedded vertices $z \in N \cap X_j$ such that $\phi(z) \in G(v)_j$ and
$$|G(v)_j \cap F_z(T_z - 1)|/|F_z(T_z - 1)| < 2^{2D}|G(v)_j|/|V_j|.$$
- Let $\Pi_{\ell,j}(t)$ be the set of embedded vertices $w \in N \cap X_\ell$ such that
$$|G(v)_j \cap F_z(T_w)|/|F_z(T_w)| \ge 2|G(v)_j \cap F_z(T_w - 1)|/|F_z(T_w - 1)| \ge d_u^2$$
for some $z \in VN_H(w) \cap N \cap X_j$.

We claim that any vertex embedded in $G(v)_j$ up to time T_I is either in the queue, or in $\Lambda_j(T_I)$, or an H-neighbour of some $w \in \Pi_{\ell,j}(T_I)$ for some ℓ. To see this, suppose z is embedded in $G(v)_j$ and is neither in the queue nor in $\Lambda_j(T_I)$. Since $z \in N$ we have $z \in VN_H(x)$ for some $x \in B$. Suppose we start embedding $VN_H(x)$ just after time T_0. Then $F_z(T_0) = V_j(T_0)$ has size at least $(1 - 2\sqrt{\delta_B})|V_j|$ by (5), so $|G(v)_j \cap F_z(T_0)|/|F_z(T_0)| \le |G(v)_j|/(1 - 2\sqrt{\delta_B})|V_j|$. We also have $|G(v)_j \cap F_z(T_z - 1)|/|F_z(T_z - 1)| \ge 2^{2D}|G(v)_j|/|V_j|$, since $z \notin \Lambda_j(T_I)$. Since $|VN_H(x) \setminus z| \le 2D - 1$, there must be some g, $1 \le g \le T_z - T_0$ so that at time $T_0 + g$ we embed some $w \in VN_H(z)$ and get $|G(v)_j \cap F_z(T_w)|/|F_z(T_w)| \ge 2|G(v)_j \cap F_z(T_w - 1)|/|F_z(T_w - 1)|$. Here we recall that we embed $VN_H(x)$ consecutively and so $w \in VN_H(x) \subseteq N$. Now for any g', $1 \le g' < T_z - T_0$, since $|F_z(T_z - 1)| \ge d_u|V_j| \ge d_u|F_z(T_0 + g')|$ we have

$$\frac{|G(v)_j \cap F_z(T_0 + g')|}{|F_z(T_0 + g')|} \ge \frac{|G(v)_j \cap F_z(T_z - 1)|}{d_u^{-1}|F_z(T_z - 1)|} \ge 2^{2D} d_u |G(v)_j|/|V_j| > 2^{2D} d_u^2 > d_u^2.$$

Therefore $w \in \Pi_{\ell,j}(T_I)$, where $\ell = i(w)$, which proves the claim. Since any w has at most $2D$ neighbours we deduce that

$$|G(v)_j \cap V_j(T_I)| \ge |G(v)_j| - |Q(T_I) \cap V_j| - |\Lambda_j(T_I)| - 2D \sum_{\ell=1}^{r} |\Pi_{\ell,j}(T_I)|. \tag{6}$$

Consider $Z_j(t) = |\Lambda_j(t)| - 2^{2D+1} d_j(G(v))|V_j \setminus V_j(t)|$. We claim that $Z_j(0), \ldots, Z_j(T_I)$ is a supermartingale. To see this, suppose we embed some vertex $z \in X_j$ at time T_z. We can assume that $z \in N$ and $|G(v)_j \cap F_z(T_z - 1)|/|F_z(T_z - 1)| < 2^{2D} d_j(G(v))$, or otherwise $z \notin \Lambda_j(T_z)$ by definition, so $|\Lambda_j(T_z)| = |\Lambda_j(T_z - 1)|$ and $Z_j(T_z) < Z_j(T_z - 1)$. Since $\phi(z)$ is chosen randomly in $OK_z(T_z - 1) \subseteq F_z(T_z - 1)$ of size at least $|F_z(T_z - 1)|/2$ (by Corollary 4.16), we have

$$\begin{aligned}\mathbb{E}[|\Lambda_j(T_z)| - |\Lambda_j(T_z - 1)|] &= \mathbb{P}(\phi(z) \in G(v)_j) = \frac{|G(v)_j \cap OK_z(T_z - 1)|}{|OK_z(T_z - 1)|} \\ &< \frac{2|G(v)_j \cap F_z(T_z - 1)|}{|F_z(T_z - 1)|} < 2^{2D+1} d_j(G(v)).\end{aligned}$$

We also have $|V_j \setminus V_j(T_z)| = |V_j \setminus V_j(T_z - 1)| + 1$, so the decrease in the second term of $Z_j(t)$ more than compensates for the increase in the first, i.e. $\mathbb{E}[Z_j(T_z) - Z_j(T_z - 1)] < 0$. Thus we have a supermartingale. Since $|Z_j(t) - Z_j(t-1)| \le 1$, $|V_j \setminus V_j(T_I)| \le 2\sqrt{\delta_B}|V_j|$ by (5), and $T_I < \delta_B^{1/3} n$, Theorem 4.17 gives

$$\begin{aligned}\mathbb{P}[|\Lambda_j(T_I)| > 2^{2D+3} d_j(G(v))\sqrt{\delta_B}|V_j|] &\le \mathbb{P}[Z_j(m) > 2^{2D+2} d_j(G(v))\sqrt{\delta_B}|V_j|] \\ &< 2\exp[-(2^{2D} d_j(G(v))\sqrt{\delta_B}|V_j|)^2/2T_I] < e^{-\sqrt{n}}, \quad \text{(say, for sufficiently large } n).\end{aligned}$$

Next consider $Y_{\ell,j}(t) = |\Pi_{\ell,j}(t)| - \epsilon_*|\Pi'_{\ell,j}(t)|$, where $\Pi'_{\ell,j}(t)$ consists of all vertices in X_ℓ with at least one H-neighbour in X_j that have been embedded at time t. We claim that $Y_{\ell,j}(0), \ldots, Y_{\ell,j}(T_I)$ is a supermartingale. To see this, suppose we embed some vertex $w \in X_\ell$ at time T_w. Consider $z \in VN_H(w) \cap N \cap X_j$: we can assume this set is non-empty, otherwise $w \notin \Pi_{\ell,j}(T_w) \cup \Pi'_{\ell,j}(t)$, so $Y_{\ell,j}(T_w) = Y_{\ell,j}(T_w - 1)$. We can also assume that $w \in N$ and $|G(v)_j \cap F_z(T_w)|/|F_z(T_w)| \ge 2|G(v)_j \cap F_z(T_w - 1)|/|F_z(T_w - 1)| \ge d_u^2$ for some $z \in VN_H(w) \cap N \cap X_j$, otherwise $w \notin \Pi_{\ell,j}(T_w)$ by definition, so $|\Pi_{\ell,j}(T_w)| = |\Pi_{\ell,j}(T_w - 1)|$ and $Y_{\ell,j}(T_w) < Y_{\ell,j}(T_w - 1)$.

By Lemmas 4.13 and Lemma 2.3, $F_{wz}(T_w - 1)[G(v)_j \cap F_z(T_w - 1)]$ is $\epsilon_{12D,2}$-regular. Applying Lemma 2.2, we see that there are at most $2\epsilon_{12D,2}|F_w(T_w - 1)|$ 'exceptional' vertices $y \in F_w(T_w - 1)$ such that embedding $\phi(w) = y$ will not satisfy

$$\begin{aligned}|G(v)_j \cap F_z(T_w)| &= |F_{zw}(T_w - 1)(y) \cap G(v)_j \cap F_z(T_w - 1)| \\ &= (1 \pm \epsilon_*) d_{zw}(F(T_w - 1))|G(v)_j \cap F_z(T_w - 1)|.\end{aligned}$$

On the other hand, the algorithm chooses $\phi(w) = y$ to satisfy $(*_{4.1})$, so $|F_z(T_w)| = (1 \pm \epsilon_*) d_{zw}(F(T_w - 1))|F_z(T_w - 1)|$. Thus we have $|G(v)_j \cap F_z(T_w)|/|F_z(T_w)| < 2|G(v)_z \cap F_z(T_w -$

$1)|/|F_z(T_w - 1)|$, unless we choose an exceptional vertex y. But y is chosen uniformly at random from $|OK_w(T_w - 1)| \geq (1 - \theta_*)|F_w(T_w - 1)|$ possibilities (by Corollary 4.16), so y is exceptional with probability at most $3\epsilon_{12D,2}$. Therefore $\mathbb{E}[|\Pi_{\ell,j}(T_w)| - |\Pi_{\ell,j}(T_w - 1)|] < 3\epsilon_{12D,2}$.

We also have $|\Pi'_{\ell,j}(T_w)| = |\Pi'_{\ell,j}(T_w - 1)| + 1$, so the decrease in the second term of $Y_{\ell,j}(t)$ more than compensates for the increase in the first, i.e. $\mathbb{E}[Y_{\ell,j}(T_w) - Y_{\ell,j}(T_w - 1)] < 0$. Thus we have a supermartingale. We also have $|Y_{\ell,j}(t) - Y_{\ell,j}(t-1)| \leq 1$. Also, $|V_j \setminus V_j(T_I)| \leq 2\sqrt{\delta_B}|V_j|$ by (5), so $|\Pi'_{\ell,j}(T_i)| \leq 2D\sqrt{\delta_B}|V_j|$, since H has maximum degree D. Then Theorem 4.17 gives

$$\mathbb{P}[|\Pi_{\ell,j}(T_I)| > 2d_u\sqrt{\delta_B}|V_j|] < \mathbb{P}[Y_{\ell,j}(t) > d_u\sqrt{\delta_B}|V_j|]$$
$$< 2\exp[-(d_u\sqrt{\delta_B}|V_\ell|)^2/2T_I] < e^{-\sqrt{n}}, \quad \text{(say, for sufficiently large } n\text{)}.$$

Taking a union bound over $v \in V$ and $1 \leq j, \ell \leq r$, with high probability we have $|\Lambda_j(T_I)| \leq 2^{2D+3}d_j(G(v))\sqrt{\delta_B}|V_j| = 2^{2D+3}\sqrt{\delta_B}|G(v)_j|$ and $|\Pi_{\ell,j}(T_I)| \leq 2d_u\sqrt{\delta_B}|V_j| \leq 2\sqrt{\delta_B}|G(v)_j|$. Therefore (6) gives

$$|G(v)_j \cap V_j(T_I)| \geq |G(v)_j| - |Q(T_I) \cap V_j| - |\Lambda_j(T_I)| - 2D\sum_{\ell=1}^{r}|\Pi_{\ell,j}(T_I)|.$$
$$\geq |G(v)_j| - \delta_Q|V_j| - 2^{2D+3}\sqrt{\delta_B}|G(v)_j| - 4Dr\sqrt{\delta_B}|G(v)_j| > (1 - \delta_B^{1/3})|G(v)_j|,$$

since $|G(v)_j| \geq d_u|V_j|$ and $\epsilon_* \ll \delta_Q \ll d_u \ll \delta_B \ll 1/r, 1/D$. ▪

For the remainder of this subsection we fix a vertex $x \in B$ and write $VN_H(x) = \{z_1, \ldots, z_g\}$, with vertices listed in the order that they are embedded. Since H has maximum degree D we have $g \leq 2D$. We let T_j be the time at which z_j is embedded. By the selection rule, $VN_H(x)$ jumps the queue and is embedded at consecutive times: $T_{j+1} = T_j + 1$ for $1 \leq j \leq g-1$. For convenience we also define $T_0 = T_1 - 1$. Note that since H is an r-partite complex, no vertex of $VN_H(x)$ lies in X_x. The selection rule also ensures that at time T_0 no vertices at distance at most 4 from x have been embedded, so for any z with distance at most 3 from x we have $F_z(T_0) = V_z(T_0)$. (The need for this property in Lemma 4.21 explains why we needed to choose the buffer vertices at mutual distance at least 9.)

Remark 4.19. Note that the argument of the Lemma 4.18 can be applied replacing the sets $G(v)_j$ by any sufficiently large subsets of G_j, provided that they are sufficiently few in number to use a union bound. For example, we can define a complex $G\langle S\rangle = \cap_{v \in S} G(v)$ for any $S \subseteq V$, and show that with high probability, for every $S \subseteq V$ with $|S| \leq 2D$ and $1 \leq j \leq r$ with $|G\langle S\rangle_j| \geq d_u|V_j|$ we have $|G\langle S\rangle_j \cap V_j(T_I)| > (1 - \delta_B^{1/3})|G\langle S\rangle_j|$. It follows that with high probability every $z \in VN_H(x)$ would have many available vertices in $G(v)_z$ throughout the initial phase, even if we did not use queue jumping in the selection rule. However, the analysis is simpler if we do use queue jumping, and then we only need the version of Lemma 4.18 stated above.

Recall that the available set A_x is obtained from $F_x(t_x^N)$ by removing all sets $M_{x,E}(t_x^N)$ for triples E containing x. Here $t_x^N = T_g$. If x is unembedded at the conclusion of the algorithm at time T we will embed x in $A'_x = A_x \cap V_x(T)$. Our second lemma shows that for that any $W \subseteq V_x$ that is not too small, the probability that W does not contain a vertex available for x is quite small.

Lemma 4.20. *For any $W \subseteq V_x$ with $|W| > \epsilon_*|V_x|$, conditional on any embedding of the vertices $\{s(u) : u < T_1\}$ that does not use any vertex of W, we have $\mathbb{P}[A_x \cap W = \emptyset] < \theta_*$.*

Proof. We apply similar arguments to those we are using for the entire embedding, defining variants of various structures that incorporate restriction to W. Suppose $1 \le j \le g$ and that we are considering the embedding of z_j. We interpret quantities at time T_j with the embedding $\phi(z_j) = y$, for some as yet unspecified $y \in F_{z_j}(T_j - 1)$. Write $W_j = W \cap F_x(T_j)$. At time T_0 we have $W \subseteq F_x(T_0) = V_x(T_0)$, so $W_0 = F_x(T_0) \cap W = W$. Since $z_j \in H(x)$ we have $F_x(T_j) = F_{xz_j}(T_j - 1)(y)$, so

$$W_j = W_{j-1} \cap F_{xz_j}(T_j - 1)(y).$$

For convenient notation we will use $[W_j]$ to denote restriction of hypergraphs and complexes to W_j, in that for $x \in S \in H$ we write $F_{S^\le}(T_j)[W_j]$ for $F_{S^\le}(T_j)[(W_j, \{\emptyset\})]$ and $F_S(T_j)[W_j]$ for $F_{S^\le}(T_j)[(W_j, \{\emptyset\})]_S$.

We define *exceptional* sets $E^W_{z_j}(T_j - 1) \subseteq F_{z_j}(T_j - 1)$ by the property that y is in $F_{z_j}(T_j - 1) \setminus E^W_{z_j}(T_j - 1)$ if and only if

$$|W_j| = (1 \pm \epsilon_{\nu'_x(T_j),1})|W||F_x(T_j)|/|F_x(T_0)|. \qquad (*_{4.20})$$

Thus if we embed z_j to $y \notin E^W_{z_j}(T_j - 1)$, then the intersection of the free set for x with W is roughly what would be 'expected'. Next, to control marked edges, we define *dangerous* vertices similarly to before, except that here we incorporate the restriction to W_j. For any triple E containing x we define

$$D^W_{z_j,E}(T_j - 1) = \{y \in F_{z_j}(T_j - 1) : |M_{E^{T_j},E}(T_j)[W_j]| > \theta_{\nu'_{E^{T_j}}(T_j)}|F_{E^{T_j}}(T_j)[W_j]|\}.$$

Now we define events as follows:

- $A_{1,j}, j \ge 0$ is the event that property $(*_{4.20})$ above holds.
- $A_{2,j}, j \ge 0$ is the event that for every triple E containing x we have
$$|M_{E^{T_j},E}(T_j)[W_j]| \le \theta_{\nu'_{E^{T_j}}(T_j)}|F_{E^{T_j}}(T_j)[W_j]|.$$
- $A_{3,j}, j \ge 1$ is the event that $y = \phi(z_j)$ is chosen in $OK^W_{z_j}(T_j - 1)$, defined to be the subset of $F_{z_j}(T_j - 1)$ obtained by deleting the sets $E^W_{z_j}(T_j - 1)$ and $D^W_{z_j,E}(T_j - 1)$ for E containing x. For convenient notation we also define $A_{3,0}$ to be the event that holds with probability 1.

We divide the remainder of the proof into a series of claims.

Claim A. The events $A_{1,0}$, $A_{2,0}$ and $A_{3,0}$ hold.

Proof. As noted above, we have $W_0 = W$, so $A_{1,0}$ holds. Also, $A_{3,0}$ holds by definition, so it remains to show $A_{2,0}$. Consider any triple $E = xzz'$ containing x. Recall that all vertices at distance at most 4 from x are unembedded. Then $E^{T_0} = E$, and by Lemma 3.12, $F_{E^\le}(T_0)$ is the restriction of $G_{E^\le}$ to the 1-complex $((F_u(T_0) : u \in E), \{\emptyset\})$, so $F_{E^\le}(T_0)[W]$ is the restriction of $G_{E^\leq}$ to $(W, F_z(T_0), F_{z'}(T_0), \{\emptyset\})$. Since G is ϵ-regular, $F_{E^\leq}(T_0)[W]$ is ϵ'-regular by regular restriction and $d_S(F_{E^\leq}(T_0)[W]) = (1 \pm \epsilon')d_S(G)$ for $S \subseteq E$, $|S| \geq 2$. Then

$$|F_E(T_0)[W]| = (1 \pm 20\epsilon')|W||F_z(T_0)||F_{z'}(T_0)| \prod_{S \subseteq E, |S| \geq 2} d_S(G)$$

by Lemma 4.10. Now

$$|M_{E,E}(T_0)[W]| = \sum_{v \in W} |M_{E,E}(T_0)(v)| \leq \sum_{v \in W} |M_E(v)| \leq \theta \sum_{v \in W} |G_E(v)|$$

by condition (ii) of super-regularity (Definition 3.16). This condition also says for any $v \in G_x$ that $G_E(v)$ is ϵ'-regular and $d_S(G(v)) = (1 \pm \epsilon')d_S(G)d_{Sx}(G)$ for $\emptyset \neq S \subseteq E \setminus x$, so

$$|G_E(v)| = d_{zz'}(G(v))|G(v)_z||G(v)_{z'}| = (1 \pm 5\epsilon')|V_z||V_{z'}| \prod_{\emptyset \neq S \subseteq zz'} d_S(G)d_{Sx}(G).$$

Here we note that $d_z(G) = d_{z'}(G) = 1$, as our hypotheses for Theorem 4.1 include the assumption $G_i = V_i$ for $1 \leq i \leq r$. Equation (5) gives $|F_z(T_0)| > (1 - 2\sqrt{\delta_B})|V_z(T_0)|$ and $|F_{z'}(T_0)| > (1 - 2\sqrt{\delta_B})|V_{z'}(T_0)|$, so

$$\frac{|M_{E,E}(T_0)[W]|}{|F_E(T_0)[W]|} < \frac{(1 + 5\epsilon')\theta|V_z||V_{z'}|}{(1 - 20\epsilon')|F_z(T_0)||F_{z'}(T_0)|} < (1 + 30\epsilon')(1 - 2\sqrt{\delta_B})^{-2}\theta < \theta_0.$$

Claim B. If $A_{3,j}$ holds then $A_{1,j}$ and $A_{2,j}$ hold.

Proof. This follows directly from the definitions: if $y \notin E^W_{z_j}(T_j - 1)$ then $(*_{4.20})$ holds, and if $y \notin D^W_{z_j,E}(T_j - 1)$ then $|M_{E^{T_j},E}(T_j)[W_j]| \leq \theta_{\nu'_{E^{T_j}}(T_j)}|F_{E^{T_j}}(T_j)[W_j]|$.

Claim C. If $A_{1,j-1}$ holds then $|E^W_{z_j}(T_j - 1) \setminus E_{z_j}(T_j - 1)| < \epsilon_*|F_{z_j}(T_j - 1)|$.

Proof. By Lemma 4.13, $F_{xz_j}(T_j - 1)$ is $\epsilon_{\nu'_{xz_j}(T_j-1),1}$-regular, so by Lemma 2.3, $F_{xz_j}(T_j - 1)[W_{j-1}]$ is $\epsilon_{\nu'_{xz_j}(T_j-1),2}$-regular. Then by Lemma 2.2, for all but at most $\epsilon_{\nu'_{xz_j}(T_j-1),2}|F_{z_j}(T_j-1)|$ vertices $y \in F_{z_j}(T_j-1)$ we have $|W_j| = |W_{j-1} \cap F_{xz_j}(T_j-1)(y)| = (1 \pm \epsilon_{\nu'_{xz_j}(T_j-1),2})d_{xz_j}(F(T_j - 1))|W_{j-1}|$. Since $A_{1,j-1}$ holds, $|W_{j-1}| = (1 \pm \epsilon_{\nu'_x(T_j-1),1})|W||F_x(T_j - 1)|/|F_x(T_0)|$. Also, by $(*_{4.1})$, for $y \notin E_{z_j}(T_j - 1)$ we have $|F_x(T_j)| = (1 \pm \epsilon_{\nu'_x(T_j-1),0})d_{xz_j}(F(T_j - 1))|F_x(T_j - 1)|$. Now $z_j \in VN_H(x)$, so $\nu_x(T_j) = \nu_x(T_j - 1) + 1$, and $\nu'_x(T_j) > \max\{\nu'_x(T_j - 1), \nu'_{xz}(T_j - 1)\}$. Combining all estimates, for all but at most $\epsilon_*|F_{z_j}(T_j-1)|$ vertices $y \in F_{z_j}(T_j-1) \setminus E_{z_j}(T_j-1)$ we have

$$\begin{aligned}
|W_j| &= (1 \pm \epsilon_{\nu'_{xz_j}(T_j-1),2})d_{xz_j}(F(T_j - 1))|W_{j-1}| \\
&= (1 \pm \epsilon_{\nu'_{xz_j}(T_j-1),2})(1 \pm \epsilon_{\nu'_x(T_j-1),1})d_{xz_j}(F(T_j - 1))|W||F_x(T_j - 1)|/|F_x(T_0)| \\
&= (1 \pm \epsilon_{\nu'_x(T_j),1})|W||F_x(T_j)|/|F_x(T_0)|, \qquad \text{i.e. } (*_{4.20}).
\end{aligned}$$

Claim D. If $A_{1,j-1}$ and $A_{2,j-1}$ hold then for any E containing x we have

$$\left|D^W_{z_j,E}(T_j-1)\right| < \theta_{\nu'_{E^{T_j}}(T_j)}|F_{z_j}(T_j-1)|.$$

Proof. Denote $\overline{E} = E^{T_j-1}$, $\nu = \nu'_{\overline{E}}(T_j-1)$ and $\nu^* = \nu'_{E^{T_j}}(T_j)$. Suppose $A_{1,j-1}$ and $A_{2,j-1}$ hold. By $A_{1,j-1}$ we have $|W_{j-1}| > \frac{1}{2}d_u|W| > \epsilon_*^2|V_x|$. Consider any triple E containing x. We bound $D^W_{z_j,E}(T_j-1)$ with a similar argument to that used in Lemma 4.15.

Case D.1. First consider the case $z_j \in E$. Then $E^{T_j} = \overline{E} \setminus z_j$ and $\nu^* > \nu$. Since $W_j = W_{j-1} \cap F_{xz_j}(T_j-1)(y)$, Lemma 3.10 gives $F_{E^{T_j}}(T_j)[W_j] = F_{\overline{E}}(T_j-1)(y)[W_j] = F_{\overline{E}}(T_j-1)[W_{j-1}](y)$. Similarly, $M_{E^{T_j},E}(T_j)[W_j] = M_{\overline{E},E}(T_j-1)(y)[W_j] = M_{\overline{E},E}(T_j-1)[W_{j-1}](y)$, so

$$D^W_{z_j,E}(T_j-1) = \{y \in F_{z_j}(T_j-1) : |M_{\overline{E},E}(T_j-1)[W_{j-1}](y)| > \theta_{\nu^*}|F_{\overline{E}}(T_j-1)[W_{j-1}](y)|\}.$$

Let B'_{z_j} be the set of $y \in F_{z_j}(T_j-1) \setminus E_{z_j}(T_j-1)$ such that we do not have

$$|F_{\overline{E}}(T_j-1)[W_{j-1}](y)| = (1 \pm \epsilon_*)|F_{\overline{E}}(T_j-1)[W_{j-1}]|/|F_{z_j}(T_j-1)|.$$

Applying Lemma 4.14(vi) with $\Gamma = (W_{j-1}, \{\emptyset\})$ gives $|B'_{z_j}| < \epsilon_*|F_{z_j}(T_j-1)|$. (Note that we need $y \notin E_{z_j}(T_j-1)$ as this is implicitly assumed to apply Lemma 4.14.) Let $B_{z_j} = B'_{z_j} \cup E_{z_j}(T_j-1)$. Then $|B_{z_j}| < 2\epsilon_*|F_{z_j}(T_j-1)|$ by Lemma 4.13. Now

$$\begin{aligned}\sum_{y\in D^W_{z_j,E}(T_j-1)} |M_{\overline{E},E}(T_j-1)[W_{j-1}](y)| &> \theta_{\nu^*} \sum_{y\in D^W_{z_j,E}(T_j-1)\setminus B_{z_j}} |F_{\overline{E}}(T_j-1)[W_{j-1}](y)| \\ &> (1-\epsilon_*)\theta_{\nu^*}\left(\left|D^W_{z_j,E}(T_j-1)\right| - 2\epsilon_*|F_{z_j}(T_j-1)|\right)|F_{\overline{E}}(T_j-1)[W_{j-1}]|/|F_{z_j}(T_j-1)|.\end{aligned}$$

We also have an upper bound

$$\begin{aligned}\sum_{y\in D^W_{z_j,E}(T_j-1)} |M_{\overline{E},E}(T_j-1)[W_{j-1}](y)| &\le \sum_{y\in F_{z_j}(T_j-1)} |M_{\overline{E},E}(T_j-1)[W_{j-1}](y)| \\ &= |M_{\overline{E},E}(T_j-1)[W_{j-1}]| < \theta_\nu|F_{\overline{E}}(T_j-1)[W_{j-1}]|\end{aligned}$$

where the last inequality holds by $A_{2,j-1}$. Therefore

$$\frac{\left|D^W_{z_j,E}(T_j-1)\right|}{|F_{z_j}(T_j-1)|} < \frac{\theta_\nu}{(1-\epsilon_*)\theta_{\nu^*}} + 2\epsilon_* < \theta_{\nu^*}.$$

Case D.2. Next consider the case $z_j \notin E$. Then $E^{T_j} = E^{T_j-1} = \overline{E}$. Also $x \in \overline{E} \cap VN_H(z_j)$, so $\nu^* > \nu$. Suppose first that $\overline{E} \in H(z_j)$. Then $F_{\overline{E}}(T_j)[W_j] = F_{\overline{E}z_j}(T_j-1)[W_{j-1}](y)$ and $M_{\overline{E},E}(T_j)[W_j] = M_{\overline{E},E}(T_j-1) \cap F_{\overline{E}}(T_j)[W_j]$, so

$$D^W_{z_j,E}(T_j-1) = \left\{y \in F_{z_j}(T_j-1) : \frac{|M_{\overline{E},E}(T_j-1) \cap F_{\overline{E}z_j}(T_j-1)[W_{j-1}](y)|}{|F_{\overline{E}z_j}(T_j-1)[W_{j-1}](y)|} > \theta_{\nu^*}\right\}.$$

Similarly to the previous case, letting B'_{z_j} be the set of $y \in F_{z_j}(T_j-1) \setminus E_{z_j}(T_j-1)$ such that we do not have

$$|F_{\overline{E}z_j}(T_j-1)[W_{j-1}](y)| = (1 \pm \epsilon_*)|F_{\overline{E}z_j}(T_j-1)[W_{j-1}]|/|F_{z_j}(T_j-1)|,$$

we have $|B'_{z_j}| < \epsilon_*|F_{z_j}(T_j-1)|$. With $B_{z_j} = B'_{z_j} \cup E_{z_j}(T_j-1)$ we have

$$\begin{aligned}\Sigma &:= \sum_{y \in D^W_{z_j,E}(T_j-1)} |M_{\overline{E},E}(T_j-1) \cap F_{\overline{E}z_j}(T_j-1)[W_{j-1}](y)| \\ &> \theta_{\nu^*} \sum_{y \in D^W_{z_j,E}(T_j-1)\setminus B_{z_j}} |F_{\overline{E}z_j}(T_j-1)[W_{j-1}](y)| \\ &> (1-\epsilon_*)\theta_{\nu^*}\left(\left|D^W_{z_j,E}(T_j-1)\right| - 2\epsilon_*|F_{z_j}(T_j-1)|\right)\frac{|F_{\overline{E}z_j}(T_j-1)[W_{j-1}]|}{|F_{z_j}(T_j-1)|}.\end{aligned}$$

We also have $\Sigma \le \sum_{y \in F_{z_j}(T_j-1)} |M_{\overline{E},E}(T_j-1) \cap F_{\overline{E}z_j}(T_j-1)[W_{j-1}](y)|$. This last sum counts all pairs (y,P) with $P \in M_{\overline{E},E}(T_j-1)[W_{j-1}]$, $y \in F_{z_j}(T_j-1)$ and $Py \in F_{\overline{E}z_j}[W_{j-1}](T_j-1)$, so we can rewrite it as $\Sigma \le \sum_{P \in M_{\overline{E},E}(T_j-1)[W_{j-1}]} |F_{\overline{E}z_j}(T_j-1)[W_{j-1}](P)|$. Then Lemma 4.14 gives

$$|F_{\overline{E}z_j}(T_j-1)[W_{j-1}](P)| = (1 \pm \epsilon_*)\frac{|F_{\overline{E}z_j}(T_j-1)[W_{j-1}]|}{|F_{\overline{E}}(T_j-1)[W_{j-1}]|}$$

for all but at most $\epsilon_*|F_{\overline{E}}(T_j-1)[W_{j-1}]|$ sets $P \in F_{\overline{E}}(T_j-1)[W_{j-1}]$. Therefore

$$\Sigma \le |M_{\overline{E},E}(T_j-1)[W_{j-1}]|(1+\epsilon_*)\frac{|F_{\overline{E}z_j}(T_j-1)[W_{j-1}]|}{|F_{\overline{E}}(T_j-1)[W_{j-1}]|} + \epsilon_*|F_{\overline{E}}(T_j-1)[W_{j-1}]||F_{z_j}(T_j-1)|.$$

Combining this with the lower bound on Σ we obtain

$$\begin{aligned}&(1-\epsilon_*)\theta_{\nu^*}\left(\left|D^W_{z_j,E}(T_j-1)\right|/|F_{z_j}(T_j-1)| - 2\epsilon_*\right) \\ &\quad < (1+\epsilon_*)\frac{|M_{\overline{E},E}(T_j-1)[W_{j-1}]|}{|F_{\overline{E}}(T_j-1)[W_{j-1}]|} + \epsilon_*\frac{|F_{\overline{E}}(T_j-1)[W_{j-1}]||F_{z_j}(T_j-1)|}{|F_{\overline{E}z_j}(T_j-1)[W_{j-1}]|}.\end{aligned}$$

Now $|M_{\overline{E},E}(T_j-1)[W_{j-1}]| < \theta_\nu|F_{\overline{E}}(T_j-1)[W_{j-1}]|$ by $A_{2,j-1}$. Also, since $x \in \overline{E}$, and since $F_{z_j}(T_j-1)[W_{j-1}] = F_{z_j}(T_j-1)$, we can apply Lemma 4.14(vi) with $\Gamma = (W_{j-1}, \{\emptyset\})$ to get $\frac{|F_{\overline{E}}(T_j-1)[W_{j-1}]||F_{z_j}(T_j-1)|}{|F_{\overline{E}z_j}(T_j-1)[W_{j-1}]|} \le 2d_u^{-1} \ll \epsilon_*^{-1}$, so

$$\frac{\left|D^W_{z_j,E}(T_j-1)\right|}{|F_{z_j}(T_j-1)|} < \frac{(1+\epsilon_*)\theta_\nu + \sqrt{\epsilon_*}}{(1-\epsilon_*)\theta_{\nu^*}} + 2\epsilon_* < \theta_{\nu^*}.$$

Case D.3. It remains to consider the case when $z_j \notin E$ and $\overline{E} \notin H(z_j)$. Since $x \in \overline{E} \cap VN_H(z_j)$ we have $|\overline{E}| \ge 2$ (otherwise we are in Case D.2). Now $F_{\overline{E}^{\le}}(T_j) = F_{\overline{E}^{\le}}(T_j -$

$1)[F_{\overline{E}^<}(T_j)]$ by Lemma 3.11, so $F_{\overline{E}^{\leq}}(T_j)[W_j] = F_{\overline{E}^{\leq}}(T_j - 1)[F_{\overline{E}^<}(T_j)[W_j]]$ by Lemma 4.5. Also, $M_{\overline{E},E}(T_j)[W_j] = M_{\overline{E},E}(T_j - 1) \cap F_{\overline{E}}(T_j)[W_j]$ by Lemma 3.14, so

$$D^W_{z_j,E}(T_j - 1) = \left\{ y \in F_{z_j}(T_j - 1) : \frac{|M_{\overline{E},E}(T_j - 1) \cap F_{\overline{E}^{\leq}}(T_j - 1)[F_{\overline{E}^<}(T_j)[W_j]]_{\overline{E}}|}{|F_{\overline{E}^{\leq}}(T_j - 1)[F_{\overline{E}^<}(T_j)[W_j]]_{\overline{E}}|} > \theta_{\nu*} \right\}.$$

Let $I = \{S \subsetneq \overline{E} : S \in H(z_j)\}$. Then $P \in F_{\overline{E}^{\leq}}(T_j - 1)[F_{\overline{E}^<}(T_j)[W_j]]$ if and only if $P \in F_{\overline{E}^{\leq}}(T_j - 1)$, for $x \notin S \in I$ we have $P_S \in F_S(T_j)$, i.e. $P_S y \in F_{Sz_j}(T_j - 1)$, and for $x \in S \in I$ we have $P_S \in F_S(T_j)[W_j]$, i.e. $P_S y \in F_{Sz_j}(T_j - 1)[W_j]$. When we choose $y \notin E_{z_j}(T_j - 1)$, Lemma 4.14 gives $d_S(F(T_j)) = (1 \pm \epsilon_*) d_S(F(T_j - 1))$ for $S \subseteq \overline{E}$ with $S \notin I$ and $d_S(F(T_j)) = (1 \pm \epsilon_*) d_S(F(T_j - 1)) d_{Sz_j}(F(T_j - 1))$ for $\emptyset \neq S \in I$. Let d'_S denote $d_S(F(T_j)[W_j]) := d_S(F_{\overline{E}^{\leq}}(T_j)[W_j])$ if $x \in S$ or $d_S(F(T_j))$ if $x \notin S$. If $y \notin E^W_{z_j}(T_j - 1)$ then $|W_j| > \frac{1}{2} d_u |W| > \epsilon_*^2 |V_x|$, so regular restriction gives $d'_S = (1 \pm \epsilon_*) d_S(F(T_j))$ for $S \subseteq \overline{E}$, $S \neq x$ and $d'_x = d(W_j) = |W_j|/|V_x|$. Applying Lemma 4.14 we have

$$\begin{aligned} d(F_{\overline{E}^{\leq}}(T_j - 1)[F_{\overline{E}^<}(T_j)[W_j]]_{\overline{E}}) &= (1 \pm \epsilon_*) \prod_{S \subseteq \overline{E}} d'_S \\ &= (1 \pm 30\epsilon_*) \frac{|W_j|}{|F_x(T_j - 1)|} \prod_{S \subseteq \overline{E}} d_S(F(T_j - 1)) \prod_{\emptyset \neq S \in I} d_{Sz_j}(F(T_j - 1)). \end{aligned}$$

Also, $|F_{\overline{E}}(T_j - 1)[W_{j-1}]| = (1 \pm \epsilon_*) \frac{|W_{j-1}|}{|F_x(T_j - 1)|} |F_{\overline{E}}(T_j - 1)|$ by Lemma 4.14. If $y \notin E^W_{z_j}(T_j - 1)$ then $\frac{|W_j|}{|F_x(T_j)|} = (1 \pm \epsilon_*) \frac{|W_{j-1}|}{|F_x(T_j - 1)|}$, so

$$|F_{\overline{E}^{\leq}}(T_j - 1)[F_{\overline{E}^<}(T_j)[W_j]]_{\overline{E}}| = (1 \pm 40\epsilon_*) |F_{\overline{E}}(T_j - 1)[W_{j-1}]| \prod_{\emptyset \neq S \in I} d_{Sz_j}(F(T_j - 1)).$$

Similarly to the previous cases, now with $B_{z_j} = E^W_{z_j}(T_j - 1) \cup E_{z_j}(T_j - 1)$, we have

$$\begin{aligned} \Sigma &:= \sum_{y \in D^W_{z_j,E}(T_j - 1)} |M_{\overline{E},E}(T_j - 1) \cap F_{\overline{E}^{\leq}}(T_j - 1)[F_{\overline{E}^<}(T_j)[W_j]]_{\overline{E}}| \\ &> \theta_{\nu*} \sum_{y \in D^W_{z_j,E}(T_j - 1) \setminus B_{z_j}} |F_{\overline{E}^{\leq}}(T_j - 1)[F_{\overline{E}^<}(T_j)[W_j]]_{\overline{E}}| \\ &> (1 - 40\epsilon_*) \theta_{\nu*} (|D^W_{z_j,E}(T_j - 1)| - 2\epsilon_* |F_{z_j}(T_j - 1)|) \\ &\quad \times |F_{\overline{E}}(T_j - 1)[W_{j-1}]| \prod_{\emptyset \neq S \in I} d_{Sz_j}(F(T_j - 1)). \end{aligned}$$

For any $P \in F_{\overline{E}}(T_j - 1)$, let $F_{P,I}$ be the set of $y \in F_{z_j}(T_j - 1)$ such that $P_S y \in F_{Sz_j}(T_j - 1)$ for all $S \in I$. Let B_I be the set of $P \in F_{\overline{E}}(T_j - 1)$ such that we do not have

$$|F_{P,I}| = (1 \pm \epsilon_*) |F_{z_j}(T_j - 1)| \prod_{\emptyset \neq S \in I} d_{Sz_j}(F(T_j - 1)).$$

Then Lemma 4.11 gives $|B_I| \leq \epsilon_* |F_{\overline{E}}(T_j - 1)|$. Also, since $F_{\overline{E}^<}(T_j)[W_j] * W_{j-1} = F_{\overline{E}^<}(T_j)[W_j]$, Lemma 4.5(iv) gives

$$F_{\overline{E}^{\leq}}(T_j - 1)[F_{\overline{E}^<}(T_j)[W_j]] = F_{\overline{E}^{\leq}}(T_j - 1)[W_{j-1}][F_{\overline{E}^<}(T_j)[W_j]].$$

Now $\Sigma \leq \sum_{y \in F_{z_j}(T_j-1)} |M_{\overline{E},E}(T_j - 1) \cap F_{\overline{E}^{\leq}}(T_j - 1)[W_{j-1}][F_{\overline{E}^{<}}(T_j)[W_j]]_{\overline{E}}|$, which counts all pairs (y, P) with $P \in M_{\overline{E},E}(T_j - 1)[W_{j-1}]$ and $y \in F_{P,I}$, so

$$\Sigma \leq |M_{\overline{E},E}(T_j - 1)[W_{j-1}]|(1 \pm \epsilon_*)|F_{z_j}(T_j - 1)| \prod_{\emptyset \neq S \in I} d_{Sz_j}(F(T_j - 1)) + \epsilon_* |F_{\overline{E}}(T_j - 1)[W_{j-1}]||F_{z_j}(T_j - 1)|.$$

Combining this with the lower bound on Σ we obtain

$$(1 - 40\epsilon_*)\theta_{v^*} \left(\frac{\left|D^W_{z_j,E}(T_j - 1)\right|}{|F_{z_j}(T_j - 1)|} - 2\epsilon_* \right) < (1 + \epsilon_*) \frac{|M_{\overline{E},E}(T_j - 1)[W_{j-1}]|}{|F_{\overline{E}}(T_j - 1)[W_{j-1}]|} + \epsilon_* \prod_{\emptyset \neq S \in I} d_{Sz_j}(F(T_j - 1))^{-1}.$$

Now $|M_{\overline{E},E}(T_j - 1)[W_{j-1}]| < \theta_v |F_{\overline{E}}(T_j - 1)[W_{j-1}]|$ by $A_{2,j-1}$, and all densities are at least $d_u \gg \epsilon_*$, so again we have

$$\frac{\left|D^W_{z_j,E}(T_j - 1)\right|}{|F_{z_j}(T_j - 1)|} < \frac{(1 + \epsilon_*)\theta_v + \sqrt{\epsilon_*}}{(1 - 40\epsilon_*)\theta_{v^*}} + 2\epsilon_* < \theta_{v^*}.$$

This proves Claim D.

Claim E. Conditional on the events $A_{i,j'}$, $1 \leq i \leq 3$, $0 \leq j' < j$ and the embedding up to time $T_j - 1$, the probability that $A_{3,j}$ does not hold is at most θ'_{12D}.

Proof. Since $A_{1,j-1}$ and $A_{2,j-1}$ hold, Claim C gives $|E^W_{z_j}(T_j-1) \setminus E_{z_j}(T_j-1)| < \epsilon_* |F_{z_j}(T_j-1)|$ and Claim D gives $|D^W_{z_j,E}(T_j - 1)| < \theta_{12D}|F_{z_j}(T_j - 1)|$ for any E containing x. We also have $|OK_{z_j}(T_j-1)| > (1-\theta_*)|F_{z_j}(T_j-1)|$ by Corollary 4.16. Since $y = \phi(z_j)$ is chosen uniformly at random in $OK_{z_j}(T_j - 1)$, the probability that $y \in E^W_{z_j}(T_j - 1)$ or $y \in D^W_{z_j,E}(T_j - 1)$ for any E containing x is at most $(\epsilon_* + D\theta_{12D})/(1 - \theta_*) < \theta'_{12D}$. This proves Claim E.

To finish the proof of the lemma, suppose that all the events $A_{i,j}$, $1 \leq i \leq 3$, $1 \leq j \leq g$ hold. Then $A_{1,g}$ gives $|F_x(T_g) \cap W| = |W_g| = (1 \pm \epsilon_*)|W||F_x(T_g)|/|F_x(T_0)| > \frac{1}{2}d_u|W| > \epsilon_*^2|V_x|$. Also, since all of $VN_H(x) = \{z_1, \ldots, z_g\}$ has been embedded at time T_g, for every triple E containing x we have $E^{T_g} = x$, and $|M_{x,E}(T_g) \cap W| < \theta'_{12D}|F_x(T_g) \cap W|$ by $A_{2,g}$. Now $A_x \cap W$ is obtained from $F_x(T_g) \cap W$ by deleting all $M_{x,E}(T_g) \cap W$ for triples E containing x, so $|A_x \cap W| > (1 - D\theta'_{12D})|F_x(T_g) \cap W|$. In particular, $A_x \cap W$ is nonempty. If any event $A_{i,j}$ fails then $A_{3,j}$ fails (by Claim B) and so by Claim E and a union bound over $1 \leq j \leq g \leq 2D$ we can bound the failure probability by θ_*. ∎

Our final lemma in this subsection is similar to the previous one, but instead of asking for a set W of vertices to contain an available vertex for x, we ask for some particular vertex v to be available for x. Recall that $x \in B$ and we start embedding $VN_H(x)$ at time T_1.

Lemma 4.21. *For any $v \in V_x$, conditional on any embedding of the vertices $\{s(u) : u < T_1\}$ that does not use v, with probability at least p we have $\phi(H(x)) \subseteq (G \setminus M)(v)$, so $v \in A_x$.*

Proof. We estimate the probability that $\phi(H(x)) \subseteq (G \setminus M)(v)$ using arguments similar to those we are using to embed H in $G \setminus M$. The structure of the proof is very similar to that of Lemma 4.20. Here we will see the purpose of properties (ii) and (iii) in the definition of super-regularity, which ensure that every $v \in V_x$ is a potential image of x. For $z \in VN_H(x)$ we write

$$\alpha_z = \frac{|F_{xz}(T_0)(v)|}{d_{xz}(F(T_0))|F_z(T_0)|} = \frac{d_z(F(T_0)(v))}{d_{xz}(F(T_0))d_z(F(T_0))}.$$

We consider a vertex z to be *allocated* if z is embedded or $z = x$. For $\emptyset \neq S \in H$ unembedded we define $\nu''_S(t)$ as follows. When $|S| = 3$ we let $\nu''_S(t) = \nu_S(t)$. When $|S| = 1, 2$ we let $\nu''_S(t) = \nu_S(t) + K$, where K is the maximum value of $\nu''_{Sx'}(t')$ over allocated vertices x' with $S \in H(x')$; if there is no such vertex x' we let $\nu''_S(t) = \nu_S(t)$. Thus $\nu''_S(t)$ is defined similarly to $\nu'_S(t)$, replacing 'embedded' with 'allocated'. We have $\nu''_S(t) \geq \nu'_S(t)$ and Lemma 4.3(iii–vi) hold replacing ν' with ν''.

Suppose $1 \leq j \leq g$ and that we are considering the embedding of z_j. We interpret quantities at time T_j with the embedding $\phi(z_j) = y$, for some as yet unspecified $y \in F_{z_j}(T_j - 1)$.

We define *exceptional* sets $E^v_{z_j}(T_j - 1) \subseteq F_{xz_j}(T_j - 1)(v)$ by $y \in F_{z_j}(T_j - 1) \setminus E^v_{z_j}(T_j - 1)$ if and only if for every unembedded $\emptyset \neq S \in H(x) \cap H(z_j)$,

$$\left.\begin{array}{l} F_{Sx}(T_j)(v) \text{ is } \epsilon_{\nu''_S(T_j),1}\text{-regular if } |S| = 2, \\ d_S(F(T_j)(v)) = (1 \pm \epsilon_{\nu''_S(T_j),1}) d_S(F(T_j)) d_{Sx}(F(T_j)) \text{ if } |S| = 2, \\ d_S(F(T_j)(v)) = (1 \pm \epsilon_{\nu''_S(T_j),1}) d_S(F(T_j)) d_{Sx}(F(T_j)) \alpha_S \text{ if } |S| = 1. \end{array}\right\} \qquad (*_{4.22})$$

Here we use the notation $d_S(F(T_j)(v)) = d_S(F_{Sx\leq}(T_j)(v))$. Let Y be the set of vertices at distance at most 3 from x in H and let $H' = \{S \in H : S \subseteq Y\}$. For any $Z \subseteq Y$ and unembedded $S \in H$ we define

$$F(T_j)^{Z*v}_{S\leq} = F_{S\leq}(T_j) \left[\bigcup_{S' \subseteq Z \cap S, S' \in H(x)} F_{S'x}(T_j)(v) \right].$$

Thus $F(T_j)^{Z*v}_S$ consists of all sets $P \in F_S(T_j)$ such that $P_{S'}v \in F_{S'x}(T_j)$ for all $S' \subseteq Z \cap S$ with $S' \in H(x)$. For any triple E, we use the notation $\overline{E} = E^{T_j - 1}$, $\nu = \nu''_{\overline{E}}(T_j - 1)$ and $\nu^* = \nu''_{E^{T_j}}(T_j)$ similarly to the previous lemma, replacing ν' with ν''. For $Z \subseteq Y$ and $E \in U(z_j)$ we define sets of *dangerous* vertices by

$$D^{Z*v}_{z_j,E}(T_j - 1) = \left\{ y \in F(T_j - 1)^{Z*v}_{z_j} : \left| M_{E^{T_j},E}(T_j) \cap F(T_j)^{Z*v}_{E^{T_j}} \right| > \theta_{\nu^*} \left| F(T_j)^{Z*v}_{E^{T_j}} \right| \right\}.$$

The strategy of the proof is to analyse the event that all the complexes $F(T_j)^{Z*v}_{E^{T_j}\leq}$ are well-behaved, meaning informally that they are regular, have roughly expected densities and do not have too many marked edges. The regularity and density properties will hold if we choose $y = \phi(z_j) \notin E^v_{z_j}(T_j - 1)$, and the marked edges will be controlled if we choose $y = \phi(z_j) \notin D^{Z*v}_{z_j,E}(T_j - 1)$.

We think of Z as the *sphere of influence*, as it defines the sets which we restrict to be in the neighbourhood of v. Our eventual goal is that all sets in $H(x)$ should be embedded in $G(v)$, but to achieve this we need to consider arbitrary choices of $Z \subseteq Y$. For later use in the proof we record here some properties of Z that follow directly from the definitions.

(i) $F(T_j)_S^{Z*v} = F(T_j)_S^{(Z\cap S)*v}$.
(ii) $F(T_j-1)_{z_j}^{Z*v}$ is $F_{z_j}(T_j-1)$ if $z_j \notin Z$ or $F_{xz_j}(T_j-1)(v)$ if $z_j \in Z$.
(iii) $F(T_j)_S^{\emptyset*v} = F(T_j)_S$, so $D_{z_j,E}^{\emptyset*v}(T_j-1) = D_{z_j,E}(T_j-1) = \{y \in F_{z_j}(T_j-1) : |M_{E^{T_j},E}(T_j)| > \theta_{v*}|F(T_j)_{E^{T_j}}|\}$, as defined in the description of the algorithm.
(iv) If $Z' = Z \cup z_j$ then $F(T_j)_{E^{T_j}}^{Z'*v} = F(T_j)_{E^{T_j}}^{Z*v}$ by (i), since $z_j \notin E^{T_j}$, so

$$D_{z_j,E}^{Z'*v}(T_j-1) = D_{z_j,E}^{Z*v}(T_j-1) \cap F_{xz_j}(T_j-1)(v).$$

(v) If $S \subseteq Z$ and $S \in H(x)$ then $F(T_j)_S^{Z*v} = F_{Sx}(T_j)(v)$.

Write $B_{z_j} = E_{z_j}^v(T_j-1) \cup E_{z_j}(T_j-1)$. Recall that $U(z_j)$ is the set of triples E with $\overline{E} \cap VN_H(z_j)z_j \neq \emptyset$. We consider the following events:

- $A_{1,j}, j \geq 1$ is the event that property $(*_{4.21})$ above holds. We also define $A_{1,0}$ to be the event that $|G(v)_z \cap V_z(T_0)| > (1-\delta_B^{1/3})|G(v)_z|$ for every $z \in VN_H(x)$.
- $A_{2,j}, j \geq 0$ is the event that for every triple $E \in H'$ and $Z \subseteq E$ we have

$$\left|M_{E^{T_j},E}(T_j) \cap F(T_j)_{E^{T_j}}^{Z*v}\right| \leq \theta'_{v*}\left|F(T_j)_{E^{T_j}}^{Z*v}\right|.$$

- $A_{3,j}, j \geq 0$ is the event that for every triple $E \in H'$, $Z \subseteq E$ and $\emptyset \neq S \subseteq E^{T_j}$,

$F(T_j)_S^{Z*v}$ is $\epsilon_{v''_S(T_j),2}$-regular for $|S| \geq 2$, with

$$d_S(F(T_j)^{Z*v}) = \begin{cases} (1 \pm \epsilon_{v''_S(T_j),2})d_S(F(T_j))d_{Sx}(F(T_j)) \text{ if } S \subseteq Z, S \in H(x), |S| = 2 \\ (1 \pm \epsilon_{v''_S(T_j),2})d_S(F(T_j))d_{Sx}(F(T_j))\alpha_S \text{ if } S \subseteq Z, S \in H(x), |S| = 1 \\ (1 \pm \epsilon_{v''_S(T_j),2})d_S(F(T_j)) \text{ otherwise.} \end{cases}$$

- $A_{4,j}, j \geq 1$ is the event that $y = \phi(z_j)$ is chosen in $OK_{z_j}^v(T_j-1)$, defined to be the subset of $F_{xz_j}(T_j-1)(v)$ obtained by deleting the sets B_{z_j} and $D_{z_j,E}^{Z'*v}(T_j-1)$ for all $E \in U(z_j)$, $Z \subseteq E$, $Z' = Z \cup z_j$. We also define $A_{4,0}$ to be the event that holds with probability 1.

By property (iv) above, an equivalent definition of $OK_{z_j}^v(T_j-1)$ is the subset of $F_{xz_j}(T_j-1)(v)$ obtained by deleting the sets B_{z_j} and $D_{z_j,E}^{Z'*v}(T_j-1)$ for all $E \in U(z_j)$ and $Z' \subseteq E \cup z_j$. Also, since $OK_{z_j}(T_j-1)$ is obtained from $F_{z_j}(T_j-1)$ by deleting $E_{z_j}(T_j-1)$ and $D_{z_j,E}(T_j-1) = D_{z_j,E}^{\emptyset*v}(T_j-1)$ for $E \in U(z_j)$ we have $OK_{z_j}^v(T_j-1) \subseteq OK_{z_j}(T_j-1)$.

We will use the following notation throughout: Z is a subset of Y, $Z' = Z \cup z_j$, $I = \{S \subseteq Z : S \in H(x)\}$, $I' = \{S \subseteq Z' : S \in H(x)\}$. We divide the remainder of the proof into a series of claims.

Claim A. The events $A_{1,0}, A_{2,0}, A_{3,0}$ and $A_{4,0}$ hold with high probability.

Proof. $A_{4,0}$ holds by definition. For any $z \in VN_H(x)$, $d_z(G(v)) = (1 \pm \epsilon')d_{xz}(G)d_z(G) > d_u$ by condition (ii) of super-regularity, so $A_{1,0}$ holds with high probability by Lemma 4.18. Next recall that no vertex at distance within 4 of x has been embedded at time T_0, so $F_z(T_0) = V_z(T_0)$ for any z within distance 3 of x. (This is why we choose the buffer vertices to be at mutual distance at least 9.) We have $|F_z(T_0)| > (1-2\sqrt{\delta_B})|V_z|$ by (5) and

$|F_z(T_0) \cap G(v)_z| > (1 - \delta_B^{1/3})|G(v)_z|$ by $A_{1,0}$. For any $S \in H'$ with $|S| \geq 2$, $F_S(T_0)$ is the restriction of G_S to $((F_z(T_0) : z \in S), \{\emptyset\})$. Since G_S is ϵ-regular, $F_S(T_0)$ is ϵ'-regular with $d_S(F(T_0)) = (1 \pm \epsilon')d_S(G)$. It also follows that $\alpha_z > 1 - 2\delta_B^{1/3}$.

Now we show that $A_{3,0}$ holds. Consider any triple $E \in H'$ and $Z \subseteq E$. Suppose $\emptyset \neq S \subseteq E$. There are two cases according to whether $S \in I$. Suppose first that $S \in I$. By property (v) above, $F(T_0)^{Z*v}_{S\leq} = F_{Sx\leq}(T_0)(v)$ is the restriction of $G_{Sx\leq}(v)$ to $((F_z(T_0) \cap G(v)_z : z \in S), \{\emptyset\})$. If $|S| = 2$ then $G_{Sx\leq}(v)$ is ϵ'-regular and $d_S(G(v)) = (1 \pm \epsilon')d_S(G)d_{Sx}(G)$ by condition (ii) of super-regularity. Then by regular restriction $F_{Sx}(T_0)(v)$ is $\epsilon_{0,0}$-regular and $d_S(F(T_0)(v)) = (1 \pm \epsilon_{0,0})d_S(G)d_{Sx}(G) = (1 \pm 2\epsilon_{0,0})d_S(F(T_0))d_{Sx}(F(T_0))$. Also, if $|S| = 1$ then $d_S(F(T_0))d_{Sx}(F(T_0))\alpha_S = d_S(F(T_0)(v))$ by definition. This gives the properties required by $A_{3,0}$ when $S \in I$. In fact, we have the stronger statements in which $\epsilon_{\nu''_S(T_0),2}$ is replaced by $\epsilon_{0,1}$, say. On the other hand, if $S \notin I$ then $F(T_0)^{Z*v}_{S\leq}$ is the restriction of $F_{S\leq}(T_0)$ to $\cup_{S' \subseteq S\cap Z, S' \in H(x)} F_{S'x}(T_0)(v)$. Since $F_{S\leq}(T_0)$ is ϵ'-regular and $F_{S'x}(T_0)(v)$ is $\epsilon_{0,1}$-regular for $S' \in I$, by regular restriction $F(T_0)^{Z*v}_S$ is $\epsilon_{0,0}$-regular with $d_S(F(T_0)^{Z*v}) = (1 \pm \epsilon')d_S(F(T_0))$. Thus $A_{3,0}$ holds.

It remains to show that $A_{2,0}$ holds. We will abuse notation and let I also denote the subcomplex $\{i(S) : S \in I\}$ of $\binom{[r]}{\leq 3}$. Then $F(0)^{Z*v}_E = G^{I_v}_E$ as defined in Definition 3.15. By property (iii) of super-regularity $|M_E \cap G^{I_v}_E| \leq \theta|G^{I_v}_E|$ and $G^{I_v}_{E\leq}$ is ϵ'-regular with S-density (for $S \subseteq E$) equal to $(1 \pm \epsilon')d_S(G)d_{Sx}(G)$ if $S \in I$ or $(1 \pm \epsilon')d_S(G)$ otherwise. Now $F(T_0)^{Z*v}_z$ is $F_z(T_0) \cap G(v)_z$ if $z \in Z \cap H(x)$ or $F_z(T_0)$ otherwise, and similarly $G^{I_v}_z$ is $G(v)_z$ if $z \in Z \cap H(x)$ or G_z otherwise. Either way we have $|F(T_0)^{Z*v}_z| > (1 - \delta_B^{1/3})|G^{I_v}_z|$ by the estimates recalled above. For $|S| \geq 2$ we showed above that $d_S(F(T_0)^{Z*v})$ is $(1 \pm 2\epsilon_{0,0})d_S(F(T_0))d_{Sx}(F(T_0))$ if $S \in I$ or $d_S(F(T_0)^{Z*v}) = (1 \pm \epsilon')d_S(F(T_0)$ if $S \notin I$. Recalling that $d_{S'}(F(T_0)) = (1 \pm \epsilon')d_{S'}(G)$ for $S' \in H'$ with $|S'| \geq 2$, Lemma 4.10 gives

$$\frac{\left|F(T_0)^{Z*v}_E\right|}{\left|G^{I_v}_E\right|} = \frac{d\left(F(T_0)^{Z*v}_E\right)}{d\left(G^{I_v}_E\right)} = \frac{(1 \pm 9\epsilon_{0,2})\prod_{z\in E}\left|F(T_0)^{Z*v}_z\right|}{(1 \pm 8\epsilon')\prod_{z\in E}|G_v|} > (1 - 10\epsilon_{0,2})\left(1 - \delta_B^{1/3}\right)^3 > 1/2.$$

Now $|M_{E,E}(T_0) \cap F(T_0)^{Z*v}_E| \leq |M_E \cap G^{I_v}_E| \leq \theta|G^{I_v}_E| \leq 2\theta|F(T_0)^{Z*v}_E|$, giving even a stronger bound on the marked edges than is needed. Thus $A_{2,0}$ holds.

Claim B. Suppose $A_{3,j-1}$ holds. If $y \notin E^v_{z_j}(T_j - 1)$ then $A_{1,j}$ and $A_{3,j}$ hold, and if $y \notin D^{Z*v}_{z_j,E}(T_j - 1)$ then $A_{2,j}$ holds. Thus $A_{3,j-1}$ and $A_{4,j}$ imply $A_{1,j}$, $A_{2,j}$ and $A_{3,j}$.

Proof. Suppose $y \notin E^v_{z_j}(T_j - 1)$. Then $A_{1,j}$ holds by definition. $A_{3,j}$ follows from $A_{1,j}$ similarly to the case $j = 0$ considered in Claim A. To see this, consider any triple $E \in H'$ and $Z \subseteq E$. Suppose $\emptyset \neq S \subseteq E$ is unembedded. If $S \subseteq Z$ and $S \in H(x) \cap H(z_j)$ then $F(T_j)^{Z*v}_S = F_{Sx}(T_j)(v)$ satisfies $(*_{4.21})$ by $A_{1,j}$, so we have the properties required by $A_{3,j}$. In fact, we have the stronger statements in which $\epsilon_{\nu''_S(T_j),2}$ is replaced by $\epsilon_{\nu''_S(T_j),1}$. For any other S we use the definition of $F(T_j)^{Z*v}_{S\leq}$ as the restriction of $F_{S\leq}(T_j)$ to $\cup_{S' \subseteq S\cap Z, S' \in H(x)} F_{S'x}(T_j)(v)$. Note that by Lemmas 3.11 and 4.5 we get the same result if we replace $F_{S'x}(T_j)(v)$ by $F_{S'x}(T_j - 1)(v)$ for those $S' \notin H(z_j)$. Now $F_{S'}(T_j)$ is $\epsilon_{\nu''_{S'}(T_j),1}$-regular for $S' \subseteq S$ by Lemma 4.13, $F_{S'x}(T_j)(v)$ is $\epsilon_{\nu''_{S'}(T_j),1}$-regular for $S' \subseteq S \cap Z$, $S' \in H(x) \cap H(z_j)$ and $F_{S'x}(T_j - 1)(v)$ is $\epsilon_{\nu''_{S'}(T_j-1),2}$-regular for $S' \in I$ by $A_{3,j-1}$. So by regular restriction $F(T_j)^{Z*v}_S$ is $\epsilon_{\nu''_S(T_j),2}$-regular with $d_S(F(T_j)^{Z*v}) = (1 \pm \epsilon_{\nu''_S(T_j),2})d_S(F(T_j))$. Thus $A_{3,j}$ holds.

Next consider any triple $E \in H'$ and $Z \subseteq E$. Suppose $y \notin D^{Z*v}_{z_j,E}(T_j - 1)$. If $E \in U(z_j)$ then by definition we have $|M_{E^{T_j},E}(T_j) \cap F(T_j)^{Z*v}_{E^{T_j}}| \leq \theta_{\nu*}|F(T_j)^{Z*v}_{E^{T_j}}|$, which is a stronger bound than required. On the other hand, if $E \notin U(z_j)$ then consider the most recent time $T_{j'} < T_j$ when we embedded $z_{j'}$ with $E \in U(z_{j'})$, setting $j' = 0$ if there is no such time. Then $E^{T_{j'}} = E^{T_j}$, and by the stronger bound at time j' we have $|M_{E^{T_j},E}(T_{j'}) \cap F(T_{j'})^{Z*v}_{E^{T_j}}| \leq \theta_{\nu*}|F(T_{j'})^{Z*v}_{E^{T_j}}|$. (Recall that we also obtained a stronger bound for $A_{2,0}$ in Claim A.) Now $F(T_j)^{Z*v}_{E^{T_j}}$ is obtained from $F(T_{j'})^{Z*v}_{E^{T_j}}$ by deleting at most $2D$ vertices $\phi(z_{j^*}), j'+1 \leq j^* \leq j$ and the sets containing them, so $|F(T_j)^{Z*v}_{E^{T_j}}| \geq (1-\epsilon_*)|F(T_{j'})^{Z*v}_{E^{T_j}}|$; this can be seen by regular restriction and Lemma 4.10, or simply from the fact that a trivial bound for the number of deleted sets has a lower order of magnitude for large n. Therefore $|M_{E^{T_j},E}(T_j) \cap F(T_j)^{Z*v}_{E^{T_j}}| \leq |M_{E^{T_j},E}(T_{j'}) \cap F(T_{j'})^{Z*v}_{E^{T_j}}| \leq \theta_{\nu*}|F(T_{j'})^{Z*v}_{E^{T_j}}| < \theta'_{\nu*}|F(T_j)^{Z*v}_{E^{T_j}}|$, so $A_{2,j}$ holds.

Claim C. If $A_{1,j-1}$ and $A_{3,j-1}$ hold then $|E^v_{z_j}(T_j-1) \setminus E_{z_j}(T_j-1)| < \epsilon_*|F_{xz_j}(T_j-1)(v)|$.

Proof. For any $S \in H$ we write $\nu''_S = \nu''_S(T_j-1)$ and $\nu^*_S = \nu''_S(T_j)$. Consider any unembedded $\emptyset \neq S \in H(x) \cap H(z_j)$.

Case C.1. Suppose first that $S = z$ has size 1. Note that $\nu^*_z > \max\{\nu''_z, \nu''_{z_jz}, \nu''_{xz}\}$ by the analogue of Lemma 4.3 for ν''. We consider two cases according to whether $xz_jz \in H$.

Case C.1.i. Suppose that $xz_jz \in H$. Then $F_{xz}(T_j)(v) = F_{xz_jz}(T_j-1)(yv)$. Since $F_{xz_jz}(T_j-1)(v) = F(T_j-1)^{z_jz*v}_{z_jz}$, using $A_{3,j-1}$ we have $d_{z_jz}(F(T_j-1)(v)) = (1 \pm \epsilon_{\nu''_{z_jz},2})d_{xz_jz}(F(T_j-1))d_{z_jz}(F(T_j-1)) > d_u^2/2$ and $F_{xz_jz}(T_j-1)(v)$ is $\epsilon_{\nu''_{z_jz},2}$-regular. We also have $d_z(F(T_j-1)(v)) = (1 \pm \epsilon_{\nu''_z,2})d_{xz}(F(T_j-1))d_z(F(T_j-1))\alpha_z$. By Lemma 2.2, for all but at most $\epsilon_{\nu''_{z_jz},3}|F_{xz_j}(T_j-1)(v)|$ vertices $y \in F_{xz_j}(T_j-1)(v)$ we have $|F_{xz}(T_j)(v)| = |F_{xz_jz}(T_j-1)(v)(y)| = (1 \pm \epsilon_{\nu''_{z_jz},3})d_{z_jz}(F(T_j-1)(v))|F_{xz}(T_j-1)(v)|$, so

$$\begin{aligned} d_z(F(T_j)(v)) &= (1 \pm \epsilon_{\nu''_{z_jz},3})d_{z_jz}(F(T_j-1)(v))d_z(F(T_j-1)(v)) \\ &= (1 \pm \epsilon_{\nu''_{z_jz},3})(1 \pm \epsilon_{\nu''_{z_jz},2})(1 \pm \epsilon_{\nu''_z,2}) \\ &\quad \times d_{xz_jz}(F(T_j-1))d_{z_jz}(F(T_j-1))d_{xz}(F(T_j-1))d_z(F(T_j-1))\alpha_z. \end{aligned}$$

Also if $y \notin E_{z_j}(T_j-1)$ then $(*_{4.1})$ gives

$$\begin{aligned} d_{xz}(F(T_j)) &= (1 \pm \epsilon_{\nu^*_{xz},0})d_{xz}(F(T_j-1))d_{xz_jz}(F(T_j-1)), \text{ and} \\ d_z(F(T_j)) &= (1 \pm \epsilon_{\nu^*_z,0})d_z(F(T_j-1))d_{z_jz}(F(T_j-1)). \end{aligned}$$

Thus for such y we have the required estimate $d_z(F(T_j)(v)) = (1 \pm \epsilon_{\nu^*_z,1})d_{xz}(F(T_j))d_z(F(T_j))\alpha_z$.

Case C.1.ii. Suppose that $xz_jz \notin H$. Since $z_jz \in H$ and $xz_j \in H$ we have

$$F_{xz\leq}(T_j) = F_{xz\leq}(T_j-1)[(F_{xz_j}(T_j-1)(y), F_{zz_j}(T_j-1)(y), \{\emptyset\})],$$

i.e. $F_{xz}(T_j)$ is the bipartite subgraph of $F_{xz}(T_j-1)$ induced by $F_{xz_j}(T_j-1)(y)$ and $F_{zz_j}(T_j-1)(y)$. Then we have $F_{xz}(T_j)(v) = F_{xz}(T_j-1)(v) \cap F_{zz_j}(T_j-1)(y)$. Now $F_{zz_j}(T_j-1)$ is $\epsilon_{\nu''_{z_jz},1}$-regular by Lemma 4.13 and $d_z(F(T_j-1)(v)) = (1 \pm \epsilon_{\nu''_z,2})d_{xz}(F(T_j-1))d_z(F(T_j-1))\alpha_z > d_u^2/2$ by $A_{3,j-1}$. Then by Lemmas 2.3 and 2.2, for all but at most $\epsilon_{\nu''_{z_jz},2}|F_{xz_j}(T_j-1)(v)|$

vertices $y \in F_{xz_j}(T_j-1)(v)$ we have $|F_{xz}(T_j)(v)| = |F_{xz}(T_j-1)(v) \cap F_{zz_j}(T_j-1)(y)| = (1 \pm \epsilon_{\nu''_{z_jz},2})d_{zz_j}(F(T_j-1))|F_{xz}(T_j-1)(v)|$. This gives

$$\begin{aligned} d_z(F(T_j)(v)) &= (1 \pm \epsilon_{\nu''_{z_jz},2})d_{zz_j}(F(T_j-1))d_z(F(T_j-1)(v)) \\ &= (1 \pm \epsilon_{\nu''_{z_jz},2})(1 \pm \epsilon_{\nu''_z,2})d_{zz_j}(F(T_j-1))d_{xz}(F(T_j-1))d_z(F(T_j-1))\alpha_z. \end{aligned}$$

We also have $d_z(F(T_j)) = (1 \pm \epsilon_{\nu^*_z,0})d_z(F(T_j-1))d_{zz_j}(F(T_j-1))$ if $y \notin E_{z_j}(T_j-1)$ by $(*_{4.1})$ and $d_{xz}(F(T_j)) = (1 \pm \epsilon^{1/2}_{\nu^*_{xz},0})d_{xz}(F(T_j-1))$ by Lemma 2.3, and using $xz_jz \notin H$. Thus for such y we have the required estimate $d_z(F(T_j)(v)) = (1 \pm \epsilon_{\nu^*_z,1})d_{xz}(F(T_j))d_z(F(T_j))\alpha_z$.

Case C.2. The remaining case is when $S = z'z$ has size 2. Note that $xz_jz'z$ is r-partite, as $z'z \in H(x) \cap H(z_j)$ and $xz_j \in H$. Note also that $\nu^*_{z'z} > \max\{\nu''_{z'z}, \nu''_{z_jz'z}, \nu''_{xz'z}\}$. Consider the complex

$$J = F_{z_jz'z^{\le}}(T_j-1)\left[\bigcup_{S' \subseteq z_jz'z, S' \in H(x)} F_{S'x}(T_j-1)(v)\right].$$

We claim that $F_{xz'z}(T_j)(v) = J(y)$. To see this, note first that $F_{xz'z^{\le}}(T_j) = F_{xz'z^{\le}}(T_j-1)[F_{xz'z^{<}}(T_j)]$ by Lemma 3.11. Then by Lemma 4.5 we can write $F_{xz'z^{\le}}(T_j) = F_{xz'z^{\le}}(T_j-1) * F_{xz'z^{<}}(T_j)$ and so

$$\begin{aligned} F_{xz'z^{\le}}(T_j)(v) &= F_{z'z^{\le}}(T_j-1) * F_{z'z^{\le}}(T_j) * F_{xz'z^{\le}}(T_j-1)(v) * F_{xz'z^{<}}(T_j)(v) \\ &= F_{z'z^{\le}}(T_j) * F_{xz'z^{\le}}(T_j-1)(v) * F_{xz'^{\le}}(T_j)(v) * F_{xz^{\le}}(T_j)(v). \end{aligned}$$

Here we used $F_{z'z^{\le}}(T_j-1) * F_{z'z^{\le}}(T_j) = F_{z'z^{\le}}(T_j-1)[F_{z'z^{\le}}(T_j)] = F_{z'z^{\le}}(T_j)$ and $F_{xz'z^{<}}(T_j)(v) = F_{xz'^{\le}}(T_j)(v) \cup F_{xz^{\le}}(T_j)(v) = F_{xz'^{\le}}(T_j)(v) * F_{xz^{\le}}(T_j)(v)$. To put the above identity in words: $F_{xz'z}(T_j)(v)$ is the bipartite subgraph of $F_{z'z^{\le}}(T_j) \cap F_{xz'z}(T_j-1)(v)$ induced by $F_{xz'}(T_j)(v)$ and $F_{xz}(T_j)(v)$. Also,

$$\begin{aligned} J &= F_{z_jz'z^{\le}}(T_j-1) * \bigodot_{S' \subseteq z_jz'z, S' \in H(x)} F_{S'x^{\le}}(T_j-1)(v), \text{ so by Lemma 4.5(ii)} \\ J(y) &= F_{z'z^{\le}}(T_j-1) * \bigodot_{S' \subseteq z'z, S' \in H(x)} F_{S'x^{\le}}(T_j-1)(v) \\ &\quad * F_{z_jz'z^{\le}}(T_j-1)(y) * \bigodot_{S' \subseteq z_jz'z, S' \in H(x)} F_{S'x^{\le}}(T_j-1)(vy) \\ &= F_{z'z^{\le}}(T_j) * F_{xz'z^{\le}}(T_j-1)(v) * \bigodot_{S' \subseteq z_jz'z, S' \in H(x)} F_{S'x^{\le}}(T_j-1)(vy). \end{aligned}$$

Here we used $F_{z_jz'z^{\le}}(T_j-1)(y) = F_{z'z^{\le}}(T_j)$ and $F_{z'z^{\le}}(T_j-1) * F_{z'z^{\le}}(T_j) = F_{z'z^{\le}}(T_j-1)[F_{z'z^{\le}}(T_j)] = F_{z'z^{\le}}(T_j)$. We also recall that $S = z'z \in H(x)$, so $S' \in H(x)$ for any $S' \subseteq z'z$. Note that $\bigodot_{S' \subseteq z_jz'z, S' \in H(x)} F_{S'x^{\le}}(T_j-1)(vy)$ is a 1-complex containing $\{\emptyset\}$, $F_{xz_jz}(T_j-1)(vy)$ if $xz_jz \in H$ and $F_{xz_jz'}(T_j-1)(vy)$ if $xz_jz' \in H$. As in Case C.1, we have $F_{xz}(T_j)(v) = F_{xz_jz}(T_j-1)(vy)$ if $xz_jz \in H$ or $F_{xz}(T_j)(v) = F_{xz}(T_j-1)(v) \cap F_z(T_j)$ if $xz_jz \notin H$, and similarly $F_{xz'}(T_j)(v) = F_{xz_jz'}(T_j-1)(vy)$ if $xz_jz' \in H$ or $F_{xz'}(T_j)(v) = F_{xz'}(T_j-1)(v) \cap F_{z'}(T_j)$ if

$xz_jz' \notin H$. Thus $J(y)$ is also equal to $F_{z'z^\le}(T_j) * F_{xz'z^\le}(T_j-1)(v) * F_{xz'^\le}(T_j)(v) * F_{xz^\le}(T_j)(v)$, which proves that $F_{xz'z^\le}(T_j)(v) = J(y)$.

For any $S' \subseteq z_jz'z$, $F_{S'}(T_j-1)$ is $\epsilon_{\nu'_{S'},1}$-regular by Lemma 4.13 if $S' \in H$ and, using property (v) above, $F_{S'x}(T_j-1)(v)$ is $\epsilon_{\nu''_{S'},2}$-regular by $A_{3,j-1}$ if $S' \in H(x)$. By Lemma 4.8, $J_{z_jz'z}$ is $\epsilon_{\nu''_{z_jz'z},2}$-regular with

$$d_{z_jz'z}(J) = (1 \pm \epsilon_{\nu''_{z_jz'z},2}) d_{z_jz'z}(F(T_j-1)).$$

Similarly, by Lemma 2.3, if $S' \subseteq z_jz'z$, $|S'| = 2$, $S' \notin H(x)$ then $J_{S'}$ is $\epsilon_{\nu''_{S'},2}$-regular with $d_{S'}(J) = (1 \pm \epsilon_{\nu''_{S'},2}) d_{S'}(F(T_j-1))$. On the other hand, if $S' \subseteq z_jz'z$, $|S'| = 2$, $S' \in H(x)$ then $J_{S'} = F_{S'x}(T_j-1)(v)$ is $\epsilon_{\nu''_S,2}$-regular with $d_{S'}(J) = (1 \pm \epsilon_{\nu''_S,2}) d_{S'x}(F(T_j-1)) d_{S'}(F(T_j-1))$ by $A_{3,j-1}$. In particular,

$$d_{z'z}(J) = (1 \pm \epsilon_{\nu''_{z'z},2}) d_{xz'z}(F(T_j-1)) d_{z'z}(F(T_j-1)).$$

Now by Lemma 4.6, for all but at most $6\epsilon_{12D,3}|F_{xz_j}(T_j-1)(v)|$ vertices $y \in F_{xz_j}(T_j-1)(v)$, $F_{xz'z}(T_j)(v) = J(y)$ is $\epsilon_{\nu^*_{z'z},0}$-regular and $d_{z'z}(F(T_j)(v)) = d_{z'z}(J(y)) = (1 \pm \epsilon_{\nu^*_{z'z},0}) d_{z_jz'z}(J) d_{z'z}(J)$. Also, $d_{z'z}(F(T_j)) = (1 \pm \epsilon_{\nu^*_{z'z},0}) d_{z_jz'z}(F(T_j-1)) d_{z'z}(F(T_j-1))$ if $y \notin E_{z_j}(T_j-1)$ by $(*_{4.1})$, and $d_{xz'z}(F(T_j)) = (1 \pm \epsilon_{\nu''_{xz'z},2}) d_{xz'z}(F(T_j-1))$ by Lemmas 4.13 and 4.8. Thus

$$\begin{aligned} d_{z'z}(F(T_j)(v)) &= (1 \pm \epsilon_{\nu^*_{z'z},0}) d_{z_jz'z}(J) d_{z'z}(J) \\ &= (1 \pm \epsilon_{\nu^*_{z'z},0})(1 \pm \epsilon_{\nu''_{z_jz'z},2})(1 \pm \epsilon_{\nu''_{z'z},2}) d_{z_jz'z}(F(T_j-1)) d_{xz'z}(F(T_j-1)) d_{z'z}(F(T_j-1)) \\ &= (1 \pm \epsilon_{\nu^*_{z'z},1}) d_{z'z}(F(T_j)) d_{xz'z}(F(T_j)), \text{ i.e. } (*_{4.21}) \text{ holds for } S = zz'. \end{aligned}$$

Combining the estimates for all cases we have at most $\epsilon_*|F_{xz_j}(T_j-1)(v)|$ exceptional vertices y, so this proves Claim C.

Claim D. If $A_{1,j-1}$, $A_{2,j-1}$ and $A_{3,j-1}$ hold then for any $E \in U(z_j)$, $Z \subseteq E$, $Z' = Z \cup z_j$ we have

$$\left| D^{Z'*v}_{z_j,E}(T_j-1) \right| < \theta_{\nu*} |F_{xz_j}(T_j-1)(v)|.$$

Proof. Note that $\nu^* > \nu$ by Lemma 4.3, since $E \in U(z_j)$.

Case D.1. First consider the case $z_j \in E$. Then $E^{T_j} = \overline{E} \backslash z_j$ and $F_{E^{T_j}\le}(T_j) = F_{\overline{E}^\le}(T_j-1)(y)$. We will show that

$$F(T_j)^{Z*v}_{E^{T_j}\le} = F(T_j-1)^{Z'*v}_{\overline{E}^\le}(y). \qquad (+_{4.21})$$

Before proving this in general we will illustrate a few cases of this statement. Suppose that $x \in E$, say $E = xz_jz$ for some z. If z is embedded then $\overline{E} = xz_j$ and $E^{T_j} = x$, so $F(T_j)^{Z*v}_x = F_x(T_j) = F_{xz_j}(T_j-1)(y) = F(T_j-1)^{Z'*v}_{xz_j}(y)$. If z is not embedded then $\overline{E} = E$ and $E^{T_j} = xz$, $F(T_j)^{Z*v}_z$ is $F(T_j)_{xz}(v) = F(T_j-1)_E(vy)$ if $z \in I$ or $F(T_j)_z = F_{z_jz}(T_j-1)(y)$ otherwise, and $F(T_j)^{Z*v}_{xz}$ is the bipartite subgraph of $F(T_j)_{xz} = F(T_j-1)_E(y)$ spanned by

$F(T_j)_x^{Z*v} = F_x(T_j) = F_{xz_j}(T_j-1)(y)$ and $F(T_j)_z^{Z*v}$. Also, we have $P \in F(T_j-1)_E^{Z'*v}(y)$ if $Py \in F(T_j-1)_E^{Z'*v}$, i.e. $Py \in F(T_j-1)_E$ and $(Py)_S v \in F_{Sx}(T_j-1)$ for all $S \subseteq Z'$, $S \in H(x)$. Equivalently, (i) $P \in F(T_j-1)_E(y) = F(T_j)_{xz}$, (ii) for $S \subseteq Z \setminus z_j$, $S \in H(x)$ we have $P_S v \in F_{Sx}(T_j-1)$, i.e. $P_S \in F_{Sx}(T_j-1)(v)$, and (iii) if $Sz_j \in H(x)$ we have $P_S yv \in F_{Sxz_j}(T_j-1)$, i.e. $P_S \in F_{Sxz_j}(T_j-1)(y)(v) = F_{Sx}(T_j)(v)$. Thus $P \in F(T_j)_{xz}^{Z*v}$.

On the other hand, suppose that $x \notin E$ and consider the case that E is unembedded, i.e. $\overline{E} = E = z_j z' z$ say. Then $F(T_j)_{z'z}^{Z*v}$ is the bipartite subgraph of $F(T_j)'_{z'z}$ spanned by $F(T_j)'_z$ and $F(T_j)'_{z'}$, where we write $F(T_j)'_{z'z}$ for $F(T_j)_{xz'z}(v)$ if $z'z \in I$ or $F(T_j)_{z'z}$ otherwise, $F(T_j)'_z$ for $F(T_j)_{xz}(v)$ if $z \in I$ or $F(T_j)_z$ otherwise, and $F(T_j)'_{z'}$ for $F(T_j)_{xz'}(v)$ if $z' \in I$ or $F(T_j)_{z'z}$ otherwise. Recall that $F(T_j)_{xz}(v)$ is $F_{xz_jz}(T_j-1)(vy)$ if $xz_jz \in H$ (see Case C.1.i) or $F_{xz}(T_j-1)(v) \cap F_{z_jz}(T_j-1)(y)$ if $xz_jz \notin H$ (see Case C.1.ii). Similar statements hold for $F_{xz'}(T_j)(v)$. Also, if $z'z \subseteq Z$ then $F(T_j-1)_{E^\le}^{Z'*v}$ is the complex J defined in Case C.2, so as shown there $F(T_j)_{xz'z^\le}(v) = F(T_j-1)_{E^\le}^{Z'*v}(y)$.

We deduce $(+_{4.21})$ from the case $A = E^{T_j}$ of the following more general statement, which will also be used in Cases D.2 and D.3:

$$F(T_j)_{A^\le}^{Z*v} = F(T_j-1)_{Az_j^\le}^{Z'*v}(y) \text{ for } A \in H(z_j). \qquad (\dagger_{4.21})$$

To see this, note that $F_{Sx^\le}(T_j) = F_{Sx^\le}(T_j-1)[F_{Sxz_j^\le}(T_j-1)(y)]$ for any $z_j \notin S \in I$ by Definition 3.7 (deleting y has no effect), so by Lemma 4.5 we have

$$\begin{aligned}
F_{Sx^\le}(T_j)(v) &= (F_{Sx^\le}(T_j-1) * F_{Sxz_j^\le}(T_j-1)(y))(v)\\
&= F_{S^\le}(T_j-1) * F_{Sz_j^\le}(T_j-1)(y)\\
&\quad * F_{Sx^\le}(T_j-1)(v) * F_{Sxz_j^\le}(T_j-1)(yv)\\
&= F_{Sx^\le}(T_j-1)(v) * F_{Sz_j^\le}(T_j-1)(y) * F_{Sxz_j^\le}(T_j-1)(yv),
\end{aligned}$$

since $F_{Sz_j^\le}(T_j-1)(y) * F_{S^\le}(T_j-1) = F_{Sz_j^\le}(T_j-1)(y)$. Now by definition and Lemma 4.5 we have

$$F(T_j-1)_{Az_j^\le}^{Z'*v} = F_{Az_j^\le}(T_j-1)[\cup_{S\in I'} F_{Sx}(T_j-1)(v)] = F_{Az_j^\le}(T_j-1) * \bigodot_{S\in I'} F_{Sx^\le}(T_j-1)(v), \text{ so}$$

$$\begin{aligned}
F(T_j-1)^{Z'*v}(y)_{A^\le} &= F_{A^\le}(T_j-1) * \bigodot_{S\in I} F_{Sx^\le}(T_j-1)(v)\\
&\qquad * F_{Az_j^\le}(T_j-1)(y) * \bigodot_{S\in I'} F_{Sx^\le}(T_j-1)(vy)\\
&= F_{Az_j^\le}(T_j-1)(y) * \bigodot_{S\in I} F_{Sx^\le}(T_j-1)(v) * \bigodot_{S\in I'} F_{Sx^\le}(T_j-1)(vy).
\end{aligned}$$

Here we used $F_{A^\le}(T_j-1) * F_{Az_j^\le}(T_j-1)(y) = F_{Az_j^\le}(T_j-1)(y)$. On the other hand,

$$\begin{aligned}
F_{A^\le}^{Z*v}(T_j) &= F_{A^\le}(T_j)[\cup_{S\in I} F_{Sx}(T_j)(v)] = F_{A^\le}(T_j) * \bigodot_{S\in I} F_{Sx^\le}(T_j)(v)\\
&= F_{A^\le}(T_j) * \bigodot_{S\in I}(F_{Sx^\le}(T_j-1)(v) * F_{Sz_j^\le}(T_j-1)(y) * F_{Sxz_j^\le}(T_j-1)(yv))\\
&= F_{Az_j^\le}(T_j-1)(y) * \bigodot_{S\in I} F_{Sx^\le}(T_j-1)(v) * \bigodot_{S'\in I'} F_{S'x^\le}(T_j-1)(vy) = F(T_j-1)^{Z'*v}(y)_{A^\le}.
\end{aligned}$$

In the second equality above we substituted for $F_{Sx^{\leq}}(T_j)(v)$, and in the third we set $S' = Sz_j$ and used $F_{Sz_j^{\leq}}(T_j - 1)(y) * F_{Az_j^{\leq}}(T_j - 1)(y) = F_{Az_j^{\leq}}(T_j - 1)(y)$. This proves ($\dagger_{4.21}$), and so ($+_{4.21}$).

Now $M_{E^{T_j},E}(T_j) = M_{\overline{E},E}(T_j - 1)(y)$ by Lemma 3.14, so we have $M_{E^{T_j},E}(T_j) \cap F(T_j)^{Z*v}_{E^{T_j}} = M_{\overline{E},E}(T_j - 1)(y) \cap F(T_j - 1)^{Z'*v}_{\overline{E}^{\leq}}(y)$. Recalling that $F(T_j - 1)^{Z'*v}_{z_j} = F_{xz_j}(T_j - 1)(v)$, we have

$$D^{Z'*v}_{z_j,E}(T_j - 1) = \left\{ y \in F_{xz_j}(T_j - 1)(v) : \frac{\left|\left(M_{\overline{E},E}(T_j - 1) \cap F(T_j - 1)^{Z'*v}_{\overline{E}^{\leq}}\right)(y)\right|}{\left|F(T_j - 1)^{Z'*v}_{\overline{E}^{\leq}}(y)\right|} > \theta_{\nu*} \right\}.$$

Also, $F(T_j - 1)^{Z'*v}_{\overline{E}^{\leq}}$ is $\epsilon_{12D,2}$-regular by $A_{3,j-1}$, so writing B'_{z_j} for the set of vertices $y \in F(T_j - 1)^{Z'*v}_{z_j}$ for which we do not have $|F(T_j - 1)^{Z'*v}_{\overline{E}}(y)| = (1 \pm \epsilon_*)|F(T_j - 1)^{Z'*v}_{\overline{E}}|/|F(T_j - 1)^{Z'*v}_{z_j}|$, we have $|B'_{z_j}| < \epsilon_* |F(T_j - 1)^{Z'*v}_{z_j}|$ by Lemma 4.12. Now

$$\begin{aligned}
\Sigma &:= \sum_{y \in D^{Z'*v}_{z_j,E}(T_j-1)} \left|\left(M_{\overline{E},E}(T_j - 1) \cap F(T_j - 1)^{Z'*v}_{\overline{E}^{\leq}}\right)(y)\right| \\
&> \theta_{\nu*} \sum_{y \in D^{Z'*v}_{z_j,E}(T_j-1)\setminus B'_{z_j}} \left|F(T_j - 1)^{Z'*v}_{\overline{E}^{\leq}}(y)\right| \\
&> (1 - \epsilon_*)\theta_{\nu*}\left(\left|D^{Z'*v}_{z_j,E}(T_j - 1)\right| - \epsilon_*|F_{xz_j}(T_j - 1)(v)|\right) \left|F(T_j - 1)^{Z'*v}_{\overline{E}}\right|/|F_{xz_j}(T_j - 1)(v)|.
\end{aligned}$$

We also have an upper bound

$$\begin{aligned}
\Sigma &\leq \sum_{y \in F_{xz_j}(T_j-1)(v)} \left|\left(M_{\overline{E},E}(T_j - 1) \cap F(T_j - 1)^{Z'*v}_{\overline{E}^{\leq}}\right)(y)\right| \\
&= \left|M_{\overline{E},E}(T_j - 1) \cap F(T_j - 1)^{Z'*v}_{\overline{E}^{\leq}}\right| < \theta'_\nu \left|F(T_j - 1)^{Z'*v}_{\overline{E}^{\leq}}\right|,
\end{aligned}$$

where the last inequality holds by $A_{2,j-1}$. Therefore

$$\frac{\left|D^{Z'*v}_{z_j,E}(T_j - 1)\right|}{|F_{xz_j}(T_j - 1)(v)|} < \frac{\theta'_\nu}{(1 - \epsilon_*)\theta_{\nu*}} + \epsilon_* < \theta_{\nu*}.$$

Case D.2. Next consider the case $z_j \notin E$ and $\overline{E} \in H(z_j)$. Then $E^{T_j} = E^{T_j-1} = \overline{E}$ and $F_{E^{T_j \leq}}(T_j) = F_{\overline{E}z_j^{\leq}}(T_j - 1)(y)$. Also, by ($\dagger_{4.21}$) we have $F(T_j)^{Z*v}_{\overline{E}^{\leq}} = F(T_j - 1)^{Z'*v}_{\overline{E}z_j^{\leq}}(y)$. Now $M_{\overline{E},E}(T_j) = M_{\overline{E},E}(T_j - 1) \cap F_{\overline{E}}(T_j)$ by Lemma 3.14 and $F(T_j)^{Z*v}_{\overline{E}} \subseteq F_{\overline{E}}(T_j)$, so $M_{\overline{E},E}(T_j) \cap F(T_j)^{Z*v}_{\overline{E}} = M_{\overline{E},E}(T_j - 1) \cap F(T_j - 1)^{Z'*v}_{\overline{E}z_j}(y)$. Since $F(T_j - 1)^{Z'*v}_{z_j} = F_{xz_j}(T_j - 1)(v)$, we have

$$D^{Z'*v}_{z_j,E}(T_j - 1) = \left\{ y \in F_{xz_j}(T_j - 1)(v) : \frac{\left|M_{\overline{E},E}(T_j - 1) \cap F(T_j - 1)^{Z'*v}_{\overline{E}z_j}(y)\right|}{\left|F(T_j - 1)^{Z'*v}_{\overline{E}z_j}(y)\right|} > \theta_{\nu*} \right\}.$$

Also, since $\overline{E} \in H(z_j)$ we have $\overline{E}z_j \subseteq E_0^{T_j-1}$ for some triple $E_0 \in H'$, so applying $A_{3,j-1}$ to E_0 we see that $F(T_j - 1)^{Z'*v}_{\overline{E}z_j^{\leq}}$ is $\epsilon_{12D,2}$-regular. Then writing B'_{z_j} for the set of vertices $y \in F(T_j-1)^{Z'*v}_{z_j}$ for which we do not have $|F(T_j-1)^{Z'*v}_{\overline{E}z_j}(y)| = (1 \pm \epsilon_*)|F(T_j - 1)^{Z'*v}_{\overline{E}z_j}|/|F(T_j-1)^{Z'*v}_{z_j}|$, we have $|B'_{z_j}| < \epsilon_*|F(T_j-1)^{Z'*v}_{z_j}|$ by Lemma 4.12. Now

$$\Sigma := \sum_{y \in D^{Z'*v}_{z_j,E}(T_j-1)} \left|M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}z_j}(y)\right| > \theta_{v*} \sum_{y \in D^{Z'*v}_{z_j,E}(T_j-1)\setminus B'_{z_j}} \left|F(T_j-1)^{Z'*v}_{\overline{E}z_j}(y)\right|$$
$$> (1-\epsilon_*)\theta_{v*}\left(\left|D^{Z'*v}_{z_j,E}(T_j-1)\right| - \epsilon_*|F_{xz_j}(T_j-1)(v)|\right)\left|F(T_j-1)^{Z'*v}_{\overline{E}z_j}\right|/|F_{xz_j}(T_j-1)(v)|.$$

We also have $\Sigma \le \sum_{y \in F_{xz_j}(T_j-1)(v)} |M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}z_j}(y)|$. This sum counts all pairs (y,P) with $P \in M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}}$, $y \in F_{xz_j}(T_j-1)(v)$ and $Py \in F(T_j-1)^{Z'*v}_{\overline{E}z_j}$, so $\Sigma \le \sum_{P \in M_{\overline{E},E}(T_j-1)\cap F(T_j-1)^{Z'*v}_{\overline{E}}} |F(T_j-1)^{Z'*v}_{\overline{E}z_j}(P)|$. By Lemma 4.14(vi) we have

$$\left|F(T_j-1)^{Z'*v}_{\overline{E}z_j}(P)\right| = (1 \pm \epsilon_*)\frac{\left|F(T_j-1)^{Z'*v}_{\overline{E}z_j}\right|}{\left|F(T_j-1)^{Z'*v}_{\overline{E}}\right|}$$

for all but at most $\epsilon_*|F(T_j-1)^{Z'*v}_{\overline{E}}|$ sets $P \in F(T_j-1)^{Z'*v}_{\overline{E}}$. Therefore

$$\Sigma \le \left|M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}}\right|(1+\epsilon_*)\frac{\left|F(T_j-1)^{Z'*v}_{\overline{E}z_j}\right|}{\left|F(T_j-1)^{Z'*v}_{\overline{E}}\right|} + \epsilon_*\left|F(T_j-1)^{Z'*v}_{\overline{E}}\right||F_{xz_j}(T_j-1)(v)|.$$

Combining this with the lower bound on Σ gives

$$(1-\epsilon_*)\theta_{v*}\left(\left|D^{Z'*v}_{z_j,E}(T_j-1)\right|/|F_{xz_j}(T_j-1)(v)| - \epsilon_*\right)$$
$$< (1+\epsilon_*)\frac{\left|M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}}\right|}{\left|F(T_j-1)^{Z'*v}_{\overline{E}}\right|} + \epsilon_*\frac{\left|F(T_j-1)^{Z'*v}_{\overline{E}}\right||F_{xz_j}(T_j-1)(v)|}{\left|F(T_j-1)^{Z'*v}_{\overline{E}z_j}\right|}.$$

Now $F(T_j-1)^{Z'*v}_{\overline{E}} = F(T_j-1)^{Z*v}_{\overline{E}}$, $|M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z*v}_{\overline{E}}| < \theta'_v|F(T_j-1)^{Z*v}_{\overline{E}}|$ by $A_{2,j-1}$, and $\frac{|F(T_j-1)^{Z'*v}_{\overline{E}}||F_{xz_j}(T_j-1)(v)|}{|F(T_j-1)^{Z'*v}_{\overline{E}z_j}|} \le 2d_u^{-4} \ll \epsilon_*^{-1}$ by Lemma 4.14 and $A_{3,j-1}$, so

$$\frac{\left|D^{Z'*v}_{z_j,E}(T_j-1)\right|}{|F_{xz_j}(T_j-1)(v)|} < \frac{(1+\epsilon_*)\theta'_v + \sqrt{\epsilon_*}}{(1-\epsilon_*)\theta_{v*}} + \epsilon_* < \theta_{v*}.$$

Case D.3. It remains to consider the case when $z_j \notin E$ and $\overline{E} \notin H(z_j)$. Since $E \in U(z_j)$ and $\overline{E} \notin H(z_j)$ we have $|\overline{E}| \ge 2$. Then $F_{\overline{E}^{\le}}(T_j) = F_{\overline{E}^{\le}}(T_j-1)[F_{\overline{E}^{<}}(T_j)]$ by Lemma 3.11. Also, we claim that

$$F(T_j)^{Z*v}_{\overline{E}^{\le}} = F(T_j-1)^{Z*v}_{\overline{E}^{\le}}\left[F(T_j)^{Z*v}_{\overline{E}^{<}}\right]. \qquad (\diamondsuit_{4.21})$$

To see this, consider $S \subseteq \overline{E}$ and $P \in F(T_j)^{Z*v}_S$. Then by definition $P \in F_S(T_j)$ and $P_{S'} \in F_{S'x}(T_j)(v)$ for all $S' \subseteq S \cap Z$ with $S' \in H(x)$. Since $F_S(T_j) \subseteq F_S(T_j-1)$ it follows that $P \in F(T_j-1)^{Z*v}_S$. Also, for any $S'' \subseteq S' \subseteq S$ with $S' \subsetneq E$, $S'' \subseteq Z$ and $S'' \in H(x)$ we have $P_{S'} \in F_{S'}(T_j)$ and $P_{S''} \in F_{S''x}(T_j)(v)$, so $P_{S'} \in F(T_j)^{Z*v}_{S'}$. Thus $P \in F(T_j-1)^{Z*v}_{\overline{E}^{\le}}[F(T_j)^{Z*v}_{\overline{E}^{<}}]$.

Conversely, suppose that $S \subseteq \overline{E}$ and $P \in F(T_j-1)^{Z*v}_{\overline{E}^{\leq}}[F(T_j)^{Z*v}_{\overline{E}^{<}}]_S$. If $S \neq \overline{E}$ then $P \in F(T_j)^{Z*v}_{\overline{E}^{<}} \subseteq F(T_j)^{Z*v}_{\overline{E}^{\leq}}$. Now suppose $S = \overline{E}$. Then $P \in F_{\overline{E}}(T_j-1)$ and $P_{S'} \in F_{S'}(T_j)$ for $S' \subsetneq S$, so $P \in F_{\overline{E}}(T_j)$. Also, for $S' \subsetneq S$ we have $P_{S'} \in F(T_j)^{Z*v}_{S'}$, so $P_{S''} \in F_{S''x}(T_j)(v)$ for all $S'' \subseteq S' \cap Z$ with $S'' \in H(x)$. Therefore $P \in F(T_j)^{Z*v}_S$. This proves $(\diamondsuit_{4.21})$. Note also that since $z \notin \overline{E}$ we can replace Z by Z' in $(\diamondsuit_{4.21})$.

Since $M_{\overline{E},E}(T_j) = M_{\overline{E},E}(T_j-1) \cap F_{\overline{E}}(T_j)$ and $F(T_j)^{Z'*v}_{\overline{E}} \subseteq F_{\overline{E}}(T_j)$ we have $M_{\overline{E},E}(T_j) \cap F(T_j)^{Z'*v}_{\overline{E}} = M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}^{\leq}}[F(T_j)^{Z'*v}_{\overline{E}^{<}}]$. Then

$$D^{Z'*v}_{z_j,E}(T_j-1) = \left\{ y \in F_{xz_j}(T_j-1)(v) : \frac{\left|M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}^{\leq}}[F(T_j)^{Z'*v}_{\overline{E}^{<}}]_{\overline{E}}\right|}{\left|F(T_j-1)^{Z'*v}_{\overline{E}^{\leq}}[F(T_j)^{Z'*v}_{\overline{E}^{<}}]_{\overline{E}}\right|} > \theta_{v^*} \right\}.$$

Next note that by $A_{3,j-1}$ and Lemma 4.10 we have

$$d\big(F(T_j-1)^{Z'*v}_{\overline{E}}\big) = (1 \pm 8\epsilon_*) \prod_{S \subseteq \overline{E}} d_S(F(T_j-1)) \prod_{S \in I} d_{Sx}(F(T_j-1)) \prod_{S \in I, |S|=1} \alpha_S.$$

Now consider $y \notin E^v_{z_j}(T_j-1)$. By Claim B we can apply the properties in $A_{3,j}$ with this choice of y, so we also have

$$d\big(F(T_j)^{Z'*v}_{\overline{E}}\big) = (1 \pm 8\epsilon_*) \prod_{S \subseteq \overline{E}} d_S(F(T_j)) \prod_{S \in I} d_{Sx}(F(T_j)) \prod_{S \in I, |S|=1} \alpha_S.$$

Now $d_S(F(T_j))$ is $(1 \pm \epsilon_*) d_S(F(T_j-1)) d_{Sz_j}(F(T_j-1)$ for $S \in H(z_j)$ by $(*_{4.1})$ or $(1 \pm \epsilon_*) d_S(F(T_j-1))$ for $S \notin H(z_j)$ by Lemma 4.14. Therefore $d(F(T_j)^{Z'*v}_{\overline{E}}) = (1 \pm 40\epsilon_*) d(F(T_j-1)^{Z'*v}_{\overline{E}}) \times d^*$, where

$$\begin{aligned} d^* &= \prod_{S \subseteq \overline{E}, S \in H(z_j)} d_{Sz_j}(F(T_j-1)) \prod_{S \in I, Sx \in H(z_j)} d_{Sxz_j}(F(T_j-1)) \\ &= (1 \pm 8\epsilon_*) \prod_{S \subseteq \overline{E}, S \in H(z_j)} d_{Sz_j}(F(T_j-1)^{Z'*v}). \end{aligned}$$

To see the second equality above, note that for any $S \in I$ with $Sx \in H(z_j)$, applying $A_{3,j-1}$ to any triple $E_0 \in H$ with $Sz_j \subseteq E_0^{T_j-1}$, since $Sz_j \subseteq Z' = Zz_j$ we have $d_{Sz_j}(F(T_j-1)^{Z'*v}) = (1 \pm \epsilon_*) d_{Sxz_j}(F(T_j-1)) d_{Sz_j}(F(T_j-1))$. We deduce that

$$\left|F(T_j-1)^{Z'*v}_{\overline{E}^{\leq}}\big[F(T_j)^{Z'*v}_{\overline{E}^{<}}\big]_{\overline{E}}\right| = (1 \pm 50\epsilon_*) \left|F(T_j-1)^{Z'*v}_{\overline{E}}\right| d^*$$

for such $y \notin E^v_{z_j}(T_j-1)$. Now

$$\begin{aligned} \Sigma &:= \sum_{y \in D^{Z'*v}_{z_j,E}(T_j-1)} \left|M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}^{\leq}}\big[F(T_j)^{Z'*v}_{\overline{E}^{<}}\big]_{\overline{E}}\right| \\ &> \theta_{v^*} \sum_{y \in D^{Z'*v}_{z_j,E}(T_j-1) \setminus B_{z_j}} \left|F(T_j-1)^{Z'*v}_{\overline{E}^{\leq}}\big[F(T_j)^{Z'*v}_{\overline{E}^{<}}\big]_{\overline{E}}\right| \\ &> (1 - 50\epsilon_*) \theta_{v^*} \Big(\left|D^{Z'*v}_{z_j,E}(T_j-1)\right| - \sqrt{\epsilon_*} |F_{xz_j}(T_j-1)(v)|\Big) \left|F(T_j-1)^{Z'*v}_{\overline{E}}\right| d^*. \end{aligned}$$

(Note that we used the estimate $|B_{z_j}| < 2\epsilon|F_{z_j}(T_j-1)| < \sqrt{\epsilon_*}|F_{xz_j}(T_j-1)(v)|$, by Lemma 4.13 and Claim C.) We also have $\Sigma \le \sum_{y\in F_{xz_j}(T_j-1)(v)} |M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}^{\le}}[F(T_j)^{Z'*v}_{\overline{E}^{<}}]_{\overline{E}}|$. This sum counts all pairs (y,P) with $P \in M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}}$, $y \in F_{xz_j}(T_j-1)(v)$ and $P_S \in F(T_j)^{Z'*v}_S$ for all $S \subsetneq \overline{E}$, $S \in H(z_j)$; there is no need to consider $S \notin H(z_j)$, as by Lemma 3.11 we get the same expression if we replace the restriction to $F(T_j)^{Z'*v}_S$ by $F(T_j-1)^{Z'*v}_S$ for such S. Note that $F(T_j)^{Z'*v}_S = F(T_j)^{Z*v}_S$ and $P_S \in F(T_j)^{Z*v}_S \Leftrightarrow P_S y \in F(T_j-1)^{Z'*v}_{Sz_j}$ by ($\dagger_{4.21}$). Given P, let $F^v_{P,Z}$ be the set of $y \in F_{xz_j}(T_j-1)(v)$ satisfying this condition, and let B^v_Z be the set of $P \in F(T_j-1)^{Z'*v}_{\overline{E}}$ such that we do not have

$$\left|F^v_{P,Z}\right| = (1\pm\epsilon_*)|F_{xz_j}(T_j-1)(v)|d^* = (1\pm\epsilon_*)\left|F(T_j-1)^{Z'*v}_{z_j}\right|d^*.$$

Lemma 4.11 applied with $G = F(T_j-1)^{Z'*v}_{\overline{E}^{\le}}$ and $I = \{i(S) : S \subsetneq \overline{E}, S \in H(z_j)\}$ gives $|B^v_Z| < \epsilon_*|F(T_j-1)^{Z'*v}_{\overline{E}}|$. Then

$$\Sigma \le \left|M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}}\right|(1+\epsilon_*)|F_{xz_j}(T_j-1)(v)|d^* + \epsilon_*\left|F(T_j-1)^{Z'*v}_{\overline{E}}\right||F_{xz_j}(T_j-1)(v)|.$$

Combining this with the lower bound on Σ we obtain

$$(1-50\epsilon_*)\theta_{v^*}\left(\frac{\left|D^{Z'*v}_{z_j,E}(T_j-1)\right|}{|F_{xz_j}(T_j-1)(v)|} - \sqrt{\epsilon_*}\right) < (1+\epsilon_*)\frac{\left|M_{\overline{E},E}(T_j-1)\cap F(T_j-1)^{Z'*v}_{\overline{E}}\right|}{\left|F(T_j-1)^{Z'*v}_{\overline{E}}\right|} + \frac{\epsilon_*}{d^*}.$$

Now $d^* \gg \epsilon_*$, and $|M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{Z'*v}_{\overline{E}}| < \theta'_v|F(T_j-1)^{Z'*v}_{\overline{E}}|$ by $A_{2,j-1}$ (since $F(T_j-1)^{Z'*v}_{\overline{E}} = F(T_j-1)^{Z*v}_{\overline{E}}$). Then

$$\frac{\left|D^{Z'*v}_{z_j,E}(T_j-1)\right|}{|F_{xz_j}(T_j-1)(v)|} < \frac{(1+\epsilon_*)\theta'_v + \sqrt{\epsilon_*}}{(1-50\epsilon_*)\theta_{v^*}} + \sqrt{\epsilon_*} < \theta_{v^*}.$$

This completes the proof of Claim D.

Claim E. Conditional on the events $A_{i,j'}$, $1 \le i \le 4$, $0 \le j' < j$ and the embedding up to time $T_j - 1$ we have $\mathbb{P}(A_{4,j}) > d_u/2$.

Proof. Suppose $A_{1,j-1}$, $A_{2,j-1}$ and $A_{3,j-1}$ hold. Then Claim D gives $|D^{Z'*v}_{z_j,E}(T_j-1)| < \theta_{12D}|F_{xz_j}(T_j-1)(v)|$ for any $E \in U(z_j)$, $Z \subseteq E$, $Z' = Z \cup z_j$. Also $F_{xz_j}(T_j-1)(v) > (1-\delta_B^{1/4})d_u|F_{z_j}(T_j-1)|$ by $A_{3,j-1}$ and since $\alpha_{z_j} > 1-2\delta_B^{1/3}$. As $|E_{z_j}(T_j-1)| < \epsilon_*|F_{z_j}(T_j-1)|$ by Lemma 4.13, $B_{z_j} = E_{z_j}(T_j-1) \cup E^v_{z_j}(T_j-1)$ has size $|B_{z_j}| < \sqrt{\epsilon_*}|F_{xz_j}(T_j-1)(v)|$ by Claim C. Since H has maximum degree at most D we have at most $2D^2$ choices for $E \in U(z_j)$ then 8 choices for $Z \subseteq E$, so $|OK^v_{z_j}(T_j-1)|/|F_{xz_j}(T_j-1)(v)| > 1 - \sqrt{\epsilon_*} - 16D^2\theta_{12D} > 1-\theta_*$. Now $y = \phi(z_j)$ is chosen uniformly at random in $OK_{z_j}(T_j-1) \subseteq F_{z_j}(T_j-1)$, and $OK^v_{z_j}(T_j-1) \subseteq OK_{z_j}(T_j-1)$. Then Claim E follows from

$$\mathbb{P}\big(y \in OK^v_{z_j}(T_j-1)\big) = \frac{OK^v_{z_j}(T_j-1)}{OK_{z_j}(T_j-1)} > \frac{(1-\theta_*)|F_{xz_j}(T_j-1)(v)|}{|F_{z_j}(T_j-1)|} > (1-\theta_*)(1-\delta_B^{1/4})d_u > d_u/2.$$

To finish the proof of the lemma, note that if all the events $A_{i,j}$, $1 \leq i \leq 4$, $1 \leq j \leq g$ hold, then the events $A_{4,j}$ imply that $\phi(H(x)) \subseteq (G \setminus M)(v)$. Multiplying the conditional probabilities given by Claim E over at most $2D$ vertices of $VN_H(x)$ gives probability at least $(d_u/2)^{2D} > p$. ■

4.6. The Conclusion of the Algorithm

The algorithm will be successful if the following two conditions hold. Firstly, it must not abort during the iterative phase because of the queue becoming too large. Secondly, there must be a system of distinct representatives for the available images of the unembedded vertices, which all belong to the buffer B. Recall that $A_x = F_x(t_x^N) \setminus \cup_{E \ni x} M_{x,E}(t_x^N)$ is the available set for $x \in B$ at the time t_x^N when the last vertex of $VN_H(x)$ is embedded. Since $VN_H(x)$ has been embedded, until the conclusion of the algorithm at time T, no further vertices will be marked as forbidden for the image x, and no further neighbourhood conditions will be imposed on the image of x, although some vertices in A_x may be used to embed other vertices in X_x. Thus the set of vertices available to embed x at time T is $A'_x = A_x \cap V_x(T) = F_x(T) \setminus \cup_{E \ni x} M_{x,E}(T)$. Therefore we seek a system of distinct representatives for $\{A'_x : x \in X(T)\}$.

We start with the 'main lemma', which is almost identical to Lemma 2.6. For completeness we repeat the proof, giving the necessary modifications.

Lemma 4.22. *Suppose $1 \leq i \leq r$, $Y \subseteq X_i$ and $A \subseteq V_i$ with $|A| > \epsilon_* n$. Let $E_{A,Y}$ be the event that (i) no vertices are embedded in A before the conclusion of the algorithm, and (ii) for every $z \in Y$ there is some time t_z such that $|A \cap F_z(t_z)|/|F_z(t_z)| < 2^{-2D}|A|/|V_i|$. Then $\mathbb{P}(E_{A,Y}) < p_0^{|Y|}$.*

Proof. We start by choosing $Y' \subseteq Y$ with $|Y'| > |Y|/(2D)^2$ so that vertices in Y' are mutually at distance at least 3 (this can be done greedily, using the fact that H has maximum degree D). It suffices to bound the probability of $E_{A,Y'}$. Note that initially we have $|A \cap F_z(0)|/|F_z(0)| = |A|/|V_i|$ for all $z \in X_i$. Also, if no vertices are embedded in A, then $|A \cap F_z(t)|/|F_z(t)|$ can only be less than $|A \cap F_z(t-1)|/|F_z(t-1)|$ for some z and t if we embed a neighbour of z at time t. It follows that if $E_{A,Y'}$ occurs, then for every $z \in Y'$ there is a first time t_z when we embed a neighbour w of z and have $|A \cap F_z(t_z)|/|F_z(t_z)| < |A \cap F_z(t_z-1)|/2|F_z(t_z-1)|$.

By Lemma 4.13, the densities $d_z(F(t_z-1))$, $d_z(F(t_z-1))$ and $d_{zw}(F(t_z-1))$ are all at least d_u and $F_{zw}(t_z-1)$ is ϵ_*-regular. Applying Lemma 2.2, we see that there are at most $\epsilon_*|F_w(t_z-1)|$ 'exceptional' vertices $y \in F_w(t_z-1)$ that do not satisfy $|A \cap F_z(t_z)| = |F_{zw}(t_z-1)(y) \cap A \cap F_z(t_z-1)| = (1 \pm \epsilon_*)d_{zw}(F(t_z-1))|A \cap F_z(t_z-1)|$. On the other hand, the algorithm chooses $\phi(w) = y$ to satisfy $(*_{4.1})$, so $|F_z(t_z)| = (1 \pm \epsilon_*)d_{zw}(F(t_z-1))|F_z(t_z-1)|$. Thus we can only have $|A \cap F_z(t_z)|/|F_z(t_z)| < |A \cap F_z(t_z-1)|/2|F_z(t_z-1)|$ by choosing an exceptional vertex y. But y is chosen uniformly at random from $|OK_w(t_z-1)| \geq (1-\theta_*)|F_w(t_z-1)|$ possibilities (by Corollary 4.16). It follows that, conditional on the prior embedding, the probability of choosing an exceptional vertex for y is at most $\epsilon_*|F_w(t_z-1)|/|OK_w(t_z-1)| < 2\epsilon_*$.

Since vertices of Y' have disjoint neighbourhoods, we can multiply the conditional probabilities over $z \in Y'$ to obtain an upper bound of $(2\epsilon_*)^{|Y'|}$. Recall that this bound is for a subset of $E_{A,Y'}$ in which we have specified a certain neighbour w for every vertex $z \in Y'$. Taking a union bound over at most $(2D)^{|Y'|}$ choices for these neighbours gives $\mathbb{P}(E_{A,Y}) \leq \mathbb{P}(E_{A,Y'}) < (4\epsilon_* D)^{|Y'|} < p_0^{|Y|}$. ■

Now we can prove the following theorem, which implies Theorem 4.1. The proof is quite similar to the graph case, except that the marked edges create an additional case when verifying Hall's criterion, which is covered by Lemma 4.20.

Theorem 4.23. *With high probability the algorithm embeds H in $G \setminus M$.*

Proof. First we estimate the probability of the iteration phase aborting with failure, which happens when the number of vertices that have ever been queued is too large. We can take a union bound over all $1 \leq i \leq r$ and $Y \subseteq X_i$ with $|Y| = \delta_Q |X_i|$ of $\mathbb{P}(Y \subseteq Q(T))$. Suppose that the event $Y \subseteq Q(T)$ occurs. Then for every $z \in Y$ there is some time t such that $|F_z(t)| < \delta'_Q |F_z(t_z)|$, where $t_z < t$ is the most recent time at which we embedded a neighbour of z. Since $A = V_i(T)$ is unused we have $A \cap F_z(t) = A \cap F_z(t_z)$, so $|A \cap F_z(t_z)|/|F_z(t_z)| = |A \cap F_z(t)|/|F_z(t_z)| \leq |F_z(t)|/|F_z(t_z)| < \delta'_Q$. However, we have $|A| \geq \delta_B n/2$ by Lemma 4.3(i), so since $\delta'_Q \ll \delta_B$ we have $|A \cap F_z(t_z)|/|F_z(t_z)| < 2^{-2D}|A|/|V_i|$. Taking a union bound over all possibilities for i, Y and A, Lemma 4.22 implies that the failure probability is at most $r \cdot 4^{Cn} \cdot p_0^{\delta_Q n} < o(1)$, since $p_0 \ll \delta_Q$.

Now we estimate the probability of the conclusion of the algorithm aborting with failure. By Hall's criterion for finding a system of distinct representatives, the conclusion fails if and only if there is $1 \leq i \leq r$ and $S \subseteq X_i(T)$ such that $|\cup_{z \in S} A'_z| < |S|$. Recall that $|X_i(T)| \geq \delta_B n/2$ by Lemma 4.3(i) and buffer vertices have disjoint neighbourhoods. We divide into cases according to the size of S.

$0 \leq |S|/|X_i(T)| \leq \gamma$**.** For every unembedded z and triple E containing z we have $|F_z(T)| \geq d_u n$ by Lemma 4.13 and $|M_{z,E}(T)| \leq \theta_* |F_z(T)|$ by Lemma 4.15. Since z has degree at most D we have $|A'_z| \geq (1 - D\theta_*) d_u n > \gamma n$, so this case cannot occur.

$\gamma \leq |S|/|X_i(T)| \leq 1/2$**.** We use the fact that $A := V_i(T) \setminus \cup_{z \in S} A'_z$ is a large set of unused vertices which cannot be used by any vertex z in S: we have $|A| \geq |V_i(T)| - |S| \geq |X_i(T)|/2 \geq \delta_B n/4$, yet $A \cap F_z(T) \subseteq \cup_{E \ni z} M_{z,E}(T)$ has size at most $D\theta_* |F_z(T)|$ by Lemma 4.15, so $|A \cap F_z(T)|/|F_z(T)| \leq D\theta_* < 2^{-2D}|A|/|V_i|$. As above, taking a union bound over all possibilities for i, S and A, Lemma 4.22 implies that the failure probability is at most $r \cdot 4^{Cn} \cdot p_0^{\gamma \delta_B n/2} < o(1)$, since $p_0 \ll \gamma, \delta_B$.

$1/2 \leq |S|/|X_i(T)| \leq 1 - \gamma$**.** We use the fact that $W := V_i(T) \setminus \cup_{z \in S} A'_z$ satisfies $W \cap A_z = W \cap A'_z = \emptyset$ for every $z \in S$. Now $|W| \geq |V_i(T)| - |S| \geq \gamma |X_i(T)| \geq \gamma \delta_B n/2$, so by Lemma 4.20, for each z the event $W \cap A_z = \emptyset$ has probability at most θ_* when we embed $VN_H(z)$, conditional on the prior embedding. Multiplying the conditional probabilities and taking a union bound over all possibilities for i, S and W, the failure probability is at most $r \cdot 4^{Cn} \cdot \theta_*^{\delta_B n/4} < o(1)$, since $\theta_* \ll \delta_B$.

$1 - \gamma \leq |S|/|X_i(T)| \leq 1$**.** We claim that with high probability $\cup_{z \in S} A'_z = V_i(T)$, so in fact Hall's criterion holds. It suffices to consider sets $S \subseteq X_i(T)$ of size exactly $(1 - \gamma)|X_i(T)|$. The claim fails if there is some $v \in V_i(T)$ such that $v \notin A'_z$ for every $z \in S$. Since v is unused we have $v \notin A_z$, and by Lemma 4.21, for each z the event $v \notin A_z$ has probability at most $1 - p$ when we embed $VN_H(z)$, conditional on the prior embedding. Multiplying the conditional probabilities and taking a union bound over all $1 \leq i \leq r$, $v \in V_i$ and $S \subseteq X_i(T)$ of size $(1-\gamma)|X_i(T)|$, the failure probability is at most $rCn\binom{Cn}{(1-\gamma)Cn}(1 - p)^{(1-\gamma)|X_i(T)|} < o(1)$. This estimate uses the bounds $\binom{Cn}{(1-\gamma)Cn} \leq 2^{\sqrt{\gamma} n}$, $(1 - p)^{(1-\gamma)|X_i(T)|} < e^{-p\delta_B n/4} < 2^{-p^2 n}$ and $\gamma \ll p$.

In all cases we see that the failure probability is $o(1)$. ∎

5. APPLYING THE BLOW-UP LEMMA

To demonstrate the utility of the blow-up lemma we will work through an application in this section. To warm up, we sketch the proof of Kühn and Osthus [31, Theorem 2] on packing bipartite graphs using the graph blow-up lemma. Then we generalise this result to packing tripartite 3-graphs. We divide this section into four subsections, organised as follows. In the first subsection we illustrate the use of the graph blow-up lemma, which is based on a decomposition obtained from Szemerédi's Regularity Lemma and a simple lemma that one can delete a small number of vertices from a regular pair to make it super-regular. The second subsection describes some more hypergraph regularity theory for 3-graphs: the Regular Approximation Lemma and Counting Lemma of Rödl and Schacht. In the third subsection we give the 3-graph analogue of the super-regular deletion lemma, which requires rather more work than the graph case. We also give a 'black box' reformulation of the blow-up lemma that will be more accessible for future applications. We then apply this in the fourth subsection to packing tripartite 3-graphs.

5.1. Applying the Graph Blow-Up Lemma

In this subsection we sketch a proof of the following result of Kühn and Osthus. First we give some definitions. For any graph F, an *F-packing* is a collection of vertex-disjoint copies of F. We say that a graph G is *$(a \pm b)$-regular* if the degree of every vertex in G lies between $a - b$ and $a + b$.

Theorem 5.1. *For any bipartite graph F with different part sizes and $0 < c \leq 1$ there is a real $\epsilon > 0$ and positive integers C, n_0 such that any $(1 \pm \epsilon)cn$-regular graph G on $n > n_0$ vertices contains an F-packing covering all but at most C vertices.*

Note that the assumption that F has different part sizes is essential. For example, if $F = C_4$ is a 4-cycle and G is a complete bipartite graph with parts of size $(1 + \epsilon)n/2$ and $(1 - \epsilon)n/2$ then any F-packing leaves at least ϵn vertices uncovered. Also, we cannot expect to cover all vertices even when the number f of vertices of F divides n, as G may be disconnected and have a component in which the number of vertices is not divisible by f.

Without loss of generality we can assume F is a complete bipartite graph $K_{r,s}$ for some $r \neq s$. It is convenient to assume that G is bipartite, having parts A and B of sizes $n/2$. This can be achieved by choosing A and B randomly: if G is $(1 \pm \epsilon)cn$-regular then with high probability the induced bipartite graph is $(1 \pm 2\epsilon)cn/2$-regular. Then we refine the partition (A, B) using the following 'degree form' of Szemerédi's Regularity Lemma.

Lemma 5.2. *Suppose $0 < 1/T \ll \epsilon \ll d < 1$ and $G = (A, B)$ is a bipartite graph with $|A| = |B| = n/2$. Then there are partitions $A = A_0 \cup A_1 \cup \cdots \cup A_t$ and $B = B_0 \cup B_1 \cup \cdots \cup B_t$ for some $t \leq T$ such that $|A_i| = |B_i| = m$ for $1 \leq i \leq t$ for some m and $|A_0 \cup B_0| \leq \epsilon n$, and a spanning subgraph G' of G such that $d_{G'}(x) > d_G(x) - (d + \epsilon)n$ for every vertex x and every pair (A_i, B_j) with $1 \leq i, j \leq k$ induces a bipartite subgraph of G' that is either empty or ϵ-regular of density at least d.*

Lemma 5.2 can be easily derived from the usual statement of Szemerédi's Regularity Lemma (see e.g. [33, Lemma 41] for a non-bipartite version). We refer to the parts A_i and B_i with $1 \leq i \leq t$ as *clusters* and the parts A_0 and B_0 as *exceptional* sets. We write $(A_i, B_j)_{G'}$ for the bipartite subgraph of G' induced by A_i and B_j. There is a naturally associated *reduced graph*, in which vertices correspond to clusters and edges to dense regular pairs: R is a weighted bipartite graph on $([t],[t])$ with an edge (i,j) of weight d_{ij} whenever $(A_i,B_j)_{G'}$ is ϵ-regular of density $d_{ij} \geq d$. We choose the parameter d to satisfy $\epsilon \ll d \ll c$.

The next step of the proof is to select a nearly-perfect matching in R, using the defect form of Hall's matching theorem. Using the fact that G is $(1 \pm \epsilon)cn$-regular one can show that for any $I \subseteq [t]$ we have $|N_R(I)| \geq (1-2(d+2\epsilon)/c)|I| > (1-\sqrt{d})|I|$ (say) so R has a matching of size $(1-\sqrt{d})t$. The details are given in Lemma 11 of [31].

Now it is straightforward to find an F-packing covering all but at most $3\sqrt{d}n$ vertices. For each edge (i,j) of the matching in R we greedily remove copies of $K_{r,s}$ while possible, alternating which of A_i and B_j contains the part of size s to maintain parts of roughly equal size. While we still have at least ϵm vertices remaining in each of A_i and B_j, the definition of ϵ-regularity implies that the remaining subgraph $(A_i,B_j)_{G'}$ has density at least $d-\epsilon$. Then the Kövari-Sós-Turán theorem [29] implies that we can choose the next copy of $K_{r,s}$. We can estimate the number of uncovered vertices by $2\sqrt{d}t \cdot m < 2\sqrt{d}n$ in clusters not covered by the matching, $2t \cdot \epsilon m < 2\epsilon n$ in clusters covered by the matching, and ϵn in $A_0 \cup B_0$, so at most $3\sqrt{d}n$ vertices are uncovered.

However, we want to prove the stronger result that there is an F-packing covering all but at most C vertices. To do this we first move a small number of vertices from each cluster to the exceptional sets so as to make the matching pairs super-regular. This is a standard property of graph regularity; we include the short proof of the next lemma for comparison with the analogous statement later for hypergraphs.

Lemma 5.3. *Suppose $G = (A,B)$ is an ϵ-regular bipartite graph of density d with $|A| = |B| = m$. Then there are $A^* \subseteq A$ and $B^* \subseteq B$ with $|A^*| = |B^*| = (1-\epsilon)m$ such that the restriction G^* of G to (A^*,B^*) is $(2\epsilon,d)$-super-regular.*

Proof. Let $A_0 = \{x \in A : d(x) < (d-\epsilon)m\}$. Then $|A_0| < \epsilon m$, otherwise (A_0,B) would contradict the definition of ϵ-regularity for (A,B). Similarly $B_0 = \{x \in B : d(x) < (d-\epsilon)m\}$ has $|B_0| < \epsilon m$. Let A^* be obtained from A by deleting a set of size ϵm containing A_0. Define B^* similarly. Then $|A^*| = |B^*| = (1-\epsilon)m$. For any $A' \subseteq A^*$, $B' \subseteq B^*$ with $|A'| > 2\epsilon|A^*|$, $|B'| > 2\epsilon|B^*|$ we have $|A'| > \epsilon m$, $|B'| > 2\epsilon m$, so $(A',B')_G$ has density $d \pm \epsilon$ by ϵ-regularity of G. Thus G^* is 2ϵ-regular. Also, for any vertex x of G^* we have $d_{G^*}(x) \geq d_G(x) - \epsilon m \geq (d-\epsilon)m - \epsilon m \geq (d-2\epsilon)(1-\epsilon)m$, so G^* is $(2\epsilon,d)$-super-regular. ∎

We make the matching pairs $(\epsilon,2d)$-super-regular and move all discarded vertices and unmatched clusters into the exceptional sets. For convenient notation, we redefine $A_1,\dots,A_{t'}$ and $B_1,\dots,B_{t'}$, where $t' = (1-\sqrt{d})t$, to be the parts of the super-regular matched pairs, and A_0, B_0 to be the new exceptional sets. Thus $|A_i| = |B_i| = (1-\epsilon)m$ for $1 \leq i \leq t'$ and $|A_0 \cup B_0| \leq \epsilon n + 2t'\epsilon m + 2\sqrt{d}tm < 3\sqrt{d}n$.

We will select vertex-disjoint copies of $K_{r,s}$ to cover $A_0 \cup B_0$. We want to do this in such a way that the matching pairs remain super-regular (with slightly weaker parameters) and the uncovered parts of the clusters all have roughly equal sizes. Then we will be able to apply the graph blow-up lemma (Theorem 2.1) to pack the remaining vertices in each matching pair almost perfectly with copies of $K_{r,s}$. (We assumed $|V_i| = |X_i| = n$ in Theorem 2.1 for simplicity, but it is easy to replace this assumption by $n \leq |V_i| = |X_i| \leq 2n$, say.) The sizes of the uncovered parts in a pair may not permit a perfect $K_{r,s}$-packing, but it is easy to see that one can cover all but at most $r+s$ vertices in each pair. Taking $C = T(r+s)$ we will thus cover all but at most C vertices.

It remains to show how to cover $A_0 \cup B_0$ with vertex-disjoint copies of $K_{r,s}$. First we set aside some vertices that we will not use so as to preserve the super-regularity of the matching pairs. For each matching pair (i,j) we randomly partition A_i as $A_i' \cup A_i''$ and B_j as $B_j' \cup B_j''$. We will only use vertices from $A' = \cup_i A_i'$ and $B' = \cup_j B_j'$ when covering $A_0 \cup B_0$. By Chernoff bounds, with high probability these partitions have the following properties:

1. all parts A_i', A_i'', B_j', B_j'' have sizes $(1-\epsilon)m/2 \pm m^{2/3}$.
2. every vertex x has at least $d(x)/2 - 4\sqrt{d}n$ neighbours in $A' \cup B'$.
3. whenever a vertex x has $K \geq \epsilon m$ neighbours in a cluster A_i it has $K/2 \pm m^{2/3}$ neighbours in each of A_i', A_i''; a similar statement holds for clusters B_j.
4. each of $N_{G'}(x) \cap B_j''$ and $N_{G'}(y) \cap A_i''$ have size at least $dm/2 - m^{2/3}$ for every matching edge (i,j) and $x \in A_i$, $y \in B_j$.

Now we cover $A_0 \cup B_0$ by the following greedy procedure. Suppose we are about to cover a vertex $x \in A_0 \cup B_0$, say $x \in A_0$. We consider a cluster to be *heavy* if we have covered more than $d^{1/4}m$ of its vertices. Since $|A_0 \cup B_0| < 3\sqrt{d}n$ we have covered at most $3(r+s)\sqrt{d}n$ vertices by copies of $K_{r,s}$, so there are at most $4(r+s)d^{1/4}t$ heavy clusters. Since G is $(1 \pm \epsilon)cn$-regular, x has at least $cn/3$ neighbours in B' by property 2. At most $4(r+s)d^{1/4}n$ of these neighbours lie in heavy clusters and at most $cn/4$ of them lie in clusters B_j where x has at most $cm/4$ neighbours. Thus we can choose a matching pair (i,j) such that x has at least $cm/4$ neighbours in B_j, so at least $cm/10$ neighbours in B_j' by property 3. Since (A_i, B_j) is 2ϵ-regular, $(A_i', N(x) \cap B_j')$ has density at least $d/2$, so by Kövari-Sós-Turán we can choose a copy of $K_{r-1,s}$ with $r-1$ vertices in A_i' and s vertices in $N(x) \cap B_j'$. Adding x we obtain a copy of $K_{r,s}$ covering x.

Thus we can cover $A_0 \cup B_0$, only using vertices from $A' \cup B'$, and by avoiding heavy clusters we never cover more than $d^{1/4}m + \max\{r,s\}$ vertices in any cluster. For each matching edge (i,j), the uncovered part of $(A_i, B_j)_{G'}$ is 3ϵ-regular, and has minimum degree at least $dm/3$ by property 4. Thus it is super-regular, and as described above we can complete the proof via the blow-up lemma.

5.2. The Regular Approximation Lemma and Dense Counting Lemma

In order to apply regularity methods to 3-graphs we need a result analogous to the Szemerédi Regularity Lemma, decomposing an arbitrary 3-graph into a bounded number of 3-complexes, most of which are regular. This was achieved by Frankl and Rödl [9], but as we mentioned in Section 3, in this sparse setting the parameters are not suitable for our blow-up lemma. We will instead use the regular approximation lemma, which provides a dense setting for an approximation of the original 3-graph. For 3-graphs this result is due to Nagle, Rödl and Schacht [37] and in general to Rödl and Schacht [41]. A similar result was proved by Tao [45]. For simplicity we will just discuss the lemma for 3-graphs, although the statement for k-graphs is very similar. Note also that we will formulate the results using the notation established in this paper. We start with a general definition.

Definition 5.4. *Suppose that $V = V_1 \cup \cdots \cup V_r$ is an r-partite set. A* partition k-system P *on V is a collection of partitions P_A of $K(V)_A$ for every $A \in \binom{[r]}{\leq k}$. We say that P is a* partition k-complex *if every two sets S, S' in the same cell of P_A are* strongly equivalent, *defined as in* [13] *to mean that S_B and S'_B belong to the same cell of P_B for every $B \subseteq A$. Given $S \in K(V)$ we write C_S^P for the cell in P_A containing S. We write $C_S = C_S^P$ when there is no danger of* *ambiguity. For any* $S' \subseteq S$ *we write* $C_{S'} \leq C_S$ *and say that* $C_{S'}$ lies under *or is* consistent *with* C_S. *We define the* cell complex $C_{S\leq} = \cup_{S' \subseteq S} C_{S'}$.

We can use a partition 2-complex to decompose a 3-graph as follows.

Definition 5.5. *Suppose* G *is an* r*-partite 3-graph on* V *and* P *is a partition 2-complex on* V. *We define* $G[P]$ *to be the coarsest partition 3-complex refining* P *and the partitions* $\{G_S, K(V)_S \setminus G_S\}$ *for* $S \in G$.

We make a few remarks here to explain the structures defined in Definitions 5.4 and 5.5. The partition 2-complex P has vertex partitions and graph partitions. The vertex partitions P_i are of the form $V_i = V_i^1 \cup \cdots \cup V_i^{a_i}$ for some a_i, for $1 \leq i \leq r$. The graph partitions P_{ij} are of the form $K(V)_{ij} = J_{ij}^1 \cup \cdots \cup J_{ij}^{a_{ij}}$ for some a_{ij}, for $1 \leq i < j \leq r$. By strong equivalence, any bipartite graph $J_{ij}^{b_{ij}}$ is spanned by some pair $(V_i^{b_i}, V_j^{b_j})$: it cannot cut across several such pairs. We also say that $V_i^{b_i}$ and $V_j^{b_j}$ lie under or are consistent with $J_{ij}^{b_{ij}}$. A choice of $i < j < k$, singleton parts $V_i^{b_i}, V_j^{b_j}, V_k^{b_k}$ and graph parts $J_{ij}^{b_{ij}}, J_{jk}^{b_{jk}}, J_{ik}^{b_{ik}}$ such that the singleton parts are consistent with the graph parts is called a *triad*. (This terminology is used by Rödl et al.) If we consider the set of triangles in a triad, then we obtain a partition of the r-partite triples of V as we range over all triads. Another way to describe this is to say as in [13] that S, S' are *weakly equivalent* when S_B, S'_B are in the same cell of P_B for every strict subset $B \subsetneq A$. Let P_A^* denote the partition of $K(V)_A$ into weak equivalence classes. Then P_{ijk}^* is the partition of $K(V)_{ijk}$ by triads as described above. The partition 3-complex $G[P]$ has two cells for each triad: for each cell C of P_{ijk}^* we have cells $G_{ijk} \cap C$ and $(K(V) \setminus G)_{ijk} \cap C$ of $G[P]_{ijk}$. For embeddings in G only the cells $G_{ijk} \cap C$ are of interest, but we include both for symmetry in the definition. We make the following further definitions.

Definition 5.6. *Suppose* P *is a partition* k*-complex. We say that* P *is* equitable *if for every* $k' \leq k$ *the* k'*-cells all have equal size, i.e.* $|T| = |T'|$ *for every* $T \in P_A$, $T' \in P_{A'}$, $A, A' \in \binom{[r]}{k'}$. *We say that* P *is* a-bounded *if* $|P_A| \leq a$ *for every* A. *We say that* P *is* ϵ-regular *if every cell complex* $C_{S\leq}$ *is* ϵ*-regular. We say that* r*-partite 3-graphs* G^0 *and* G *on* V *are* ν-close *if* $|G_A^0 \Delta G_A| < \nu |K(V)_A|$ *for every* $A \in \binom{[r]}{3}$. *Here* Δ *denotes symmetric difference.*

The following is a slightly modified version of the Regular Approximation Lemma [41, Theorem 14].[2] The reader should note the key point of the constant hierarchy: although the closeness of approximation ν may be quite large (it will satisfy $d_2 \ll \nu \ll d_3$), the regularity parameter ϵ will be much smaller.

Theorem 5.7 (Rödl-Schacht [41]**).** *Suppose integers* n, a, r *and reals* ϵ, ν *satisfy* $0 < 1/n \ll \epsilon \ll 1/a \ll \nu, 1/r$ *and that* G^0 *is an* r*-partite 3-graph on an equitable* r*-partite set of* n *vertices* $V = V_1 \cup \cdots \cup V_r$, *where* $a!|n$. *Then there is an* a*-bounded equitable* r*-partite partition 2-complex* P *on* V *and an* r*-partite 3-graph* G *on* V *that is* ν*-close to* G^0 *such that* $G[P]$ *is* ϵ*-regular.*

---

[2] The differences are: (i) we are starting with an initial partition of V, so technically we are using a simplified form of [41, Lemma 25], (ii) a weaker definition of 'equitable' is given in [41], that the singleton cells have equal sizes, but in fact they prove their result with the definition used here, and (iii) we omit the parameter η in our statement, as by increasing r we can ensure that all but at most ηn^3 edges are r-partite.

As we mentioned earlier, ϵ-regular 3-complexes are useful because of a counting lemma that allows one to estimate the number of copies of any fixed complex J, using a suitable product of densities. First we state a counting lemma for tetrahedra, analogous to the triangle counting lemma in (1). Suppose $0 < \epsilon \ll d, \gamma$ and G is an ϵ-regular 4-partite 3-complex on $V = V_1 \cup \cdots \cup V_4$ with all relative densities $d_S(G) \geq d$. Then $G^*_{[4]}$ is the set of tetrahedra in G. We have the estimate

$$d\big(G^*_{[4]}\big) := \frac{\big|G^*_{[4]}\big|}{|V_1||V_2||V_3||V_4|} = (1 \pm \gamma) \prod_{S \subseteq [4]} d_S(G). \tag{7}$$

This follows from a result of Kohayakawa, Rödl and Skokan [22, Theorem 6.5]: they proved a counting lemma for cliques in regular k-complexes.

More generally, suppose J is an r-partite 3-complex on $Y = Y_1 \cup \cdots \cup Y_r$ and G is an r-partite 3-complex on $V = V_1 \cup \cdots \cup V_r$. We let $\Phi(Y, V)$ denote the set of all r-partite maps from Y to V: these are maps $\phi : Y \to V$ such that $\phi(Y_i) \subseteq V_i$ for each i. We say that ϕ is a *homomorphism* if $\phi(J) \subseteq G$. For $I \subseteq [r]$ we let $G_I : K(V)_I \to \{0, 1\}$ also denote the characteristic function of G_I, i.e. $G_I(S)$ is 1 if $S \in G_I$ and 0 otherwise. The following general dense counting lemma from [42] gives an estimate for *partite homomorphism density* $d_J(G)$ of J in G, by which we mean the probability that a random r-partite map from Y to V is a homomorphism from J to G. We use the language of homomorphisms for convenient notation, but note that we can apply the same estimate to the density of embedded copies of J in G, as most maps are injective.

Theorem 5.8 (Rödl-Schacht [42]**, see Theorem 13).**[3] *Suppose* $0 < \epsilon \ll d, \gamma, 1/r, 1/j$, *that J and G are r-partite 3-complexes with vertex sets $Y = Y_1 \cup \cdots \cup Y_r$ and $V = V_1 \cup \cdots \cup V_r$ respectively, that $|J| = j$, and G is ϵ-regular with all densities $d_S(G) \geq d$. Then*

$$d_J(G) = \mathbb{E}_{\phi \in \Phi(Y,V)} \left[\prod_{A \in J} G_A(\phi(A)) \right] = (1 \pm \gamma) \prod_{A \in J} d_A(G).$$

5.3. Obtaining Super-Regularity

Suppose that we want to embed some bounded degree 3-graph H in another 3-graph G^0 on a set V of n vertices, where n is large. We fix constants with hierarchy $0 < 1/n \ll \epsilon \ll 1/a \ll \nu, 1/r \ll 1$. We delete at most $a!$ vertices so that the number remaining is divisible by $a!$, take an equitable r-partition $V = V_1 \cup \cdots \cup V_r$, and apply Theorem 5.7 to obtain an a-bounded equitable r-partite partition 2-complex P on V and an r-partite 3-graph G on V that is ν-close to G^0 such that $G[P]$ is ϵ-regular. Since G is so regular, our strategy for embedding H in G^0 will be to think about embedding it in G, subject to the rule that the edges $M = G \setminus G^0$ are marked as 'forbidden'. Recall that we refer to the pair (G, M) as a marked complex. To apply the 3-graph blow-up lemma (Theorem 4.1) we need the following analogue of Lemma 5.3, showing that we can enforce super-regularity by deleting a small number of vertices.

[3] We have rephrased their statement and slightly generalised it by allowing the sets Y_i to have more than one vertex: this version can easily be deduced from the case $|Y_i| = 1$, $1 \leq i \leq r$ by defining an auxiliary complex with the appropriate number of copies of each V_i (see [4]).

Lemma 5.9. *Suppose that* $0 < \epsilon_0 \ll \epsilon \ll \epsilon' \ll d_2 \ll \theta \ll d_3, 1/r$, *and* (G, M) *is a marked r-partite 3-complex on* $V = V_1 \cup \cdots \cup V_r$ *such that when defined* G_S *is* ϵ_0*-regular,* $|M_S| \le \theta|G_S|$ *and* $d_S(G) \ge d_{|S|}$ *if* $|S| = 2, 3$. *Then we can delete at most* $2\theta^{1/3}|G_i|$ *vertices from each* G_i, $1 \le i \le r$ *to obtain an* $(\epsilon, \epsilon', d_2/2, 2\sqrt{\theta}, d_3/2)$*-super-regular marked complex* $(G^\sharp, M^\sharp)$.

Proof. The idea is to delete vertices which cause failure of the regularity, density or marking conditions in Definition 3.16 (super-regularity). However, some care must be taken to ensure that this process terminates. There are three steps in the proof: firstly, we identify sets Y_i of vertices in G_i that cause the conditions on marked edges to fail; secondly, we identify sets Z_i of vertices in G_i that either cause the regularity and density conditions to fail or have atypical neighbourhood in some Y_j; thirdly, we delete the sets Y_i and Z_i and show that what remains is a super-regular pair.

Step 1. Fix $1 \le i \le r$. We will identify a set Y_i of vertices in G_i that are bad with respect to the conditions on marked edges in the definition of super-regularity. For any j, k such that G_{ijk} is defined we let $Y_{i,jk}$ be the set of vertices $v \in G_i$ for which $|M(v)_{jk}| > \sqrt{\theta}|G(v)_{jk}|$. For any triple S such that G_S is defined and subcomplex I of $S^{\le}$ such that $G_{S'i}$ is defined for all $S' \in I$ we let $Y^I_{i,S}$ be the set of vertices $v \in G_i$ for which $|(M \cap G^{I_v})_S| > \sqrt{\theta}|G^{I_v}_S|$. Let Y_i be the union of all such sets $Y_{i,jk}$ and $Y^I_{i,S}$. We will show that $|Y_i| < \theta^{1/3}|G_i|$.

First we bound the sets $Y_{i,jk}$. Let $Z_{i,jk}$ be the set of vertices $v \in G_i$ such that we do not have $|G(v)_{jk}| = (1 \pm \epsilon)|G_{ijk}|/|G_i|$ and $G(v)_S$ is ϵ-regular with $d_S(G(v)) = (1 \pm \epsilon)d_S(G)d_{Si}(G)$ for $\emptyset \ne S \subseteq jk$. Since G_{ijk} is ϵ_0-regular we have $|Z_{i,jk}| < \epsilon|G_i|$ by Lemma 4.12 and Lemma 4.6. Therefore $\sum_{v \in Y_{i,jk}} |M(v)_{jk}| > \sqrt{\theta} \sum_{v \in Y_{i,jk} \setminus Z_{i,jk}} |G(v)_{jk}| > \sqrt{\theta}(|Y_{i,jk}| - \epsilon|G_i|)(1 - \epsilon)|G_{ijk}|/|G_i|$. We also have an upper bound $\sum_{v \in Y_{i,jk}} |M(v)_{jk}| \le |M_{ijk}| \le \theta|G_{ijk}|$ by the hypotheses of the lemma. This gives $|Y_{i,jk}|/|G_i| < \sqrt{\theta}/(1 - \epsilon) + \epsilon < 2\sqrt{\theta}$.

Now we bound $Y^I_{i,S}$. Define $Z_{i,S}$ to equal $Z_{i,jk}$ if $S = ijk$ or $Z_{i,ab} \cup Z_{i,bc} \cup Z_{i,ac}$ if $i \notin S = abc$. If $v \in G_i \setminus Z_{i,S}$ then by regular restriction $G^{I_v}_{S^{\le}}$ is $\sqrt{\epsilon}$-regular and $d_{S'}(G^{I_v})$ is $(1 \pm \epsilon)d_{S'}(G)d_{S'i}(G)$ if $\emptyset \ne S' \in I$ or $(1 \pm \sqrt{\epsilon})d_{S'}(G)$ otherwise. By Lemma 4.10 we have $d(G_S) = (1 \pm 8\epsilon)\prod_{S' \subseteq S} d_{S'}(G)$ and $d(G^{I_v}_S) = (1 \pm 8\sqrt{\epsilon})\prod_{S' \subseteq S} d_{S'}(G^{I_v}) = (1 \pm 20\sqrt{\epsilon})d(G_S)\prod_{\emptyset \ne S' \in I} d_{S'i}(G)$.

Write $\Sigma = \sum_{v \in Y^I_{i,S}} |(M \cap G^{I_v})_S|$. Then

$$\Sigma > \sqrt{\theta} \sum_{v \in Y^I_{i,S} \setminus Z_{i,S}} \left|G^{I_v}_S\right| > \sqrt{\theta}\left(\left|Y^I_{i,S}\right| - 3\epsilon|G_i|\right)(1 - 20\sqrt{\epsilon})|G_S| \prod_{\emptyset \ne S' \in I} d_{S'i}(G).$$

For any $P \in G_S$, let $G_{P,I}$ be the set of $v \in G_i$ such that $P_{S'}v \in G_{S'i}$ for all $\emptyset \ne S' \in I$. Let B_I be the set of $P \in G_S$ such that we do not have $|G_{P,I}| = (1 \pm \epsilon')|G_i|\prod_{\emptyset \ne S' \in I} d_{S'i}(G)$. Then $|B_I| \le \epsilon'|G_S|$ by Lemma 4.11. Now $\Sigma \le \sum_{v \in G_i} |(M \cap G^{I_v})_S|$, which counts all pairs (v, P) with $P \in M_S$ and $v \in G_{P,I}$, so

$$\Sigma \le |M_S|(1 + \epsilon')|G_i| \prod_{\emptyset \ne S' \in I} d_{S'i}(G) + \epsilon'|G_S||G_i|.$$

Combining this with the lower bound on Σ we obtain

$$\sqrt{\theta}\left(\left|Y^I_{i,S}\right|/|G_i| - 3\epsilon\right)(1 - 20\sqrt{\epsilon}) < \frac{|M_S|}{|G_S|}(1 + \epsilon') + \epsilon' \prod_{\emptyset \ne S' \in I} d_{S'i}(G)^{-1}.$$

Since $|M_S| < \theta|G_S|$ and $\epsilon' \ll d_2$ we deduce that $|Y_{i,jk}|/|G_i| < 2\sqrt{\theta}$.

In total we deduce that $|Y_i|/|G_i| < \left(\binom{r-1}{2} + 2^8\binom{r}{3}\right)2\sqrt{\theta} < \theta^{1/3}$.

Step 2. Next we come to the regularity and density conditions. Recall that $G(v)$ is ϵ-regular with $d_S(G(v)) = (1 \pm \epsilon)d_S(G)d_{Si}(G)$ for $\emptyset \neq S \subseteq jk$ when $v \notin Z_{i,jk}$, where $|Z_{i,jk}| < \epsilon|G_i|$. Now suppose G_{ij} is defined and let $Z'_{i,j}$ be the set of $v \in G_i$ such that $|G(v)_j \cap Y_j| \neq d_{ij}(G)|Y_j| \pm \epsilon|G_j|$. We claim that $|Z'_{i,j}| < 2\epsilon|G_i|$. To prove this, we can assume that $|Y_j| > \epsilon|G_j|$, otherwise $Z'_{i,j}$ is empty. Since G_{ij} is ϵ_0-regular, $G_{ij}[Y_j]$ is ϵ-regular by Lemma 2.3. Then the bound on $Z'_{i,j}$ follows from Lemma 2.2. Let Z_i be the union of all the sets $Z_{i,jk}$ and $Z'_{i,j}$. Then $|Z_i|/|G_i| < \binom{r}{2}\epsilon + 2(r-1)2\epsilon < \sqrt{\epsilon}$, say.

Step 3. Now we show that deleting $Y_i \cup Z_i$ from G_i for every $1 \le i \le r$ gives an $(\epsilon, \epsilon', d_2/2, 2\sqrt{\theta}, d_3/2)$-super-regular marked complex $(G^\sharp, M^\sharp)$. To see this, note first that the above upper bounds on Y_i and Z_i show that we have deleted at most $2\theta^{1/3}$-proportion of each G_i. Since G is ϵ_0-regular with $d_S(G) \ge d_{|S|}$ if $|S| = 2, 3$, regular restriction implies that $G^\sharp$ is ϵ-regular with $d_S(G^\sharp) = (1 \pm \epsilon)d_S(G)$ if $|S| = 2, 3$ and $d_i(G^\sharp) > (1 - 2\theta^{1/3})d_i(G)$ for $1 \le i \le r$. This gives property (i) of super-regularity.

Now suppose G_{ij} is defined and $v \in G_i^\sharp$. Then $|G(v)_j| = (1 \pm \epsilon)d_{ij}(G)|G_j|$ and $|G(v)_j \cap Y_j| = d_{ij}(G)|Y_j| \pm \epsilon|G_j|$ since $v \notin Z'_{i,j}$. Since $G_j^\sharp = G_j \setminus (Y_j \cup Z_j)$ and $|Z_j| < \sqrt{\epsilon}|G_j|$ we have

$$|G^\sharp(v)_j| = |G(v)_j \setminus (Y_j \cup Z_j)| = |G(v)_j| - d_{ij}(G)|Y_j| \pm 2\sqrt{\epsilon}|G_j| = (1 \pm \epsilon')d_{ij}(G^\sharp)\big|G_j^\sharp\big|.$$

Next suppose that G_{ijk} is defined and $v \in G_i^\sharp$. Then $d_{jk}(G(v)) = (1 \pm \epsilon)d_{jk}(G)d_{ijk}(G)$ and $G(v)_{jk}$ is ϵ-regular, so $d_{jk}(G^\sharp(v)) = (1 \pm \epsilon')d_{jk}(G^\sharp)d_{ijk}(G^\sharp)$ and $G^\sharp(v)_{jk}$ is ϵ'-regular by regular restriction. Also, since $v \notin Y_{i,jk}$ we have $|M(v)_{jk}| \le \sqrt{\theta}|G(v)_{jk}|$. Since

$$|G^\sharp(v)_{jk}| = d_{jk}(G^\sharp(v))|G^\sharp(v)_j||G^\sharp(v)_k| > (1 - \epsilon')(1 - 2\theta^{1/3})^2 d_{jk}(G(v))|G(v)_j||G(v)_k| > \frac{1}{2}|G(v)_{jk}|$$

we have $|M^\sharp(v)_{jk}| \le |M(v)_{jk}| \le 2\sqrt{\theta}|G^\sharp(v)|$. This gives property (ii) of super-regularity.

Finally, consider any triple S such that G_S is defined and subcomplex I of $S^\le$ such that $G_{S'i}$ is defined for all $S' \in I$. Since $v \notin Z_{i,S}$, $G_{S^\le}^{I_v}$ is $\sqrt{\epsilon}$-regular and $d_{S'}(G^{I_v})$ is $(1 \pm \epsilon)d_{S'}(G)d_{S'i}(G)$ if $\emptyset \neq S' \in I$ or $(1 \pm \sqrt{\epsilon})d_{S'}(G)$ otherwise. For $j \in S$, we have $d_j(G^{\sharp I_v}) = d_j(G^\sharp) > (1 - 2\theta^{1/3})d_j(G)$ if $j \notin I$ or $d_j(G^{\sharp I_v}) = d_j(G^\sharp(v)) = (1 \pm \epsilon')d_{ij}(G^\sharp)d_j(G^\sharp) > \frac{1}{2}d_2 d_j(G)$ if $j \in I$. Then by regular restriction, for $|S'| \ge 2$ with $S' \subseteq S$, $G_{S'}^{\sharp I_v}$ is ϵ'-regular with $d_{S'}(G^{\sharp I_v}) = (1 \pm \epsilon^{1/4})d_{S'}(G^{I_v})$ equal to $(1 \pm \epsilon')d_{S'}(G^\sharp)d_{S'i}(G^\sharp)$ if $\emptyset \neq S' \in I$ or $(1 \pm \epsilon')d_{S'}(G^\sharp)$ otherwise. Also, by Lemma 4.10 we have

$$\frac{\big|G_S^{\sharp I_v}\big|}{\big|G_S^{I_v}\big|} = \frac{d\big(G_S^{\sharp I_v}\big)}{d\big(G_S^{I_v}\big)} = (1 \pm 10\epsilon') \prod_{S' \subseteq S} \frac{d_{S'}(G^{\sharp I_v})}{d_{S'}(G^{I_v})} > (1 - 20\epsilon')(1 - 2\theta^{1/3})^3 > 1/2.$$

Since $v \notin Y_{i,S}^I$ we have $|(M \cap G^{I_v})_S| \le \sqrt{\theta}|G_S^{I_v}|$, so $|(M^\sharp \cap G^{\sharp I_v})_S| \le 2\sqrt{\theta}|G_S^{\sharp I_v}|$. This gives property (iii) of super-regularity, so the proof is complete. ∎

Remark 5.10. For some applications it may be important to preserve exact equality of the part sizes, i.e. start with $|G_i| = n$ for $1 \le i \le r$ and delete exactly $2\theta^{1/3}n$ vertices from each G_i to obtain super-regularity. This can be achieved by deleting the sets Y_i and Z_i of Lemma 5.9, together with some randomly chosen additional vertices to equalise the numbers. With high probability these additional vertices intersect all vertex neighbourhoods in approximately the correct proportion, and then the same proof shows that the resulting marked complex is super-regular. We omit the details.

For applications it is also useful to know that super-regularity is preserved when one restricts to subsets that are both large and have large intersection with every vertex neighbourhood.

Lemma 5.11 (Super-regular restriction). *Suppose* $0 < \epsilon \ll \epsilon' \ll \epsilon'' \ll d_2 \ll \theta \ll d_3, d'$ *and* (G, M) *is a* $(\epsilon, \epsilon', d_2, \theta, d_3)$*-super-regular marked r-partite 3-complex on* $V = V_1 \cup \cdots \cup V_r$ *with* $G_i = V_i$ *for* $1 \leq i \leq r$. *Suppose also that we have* $V_i' \subseteq V_i$ *for* $1 \leq i \leq r$, *write* $V' = V_1' \cup \cdots \cup V_r'$, $G' = G[V']$, $M' = M[V']$, *and that* $|V_i'| \geq d'|V_i|$ *and* $|G(v)_i \cap V_i'| \geq d'|G(v)_i|$ *whenever* $1 \leq i, j \leq r$, $v \in V_j'$ *and* G_{ij} *is defined. Then* (G', M') *is* $(\epsilon', \epsilon'', d_2/2, \sqrt{\theta}, d_3/2)$*-super-regular.*

Proof. The argument is similar to Step 3 of the previous lemma. Suppose $|S| = 3$, G_S is defined, $i \in S$, $v \in G_i$. By Definition 3.16(i) for (G, M), $G_{S^\leq}$ is ϵ-regular with $d_{S'}(G) \geq d_{|S'|}$ for $S' \subseteq S$, $|S'| = 2, 3$. By assumption we have $d_j(G') \geq d' d_j(G)$ for $j \in S$, so regular restriction implies that $G'_{S^\leq}$ is ϵ'-regular with $d_{S'}(G') \geq d_{|S'|}/2$ for $S' \subseteq S$, $|S'| = 2, 3$. This gives Definition 3.16(i) for (G', M'). Similarly, by Definition 3.16(ii) for (G, M), writing $S = ijk$, $G(v)_{jk^\leq}$ is ϵ'-regular with $d_{S'}(G(v)) = (1 \pm \epsilon')d_{S'}(G)d_{S'i}(G)$ for $\emptyset \neq S' \subseteq jk$ and $|M(v)_{jk}| \leq \theta|G(v)_{jk}|$. By assumption we have $d_j(G'(v)) \geq d'd_j(G(v))$ and $d_k(G'(v)) \geq d'd_k(G(v))$, so regular restriction implies that $G'(v)_{jk^\leq}$ is ϵ''-regular with $d_{S'}(G'(v)) = (1 \pm \epsilon'')d_{S'}(G')d_{S'i}(G')$ for $\emptyset \neq S' \subseteq jk$. Also $|G'(v)_{jk}|/|G(v)_{jk}| = d(G'(v)_{jk})/d(G(v)_{jk}) > (d')^2/2$ so $|M'(v)_{jk}|/|G'(v)_{jk}| \leq 2\theta/(d')^2 < \sqrt{\theta}$. This gives Definition 3.16(ii) for (G', M').

Finally, consider any triple S such that G_S is defined and subcomplex I of $S^\leq$ such that $G_{S'i}$ is defined for all $S' \in I$. By Definition 3.16(iii) for (G, M), $|(M \cap G^{I_v})_S| \leq \theta|G_S^{I_v}|$, $G^{I_v}_{S^\leq}$ is ϵ'-regular and $d_{S'}(G^{I_v})$ is $(1 \pm \epsilon')d_{S'}(G)d_{S'i}(G)$ if $\emptyset \neq S' \in I$ or $(1 \pm \epsilon')d_{S'}(G)$ otherwise. By assumption $d_j(G'^{I_v})$ is $d_j(G') > d'd_j(G)$ if $j \notin I$ or $d_j(G'(v)) \geq d'd_j(G(v))$ if $j \in I$. Then by regular restriction, for $|S'| \geq 2$ with $S' \subseteq S$, $G'^{I_v}_{S'}$ is ϵ''-regular with $d_{S'}(G'^{I_v}) = (1 \pm \sqrt{\epsilon'})d_{S'}(G^{I_v})$ equal to $(1 \pm \epsilon'')d_{S'}(G')d_{S'i}(G')$ if $\emptyset \neq S' \in I$ or $(1 \pm \epsilon'')d_{S'}(G')$ otherwise. Also, by Lemma 4.10 we have $|G'^{I_v}_S|/|G^{I_v}_S| = d(G'^{I_v}_S)/d(G^{I_v}_S) = (1 \pm 10\epsilon')\prod_{S' \subseteq S} d_{S'}(G'^{I_v})/d_{S'}(G^{I_v}) > d'^3/2$. Therefore $|(M \cap G'^{I_v})_S|/|G'^{I_v}_S| \leq 2\theta/d'^3 < \sqrt{\theta}$. This gives Definition 3.16(iii) for (G', M'). ∎

Now we will present a 'black box' reformulation that goes straight from regularity to embedding, bypassing super-regularity and the blow-up lemma. This more accessible form of our results will be more convenient for future applications of the method. First we give a definition.

Definition 5.12 (Robustly universal). *Suppose* J *is an r-partite 3-complex on* $Y = Y_1 \cup \cdots \cup Y_r$ *with* $J_i = Y_i$ *for* $1 \leq i \leq r$. *We say that* J *is* $c^\sharp$-robustly D-universal *if whenever*

(i) $Y_i' \subseteq Y_i$ *with* $|Y_i'| \geq c^\sharp|Y_i|$ *such that* $Y' = \cup_{i=1}^r Y_i'$, $J' = J[Y']$ *satisfy* $|J_S'(v)| \geq c^\sharp|J_S(v)|$ *whenever* $|S| = 3$, J_S *is defined,* $i \in S$, $v \in J_i'$,

(ii) *H' is an r-partite 3-complex on $X' = X'_1 \cup \cdots \cup X'_r$ of maximum degree at most D with $|X'_i| = |Y'_i|$ for $1 \le i \le r$,*

then J' contains a copy of H', in which vertices of X'_i correspond to vertices of Y'_i for $1 \le i \le r$.

Now we show that one can delete a small number of vertices from a regular complex with a small number of marked triples to obtain a robustly universal pair.

Theorem 5.13. *Suppose $0 \le 1/n \ll \epsilon \ll d^\sharp \ll d_2 \ll \theta \ll d_3, c^\sharp, 1/D, 1/C$, that G is an ϵ-regular r-partite 3-complex on $V = V_1 \cup \cdots \cup V_r$ with $d_S(G) \ge d_{|S|}$ for $|S| = 2, 3$ when defined and $n \le |G_i| = |V_i| \le Cn$ for $1 \le i \le r$, and $M \subseteq G_=$ with $|M_S| \le \theta |G_S|$ when defined. Then we can delete at most $2\theta^{1/3}|G_i|$ vertices from G_i for $1 \le i \le r$ to obtain $G^\sharp$ and $M^\sharp$ so that*

(i) *$d(G^\sharp_S) > d^\sharp$ and $|G^\sharp_S(v)| > |G^\sharp_S|/2|G^\sharp_i|$ whenever $|S| = 3$, G_S is defined, $i \in S$, $v \in G_i$, and*
(ii) *$G^\sharp \setminus M^\sharp$ is $c^\sharp$-robustly D-universal.*

Proof. Define additional constants with $\epsilon \ll \epsilon_1 \ll \epsilon_2 \ll \epsilon_3 \ll d^\sharp$. Applying Lemma 5.9, we can delete at most $2\theta^{1/3}|G_i|$ vertices from each G_i to obtain an $(\epsilon_1, \epsilon_2, d_2/2, 2\sqrt{\theta}, d_3/2)$-super-regular marked complex $(G^\sharp, M^\sharp)$ on some $V^\sharp = V^\sharp_1 \cup \cdots \cup V^\sharp_r$. We will show that $J = G^\sharp \setminus M^\sharp$ is $(c^\sharp, d^\sharp)$-robustly D-universal. To see this suppose $|S| = 3$, G_S is defined, $i \in S$, $v \in G'_i$. By Definition 3.16(i) and Lemma 4.10 we have $d(G^\sharp_S) = (1 \pm 8\epsilon_1)\prod_{S' \subseteq S} d_{S'}(G^\sharp) > (1 - 8\epsilon_1)(1 - 2\theta^{1/3})^3 (d_2/2)^3 (d_3/2) > d^\sharp$. Writing $S = ijk$, by Definition 3.16(ii) we have $d(G^\sharp_S(v)) = d_{jk}(G^\sharp(v)) d_j(G^\sharp(v)) d_k(G^\sharp(v)) = \prod_{\emptyset \ne S' \subseteq jk}(1 \pm \epsilon_2) d_{S'}(G^\sharp) d_{S'i}(G^\sharp)$ so $|G^\sharp_i||G^\sharp_S(v)|/|G^\sharp_S| = d_i(G^\sharp) d(G^\sharp_S(v))/d(G^\sharp_S) = (1 \pm 8\epsilon_1)/(1 \pm \epsilon_2)^3 > 1/2$. Now suppose that $V'_i \subseteq V^\sharp_i$ and H' are given as in Definition 5.12 applied to J. Then $(G^\sharp[V'], M^\sharp[V'])$ is $(\epsilon_2, \epsilon_3, d_2/4, 2\theta^{1/4}, d_3/4)$-super-regular by Lemma 5.11. Applying Theorem 4.1, $J' = J[V']$ contains a copy of H', in which vertices of X'_i correspond to vertices of V'_i for $1 \le i \le r$. ∎

5.4. Applying the 3-Graph Blow-Up Lemma

In this subsection we illustrate the 3-graph blow-up lemma by proving the following generalisation of Theorem 5.1 to packings of tripartite 3-graphs.

Theorem 5.14. *For any 3-partite 3-graph F in which not all part sizes are equal and $0 < c_1, c_2 < 1$ there is a real $\epsilon > 0$ and positive integers C, n_0 such that if G is a 3-graph on $n > n_0$ vertices V such that every vertex v has degree $|G(v)| = (1 \pm \epsilon)c_1 n^2$ and every pair of vertices u, v has degree $|G(uv)| > c_2 n$ then G contains an F-packing that covers all but at most C vertices.*

We start by recording some auxiliary results that will be used in the proof. First we need a result of Erdős on the number of copies of a k-partite k-graph.

Theorem 5.15 (Erdős [8]). *For any $a > 0$ and k-partite k-graph F on f vertices there is $b > 0$ so that if H is a k-graph on n vertices with at least an^k edges then H contains at least bn^f copies of F.*

Next we need Azuma's inequality on martingale deviations.

Theorem 5.16 (Azuma [2]**).** *Suppose $Z_0, \dots, Z_n$ is a martingale, i.e. a sequence of random variables satisfying $\mathbb{E}(Z_{i+1}|Z_0, \dots, Z_i) = Z_i$, and that $|Z_i - Z_{i-1}| \le c_i$, $1 \le i \le n$, for some constants c_i. Then for any $t \ge 0$,*

$$\mathbb{P}(|Z_n - Z_0| \ge t) \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right).$$

We also need the following theorem of Kahn, which is a linear programming generalisation of a theorem of Pippenger on matchings in regular hypergraphs with small codegrees. (A *matching* is a set of vertex-disjoint edges.) Suppose H is a k-graph and $t : E(H) \to \mathbb{R}^+$. Write $t(H) = \sum_{A \in E(H)} t(A)$ and $t'(S) = \sum_{A \in E(H), S \subseteq A} t(A)$ for $S \subseteq V(H)$. Let $\text{co}(t) = \max_{S \in \binom{V(H)}{2}} t'(S)$. Say that t is a *fractional matching* if $t'(x) = \sum_{A \in E(H): x \in A} t(A) \le 1$ for every $x \in V(H)$.

Theorem 5.17 (Kahn [17]**).** *For any $\epsilon > 0$ there is $\delta > 0$ so that if H is a k-graph on n vertices and t is a fractional matching of H with $co(t) < \delta$ then H has a matching of size at least $t(H) - \epsilon n$.*

Now we will prove Theorem 5.14. We may suppose that F is complete 3-partite, say $F = K(Y)_{123}$ on $Y = Y_1 \cup Y_2 \cup Y_3$. We introduce parameters with the hierarchy

$$0 \ll 1/n_0 \ll \epsilon \ll \epsilon' \ll \epsilon'' \ll d^\sharp \ll d_2 \ll 1/a \ll \nu \ll 1/r \ll d_3 \ll \delta \ll \gamma \ll \beta \ll \alpha \ll c_1, c_2, 1/|Y|.$$

We delete at most $a!$ vertices of G so that the number remaining is divisible by $a!$, take an equitable r-partition $V = V_1 \cup \cdots \cup V_r$, and apply Theorem 5.7 to obtain an a-bounded equitable r-partite partition 2-complex P on V and an r-partite 3-graph G' on V that is ν-close to G such that $G'[P]$ is ϵ-regular. Since P is a-bounded, for every graph $J_{ij} \in P_{ij}$ with $1 \le i < j \le r$, the densities $d(J_i)$, $d(J_j)$ and $d(J_{ij})$ are all at least $1/a$. We refer to singleton parts in P as *clusters*. Let n_1 be the size of each cluster. By means of an initial partition we may also assume that $n_1 < \nu n$. Let $M = G' \setminus G$ be the edges marked as 'forbidden'.

Next we define the *reduced 3-graph* R, a weighted 3-graph in which vertices correspond to clusters and triples correspond to cells of $G'[P]$ that are useful for embedding, in that they have large density and few marked edges. Let $Z = Z_1 \cup \cdots \cup Z_r$ be an r-partite set with $|Z_i| = a_1 := |P_i|$ for $1 \le i \le r$, where $a_1 \le a$ and $n - a! < rn_1a_1 \le n$. We identify Z_i with $[a_1]$, although it is to be understood that Z_i and Z_j are disjoint for $i \ne j$, and label the cells of P_i as $C_{i,1}, \dots, C_{i,a_1}$. We identity an r-partite triple $S \in K(Z)$ with $S' = \cup_{i \in S} C_{i,S_i}$, where $S_i = S \cap Z_i$. Write $N = n^2 n_1$ and $K(S') = K(V)[S']$. We say that S is an edge of R with *weight* $w(S) = |G'[S']_S|/N$ if $|G'[S']_S| > d_3|K(S')_S|$ and $|M[S']_S| < \sqrt{\nu}|K(S')_S|$.

We define the *weighted degree* $d_w(j)$ of a vertex j in R to be the sum of $w(S)$ over all $j \in S \in R$. We will delete a small number of vertices from R to obtain a 3-graph R' that is almost regular with respect to weighted degrees. For any $i \in A \in \binom{[r]}{3}$ and $j \in Z_i$ we define

$$B^j_{i,A} = \{S : j \in S \in K(Z)_A, |M[S']_S| > \sqrt{\nu}|K(S')_S|\} \text{ and } Z_{i,A} = \left\{j \in Z_i : \left|B^j_{i,A}\right| > \nu^{1/4} a_1^2\right\}.$$

Then

$$|M_A| = \sum_{j \in Z_i} \sum_{S: j \in S \in K(Z)_A} |M[S']| > \sum_{j \in Z_{i,A}} \sum_{S \in B^j_{i,A}} \sqrt{\nu}|K(S')_S| > |Z_{i,A}| \cdot \nu^{1/4} a_1^2 \cdot \sqrt{\nu} n_1^3.$$

Since G' is ν-close to G (see Definition 5.6) we have $|M_A| < \nu|K(V)_A| < \nu(n/r)^3$, so $|Z_{i,A}| < \nu^{1/4}(n/rn_1)^3/a_1^2 < 2\nu^{1/4}a_1$. Let $Z' = Z'_1 \cup \cdots \cup Z'_r$ be obtained by deleting all sets $Z_{i,A}$ from Z and let $R' = R[Z']$. Since $\nu \ll 1/r$ we have $|Z'_i| > (1-\nu^{1/5})a_1$ for $1 \le i \le r$.

Now we estimate the weighted degrees $d'_w(j)$ in R'. Suppose $j \in Z'$. We have $d'_w(j) = N^{-1}\sum_{S:j\in S\in R}|G'[S']_S|$. There are at most a_1^2 triples $S \in K(V)$ containing j, so at most $d_3a_1^2n_1^3 \le d_3N$ triples in $G'[S']_S$ for such S with $|G'[S']_S| < d_3|K(S')_S|$. There are at most $\nu^{1/5}a_1^2$ triples $S \nsubseteq Z'$ with $j \in S \in K(V)$, so at most $\nu^{1/5}N$ triples in $G'[S']_S$ for such S. Since $j \in Z'$ we have $|B^j_{i,A}| \le \nu^{1/4}a_1^2$, so there are at most $\nu^{1/4}N$ triples in $G'[S']_S$ for triples S with $j \in S \in K(V)$ and $|M[S']_S| > \sqrt{\nu}|K(S')_S|$. Finally, there are at most $r(n/r)^2n_1 = N/r$ triples that are not r-partite. Altogether, at most $\frac{3}{2}d_3N$ triples contributing to $d'_w(j)$ do not belong to $G'[S']_S$ with $j \in S \in R$. Since $|G(v)| = (1\pm\epsilon)c_1n^2$ for all $v \in G$ and $\epsilon \ll 1/a \ll \nu \ll 1/r \ll d_3$ we have

$$d'_w(j) = N^{-1}\sum_{S:j\in S\in R}|G'[S']_S| = N^{-1}\sum_{v\in C_{i,j}}|G(v)| \pm 3d_3/2 = c_1 \pm 2d_3.$$

Define $t : E(R') \to \mathbb{R}^2$ by $t(S) = w(S)/(c_1+2d_3)$. Then $t'(j) = d'_w(j)/(c_1+2d_3) \le 1$ for $j \in R'$, so t is a fractional matching. We have

$$t(R') = \sum_{j\in R'}\sum_{S:j\in S}t'(j)/3 = \sum_{j\in R'}d'_w(j)/3(c_1+2d_3) > (1-\sqrt{d_3})|Z'|/3.$$

Also, the trivial bound $w(S) < n_1^3/N$ gives $\mathrm{co}(t) < n/n_1 \cdot n_1^3/(c_1+2d_3)N < n_1/c_1n < d_3$. Applying Theorem 5.17, there is a matching in R' of size at least $|Z'|/3 - \frac{1}{6}\delta|Z'|$, i.e. at most $\frac{1}{2}\delta|Z'|$ vertices are not covered by the matching.

We can use an edge S of the matching as follows. Consider the partition P^*_S of $K(V)_S$ by weak equivalence, i.e. we have a cell of P^*_S corresponding to each triad of consistent bipartite graphs indexed by S. The cells lying over $S' = \cup_{i\in S}C_{i,S_i}$ give a partition of $K(S')$, which we denote by $C^{S,1} \cup \cdots \cup C^{S,a_S}$. Since P is a-bounded we have $a_S \le a^3$. Furthermore, since P is equitable, the triangle counting lemma (1) gives $|C^{S,i}| = (1\pm\epsilon')|C^{S,j}|$ for any $1 \le i,j \le a_S$. Now at most $2\nu^{1/4}|K(S')_S|$ triples of $G'[S']_S$ can lie in cells $C^{S,i}$ with $|M \cap C^{S,i}| > \nu^{1/4}|C^{S,i}|$; otherwise we would have at least $(1-\epsilon')2\nu^{1/4}a_S$ such cells, giving $|M[S']| \ge (1-\epsilon')2\nu^{1/4}a_S \cdot \nu^{1/4}(1-\epsilon')|K(S')_S|/a_S > (1-3\epsilon')2\sqrt{\nu}|K(S')_S|$, contradicting $S \in R$. Since $|G'[S']_S| > d_3|K(S')_S|$ and $\nu \ll d_3$, more than $\frac{1}{2}d_3|K(S')_S|$ triples of $G'[S']_S$ lie in cells $C^{S,i}$ with $|M \cap C^{S,i}| < \nu^{1/4}|C^{S,i}|$, so we can choose such a cell $C^{S,i}$ with $|G' \cap C^{S,i}| > \frac{1}{2}d_3|C^{S,i}|$.

Fix such a cell $C^{S,i}$ for each matching edge S and let G^S be the associated cell complex of $G'[P]$, i.e. $G^S_S = G' \cap C^{S,i}$ and for $S' \subsetneq S$, $G^S_{S'}$ is the cell of $P_{S'}$ underlying $C^{S,i}$. Then G^S is ϵ-regular. Also, $|G^S_S| > \frac{1}{2}d_3|C^{S,i}|$, so writing $M^S = M \cap G^S$ we have $|M^S| < \nu^{1/4}|C^{S,i}| < \nu^{1/5}|G^S_S|$. At this stage, if we were satisfied with an F-packing covering all but $o(n)$ vertices, we could just repeatedly remove copies of F from each $G^S \setminus M^S$.[4] However, we want to cover all but at most C vertices, so we will apply the blow-up lemma, using the black box form in the previous subsection. By Lemma 5.13 we can delete at most $2\nu^{1/15}n_1$ vertices

[4]This could be achieved using the counting lemma to count copies of F in G^S and the 'extension lemma' to bound the number of copies of F using an edge in M^S. Alternatively one could start the proof with the 'regular decomposition lemma' instead of the regular approximation lemma, then find F using the sparse counting lemma.

from each cluster so that each (G^S, M^S) becomes a pair $(G^{\sharp S}, M^{\sharp S})$ such that $J^S = G^{\sharp S} \setminus M^{\sharp S}$ is $1/2$-robustly $|Y|^2$-universal.

Now we gather together all the removed vertices into an exceptional set A_0. This includes at most $a!$ vertices removed at the start of the proof, at most $\nu^{1/5}n$ vertices in parts indexed by $Z \setminus Z'$, at most $\frac{1}{2}\delta n$ vertices in parts not covered by the matching, and at most $2\nu^{1/15}n$ vertices deleted in making the pairs robustly universal. Therefore $|A_0| < \delta n$. For convenient notation we denote the clusters of $G^{\sharp S}$ by $A_{S,1}$, $A_{S,2}$, $A_{S,3}$. Thus A_0 and $A_{S,1}$, $A_{S,2}$, $A_{S,3}$ for matching edges S partition the vertex set of G. To cover the vertices of A_0 by copies of F we need the following claim.

Claim. *Any vertex v belongs to at least $\beta n^{|Y|-1}$ copies of F in G.*

Proof. Let Φ be the set of all pairs (ab, T) such that $ab \in G(v)$ and $T \in \binom{G(ab)\setminus v}{|Y_3|-1}$. There are $|G(v)| > (1-\epsilon)c_1 n^2$ choices for ab, and for each ab the minimum degree property gives at least $\binom{c_2 n-1}{|Y_3|-1}$ choices for T. Therefore $|\Phi| \geq (1-\epsilon)c_1 n^2\binom{c_2 n-1}{|Y_3|-1}$. Let Ψ be the set of all $T \in \binom{V(G)\setminus v}{|Y_3|-1}$ such that there are at least $\frac{1}{3}c_1 c_2^{|Y_3|-1}n^2$ pairs $ab \in G(v)$ with $(ab, T) \in \Phi$. Then $|\Phi| < |\Psi|n^2 + \binom{n}{|Y_3|-1} \cdot \frac{1}{3}c_1 c_2^{|Y_3|-1}n^2$, so

$$|\Psi| > (1-\epsilon)c_1\binom{c_2 n-1}{|Y_3|-1} - \binom{n}{|Y_3|-1} \cdot \frac{1}{3}c_1 c_2^{|Y_3|-1} > \frac{1}{3}c_1 c_2^{|Y_3|-1}\binom{n}{|Y_3|-1}.$$

For each T in Ψ, since $\alpha \ll c_1, c_2$, Theorem 5.15 implies that the sets $ab \in G(v)$ with $(ab, T) \in \Phi$ span at least $\alpha n^{|Y_1|+|Y_2|}$ copies of $K_{|Y_1|,|Y_2|}$. Each of these gives a copy of F containing v when we add $T \cup v$. Summing over T in Ψ and dividing by $|Y|!$ (a crude estimate for overcounting) we obtain (since $\beta \ll \alpha$) at least $\beta n^{|Y|-1}$ copies of F containing v. ∎

Next we randomly partition each set $A_{S,j}$ as $A'_{S,j} \cup A''_{S,j}$, each vertex being placed independently into either class with probability $1/2$. The reason for this partition is that, as in the proof of Theorem 5.1, we will be able to use the sets $A'_{S,j}$ when covering the vertices in A_0, whilst preserving the vertices in $A''_{S,j}$ so as to maintain super-regularity. Theorem 5.16 gives the following properties with high probability:

1. $|A'_{S,j}|$ and $|A''_{S,j}|$ are $|A_{S,j}|/2 \pm n^{2/3}$ for every S and j,
2. for every S and j and each vertex $v \in A_{S,j}$, letting $\{T_i\}_{i \neq j}$ denote the singleton classes of $G^{\sharp S}(v)$, $|A'_{S,j} \cap T_i|$ and $|A''_{S,j} \cap T_i|$ are $|T_i|/2 \pm n^{2/3}$ for $i \neq j$, and
3. for any vertex v of G, there are at least $\gamma n^{|Y|-1}$ copies of F in which all vertices, except possibly v, are in $\cup_{S,j}A'_{S,j}$.

In fact, the first two properties are simple applications of Chernoff bounds (in which the martingale is just a sum of independent variables). For the third property we use a vertex exposure martingale. Fix v and let Z be the random variable which is the number of copies of F in which all vertices, except possibly v, are in $\cup_{S,j}A'_{S,j}$. Since $|A_0| < \delta n$, the Claim gives $\mathbb{E}Z > (1/2)^{|Y|-1}(\beta - \delta)n^{|Y|-1}$. Order the vertices of $\cup_{S,j}A_{S,j}$ as $v_1, \ldots, v_{n'}$, where $n' > (1-\delta)n$, and define the random variable Z_i as the conditional expectation of Z given whether $v_{i'}$ is in $A'_{S,j}$ or $A''_{S,j}$ for $i' \leq i$. Then $Z_0 = \mathbb{E}Z$ and $Z_{n'} = Z$. Also $|Z_i - Z_{i-1}| < n^{|Y|-2}$, using a crude upper bound on the number of copies of F containing v and some other vertex v_j. Now by Theorem 5.16 we have $\mathbb{P}(Z < \gamma n^{|Y|-1}) < \mathbb{P}(|Z_{n'} - Z_0| > 2^{-|Y|}\beta n^{|Y|-1})$ $< e^{-\beta^3 n}$.

Now we cover A_0 by the following greedy procedure. Suppose we are about to cover a vertex $v \in A_0$. We consider a cluster to be *heavy* if we have covered more than γn_1 of its vertices. Since $|A_0| < \delta n$ we have covered at most $|Y|\delta n$ vertices by copies of F, so there are at most $|Y|\delta n/\gamma n_1 < \frac{1}{2}\gamma r a_1$ heavy clusters. As shown above, there at least $\gamma n^{|Y|-1}$ copies of F that we can use to cover v. At most $\frac{1}{2}\gamma r a_1 n_1 n^{|Y|-2} < \frac{1}{2}\gamma n^{|Y|-1}$ of these use a heavy cluster, so we can cover A_0 while avoiding heavy clusters.

Next we restrict to the vertices not already covered by the copies of F covering A_0, where we will use robust universality to finish the packing. Recall that each J^S is 1/2-robustly $|Y|^2$-universal. By properties (1) and (2) of the partitions $A_{S,j} = A'_{S,j} \cup A''_{S,j}$, on restricting to the uncovered vertices we obtain J'^S that satisfies conditions (i) in Definition 5.12. (Property (i) of Theorem 5.13 and $d^\sharp \gg 1/n$ shows that the $\pm n^{2/3}$ errors are negligible.) Also, any F-packing has maximum degree less than $|Y|^2$, so satisfies condition (ii) in Definition 5.12. Thus we can assume that J'^S is complete, in that we can embed any F-packing in J'^S, subject only to the constraints given by the sizes of the uncovered parts of each cluster.

Now it is not hard to finish the proof with a slightly messy ad hoc argument, but we prefer to use the elegant argument of Kómlos [23, Lemma 12]. Denote the classes of J'^S by J_1, J_2, J_3. Since we avoided heavy clusters we have $(1-2\gamma)n_1 \le |J_i| \le n_1$ for $1 \le i \le 3$. Let $P^3 = \{\alpha \in [0,1]^3 : \alpha_1 \le \alpha_2 \le \alpha_3, \sum_{i=1}^3 \alpha_i = 1\}$. We can associate a 'class vector' $\alpha(X) \in P^3$ to a 3-partite set $X = X_1 \cup X_2 \cup X_3$ by $\alpha(X)_i = |X_{\sigma(i)}|/|X|$, for some permutation $\sigma \in S_3$ chosen to put the classes in increasing order by size. For $\alpha, \beta \in P^3$ write $\alpha \prec \beta$ if $\alpha_1 \le \beta_1$ and $\alpha_1 + \alpha_2 \le \beta_1 + \beta_2$. Since the classes of F are not all of equal size and $\gamma \ll 1/|Y|$ we have $\alpha(F) \prec \alpha(J)$. By a theorem of Hardy, Littlewood and Pólya, this implies that there is a doubly stochastic matrix M such that $\alpha(J) = M\alpha(F)$. By Birkhoff's theorem M is a convex combination of permutation matrices $M = \sum_i \lambda_i P_i$, $\sum \lambda_i = 1$. Thus we can write the class vector of J as $\alpha(J) = \sum_i \lambda_i P_i \alpha(F)$, which is a convex combination of the permutations of the class vector of F. In fact, although the constant is not important, since P^3 has dimension 3, we can apply Carathéodory's theorem to write $\alpha(J) = \sum_{i=1}^3 \mu_i P_i \alpha(F)$, $\sum_{i=1}^3 \mu_i = 1$ as a convex combination using only 3 permutations of $\alpha(F)$.[5] Finally, to pack copies of F in J'^S we can use $\lfloor \mu_i |J|/|Y| \rfloor$ copies of F permuted according to P_3, for $1 \le i \le 3$. At most $3|Y|$ vertices of any J_i are left uncovered because of the rounding, so in total at most $C = 3|Y|ra_1$ vertices will remain uncovered. This completes the proof. ▪

As for Theorem 5.1, we needed to assume that not all part sizes of F are equal and we could not expect to cover all vertices. Some assumption on the degrees of pairs was convenient, as without it a nearly regular 3-graph can have some vertices that do not belong to any copies of F. For example, let G_0 be a tripartite 3-graph on $V = V_1 \cup V_2 \cup V_3$ with $|V_1| = |V_2| = |V_3| = n_0/3$ such that every vertex v has degree $|G_0(v)| = (1 \pm \epsilon)c_1 n_0^2$. Form G from G_0 by adding new vertices $v_1, \dots, v_t$ where $t \approx (3c_1)^{-1}$ and edges so that $G(v_i)$ are pairwise disjoint graphs of size $c_1 n_0^2$ contained in $\binom{V_1}{2} \cup \binom{V_2}{2} \cup \binom{V_3}{2}$. Then G has $n = n_0 + t$ vertices and $|G(v)| = (1 \pm 2\epsilon)c_1 n^2$ for every vertex v. However, for every new vertex v_i, every pair $ab \in G(v_i)$ is only contained in the edge $v_i ab$, so v_i is not contained in any $K_{2,2,3}$ (say). This example does not show that the assumption on pairs is necessary, as we can still cover all but t vertices, but it at least indicates that it may not be so easy to remove the assumption. For simplicity we assumed that every pair has many neighbours, but it is clear from the proof that this assumption can be relaxed somewhat. The bottleneck is the Claim,

[5] This last remark is attributed to Endre Boros in [23].

which can be established under the weaker assumption that for every vertex v there are at least $n^{2-\theta}$ pairs ab in $G(v)$ with $|G(ab)| > n^{1-\theta}$, for some $\theta > 0$ depending on F.

Finally, we remark that one can apply the general hypergraph blow-up lemma in the next section and the same proof to obtain the following result (we omit the details). For any k-partite k-graph F in which not all part sizes are equal and $0 < c_1, c_2 < 1$ there is a real $\epsilon > 0$ and positive integers C, n_0 such that if G is an k-graph on $n > n_0$ vertices V such that every vertex v has degree $|G(v)| = (1 \pm \epsilon)c_1 n^{k-1}$ and every $(k-1)$-tuple S of vertices has degree $|G(S)| > c_2 n$ then G contains an F-packing that covers all but at most C vertices.

6. GENERAL HYPERGRAPHS

In this section we present the general blow-up lemma. Besides working with k-graphs for any $k \geq 3$, we will introduce the following further generalisations:

(i) Restricted positions: a small number of sets in H may be constrained to use a certain subset of their potential images in G (provided that these constraints are regular and not too sparse).
(ii) Complex-indexed complexes: a structure that provides greater flexibility, in particular the possibility of embedding spanning hypergraphs (such as Hamilton cycles).

We divide this section into five subsections organised as follows. The first subsection contains various definitions needed for the general case, some of which are similar to those already given for 3-complexes and some of which are new. In the second subsection we state the general blow-up lemma and the algorithm that we use to prove it. The third subsection contains some properties of hypergraph regularity, analogous to those proved earlier for 3-graphs. We give the analysis of the algorithm in the fourth subsection, thus proving the general blow-up lemma. Since much of the analysis is similar to that for 3-graphs we only give full details for those aspects of the general case that are different. The last subsection contains the general cases of the lemmas to be used in applications of the blow-up lemma, namely Lemmas 5.9, 5.11 (super-regular restriction) and 5.13 (robust universality).

6.1. Definitions

We start by defining complex-indexed complexes.

Definition 6.1. *We say R is a* multicomplex *on $[r]$ if it consists some number of copies (possibly 0) of every $I \subseteq [r]$ which are partially ordered by some relation, which we denote by $\subseteq$, such that whenever $I^* \in R$ is a copy of some subset I of $[r]$ and J is a subset of I, there is a unique copy J^* of J with $J^* \subseteq I^*$. We say that R is a* multi-k-complex *if $|I| \leq k$ for all $I \in R$.*

Suppose V is a set partitioned as $V = V_1 \cup \cdots \cup V_r$. Suppose each V_i, $1 \leq i \leq r$ is further partitioned as $V_i = \cup_{i^*} V_{i^*}$, where i^* ranges over all copies of i in R. We say G is an R-indexed complex *on V if it consists of disjoint parts G_I for $I \in R$ (possibly undefined), such that $G_i \subseteq V_i$ for singletons $i \in R$, and $G_{I^\leq} := \cup_{I' \subseteq I} G_{I'}$ is a complex whenever G_I is defined. We say that an R-indexed complex J on V is an* R-indexed subcomplex *of G if $J_I \subseteq G_I$ when defined. We say $S \subseteq V$ is* r-partite *if $|S \cap V_i| \leq 1$ for $1 \leq i \leq r$. The*

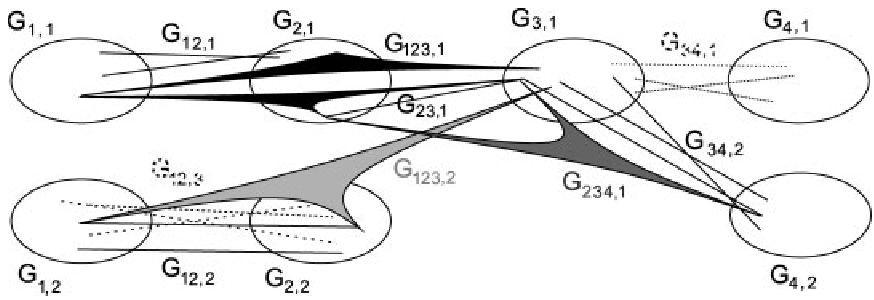

Fig. 1. A complex-indexed complex.

multi-index $i^*(S)$ *of* S *is that* $I \in R$ *with* $S \in G_I$. *If* $S \in G$ *we write* $G_S = G_{i^*(S)}$ *for the part of* G *containing* S.

We emphasise that $\subseteq$ is more restrictive than the inclusion relation (also denoted $\subseteq$) between copies considered merely as subsets of $[r]$. To avoid confusion we never 'mix' subsets with copies of subsets. Thus, if $I \in R$ then $J \subseteq I$ means $J \in R$ and (J, I) is in the relation $\subseteq$. Also, if $I \in R$ then $i \in I$ means $\{i\} \in R$ and $(\{i\}, I)$ is in the relation $\subseteq$. We also write $i \in R$ to mean $\{i\} \in R$ when the meaning is clear from the context. As in Definition 3.2, we henceforth simplify notation by writing i instead of $\{i\}$.

Remark 6.2. We are adopting similar notation for complex-indexed complexes as for usual complexes for ease of discussing analogies between the two situations. Thus we typically denote a singleton multi-index by i and a set multi-index by I. If we need to distinguish a multi-index from the index of which it is a copy, we typically use the notation that i^* is a copy of i and I^* is a copy of I.

We illustrate Definition 6.1 with the following example.

Example 6.3. Figure 1 depicts an example of an R-indexed complex G in which R is a multi-3-complex on $[4]$ (not all parts have been labelled). The multi-indices have been represented as ordered pairs (A, t), where $A \in \binom{[4]}{\leq 3}$ and t is a number (arbitrarily chosen) to distinguish different copies of A. An example of the inclusion structure is $(34, 2) \subseteq (234, 1)$, since the intended interpretation of our picture is that for every triple in $G_{234,1}$ its restriction to index 34 lies in $G_{34,2}$. Other examples are $(1, 2) \subseteq (123, 2)$ and $(2, 2) \subseteq (123, 2)$, but $(1, 1) \not\subseteq (123, 2)$ and $(12, 3) \not\subseteq (123, 2)$, since the intended interpretation of our picture is that there are triples in $G_{123,2}$ such that their restriction to index 12 is a pair in $G_{1,2} \times G_{2,2}$ not belonging to $G_{12,3}$.

Complex-indexed complexes arise naturally from the partition complexes needed for regular decompositions of hypergraphs, as in Theorem 5.7. Suppose P is an r-partite partition k-complex on V. Recall that C_S denotes the cell containing a set S. We can index the cells of P by a multicomplex R on $[r]$, where for each cell C_S we form a copy $i^*(S)$ of its (usual) index $i(S)$. The elements of R are partially ordered by a relation $\subseteq$ which corresponds to the consistency relation $\leq$ discussed above, i.e. $i^*(S') \subseteq i^*(S)$ exactly when $C_{S'} \leq C_S$. One could think of P as a 'complete R-indexed k-complex', in that it contains every r-partite set of size at most k, although we remark that many partition k-complexes P will give rise to the same multicomplex R, so the phrase is ambiguous.

Note that if we do not allow sets in R to have multiplicity more than 1 then an R-indexed complex is precisely an r-partite complex. The reason for working in the more general context is that R-indexed complexes are the structures that naturally arise from an application of Theorem 5.7, so a general theory of hypergraph embedding will need to take this into account. In particular, in order to use every vertex of V we need to consider every part of the partition P_i of V_i for $1 \leq i \leq r$, i.e. we need multi-index copies of each index i. The multi-index copies of larger index sets are useful because it may not be possible to choose mutually consistent cells. To illustrate this point, it may be helpful to consider an example of a 4-partite 3-complex where each triple has constant density (say $1/10$) and there is no tetrahedron K_4^3. A well-known example of Rödl is obtained by independently orienting each pair of vertices at random and taking the edges to be all triples that form cyclic triangles. (A similar example is described in [36].) Then we cannot make a consistent choice of cells in any 4-partite subcomplex so that each cell has good density. However, by working with indexed complexes one can embed using cells from each of the four triples, provided that the choice of cells is locally consistent.

Much of the notation we set up for r-partite 3-complexes extends in a straightforward manner to R-indexed complexes for some multi-k-complex R on $[r]$. Throughout we make the following replacements: replace '3' by 'k', 'r-partite' by 'R-indexed', '$I \subseteq [r]$' by '$I \in R$', 'index' by 'multi-index', $i(S)$ by $i^*(S)$, and understand $\subseteq$ as the partial order of multi-indices. Thus we define $G_{I^\leq} = \cup_{I' \subseteq I} G_{I'}$, $G_{I^<} = \cup_{I' \subsetneq I} G_{I'}$, etc. as in Definition 3.2. We define restriction of R-indexed complexes as in Definition 3.5, and more generally composition of R-indexed complexes G and G' as in Definition 4.4: we define $(G * G')_S$ if $(G \cup G')_S$ is defined and say that $S \in (G * G')_S$ if $A \in^* G_A$ and $A \in^* G'_A$ for any $A \subseteq S$. Lemma 4.5 applies to R-indexed complexes, with the same proof.

As in Definition 3.9, if $S \subseteq X$ is r-partite and $I \subseteq i^*(S)$ we write $S_I = S \cap \cup_{i \in I} X_i$. We also write $S_T = S_{i(T)}$ for any r-partite set T with $i^*(T) \subseteq i^*(S)$. As in Definition 3.6 we write G_I^* for the set of r-partite $|I|$-tuples S with $i^*(S) = I$ such that $T \in G_T$ when defined for all $T \subsetneq S$. When defined we have relative densities $d_I(G) = |G_I|/|G_I^*|$ and absolute densities $d(G_I) = |G_I|/\prod_{i \in I} |V_i|$. (Recall that $i \in I$ means $\{i\} \subseteq I$ according to R.)

To define regularity we adopt the reformulation using restriction notation already discussed for 3-complexes. For any $I \in R$ such that G_I is defined we say that G_I is *ϵ-regular* if for every subcomplex J of G with $|J_I^*| > \epsilon |G_I^*|$ and J_I undefined we have $d_I(G[J]) = d_I(G) \pm \epsilon$. Note that if $|I| = 0, 1$ we have $G[J]_I = G_I$, so G_I is automatically ϵ-regular for any ϵ. We say that G is *ϵ-regular* if whenever any G_I is defined it is ϵ-regular.

Remark 6.4. Some care is needed when forming neighbourhoods in complex-indexed complexes. The usual definition defines $G_I(v)$ and $G_{I^\leq}(v)$. However the expression $G(v)_I$ may be ambiguous, as there may be several multi-indices I' such that $I = I' \setminus i$, where $i = i^*(v)$. We avoid this ambiguous expression unless the meaning is clear from the context. The expression $P \in G(v)$ is unambiguous: it means $Pv \in G$.

Henceforth we suppose R is a multi-k-complex on $[r]$, H is an R-indexed complex on $X = \cup_{i \in R} X_i$ and G is an R-indexed complex on $V = \cup_{i \in R} V_i$, with $|V_i| = |X_i|$ for $i \in R$. As before we want to find an embedding ϕ of H in G, via an algorithm that considers the vertices of X in some order and embeds them one at a time. At some time t in the algorithm, for each $S \in H$ there will be some $|S|$-graph $F_S(t) \subseteq G_S$ consisting of those sets $P \in G_S$ that are 'free' for S, in that mapping S to P is 'locally consistent' with the embedding so far. These free sets will be 'mutually consistent', in that $F_{S^\leq}(t) = \cup_{S' \subseteq S} F_{S'}(t)$ is a complex. With the modifications already mentioned, we can apply the following definitions and lemmas (and their proofs) to the general case: Definition 3.7 (the update rule), Lemma 3.10 (consistency), Lemma 3.11 (iterative construction) and Lemma 3.12 (localisation). Just as some triples were marked as forbidden for 3-graphs, in the general case some k-tuples M will be marked as forbidden. Definition 3.13 ($M_{E^t,E}(t)$) applies in general when $E \in H$ is a k-tuple.

Definition 3.15 is potentially ambiguous for R-indexed complexes (see Remark 6.4), so we adopt the following modified definition.

Definition 6.5. *Suppose G is an R-indexed complex on $V = \cup_{i \in R} V_i$, $i \in R$, $v \in G_i$ and I is a submulticomplex of R. We define $G^{I_v} = G[\cup_{S \in I} G_S(v)]$.*

Now we give the general definition of super-regularity, which is very similar to that used for 3-graphs.

Definition 6.6 (Super-regularity). *Suppose R is a multi-k-complex on $[r]$, G is an R-indexed complex on $V = \cup_{i \in R} V_i$ and $M \subseteq G_= := \{P \in G : |P| = k\}$. We say that (G, M) is $(\epsilon, \epsilon', d_a, \theta, d)$-super-regular if*

(i) *G is ϵ-regular, and if G_S is defined then $d_S(G) \geq d$ if $|S| = k$ or $d_S(G) \geq d_a$ if $|S| < k$,*
(ii) *if G_S is defined, $i \in S$ and $v \in G_i$ then $|M_S(v)| \leq \theta |G_S(v)|$ if $|S| = k$ and $G_{S^\leq}(v)$ is ϵ'-regular with $d_{S' \setminus i}(G_{S^\leq}(v)) = (1 \pm \epsilon') d_{S' \setminus i}(G) d_{S'}(G)$ for $i \subsetneq S' \subseteq S$,*
(iii) *for every submulticomplex I of R, if G_S is defined, $i \in R$ and $v \in G_i$, then $|(M \cap G^{I_v})_S| \leq \theta |G^{I_v}_S|$ if $|S| = k$, and G^{I_v} is ϵ'-regular with densities (when defined)*

$$d_S(G^{I_v}) = \begin{cases} (1 \pm \epsilon') d_S(G) d_T(G) & \text{if } \emptyset \neq S = T \setminus i, T \in I, G_T \text{ defined}, \\ (1 \pm \epsilon') d_S(G) & \text{otherwise}. \end{cases}$$

We remark that the parameters in Definition 6.6 will satisfy the hierarchy $\epsilon \ll \epsilon' \ll d_a \ll \theta \ll d$. Thus we work in a dense setting with regularity parameters much smaller than density parameters. Note that we have two density thresholds d_a and d, where d_a is a lower bound on the S-densities when $|S| < k$ and d is a lower bound on the S-densities when $|S| = k$. The marking parameter θ again lies between these thresholds. Next we will formalise the notation $d_S(F(t)) = d_S(F_{S^\leq}(t))$ used for 3-complexes by defining $F(t)$ as an object in its own right: a *complex-coloured complex*.

Definition 6.7. *Suppose R is a multi-k-complex on $[r]$, H is an R-indexed complex on $X = \cup_{i \in R} X_i$ and G is an R-indexed complex on $V = \cup_{i \in R} V_i$. An H-*coloured complex F (in *G) consists of $|S|$-graphs $F_S \subseteq G_S$ such that $F_{S^\leq} = \cup_{S' \subseteq S} F_{S'}$ is a complex for $S \in H$. We allow F_S to be undefined for some $S \in H$. We say that an H-coloured complex J is an H-*coloured subcomplex *of F if $J_S \subseteq F_S$ when defined. The* restriction $F[J]$ *is the H-coloured complex with $F[J]_S = F_{S^\leq}[J_{S^\leq}]_S$. When $F(t)$ is an H-coloured complex at time t we use $F(t)_S$ and $F_S(t)$ interchangeably. We let G also denote the H-coloured complex F in G such that $F_S = G_S$ for all $S \in H$. When I is a subcomplex of H we let F_I denote the H-coloured complex that consists of F_S for every $S \in I$ and is otherwise undefined.*

We use the terminology 'coloured' in analogy with various combinatorial questions involving hypergraphs in which each set can be assigned several colours. In our case a set

$E \in G$ is assigned as colours all those $S \in H$ for which $E \in F_S$. Note also that if we had the additional property that the parts F_S, $S \in H$ were mutually disjoint, i.e. any set in G has at most one colour from H, then F would be an H-indexed complex. (We make this comment just to illustrate the definition: we will not have cause to consider any H-indexed complexes.)

In the proof of Theorem 4.1 there were several places where we divided the argument into separate cases. This will not be feasible for general k, so we will introduce some more notation, which may at first appear somewhat awkward, but will repay us by unifying cases into a single argument.

Definition 6.8. *Suppose R is a multi-k-complex on $[r]$, H is an R-indexed complex on $X = \cup_{i\in R} X_i$ and G is an R-indexed complex on $V = \cup_{i\in R} V_i$. Fix $x \in X$.*

(i) *We define a multi-$(k+1)$-complex R^+ on $[r+1]$ as follows. There is a single multi-index copy of $r+1$, also called $r+1$. Suppose $x \in X_{i^*}$, where $i^* \in R$ is a copy of some $i \in [r]$. Consider $I^* \in R$ such that I^* is a copy of $I \subseteq [r]$. If $i^* \in I^*$ we let I^{*c} be a copy of $(I \setminus i) \cup \{r+1\}$. If $i^* \notin I^*$ and $I^* \neq J \setminus i^*$ for any $J \in R$ we let I^{*+} be a copy of $I \cup \{r+1\}$. We extend $\subseteq$ by the rules (when defined) $J \subseteq I^{*c}$ for $i^* \notin J \subseteq I^*$; $J^c \subseteq I^{*c}$ for $i^* \in J \subseteq I^*$; $J^c \subseteq I^{*+}$ for $i^* \in J$ and $J \setminus i^* \subseteq I^*$; $I^* \subseteq I^{*+}$; and $J^+ \subseteq I^{*+}$ for $J \subseteq I^*$.*

(ii) *Let $X_{r+1} = \{x^c\}$ consist of a single new vertex that we consider to be a copy of x.[6] Let H^+ be the R^+-indexed complex $H \cup \{Sx^c : S \in H\}$ on $X^+ = X \cup X_{r+1}$, where $i^*(Sx^c)$ is $(i^*(Sx))^c$ if $Sx \in H$ or $i^*(S)^+$ if $Sx \notin H$. If $Sx \in H$ we write $(Sx)^c = Sx^c$. If $x \notin S \in H$ we write $S^c = S$. If $S \in H \setminus H(x)$ we write $S^+ = Sx^c$. Note that $i^*((Sx)^c) = (i^*(Sx))^c$ and $i^*(S^+) = (i^*(S))^+$.*

(iii) *Let V_{r+1} be a new set of vertices disjoint from V having the same size as $V_x = V_{i^*}$. We think of V_{r+1} as a copy of V_x, in that for each $v \in V_x$ there is a copy $v^c \in V_{r+1}$. Let G^+ be the R^+-indexed complex $G \cup \{Pv^c : P \in G, v \in V_x\}$ on $V^+ = V \cup V_{r+1}$, where $i^*(Pv^c)$ is $i^*(Pv)^c$ if $Pv \in G$ or $i^*(P)^+$ if $Pv \notin G$. If $Pv \in G$ we write $(Pv)^c = Pv^c$. If $v \notin P \in G$ we write $P^c = P$. If $P \in G \setminus G(v)$ we write $P^+ = Pv^c$. Note that $i^*((Pv)^c) = (i^*(Pv))^c$ and $i^*(P^+) = (i^*(P))^+$.*

(iv) *If $I \subseteq H$ or $I \subseteq G$ we write $I^c = \{A^c : A \in I\}$.*

(v) *Suppose F is an H-coloured complex in G. We define H^+-coloured complexes $F^c = \bigcup_{S\in H} F^c_S$ and $F^+ = \bigcup_{S\in H}(F_S \cup F^c_S) = F \cup F^c$. Suppose I is an R-indexed subcomplex of H. The* plus complex *is $F^{I+x} = G^+[F \cup F^+_{I^c}]$.*

We give the following example to illustrate Definition 6.8.

Example 6.9. As in Example 3.8, suppose that H and G are 4-partite 3-complexes, that we have 4 vertices $x_i \in X_i$, $1 \le i \le 4$ that span a tetrahedron K^3_4 in H, that we have the edges $x'_1x'_2x_3$ and $x'_1x'_3x'_4$ and all their subsets for some other 4 vertices $x'_i \in X_i$, $1 \le i \le 4$, and that there are no other edges of H containing any x_i or x'_i, $1 \le i \le 4$. We can think of H and G as R-indexed complexes with $R = \binom{[4]}{\le 3}$, i.e. we have one copy of each subset of [4] of size at most 3.

[6] To avoid confusion we should point out that the use of 'copy' here is different to the sense in which multi-indices are copies of normal indices.

We work through Definition 6.8, setting $x = x_1$. R^+ is the subcomplex of [5] that contains $R = \binom{[4]}{\le 3}$ and all sets $S \cup 5$ with $S \in \binom{[4]}{\le 3}$. H^+ is the R^+-indexed complex on $X^+ = X \cup X_5$ where $X_5 = \{x_1^c\}$ and H^+ consists of all sets S and Sx_1^c with $S \in H$. G^+ is the R^+-indexed complex on $V^+ = V \cup V_5$ where $V_5 = \{v^c : v \in V_1\}$ and G^+ consists of all sets S and Sv^c with $S \in G$ and $v \in V_1$. Note that H^+ and G^+ are 5-partite 4-complexes.

Let $F(0)$ be the H-coloured 3-complex in which $F(0)_S = G_S$ for all $S \in H$. We will describe the plus complex $F(0)^{H+x_1}$. For any $S \in H$ we have $F(0)_S^{H+x_1} = F(0)_S = G_S$ by equation (2). Similarly, for any $S \in H(x_1)$ we have $F(0)_{Sx_1^c}^{H+x_1} = (F(0)_{H^c}^+)_{Sx_1^c} = G_{Sx_1}^c$. (Recall that $S \in H(x_1)$ iff $Sx_1 \in H$.) For example, $F(0)_{x_2x_1^c}^{H+x_1} = G_{x_2x_1}^c = \{v_2v_1^c : v_2v_1 \in G_{x_2x_1}\}$. If $S \in H \setminus H(x_1)$ then $F(0)_{Sx_1^c}^{H+x_1}$ consists of all $P \in G_{Sx_1^c}^+$ such that for all $S' \subseteq S$ we have $P_{S'} \in G_{S'}$ and $P_{S'x_1^c} \in G_{S'x_1}^c$ if $S' \in H(x_1)$. For example, $F(0)_{x_1x_2x_3x_1^c}^{H+x_1}$ consists of all 4-tuples $v_1'v_2v_3v_1^c$ where $v_1v_2v_3$ and $v_1'v_2v_3$ are in G_{123}.

Now suppose, as in Example 3.8, that we start the embedding by mapping x_1 to some $v_1 \in V_1$. Let $F(1)$ be the H-coloured 3-complex given by the update rule: this is worked out in Example 3.8 and formally defined in Definition 3.7. We can also describe it using the plus complex. For example, we saw that $F_{x_2x_3x_4}(1) = F(1)_{x_2x_3x_4}$ consists of all triples in G_{234} that form a triangle in the neighbourhood of v_1, i.e. form a tetrahedron with v_1. We can write this as $F(1)_{x_2x_3x_4} = F(0)^{H+x_1}(v_1^c)_{x_2x_3x_4}$, as by definition $v_2v_3v_4v_1^c \in F(0)_{x_2x_3x_4x_1^c}^{H+x_1}$ exactly when $v_1v_2v_3v_4$ is a tetrahedron. Another example from Example 3.8 is that $F(1)_{x_1'x_2'x_3}$ consists of all triples $P \in G_{123}$ not containing v_1 such that $P_3 = P \cap V_3$ is a neighbour of v_1. We can write this as $F(1)_{x_1'x_2'x_3} = F(0)^{H+x_1}(v_1^c)_{x_1'x_2'x_3} \setminus v_1$, as by definition $v_1'v_2v_3v_1^c \in F(0)_{x_1'x_2'x_3x_1^c}^{H+x_1}$ exactly when $v_1'v_2v_3 \in G_{123}$ and $v_1v_3 \in G_{13}$.

Note that the plus complex in Definition 6.8(v) depends on H, but we suppress this in the notation, as H will always be clear from the context. We have also suppressed x from the notation in R^+ (etc). One should note that the definition of F^+ is rather different than G^+ and H^+. To clarify this definition, we note that $F_{H^c}^+ = F^c$, and also that $F_{I^c}^+ = F_{I^c}^c$, so one could also write $F^{I+x} = G^+[F \cup F_{I^c}^c]$. To justify the definition of the plus complex, we note that F^c and F^+ are H^+-coloured complexes in G^+, as $F_{(Sx^c)\le}^c$ is the copied version of the complex $F_{Sx\le}$ for any $S \in H$. Then $F \cup F_{I^c}^+$ is an H^+-coloured complex in G^+, with $(F \cup F^+)_S = F_S$ for $S \in H$ and $(F \cup F_{I^c}^+)_{S^c} = F_S^c$ for $S \in I$. Regarding G^+ as an H^+-coloured complex in G^+, the plus complex F^{I+x} is a well-defined H^+-coloured complex in G^+. By equation (2) we have

$$F_S^{I+x} = F_S \text{ for } S \in H \text{ and } F_{S^c}^{I+x} = F_S^c \text{ for } S \in I.$$

As noted in Remark 6.4, some care must be taken to avoid ambiguity when defining neighbourhoods. We adopt the following convention:

$$F^{I+x}(v^c)_S = F_{Sx^c}^{I+x}(v^c) \text{ for } S \in H.$$

Note that we set $I = H$ in Example 6.9, and indeed this is the typical application of this definition. The reason for allowing general I is for proving the analogue of Lemma 4.21 in the general case. Next we prove a lemma which confirms that the plus complex does describe the update rule in general, when we map x to $\phi(x) = y$ at time t.

Lemma 6.10. *If $x \notin S \in H$ then $F(t)_{S\le} = F(t-1)^{H+x}(y^c)_{S\le} \setminus y$.*

Proof. By Definition 3.7 we have $F_{S^\leq}(t) = F_{S^\leq}(t-1)[F_{S.x^\leq}(t-1)(y)] \setminus y$. Thus $P \in F_S(t)$ exactly when $P \in F_S(t-1)$, $y \notin P$ and $P_{S'}y \in F_{S'x}(t-1)$ for all $S' \subseteq S$ with $S' \in H(x)$. Since $P_{S'}y \in F_{S'x}(t-1) \Leftrightarrow P_{S'}y^c \in F_{S'x}(t-1)^c$, Definition 6.8 gives $P \in F_S(t)$ exactly when $y \notin P$ and $Py^c \in F(t-1)^{H+x}_{Sx^c}$. ∎

We also note that the plus complex can describe the construction G^{I_v} in a similar manner. For the following identity we could take $H = R$, considered as an R-indexed complex with exactly one set $S \in H_S$ for every $S \in R$. Actually, the identity makes sense for any R-indexed complex H, when we interpret each R-indexed complex as an H-coloured complex as in Definition 6.7, i.e. G^{I_v} is the H-coloured complex F in G^{I_v} with $F_S = G^{I_v}_S$ for all $S \in H$, and similarly for $G^{I+v}(v^c)$.

Lemma 6.11. *$G^{I+x}(v^c)_S = G^{I_v}_S$ for $S \in H$ and $v \in G_x$.*

Proof. We have $P \in G^{I_v}_S$ exactly when $P \in G_S$ and $P'v \in G$ for all $P' \subseteq P$ with $i^*(P'v) \in I$. Since $P'v \in G \Leftrightarrow P'v^c \in G^c$ this is equivalent to $Pv^c \in G^{I+x}_{Sx^c}$. ∎

6.2. The General Blow-Up Lemma

Now we come to the general blow-up lemma. First we give a couple of definitions. Suppose R is a multi-k-complex on $[r]$. We write $|R|$ for the number of multi-indices in R. For $S \in R$, the *degree* of S is the number of $T \in R$ with $S \subseteq T$.

Theorem 6.12 (Hypergraph blow-up lemma). *Suppose that*

(i) $0 \ll 1/n \ll 1/n_R \ll \epsilon \ll \epsilon' \ll c \ll d_a \ll \theta \ll d, c', 1/D_R, 1/D, 1/C, 1/k$,
(ii) *R is a multi-k-complex on $[r]$ of maximum degree at most D_R with $|R| \leq n_R$,*
(iii) *H is an R-indexed complex on $X = \cup_{i\in R} X_i$ of maximum degree at most D, G is an R-indexed complex on $V = \cup_{i\in R} V_i$, G_S is defined whenever H_S is defined, and $n \leq |X_i| = |V_i| = |G_i| \leq Cn$ for $i \in R$,*
(iv) *$M \subseteq G_= = \{S \in G : |S| = k\}$ and (G, M) is $(\epsilon, \epsilon', d_a, \theta, d)$-super-regular,*
(v) *Γ is an H-coloured complex in G with Γ_x defined only when $x \in X_*$, where $|X_* \cap X_i| \leq c|X_i|$ for all $i \in R$, and for $S \in H$, when defined Γ_S is ϵ'-regular with $d_S(\Gamma) > c'd_S(G)$,*

Then there is a bijection $\phi : X \to V$ with $\phi(X_i) = V_i$ for $i \in R$ such that for $S \in H$ we have $\phi(S) \in G_S$, $\phi(S) \in G_S \setminus M_S$ when $|S| = k$ and $\phi(S) \in \Gamma_S$ when defined.

We make some comments here to explain the statement of Theorem 6.12. An informal statement is that we can embed any bounded degree R-indexed complex in any super-regular marked R-indexed complex, even with some restricted positions. The restricted positions are described by assumption (v): for some sets $S \in H$ we constrain the embedding to satisfy $\phi(S) \in \Gamma_S$, for some H-coloured complex Γ that is regular and dense and is not defined for too many vertices. Note that we now allow the embedding to use all $|R|$ parts of V, provided that R is of bounded degree. Thus this theorem could be used for embedding spanning hypergraphs, such as Hamilton cycles. Even in the graph case, embedding spanning subgraphs is a generalisation of the graph blow-up lemma in [24]. A blow-up lemma for spanning subgraphs was previously given by Csaba [5] (see [20] for another application). Restricted positions have arisen naturally in many applications of the graph blow-up lemma, and will no doubt be similarly useful in future applications of the hypergraph blow-up lemma. In particular, a simplified form of the condition (where Γ_S is only defined when $|S| = 1$) is used in a forthcoming work [19] on embedding loose Hamilton cycles in hypergraphs. We allow a general H-coloured complex Γ as the proof is the same, and it would be needed for embedding general Hamilton cycles.[7]

We prove Theorem 6.12 with an embedding algorithm that is very similar to that used for Theorem 4.1. We introduce more parameters with the hierarchy

$$\begin{aligned}&0 \le 1/n \ll 1/n_R \ll \epsilon \ll \epsilon' \ll \epsilon_{0,0} \ll \cdots \ll \epsilon_{k^3D,3} \ll \epsilon_* \\ &\quad \ll p_0 \ll c \ll \gamma \ll \delta_Q \ll p \ll d_u \ll d_a \ll \theta \ll \theta_0 \ll \theta'_0 \ll \cdots \ll \theta_{k^3D} \ll \theta'_{k^3D} \\ &\quad \ll \theta_* \ll \delta'_Q \ll \delta_B \ll d, c', 1/D, 1/D_R, 1/C, 1/k.\end{aligned}$$

The roles of the parameters from Theorem 4.1 are exactly as before. Our generalisations to R-indexed complexes and restricted positions have introduced some additional parameters, so one should note how they fit into the hierarchy. The restricted positions hypothesis has two parameters c and c'. Parameter c controls the number of restricted positions and satisfies $p_0 \ll c \ll \gamma$. Parameter c' gives a lower bound on the density relative to G of the constraints and satisfies $c' \gg \delta_B$. The indexing complex R has two parameters n_R and D_R. Parameter n_R is a bound for $|R|$ and can be very large, provided that n is even larger. Parameter D_R bounds the maximum degree of R and satisfies $D_R \ll 1/\delta_B$.

Initialisation and notation. Write $X'_* = X_* \cup \bigcup_{x \in X_*} VN_H(x)$. We choose a buffer set $B \subset X$ of vertices at mutual distance at least 9 in H so that $|B \cap X_i| = \delta_B |X_i|$ for $i \in R$ and $B \cap X'_* = \emptyset$. Since $n \le |X_i| \le Cn$ for $i \in R$ and H has maximum degree D we can construct B by selecting vertices one-by-one greedily. Every vertex neighbourhood in H has size less than kD, so there are at most $(kD)^8$ vertices at distance less than 9 from any vertex of H. Similarly, there are at most $(kD_R)^8$ multi-indices $j \in R$ at distance less than 9 from any fixed multi-index $i \in R$. Thus at any stage we have excluded at most $(kD_R)^8(kD)^8\delta_B Cn < n/2$ vertices from X_i, since $\delta_B \ll 1/D_R$. Similarly, since $|X_* \cap X_j| < c|X_j|$ for all $j \in R$ we have $|X'_* \cap X_i| < (kD_R)^8(kD)^8 cCn < \sqrt{c}n$. Since $|X_i| \ge n$ we can construct B greedily. Let $N = \cup_{x \in B} VN_H(x)$ be the set of all vertices that have a neighbour in the buffer. Then N is disjoint from X_*, as we chose B disjoint from X'_*. Also, $|N \cap X_i| < (kD_R)(kD)\delta_B cn < \sqrt{\delta_B}|X_i|$ for any $i \in R$.

For $S \in H$ we set $F_S(0) = G[\Gamma]_S$. We define $L = L(0)$, $q(t)$, $Q(t)$, $j(t)$, $J(t)$, $X_i(t)$, $V_i(t)$ as in the 3-graph algorithm. We let $X(t) = \cup_{i \in R} X_i(t)$ and $V(t) = \cup_{i \in R} V_i(t)$.

Iteration. At time t, while there are still some unembedded non-buffer vertices, we select a vertex to embed $x = s(t)$ according to the same selection rule as for the 3-graph algorithm. We choose the image $\phi(x)$ of x uniformly at random among all elements $y \in F_x(t-1)$ that are 'good' (a property defined below). Note that all expressions at time t are to be understood with the embedding $\phi(x) = y$, for some unspecified vertex y.

[7]While [19] and the current paper have been under review, more general results on Hamilton cycles have been obtained in [28] and [21] without using the hypergraph blow-up lemma. However, it seems most likely that more complicated embedding problems will require the hypergraph blow-up lemma.

Definitions.

1. For a vertex x we write $\nu_x(t)$ for the number of elements in $VN_H(x)$ that have been embedded at time t. For a set S we write $\nu_S(t) = \sum_{y \in S} \nu_y(t)$. We also define $\nu'_S(t)$ as follows. When $|S| = k$ we let $\nu'_S(t) = \nu_S(t)$. When $|S| < k$ we let $\nu'_S(t) = \nu_S(t) + K$, where K is the maximum value of $\nu'_{Sx'}(t')$ over vertices x' embedded at time $t' \le t$ with $S \in H(x')$; if there is no such vertex x' we let $\nu'_S(t) = \nu_S(t)$.
2. For any r-partite set S we define $F_S(t) = F_S(t-1)^y$ if $x \in S$ or $F_S(t) = F_{S^{\le}}(t-1)[F_{S,x^{\le}}(t-1)(y)]_S \setminus y$ if $x \notin S$. We define an *exceptional* set $E_x(t-1) \subseteq F_x(t-1)$ by saying y is in $F_x(t-1) \setminus E_x(t-1)$ if and only if for every unembedded $\emptyset \ne S \in H(x)$,

$$d_S(F(t)) = (1 \pm \epsilon_{\nu'_S(t),0}) d_S(F(t-1)) d_{Sx}(F(t-1)), \text{ and } F_S(t) \text{ is } \epsilon_{\nu'_S(t),0}\text{-regular.} \quad (*_{6.2})$$

3. We define $E^t, M_{E^t,E}(t), D_{x,E}(t-1)$ and $U(x)$ as in the 3-graph algorithm, replacing 'triple' by 'k-tuple'. We obtain the set of *good* elements $OK_x(t-1)$ from $F_x(t-1)$ by deleting $E_x(t-1)$ and $D_{x,E}(t-1)$ for every $E \in U(x)$.

We embed x as $\phi(x) = y$ where y is chosen uniformly at random from the good elements of $F_x(t-1)$. We update $L(t)$, $q(t)$ and $j(t)$ as before, using the same rule for adding vertices to the queue. We repeat until the only unembedded vertices are buffer vertices, but abort with failure if at any time we have $|Q(t) \cap X_i| > \delta_Q |X_i|$ for some $i \in R$. Let T denote the time at which the iterative phase terminates (whether with success or failure).

Conclusion. When all non-buffer vertices have been embedded, we choose a system of distinct representatives among the available slots A'_x (defined as before) for $x \in X(T)$ to complete the embedding, ending with success if possible, otherwise aborting with failure.

Similarly to Lemma 4.2, the algorithm embeds H in $G \setminus M$ unless it aborts with failure. Furthermore, when Γ_S is defined we have $F_S(0) = G[\Gamma]_S = \Gamma_S$, so we ensure that $\phi(S) \in \Gamma_S$. Note that any vertex neighbourhood contains at most $(k-1)D$ vertices. Thus in the selection rule, any element of the queue can cause at most $(k-1)D$ vertices to jump the queue. Note also that when a vertex neighbourhood jumps the queue, its vertices are immediately embedded at consecutive times before any other vertices are embedded. We collect here a few more simple observations on the algorithm.

Lemma 6.13.

(i) *For any $i \in R$ and time t we have $|V_i(t)| \ge \delta_B n/2$.*
(ii) *For any $i \in R$ and time t we have $|J(t) \cap X_i| < \sqrt{\delta_Q} n$.*
(iii) *We have $\nu_x(t) \le (k-1)D$ for any vertex and $\nu'_S(t) \le k^3 D$ for any $S \in H$. Thus the ϵ-subscripts are always defined in $(*_{6.2})$.*
(iv) *For any $z \in VN_H(x)$ we have $\nu_z(t) = \nu_z(t-1) + 1$, so for any $S \in H$ that intersects $VN_H(x)$ we have $\nu_S(t) > \nu_S(t-1)$.*
(v) *If $\nu_S(t) > \nu_S(t-1)$ then $\nu'_S(t) > \nu'_S(t-1)$.*
(vi) *If z is embedded at time $t' \le t$ and $S \in H(z)$ then $\nu'_S(t) \ge \nu'_{Sz}(t) > \nu'_{Sz}(t'-1)$.*

Proof. The proofs of (i) and (iii–vi) are similar to those in Lemma 4.3 so we omit them. For (ii) we have to be more careful to get a good bound inside each part. Note that we only obtain a new element x of $J(t) \cap X_i$ when x is a neighbour of some $b \in B$ and some z within distance 4 of x is queued. In particular z is within distance 5 of x. Given i, there are at most $(kD_R)^5$ choices for $j = i^*(z) \in R$, at most $|Q(t) \cap X_j| < \delta_Q Cn$ choices for z, then at most $(kD)^5$ choices for x. Therefore $|J(t) \cap X_i| < (kD_R)^5 (kD)^5 \delta_Q Cn < \sqrt{\delta_Q} n$. ▪

6.3. Hypergraph Regularity Properties

This subsection contains various properties of hypergraph regularity analogous to those described earlier for 3-graphs. We start with the general counting lemma, analogous to Theorem 5.8.

Theorem 6.14 (Rödl-Schacht [42]**, see Theorem 13).** *Suppose* $0 < \epsilon \ll d, \gamma, 1/r, 1/j, 1/k$, *that* J *and* G *are* r-*partite* k-*complexes with vertex sets* $Y = Y_1 \cup \cdots \cup Y_r$ *and* $V = V_1 \cup \cdots \cup V_r$ *respectively, that* $|J| = j$, *and* G *is* ϵ-*regular with all densities at least* d. *Then*

$$d(J, G) = \mathbb{E}_{\phi \in \Phi(Y,V)} \left[\prod_{A \in J} G_A(\phi(A)) \right] = (1 \pm \gamma) \prod_{A \in J} d_A(G).$$

A useful case of Theorem 6.14 is when $r = k$ and $J = [k]^{\leq}$ consists of all subsets of a k-tuple; this gives the following analogue of Lemma 4.10.

Lemma 6.15. *Suppose* $0 < \epsilon \ll \epsilon' \ll d, 1/k$, G *is a* k-*partite* k-*complex on* $V = V_1 \cup \cdots \cup V_k$ *with all densities* $d_S(G) > d$ *and* G *is* ϵ-*regular. Then* $d(G_{[k]}) = (1 \pm \epsilon') \prod_{S \subseteq [k]} d_S(G)$.

Next we give the analogue of Lemma 4.6.

Lemma 6.16 (Vertex neighbourhoods). *Suppose* G *is a* k-*partite* k-*complex on* $V = V_1 \cup \cdots \cup V_k$ *with all densities* $d_S(G) > d$ *and* $0 < \eta_I \ll \eta'_I \ll d, 1/k$ *for each* $I \subseteq [k]$. *Suppose that each* G_I *is* η_I-*regular. Then for all but at most* $2 \sum_{I \subseteq [k-1]} \eta'_{Ik} |G_k|$ *vertices* $v \in G_k$, *for every* $\emptyset \neq I \subseteq [k-1]$, $G(v)_I$ *is* $(\eta'_I + \eta'_{Ik})$-*regular with* $d_I(G(v)) = (1 \pm \eta'_I \pm \eta'_{Ik}) d_I(G) d_{Ik}(G)$.

Proof. The argument is similar to that in Lemmas 2.2 and 4.6. We show the following statement by induction on $|\mathcal{C}|$: for any subcomplex $\mathcal{C}$ of $[k-1]^{\leq}$, for all but at most $2 \sum_{I \in \mathcal{C}} \eta'_{Ik} |G_k|$ vertices $v \in G_k$, for every $I \in \mathcal{C}$, $G(v)_I$ is $(\eta'_I + \eta'_{Ik})$-regular with $d_I(G(v)) = (1 \pm \eta'_I \pm \eta'_{Ik}) d_I(G) d_{Ik}(G)$. The base case is $\mathcal{C} = \emptyset$, or less trivially any $\mathcal{C}$ with $|I| \leq 2$ for all $I \in \mathcal{C}$, by Lemmas 2.2 and 4.6.

For the induction step, fix any maximal element I of $\mathcal{C}$. By induction hypothesis, for all but at most $2 \sum_{I' \in \mathcal{C} \setminus I} \eta'_{I'k} |G_k|$ vertices $v \in G_k$, for every $I' \in \mathcal{C} \setminus I$, $G(v)_{I'}$ is $(\eta'_{I'} + \eta'_{I'k})$-regular with $d_{I'}(G(v)) = (1 \pm \eta'_{I'} \pm \eta'_{I'k}) d_{I'}(G) d_{I'k}(G)$. Let G'_k be the set of such vertices. It suffices to show the claim that all but at most $2\eta'_{Ik}|G_k|$ vertices $v \in G'_k$ have the following property: if J^v is a subcomplex of $G(v)_{I^<}$ with $|(J^v)^*_I| > (\eta'_I + \eta'_{Ik}) |G(v)^*_I|$ then $|G[J^v]_I| = (1 \pm \eta'_I/2) d_I(G) |(J^v)^*_I|$ and $|G(v)[J^v]_I| = (1 \pm \eta'_{Ik}/2) d_{Ik}(G) |G[J^v]_I|$.

Suppose for a contradiction that this claim is false. Let $\gamma = \max_{I'\subseteq I} \eta'_{I'}$. By Theorem 6.14, for any $v \in G'_k$ we have

$$\left|G(v)^*_I\right| \prod_{i\in I} |V_i|^{-1} = d\left(G(v)^*_I\right) > (1-\gamma) \prod_{I'\subsetneq I} \left(1 - \eta'_{I'} - \eta'_{I'k}\right) d_{I'}(G) d_{I'k}(G) > \frac{1}{2} d^{2^k}.$$

Then for any $J^v \subseteq G(v)_{I^<}$ with $|(J^v)^*_I| > (\eta'_I + \eta'_{Ik})|G(v)^*_I|$ we have $|(J^v)^*_I| > \eta_I \prod_{i\in I} |V_i| \geq \eta_I |G^*_I|$, so $|G[J^v]_I| = (d_I(G) \pm \eta_I)|(J^v)^*_I| = (1 \pm \eta'_I/2) d_I(G)|(J^v)^*_I|$, since G_I is η_I-regular. So without loss of generality, we can assume that we have vertices $v_1, \ldots, v_t \in G'_k$ with $t > \eta'_{Ik}|G_k|$, and subcomplexes $J^{v_i} \subseteq G(v_i)_{I^<}$ with $|(J^{v_i})^*_I| > (\eta'_I + \eta'_{Ik})|G(v_i)^*_I|$ such that $|G(v_i)[J^{v_i}]_I| < (1 - \eta'_{I'k}/2) d_{Ik}(G)|G[J^{v_i}]_I|$ for $1 \leq i \leq t$. Define complexes $A^i = G[J^{v_i}]_I \cup \{v_i S : S \in J^{v_i}\}$ and $A = \cup_{i=1}^t A_i$.

We have $|A^*_{Ik}| = \sum_{i=1}^t |(A^i)^*_{Ik}| = \sum_{i=1}^t |G[J^{v_i}]_I|$. Now $|G[J^{v_i}]_I| > (d_I(G) - \eta_I)|(J^{v_i})^*_I|$, $d_I(G) > d$, $|(J^{v_i})^*_I| > (\eta'_I + \eta'_{Ik})|G(v_i)^*_I| > \eta'_{Ik} \cdot \frac{1}{2} d^{2^k} \prod_{i\in I} |V_i|$ and $t > \eta'_{Ik}|G_k|$, so

$$\left|A^*_{Ik}\right| > \eta'_{Ik}|G_k| \cdot (d - \eta_I) \cdot \eta'_{Ik} \cdot \frac{1}{2} d^{2^k} \prod_{i\in I} |V_i| > \eta_{Ik} \prod_{i\in Ik} |V_i| \geq \eta_{Ik}\left|G^*_{Ik}\right|.$$

Since G_{Ik} is η_{Ik}-regular we have $d_{Ik}(G[A])_{Ik} = d_{Ik}(G) \pm \eta_{Ik}$. Therefore $|G \cap A^*_{Ik}| > (d_{Ik}(G) - \eta_{Ik})|A^*_{Ik}| = (d_{Ik}(G) - \eta_{Ik}) \sum_{i=1}^t |G[J^{v_i}]_I|$. But we also have

$$\left|G \cap A^*_{Ik}\right| = \sum_{i=1}^t |G(v_i)[J^{v_i}]_I| < \sum_{i=1}^t \left(1 - \eta'_{I'k}/2\right) d_{Ik}(G)|G[J^{v_i}]_I| < (d_{Ik}(G) - \eta_{Ik}) \sum_{i=1}^t |G[J^{v_i}]_I|,$$

contradiction. This proves the claim, and so completes the induction. ∎

We apply Lemma 6.16 in the next lemma showing that arbitrary neighbourhoods are typically regular.

Lemma 6.17 (Set neighbourhoods). *Suppose* $0 < \epsilon \ll \epsilon' \ll d, 1/k$ *and* G *is an* ϵ*-regular* k*-partite* k*-complex on* $V = V_1 \cup \cdots \cup V_k$ *with all densities* $d_S(G) > d$. *Then for any* $A \subseteq [k]$ *and for all but at most* $\epsilon'|G_A|$ *sets* $P \in G_A$, *for any* $\emptyset \neq I \subseteq [k] \setminus A$, $G(P)_I$ *is* ϵ'*-regular with* $d_I(G(P)) = (1 \pm \epsilon') \prod_{A'\subseteq A} d_{A'I}(G)$ *(and* $d_\emptyset(G(P)) = 1$ *as usual).*

Proof. For convenient notation suppose that $A = [k']$ for some $k' < k$. Introduce additional constants with the hierarchy $\epsilon \ll \epsilon_1 \ll \epsilon'_1 \ll \cdots \ll \epsilon_{k'} \ll \epsilon'_{k'} \ll \epsilon'$. We prove inductively for $1 \leq t \leq k'$ that for all but at most $\epsilon_t|G_{[t]}|$ sets $S \in G_{[t]}$, for any $\emptyset \neq I \subseteq [k] \setminus [t]$, $G(S)$ is ϵ_t-regular with $d_I(G(S)) = (1 \pm \epsilon_t) \prod_{T\subseteq[t]} d_{TI}(G)$. The base case $t = 1$ is immediate from Lemma 6.16 with ϵ' replaced by ϵ_1. For the induction step, consider $S \in G_{[t]}$ such that for any $\emptyset \neq I \subseteq [k] \setminus [t]$, $G(S)$ is ϵ_t-regular with $d_I(G(S)) = (1 \pm \epsilon_t) \prod_{T\subseteq[t]} d_{TI}(G)$. We have $d_I(G(S)) > \frac{1}{2} d^{2^t}$, so we can apply Lemma 6.16 to $G(S)$ with $\eta_I = \epsilon_t$, $\eta'_I = 2^{-k}\epsilon'_t$ and d replaced by $\frac{1}{2} d^{2^t}$. Then for all but at most $\epsilon'_t|G(S)_{t+1}|$ vertices $v \in G(S)_{t+1}$, for every $\emptyset \neq I \subseteq [k] \setminus [t+1]$, $G(S)(v)_I = G(Sv)_I$ is ϵ'_t-regular with $d_I(G(Sv)) = (1 \pm \epsilon'_t) d_I(G(S)) d_{Ik}(G(S)) = (1 \pm 2\epsilon'_t) \prod_{T\subseteq[t+1]} d_{TI}(G)$. Also, since $|G_{[t+1]}| > \frac{1}{2} d^{2^{t+1}} \prod_{i=1}^{t+1} |V_i|$ by Theorem 6.14, the number of pairs (S, v) for which this fails is at most $\epsilon_t|G_{[t]}| \cdot |V_{t+1}| + |G_{[t]}| \cdot \epsilon'_t|G(S)_{t+1}| < \epsilon_{t+1}|G_{[t+1]}|$. ∎

We omit the proofs of the next three lemmas, as they are almost identical to those of the corresponding Lemmas 4.8, 4.9 and 4.12, replacing 123 by $[k]$ and using Theorem 6.14 instead of the triangle-counting lemma.

Lemma 6.18 (Regular restriction). *Suppose G is a k-partite k-complex on $V = V_1 \cup \cdots \cup V_k$ with all densities $d_S(G) > d$, $G_{[k]}$ is ϵ-regular, where $0 < \epsilon \ll d, 1/k$, and $J \subseteq G$ is a $(k-1)$-complex with $|J^*_{[k]}| > \sqrt{\epsilon}|G^*_{[k]}|$. Then $G[J]_{[k]}$ is $\sqrt{\epsilon}$-regular and $d_{[k]}(G[J]) = (1 \pm \sqrt{\epsilon})d_{[k]}(G)$.*

Lemma 6.19. *Suppose G is a k-partite k-complex on $V = V_1 \cup \cdots \cup V_k$ with all densities $d_S(G) > d$ and $G_{[k]}$ is ϵ-regular, where $0 < \epsilon \ll d, 1/k$. Suppose also that $J \subseteq G$ is a $(k-1)$-complex, and when defined, $d_I(J) > d$ and J_I is η-regular, where $0 < \eta \ll d$. Then $G[J]_{[k]}$ is $\sqrt{\epsilon}$-regular and $d_{[k]}(G[J]) = (1 \pm \sqrt{\epsilon})d_{[k]}(G)$.*

Lemma 6.20. *Suppose G is a k-partite k-complex on $V = V_1 \cup \cdots \cup V_k$ with all densities $d_S(G) > d$ and G is ϵ-regular, where $0 < \epsilon \ll \epsilon' \ll d, 1/k$. Then for any $A \subseteq [k]$ and for all but at most $\epsilon'|G_A|$ sets $P \in G_A$ we have $|G(P)_{[k]\setminus A}| = (1 \pm \epsilon')|G_{[k]}|/|G_A|$.*

Note that we will not need an analogue of the technical Lemma 4.11.

6.4. Analysis of the Algorithm

We start the analysis of the algorithm by showing that most free vertices are good. First we record some properties of the initial sets $F_S(0)$, taking into account the restricted positions.

Lemma 6.21. *$F_S(0)$ is ϵ'-regular with $d_S(F(0)) > c'd_S(G)$ and $|F_S(0)| > (c')^{2^{|S|}}|G_S|$.*

Proof. By definition we have $F_S(0) = G[\Gamma]_S$. Condition (i) of Definition 6.6 tells us that G_S is ϵ-regular with $d_S(G) \geq d_a$. Hypothesis (v) of Theorem 6.12 says that when defined $\Gamma_{S'}$ is ϵ'-regular with $d_{S'}(\Gamma) > c'd_{S'}(G)$ for $S' \subseteq S$. If Γ_S is defined then $F_S(0) = \Gamma_S$ is ϵ'-regular with $d_S(F(0)) = d_S(\Gamma) > c'd_S(G)$. Otherwise, by Lemma 6.19, Γ_S is $\sqrt{\epsilon}$-regular with $d_S(F(0)) = (1 \pm \epsilon)d_S(G) > c'd_S(G)$. The estimate $|F_S(0)| > (c')^{2^{|S|}}|G_S|$ follows by applying Theorem 6.14 to $F(0)_{S\leq}$ and $G_{S\leq}$. Note that $d_\emptyset(F(0)) = d_\emptyset(G) = 1$, so one of the c' factors compensates for the error terms in Theorem 6.14. ▪

Our next lemma handles the definitions for regularity and density in the algorithm.

Lemma 6.22. *The exceptional set $E_x(t-1)$ defined by $(*_{6.2})$ satisfies $|E_x(t-1)| < \epsilon_*|F_x(t-1)|$, and $F_S(t)$ is $\epsilon_{\nu'_S(t),1}$-regular with $d_S(F(t)) \geq d_u$ for every $S \in H$.*

Proof. We argue by induction on t. At time $t = 0$ the first statement is vacuous and the second follows from Lemma 6.21, since $d_S(G) \geq d_a$ for $S \in H$. Now suppose $t \geq 1$ and $\emptyset \neq S \in H$ is unembedded, so $x \notin S$. We consider various cases for S to establish the bound on the exceptional set and the regularity property, postponing the density bound until later in the proof.

We start with the case when $S \in H(x)$. By induction $F_{S'}(t-1)$ is $\epsilon_{\nu'_{S'}(t-1),1}$-regular and $d_{S'}(F(t-1)) \geq d_u$ for every $S' \subseteq Sx$. Write $\nu = \max_{S' \subseteq S} \nu'_{S'x}(t-1)$ and $\nu^* = \max\{\nu'_S(t-1), \nu'_{Sx}(t-1)\}$. By Lemma 6.16, for all but at most $\epsilon_{\nu,2}|F_x(t-1)|$ vertices $y \in F_x(t-1)$,

$F_S(t) = F_{Sx}(t-1)(y)$ is $\epsilon_{\nu^*,2}$-regular with $d_S(F(t)) = (1 \pm \epsilon_{\nu^*,2}) d_S(F(t-1)) d_{Sx}(F(t-1))$. Also, we claim that $\nu'_S(t) > \nu^*$. This holds by Lemma 6.13: (iv) gives $\nu_S(t) > \nu_S(t-1)$, (v) gives $\nu'_S(t) > \nu'_S(t-1)$, and (vi) gives $\nu'_S(t) > \nu'_{Sx}(t-1)$. Thus we have $(*_{6.2})$ for such y. We also note for future reference that $d_S(F(t)) > d_u^2/2$. In the argument so far we have excluded at most $\epsilon_{k^3 D,2}|F_x(t-1)|$ vertices $y \in F_x(t-1)$ for each of at most $2^{k-1}D$ sets $S \in H(x)$; this gives the required bound on $E_x(t-1)$. We also have the required regularity property of $F_S(t)$, but for now we postpone showing the density bounds.

Next we consider the case when $S \in H$ and $S \notin H(x)$. We show by induction on $|S|$ that $F_S(t)$ is $\epsilon_{\nu'_S(t),1}$-regular with $d_S(F(t)) > d_u/2$, and moreover $F_S(t)$ is $\epsilon_{\nu'_S(t-1),2}$-regular when S intersects $VN_H(x)$. Note that if S intersects $VN_H(x)$ then we have $\nu'_S(t) > \nu'_S(t-1)$ by Lemma 6.13(v). For the base case when $S = \{v\}$ has size 1 we have $|F_v(t)| = |F_v(t-1) \setminus y| \geq |F_v(t-1)| - 1$, so $d_v(F(t)) \geq d_v(F(t-1)) - 1/n > d_u/2$. For the induction step, suppose that $|S| \geq 2$. Recall that Lemma 3.11 gives $F_S(t) = F_{S^\leq}(t-1)[F_{S^<}(t)]_S$. Since $F_S(t-1)$ is $\epsilon_{\nu'_S(t-1),1}$-regular, by Lemma 6.18, $F_S(t)$ is $\epsilon_{\nu'_S(t-1),2}$-regular with $d_S(F(t)) = (1 \pm \epsilon_{\nu'_S(t-1),2}) d_S(F(t-1))$. This gives the required property in the case when S intersects $VN_H(x)$. Next suppose that S and $VN_H(x)$ are disjoint. Let t' be the most recent time at which we embedded a vertex x' with a neighbour in S. Then by Lemma 3.12, $F_{S^\leq}(t)$ is obtained from $F_{S^\leq}(t')$ just by deleting all sets containing any vertices that are embedded between time $t'+1$ and t. Equivalently, $F_S(t) = F_S(t')[((F_z(t) : z \in S), \{\emptyset\})]$. Now $F_S(t')$ is $\epsilon_{\nu'_S(t'-1),2}$-regular with $d_S(F(t')) > d_u$ and $\nu'_S(t) \geq \nu'_S(t') > \nu'_S(t'-1)$, so $F_S(t)$ is $\epsilon_{\nu'_S(t),1}$-regular with $d_S(F(t)) > d_u/2$ by Lemma 6.18.

Now we have established the bound on $E_x(t-1)$ and the regularity properties, so it remains to show the density bounds. First we consider any unembedded $S \in H$ with $|S| = k$. Then $F_S(t) = F(0)[F_{S^<}(t)]_S$, similarly to (4) in Lemma 4.13. By Lemma 6.21, $F(0)_S$ is ϵ'-regular with $d_S(F(0)) > c'd$. Also, we showed above for every $S' \subsetneq S$ that $F_{S'}(t)$ is $\epsilon_{\nu'_{S'}(t),1}$-regular with $d_{S'}(F(t)) > d_u/2$. Now Lemma 6.19 shows that $F_S(t)$ is $\sqrt{\epsilon'}$-regular with $d_S(F(t)) = (1 \pm \sqrt{\epsilon'}) d_S(F(0)) > c'd/2$. In particular we have $d_S(F(t)) > d_u$, though we also use the bound $c'd/2$ below.

Next we show for $k-1 \geq |S| \geq 2$ that $d_S(F(t)) > 4d_a^{2D^{2(k-|S|)}}$. We argue by induction on t and reverse induction on $|S|$, i.e. we assume that the bound holds for larger sets than S. When $|S| = k$ we have already proved $d_S(F(t)) > c'd/2 > 4d_a^2$. Let $t' \leq t$ be the most recent time at which we embedded a vertex x' with $S \in H(x')$, or let $t' = 0$ if there is no such vertex x'. Note that we may have $t' = t$ if $S \in H(x)$. Let $J(t)$ be the 1-complex $((F_z(t) : z \in S), \{\emptyset\})$. As in Lemma 4.13 we have $F_S(t) = F(t')[J(t)]_S$, so Lemma 6.18 gives $d_S(F(t)) > d_S(F(t'))/2$. Also, $(*_{6.2})$ gives $d_S(F(t')) = (1 \pm \epsilon_{\nu'_S(t'),0}) d_S(F(t'-1)) d_{Sx'}(F(t'-1))$, where $d_{Sx'}(F(t'-1)) > 4d_a^{2D^{2(k-|S|-1)}}$ by induction. Thus the S-density starts with $d_S(F(0)) > c'd_a$, loses a factor no worse than $1/2$ before we embed some x' with $S \in H(x')$, then loses a factor no worse than $d_a^{2D^{2(k-|S|-1)}}$ on at most D occasions when we embed some x' with $S \in H(x')$. This gives $d_S(F(t)) > c'd_a/2 \cdot d_a^{2D^{2(k-|S|-1)+1}} > 4d_a^{2D^{2(k-|S|)}}$. In particular we have $d_S(F(t)) > d_u$, though we also use the bound $4d_a^{2D^{2(k-|S|)}}$ below.

Finally we consider any unembedded vertex z. Suppose that the current z-regime started at some time $t_z \leq t$. If $t_z > 0$ then we embedded some neighbour $w = s(t_z)$ of z at time t_z. By $(*_{6.2})$ and the above bound for pair densities we have $d_z(F(t_z)) > d_{wz}(F(t_z-1)) d_z(F(t_z-1))/2 > d_a^{2D^{2(k-2)}} d_z(F(t_z-1))$. Now we consider cases according to whether z is in the list $L(t-1)$, the queue $q(t-1)$ or the queue jumpers $j(t-1)$. Suppose first that $z \in L(t-1)$. Then the rule for updating the queue in the algorithm gives $|F_z(t)| \geq \delta'_Q |F_z(t_z)|$. Next suppose that $z \in j(t-1) \cup q(t-1)$, and z first joined $j(t') \cup q(t')$ at some time $t' < t$. Since z did not join the queue at time $t'-1$ we have $|F_z(t'-1)| \geq \delta'_Q |F_z(t_z)|$. Also, between times t' and t we only embed vertices that are in the queue or jumping the queue, or otherwise we would have embedded z before x. Now $|Q(t) \cap X_i| \leq \delta_Q Cn$ for $i \in R$, or otherwise we abort the algorithm, and $|J(t) \cap X_z| < \sqrt{\delta_Q} n$ by Lemma 6.13(ii), so we embed at most $2\sqrt{\delta_Q}|V_z|$ vertices in V_z between times t' and t. Thus we have catalogued all possible ways in which the number of vertices free for z can decrease. It may decrease by a factor of $d_a^{2D^{2(k-2)}}$ when a new z-regime starts, and by a factor δ'_Q during a z-regime before z joins the queue. Also, if z joins the queue or jumps the queue it may decrease by at most $2\sqrt{\delta_Q}|V_z|$ in absolute size. Since z has at most $2D$ neighbours, and $|F_z(0)| > c'|V_z|$, we have $|F_z(t)| \geq (d_a^{2D^{2(k-2)}} \delta'_Q)^{kD} c'|V_z| - 2\sqrt{\delta_Q}|V_z| > d_u|V_z|$. ■

From now on it will often suffice and be more convenient to use a crude upper bound of ϵ_* for any epsilon parameter. The estimates in Lemma 4.14 hold in general (we can replace $12D$ by k^3D in (vi)). We also need some similar properties for plus complexes. In the following statement H^+ is to be understood as in Definition 6.8 but with x replaced by z.

Lemma 6.23. *Suppose $z \in X$ and $S' \subseteq S \in H^+$ are unembedded and I is a subcomplex of H. Then*

(i) *$F(t)^{I+z}_{S\leq}$ is $\epsilon_{k^3D,3}$-regular.*
(ii) *If $S \in H$ then $d_S(F(t)^{I+z}) = d_S(F(t))$.*
If $S \in H^+ \setminus H$ then $d_S(F(t)^{I+z})$ is $d_T(F(t))$ if $S = T^c$, $T \in I$ or 1 otherwise.
(iii) *For all but at most $\epsilon_*|F(t)^{I+z}_{S'}|$ sets $P \in F(t)^{I+z}_{S'}$ we have*

$$\left|F(t)^{I+z}_S(P)\right| = (1 \pm \epsilon_*)\left|F(t)^{I+z}_S\right| / \left|F(t)^{I+z}_{S'}\right|.$$

(iv) $d(F(t)^{I+z}_S) = (1 \pm \epsilon_*) \prod_{T \subseteq S} d_T(F(t)^{I+z})$.
(v)

$$\frac{\left|F(t)^{I+z}_S\right|}{\left|F(t)^{I+z}_{S'}\right|\left|F(t)^{I+z}_{S\setminus S'}\right|} = \frac{d\left(F(t)^{I+z}_S\right)}{d\left(F(t)^{I+z}_{S'}\right)d\left(F(t)^{I+z}_{S\setminus S'}\right)} = (1 \pm 4\epsilon_*) \prod_{T:T\subseteq S, T\nsubseteq S', T\nsubseteq S\setminus S'} d_T(F(t)^{I+z}).$$

(vi) *Statements (iii–v) hold replacing $F(t)^{I+z}_{S\leq}(t)$ by $F(t)^{I+z}_{S\leq}(t)[\Gamma]$ for any $\epsilon_{k^3D,3}$-regular subcomplex Γ of $F(t)^{I+z}_{S\leq}(t)$, such that $d_T(\Gamma) \geq \epsilon_*$ when defined.*

Proof. If $S \in H$ then $F(t)^{I+z}_{S\leq} = F(t)_{S\leq}$. Now suppose $S \in H^+ \setminus H$, say $S = Tz^c$ with $T \in H$. Then by Definition 6.8, $F(t)^{I+z}_{S\leq} = G^+[F(t) \cup F(t)^+_{I^c}]_{S\leq}$ consists of all Py^c with $P \in F(t)_T$ and $P_{S'}y^c \in F_{S'z}(t)^c$ when defined for all $S'z \in I$. Thus we can write $F(t)^{I+z}_{S\leq} = K(V^+)[F(t) \cup F(t)^+_{I^c}]_{S\leq}$. Note that $K(V^+)_{S'}$ is ϵ-regular with $d_{S'}(K(V^+)) = 1$ for any $S' \in H^+$. Thus (i) and (ii) follow by regular restriction. The other statements of the Lemma can be proved as in Lemma 4.14. ■

Our next lemma concerns the definitions for marked edges in the algorithm.

Lemma 6.24.

(i) *For every k-tuple* $E \in H$ *we have* $|M_{E^t,E}(t)| < \theta'_{\nu'_{E^t}(t)}|F_{E^t}(t)|$, *and in fact*

$$|M_{E^t,E}(t)| \leq \theta_{\nu'_{E^t}(t)}|F_{E^t}(t)| \text{ for } E \in U(x).$$

(ii) *For every* x *and k-tuple* $E \in U(x)$ *we have* $|D_{x,E}(t-1)| < \theta_{\nu'_{E^t}(t)}|F_x(t-1)|$.

Proof. Throughout we use the notation $\overline{E} = E^{t-1}$, $\nu = \nu'_{\overline{E}}(t-1)$, $\nu^* = \nu'_{E^t}(t)$.

(i) To verify the bound for $t = 0$ we use our assumption that (G, M) is $(\epsilon, \epsilon', d_a, \theta, d)$-super-regular. We take $I = (\{\emptyset\})$, when for any ν we have $G^{I_\nu} = G$ by Definition 6.5. Then condition (iii) in Definition 6.6 gives $|M_E| \leq \theta|G_E|$. Since $E^0 = E$ we have $M_{E,E}(0) = M_E \cap F_E(0)$, where $|F_E(0)| > (c')^{2^k}|G_E|$ by Lemma 6.21. Since $\theta \ll c'$ we have $|M_{E,E}(0)| \leq |M_E| \leq \theta|G_E| < \theta(c')^{-2^k}|F_E(0)| < \theta_0|F_E(0)|$. Now suppose $t > 0$. When $E \in U(x)$ we have $|M_{E^t,E}(t)| \leq \theta_{\nu^*}|F_{E^t}(t)|$ by definition, since the algorithm chooses $y = \phi(x) \notin D_{x,E}(t-1)$. Now suppose $E \notin U(x)$, and let $t' < t$ be the most recent time at which we embedded a vertex x' with $E \in U(x')$. Then $E^{t'} = E^t$, $\nu'_{E^t}(t') = \nu^*$, and $|M_{E^{t'},E}(t')| \leq \theta_{\nu'_{E^t}(t')}|F_{E^{t'}}(t')|$ by the previous case. For any $z \in E^t$, we can bound $|F_z(t)|$ using the same argument as that used at the end of the proof of Lemma 6.22. We do not embed any neighbour of z between time $t'+1$ and t, so the size of the free set for z can only decrease by a factor of δ'_Q and an absolute term of $2\sqrt{\delta_Q}n$. Since $d_z(F(t')) \geq d_u \gg \delta_Q$ we have $|F_z(t)| \geq \delta'_Q|F_z(t')| - 2\sqrt{\delta_Q}n \geq \frac{1}{2}\delta'_Q|F_z(t')|$. By Lemma 3.11, for every $\emptyset \neq S \subseteq E^t$, $F_S(t)$ is obtained from $F_S(t')$ by restricting to the 1-complex $((F_z(t) : z \in S), \{\emptyset\})$. If $|S| \geq 2$ then Lemma 6.18 gives $d_S(F(t)) = (1 \pm \epsilon_*)d_S(F(t'))$. Now $d(F_{E^t}(t)) = (1 \pm \epsilon_*)\prod_{S \subseteq E^t} d_S(F(t))$ by Lemma 6.15, so

$$\frac{|F_{E^t}(t)|}{|F_{E^t}(t')|} = \left(1 \pm 2^{k+1}\epsilon_*\right)\prod_{S \subseteq E^t}\frac{d_S(F(t))}{d_S(F(t'))} = \left(1 \pm 2^{k+2}\epsilon_*\right)\prod_{z \in E^t}\frac{d_z(F(t))}{d_z(F(t'))} > \left(\delta'_Q/2\right)^k/2.$$

Therefore $|M_{E^t,E}(t)| \leq |M_{E^t,E}(t')| \leq \theta_{\nu^*}|F_{E^t}(t')| < 2(\delta'_Q/2)^{-k}\theta_{\nu^*}|F_{E^t}(t)| < \theta'_{\nu^*}|F_{E^t}(t)|$.

(ii) The argument when $x \in E$ is identical to that in Lemma 4.15, so we just consider the case when $x \notin E$. Then $E^t = E^{t-1} = \overline{E}$. Note that when $E \in U(x)$ we have $\overline{E} \cap VN_H(x) \neq \emptyset$, so $\nu^* > \nu$. By Lemma 6.10 we have $F(t)_{\overline{E}} = F(t-1)^{H+x}(y^c)_{\overline{E}} \setminus y = (F(t-1)^{H+x}_{\overline{E}x^c} \setminus y)(y^c)$. Since $M_{\overline{E},E}(t) = M_{\overline{E},E}(t-1) \cap F_{\overline{E}}(t)$ we have

$$D_{x,E}(t-1) = \left\{ y \in F_x(t-1) : \frac{\left|M_{\overline{E},E}(t-1) \cap \left(F(t-1)^{H+x}_{\overline{E}x^c} \setminus y\right)(y^c)\right|}{\left|\left(F(t-1)^{H+x}_{\overline{E}x^c} \setminus y\right)(y^c)\right|} > \theta_{\nu^*} \right\}.$$

Then

$$\begin{aligned}
\Sigma &:= \sum_{y \in D_{x,E}(t-1)} \left|M_{\overline{E},E}(t-1) \cap \left(F(t-1)^{H+x}_{\overline{E}x^c} \setminus y\right)(y^c)\right| \\
&> \theta_{\nu^*} \sum_{y \in D_{x,E}(t-1) \setminus E_x(t-1)} \left|\left(F(t-1)^{H+x}_{\overline{E}x^c} \setminus y\right)(y^c)\right| \\
&> (1-2\epsilon_*)\theta_{\nu^*}(|D_{x,E}(t-1)| - \epsilon_*|F_x(t-1)|)\left|F(t-1)^{H+x}_{\overline{E}x^c}\right|/|F_{x^c}(t-1)| \\
&= (1-2\epsilon_*)\theta_{\nu^*}(|D_{x,E}(t-1)|/|F_x(t-1)| - \epsilon_*)\left|F(t-1)^{H+x}_{\overline{E}x^c}\right|.
\end{aligned}$$

Here we used $|F_{x^c}(t-1)| = |F_x(t-1)|$, and in the second inequality we applied Lemma 6.23, with a factor $1-2\epsilon_*$ rather than $1-\epsilon_*$ to account for the error from deleting y (which has a lower order of magnitude). We also have $\Sigma \le \sum_{y\in F_x(t-1)} |M_{\overline{E},E}(t-1)\cap F(t-1)^{H+x}_{\overline{E}x^c}(y^c)|$. This counts all pairs (y,P) with $P\in M_{\overline{E},E}(t-1)$, $y\in F_x(t-1)$ and $Py^c\in F(t-1)^{H+x}_{\overline{E}x^c}$, so we can rewrite it as $\Sigma \le \sum_{P\in M_{\overline{E},E}(t-1)} |F(t-1)^{H+x}_{\overline{E}x^c}(P)|$. By Lemma 6.23 we have

$$\left|F(t-1)^{H+x}_{\overline{E}x^c}(P)\right| = (1\pm\epsilon_*)\frac{\left|F(t-1)^{H+x}_{\overline{E}x^c}\right|}{\left|F(t-1)^{H+x}_{\overline{E}}\right|}$$

for all but at most $\epsilon_*|F(t-1)^{H+x}_{\overline{E}}|$ sets $P\in F(t-1)^{H+x}_{\overline{E}}$. Since $F(t-1)^{H+x}_{\overline{E}} = F_{\overline{E}}(t-1)$ we have

$$\Sigma \le |M_{\overline{E},E}(t-1)|(1+\epsilon_*)\frac{\left|F(t-1)^{H+x}_{\overline{E}x^c}\right|}{|F_{\overline{E}}(t-1)|} + \epsilon_*|F_{\overline{E}}(t-1)||F_x(t-1)|.$$

Combining this with the lower bound on Σ we obtain

$$(1-2\epsilon_*)\theta_{v*}\left(\frac{|D_{x,E}(t-1)|}{|F_x(t-1)|}-\epsilon_*\right) < (1+\epsilon_*)\frac{|M_{\overline{E},E}(t-1)|}{|F_{\overline{E}}(t-1)|} + \epsilon_*\frac{|F_{\overline{E}}(t-1)||F_x(t-1)|}{\left|F(t-1)^{H+x}_{\overline{E}x^c}\right|}.$$

Now $|M_{\overline{E},E}(t-1)| < \theta'_v|F_{\overline{E}}(t-1)|$ by (i) and $\frac{|F_{\overline{E}}(t-1)||F_x(t-1)|}{|F(t-1)^{H+x}_{\overline{E}x^c}|} \le d_u^{-2^k} \ll \epsilon_*^{-1}$, by Lemma 6.23, so

$$\frac{|D_{x,E}(t-1)|}{|F_x(t-1)|} < \frac{(1+\epsilon_*)\theta'_v+\sqrt{\epsilon_*}}{(1-2\epsilon_*)\theta_{v*}} + \epsilon_* < \theta_{v*}.$$

■

The following corollary is now immediate from Lemmas 6.22 and 6.24. Recall that $OK_x(t-1)$ is obtained from $F_x(t-1)$ by deleting $E_x(t-1)$ and $D_{x,E}(t-1)$ for $E\in U(x)$, and note that since H has maximum degree at most D we have $|U(x)|\le(k-1)D^2$.

Corollary 6.25. $|OK_x(t-1)| > (1-\theta_*)|F_x(t-1)|$.

Next we consider the initial phase of the algorithm, during which we embed the neighbourhood N of the buffer B. We give three lemmas that are analogous to those used for the 3-graph blow-up lemma. First we recall the key properties of the selection rule during the initial phase. Since H has maximum degree D we have $|VN_H(x)|\le(k-1)D$ for all x. We embed all vertex neighbourhoods $VN_H(x)$, $x\in B$ at consecutive times, and before x or any other vertices at distance at most 4 from x. Then Lemma 3.12 implies that if we start embedding $VN_H(x)$ just after some time T_0 then $F_z(T_0) = F_z(0)\cap V_z(T_0)$ consists of all vertices in $F_z(0)$ that have not yet been used by the embedding, for every z at distance at most 3 from x. Recall that $F_z(0) = G[\Gamma]_z$ is either a set of restricted positions Γ_z with $|\Gamma_z| > c'|G_z| = c'|V_z|$ or is $G_z = V_z$ if Γ_z is undefined. We chose B disjoint from $X'_* = X_*\cup\bigcup_{x\in X_*} VN_H(x)$, so $B\cup N$ is disjoint from X_*. Thus for $z\in VN_H(x)\cup\{x\}$ we have $F_z(T_0) = V_z(T_0)$. We also recall that $|B\cap V_z| = \delta_B|V_z|$, $|N\cap V_z| < \sqrt{\delta_B}|V_z|$, $|Q(T_0)\cap V_z|\le\delta_Q|V_z|$ and $|J(T_0)\cap V_z|\le\sqrt{\delta_Q}|V_z|$ by Lemma 4.3(ii). Since $\delta_Q\ll\delta_B\ll c'$, for any z at distance at most 3 from x we have

$$|F_z(T_0)| = |F_z(0)\cap V_z(T_0)| > \left(1-\delta_B^{1/3}\right)|F_z(0)|. \qquad (8)$$

Our first lemma is analogous to Lemma 4.18. We omit the proof, which is almost identical to that for 3-graphs. The only modifications are to replace $2D$ by $(k-1)D$, $12D$ by k^3D, and $\sum_{\ell=1}^{r}$ by a sum over at most D_R neighbours $\ell \in R$ of i; the estimates are still valid as $\delta_B \ll 1/D_R$.

Lemma 6.26. *With high probability, for every $S \in R$ with $|S| = 2$ lying over some $i, j \in R$, and vertex $v \in G_i$ with $|G_S(v)| \geq d_u|V_j|$, we have $|G_S(v) \cap V_j(T_I)| > (1 - \delta_B^{1/3})|G_S(v)|$.*

Next we fix a vertex $x \in B$ and write $VN_H(x) = \{z_1, \dots, z_g\}$, with vertices listed in the order that they are embedded. We let T_j be the time at which z_j is embedded. By the selection rule, $VN_H(x)$ jumps the queue and is embedded at consecutive times: $T_{j+1} = T_j + 1$ for $1 \leq j \leq g-1$. For convenience we also define $T_0 = T_1 - 1$. Note that no vertex of $VN_H(x)$ lies in X_x. Our second lemma shows that for that any $W \subseteq V_x$ that is not too small, the probability that W does not contain a vertex available for x is quite small.

Lemma 6.27. *For any $W \subseteq V_x$ with $|W| > \epsilon_*|V_x|$, conditional on any embedding of the vertices $\{s(u) : u < T_1\}$ that does not use any vertex of W, we have $\mathbb{P}[A_x \cap W = \emptyset] < \theta_*$.*

Proof. The proof is very similar to that of Lemma 4.20, so we will just describe the necessary modifications. We note that since $B \cup N$ is disjoint from X_* we do not need to consider restricted positions in this proof. Suppose $1 \leq j \leq g$ and that we are considering the embedding of z_j. We interpret quantities at time T_j with the embedding $\phi(z_j) = y$, for some as yet unspecified $y \in F_{z_j}(T_j - 1)$. We define W_j, $[W_j]$, $E^W_{z_j}(T_j - 1)$, $D^W_{z_j,E}(T_j - 1)$ and the events $A_{i,j}$ as before (replacing triples with k-tuples). The proofs of Claims A, B, C, E and the conclusion of the proof are almost identical to before. We need to modify various absolute constants to take account of the dependence on k, e.g. changing 20 to 2^{k+2} in Claim A, 12 to k^3 in Claim E, and $g \leq 2D$ to $g \leq (k-1)D$. Also, when we apply equation (8) instead of (5) we will replace $2\sqrt{\delta_B}$ by $\delta_B^{1/3}$.

To complete the proof of the lemma it remains to establish Claim D. This requires more substantial modifications, similar to those in Lemma 6.24, so we will give more details here. Suppose that $A_{1,j-1}$ and $A_{2,j-1}$ hold and E is a k-tuple containing x. As before we write $\overline{E} = E^{T_j - 1}$, $\nu = \nu'_{\overline{E}}(T_j - 1)$, $\nu^* = \nu'_{E^{T_j}}(T_j)$ and $B_{z_j} = E_{z_j}(T_j - 1) \cup E^W_{z_j}(T_j - 1)$. Again we have $|B_{z_j}| < 2\epsilon_*|F_{z_j}(T_j - 1)|$. We are required to prove that $|D^W_{z_j,E}(T_j - 1)| < \theta_{\nu^*}|F_{z_j}(T_j - 1)|$. The proof of Case D.1 when $z_j \in E$ is exactly as before, so we just consider the case $z_j \notin E$. In Lemma 4.20 we divided this into Cases D.2 and D.3, but here we will give a unified argument.

Suppose that $z_j \notin E$. Then $E^{T_j} = E^{T_j - 1} = \overline{E}$. Also $x \in \overline{E} \cap VN_H(z_j)$, so $\nu^* > \nu$. By Lemma 6.10 we have $F(T_j)_{\overline{E}} = F(T_j - 1)^{H+z_j}(y^c)_{\overline{E}} \setminus y = (F(T_j - 1)^{H+z_j}_{\overline{E}z_j^c} \setminus y)(y^c)$. Now $W_j = W_{j-1} \cap F_{xz_j}(T_j - 1)(y) = W_{j-1} \cap F(T_j - 1)^{H+z_j}_{xz_j^c}(y^c)$, so

$$F_{\overline{E}}(T_j)[W_j] = \big(F(T_j - 1)^{H+z_j}_{\overline{E}z_j^c} \setminus y\big)(y^c)[W_j] = \big(F(T_j - 1)^{H+z_j}_{\overline{E}z_j^c}[W_{j-1}] \setminus y\big)(y^c).$$

Since $M_{\overline{E},E}(T_j) = M_{\overline{E},E}(T_j - 1) \cap F_{\overline{E}}(T_j)$ we have

$$D^W_{z_j,E}(T_j - 1) = \left\{ y \in F_{z_j}(T_j - 1) : \frac{\left|M_{\overline{E},E}(T_j - 1) \cap \big(F(T_j - 1)^{H+z_j}_{\overline{E}z_j^c}[W_{j-1}] \setminus y\big)(y^c)\right|}{\left|\big(F(T_j - 1)^{H+z_j}_{\overline{E}z_j^c}[W_{j-1}] \setminus y\big)(y^c)\right|} > \theta_{\nu^*} \right\}.$$

Then

$$\begin{aligned}\Sigma &:= \sum_{y \in D^W_{z_j,E}(T_j-1)} \left|M_{\overline{E},E}(T_j-1) \cap \left(F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}] \setminus y\right)(y^c)\right| \\ &> \theta_{\nu^*} \sum_{y \in D^W_{z_j,E}(T_j-1)\setminus B'_{z_j}} \left|\left(F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}] \setminus y\right)(y^c)\right| \\ &> (1-2\epsilon_*)\theta_{\nu^*}\left(\left|D^W_{z_j,E}(T_j-1)\right| - 2\epsilon_*|F_{z_j}(T_j-1)|\right) \left|F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}]\right|/|F_{z^c_j}(T_j-1)| \\ &= (1-2\epsilon_*)\theta_{\nu^*}\left(\left|D^W_{z_j,E}(T_j-1)\right|/\left|F_{z_j}(T_j-1)\right| - 2\epsilon_*\right)\left|F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}]\right|.\end{aligned}$$

Here we used $|F_{z^c_j}(T_j-1)| = |F_{z_j}(T_j-1)|$ and Lemma 6.23 with $\Gamma = (W_{j-1}, \{\emptyset\})$, as usual denoting the exceptional set by B'_{z_j}; the factor $1-2\epsilon_*$ rather than $1-\epsilon_*$ accounts for the error from deleting y (which has a lower order of magnitude). We also have $\Sigma \le \sum_{y \in F_{z_j}(T_j-1)} |M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}](y^c)|$. This counts all pairs (y, P) with $P \in M_{\overline{E},E}(T_j-1)[W_{j-1}]$, $y \in F_{z_j}(T_j-1)$ and $Py^c \in F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}]$, so we can rewrite it as $\Sigma \le \sum_{P \in M_{\overline{E},E}(T_j-1)[W_{j-1}]} |F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}](P)|$. By Lemma 6.23, we have

$$\left|F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}](P)\right| = (1 \pm \epsilon_*)\frac{\left|F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}]\right|}{\left|F(T_j-1)^{H+z_j}_{\overline{E}}[W_{j-1}]\right|}$$

for all but at most $\epsilon_*|F(T_j-1)^{H+z_j}_{\overline{E}}[W_{j-1}]|$ sets $P \in F(T_j-1)^{H+z_j}_{\overline{E}}[W_{j-1}]$.

Since $F(T_j-1)^{H+z_j}_{\overline{E}} = F_{\overline{E}}(T_j-1)$ we have

$$\Sigma \le |M_{\overline{E},E}(T_j-1)[W_{j-1}]|(1+\epsilon_*)\frac{\left|F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}]\right|}{|F_{\overline{E}}(T_j-1)[W_{j-1}]|} + \epsilon_*|F_{\overline{E}}(T_j-1)[W_{j-1}]||F_{z_j}(T_j-1)|.$$

Combining this with the lower bound on Σ we obtain

$$\begin{aligned}&(1-2\epsilon_*)\theta_{\nu^*}\left(\frac{\left|D^W_{z_j,E}(T_j-1)\right|}{|F_{z_j}(T_j-1)|} - 2\epsilon_*\right) \\ &\quad < (1+\epsilon_*)\frac{|M_{\overline{E},E}(T_j-1)[W_{j-1}]|}{|F_{\overline{E}}(T_j-1)[W_{j-1}]|} + \epsilon_*\frac{|F_{\overline{E}}(T_j-1)[W_{j-1}]||F_{z_j}(T_j-1)|}{\left|F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}]\right|}.\end{aligned}$$

Now $|M_{\overline{E},E}(T_j-1)[W_{j-1}]| < \theta_\nu|F_{\overline{E}}(T_j-1)|$ by $A_{2,j-1}$ and

$$\frac{|F_{\overline{E}}(T_j-1)[W_{j-1}]||F_{z_j}(T_j-1)|}{\left|F(T_j-1)^{H+z_j}_{\overline{E}z^c_j}[W_{j-1}]\right|} \le d_u^{-2^k} \ll \epsilon_*^{-1},$$

by Lemma 6.23, so

$$\frac{\left|D^{W}_{z_j,E}(T_j-1)\right|}{|F_{z_j}(T_j-1)|} < \frac{(1+\epsilon_*)\theta_\nu + \sqrt{\epsilon_*}}{(1-2\epsilon_*)\theta_{\nu*}} + 2\epsilon_* < \theta_{\nu*}.$$

∎

Our final lemma for the initial phase is similar to the previous one, but instead of asking for a set W of vertices to contain an available vertex for x, we ask for some particular vertex v to be available for x.

Lemma 6.28. *For any $v \in V_x$, conditional on any embedding of the vertices $\{s(u) : u < T_1\}$ that does not use v, with probability at least p we have $\phi(H(x)) \subseteq (G \setminus M)(v)$, so $v \in A_x$.*

Proof. We follow the proof of Lemma 4.21, indicating the necessary modifications. We note again that since $B \cup N$ is disjoint from X_*, restricted positions have no effect on any $z \in VN_H(x) \cup \{x\}$. However, we need to consider all vertices within distance 3 of x, so some of these may have restricted positions. The bound in equation (8) will be adequate to deal with these. By Remark 6.4 we also need to clarify the meaning of neighbourhood constructions, which are potentially ambiguous: $F(T_j)(v)_S$ is $F(T_j)_{Sx}(v)$ when $Sx \in H$ or undefined when $Sx \notin H$.

For $z \in VN_H(x)$ we define α_z as before. We also define $\alpha_S = 1$ for $S \in H$ with $|S| \geq 2$. As before, we define $v''_S(t)$ similarly to $v'_S(t)$, replacing 'embedded' with 'allocated'. Suppose $1 \leq j \leq g$ and that we are considering the embedding of z_j. We interpret quantities at time T_j with the embedding $\phi(z_j) = y$, for some as yet unspecified $y \in F_{z_j}(T_j - 1)$. We define $E^v_{z_j}(T_j - 1)$ as before, except that the condition for $|S| = 2$ now applies whenever $|S| \geq 2$. We define Y, H', $F(T_j)^{Z*v}_{S\leq}$ and $D^{Z*v}_{z_j,E}(T_j - 1)$ as before. Properties (i–iv) of Z hold as before. We define the events $A_{i,j}$ as before.

Recall that we used the notation $Z \subseteq Y$, $Z' = Zz_j$, $I = \{S \subseteq Z : S \in H(x)\}$, $I' = \{S \subseteq Z' : S \in H(x)\}$. Here we also define $J = \{S \subseteq Zx : S \in H\}$ and $J' = \{S \subseteq Z'x : S \in H\}$. Using the plus complex notation we can write

$$F(T_j)^{Z*v}_{S\leq} = F(T_j)^{J+v}(v^c)_{S\leq}. \qquad (+_{6.28})$$

To see this, we need to show that $P \in F(T_j)^{Z*v}_S \Leftrightarrow Pv^c \in F(T_j)^{J+v}_{Sx^c}$. Recall that $F(T_j)^{Z*v}_S$ consists of all sets $P \in F_S(T_j)$ such that $P_{S'}v \in F_{S'x}(T_j)$ for all $S' \subseteq S$ with $S' \in I$. Also, from Definition 6.8(v) we have $Pv^c \in F(T_j)^{J+v}_{Sx^c}$ if and only if $P \in F(T_j)_S$ (this is the restriction from F) and $P_{S'}v^c \in (F(T_j)^+_{J^c})_{S'x^c} = F(T_j)^c_{S'x^c}$ for every $S' \subseteq S$ with $S'x \in J$. Since $S' \in I \Leftrightarrow S'x \in J$, this is equivalent to the condition for $F(T_j)^{Z*v}_S$, as required.

The proof of Claim A is similar to before. Instead of the bound $|F_z(T_0)| > (1-2\sqrt{\delta_B})|V_z|$ from equation (5) we use $|F_z(T_0)| > (1 - \delta_B^{1/3})|V_z|$ from equation (8). We again have $|F_z(T_0) \cap G(v)_z| > (1 - \delta_B^{1/3})|G(v)_z|$ for $z \in VN_H(x)$ by Lemma 6.26, using the fact that $z \notin X_*$. Again, $A_{4,0}$ holds by definition and Lemma 6.26 implies that $A_{1,0}$ holds with high probability. The arguments for $A_{2,0}$ and $A_{3,0}$ are as before, modifying the absolute constants to take account of their dependence on k. (In the $A_{2,0}$ argument we have $F(0)^{Z*v}_E = G^{J_v}_E$, where we let J also denote the submulticomplex $\{i^*(S) : S \in J\}$ of R.) The proofs of Claims B and E and the conclusion of the lemma are also similar to before. As usual we replace triple by k-tuple, $2D$ by $(k-1)D$ and $12D$ by k^3D. Also, in Claim E we previously estimated

$2D^2$ choices for E then 8 choices for Z; here we estimate $(k-1)D^2$ choices for E then 2^k choices for Z. To complete the proof of the lemma it remains to establish Claims C and D. These require more substantial modifications, so we will give the details here.

We start with Claim C. Suppose that $A_{1,j-1}$ and $A_{3,j-1}$ hold. We are required to prove that $|E^v_{z_j}(T_j-1) \setminus E_{z_j}(T_j-1)| < \epsilon_* |F_{xz_j}(T_j-1)(v)|$. For any $S \in H$ we write $\nu''_S = \nu''_S(T_j-1)$ and $\nu^*_S = \nu''_S(T_j)$. Consider any unembedded $\emptyset \neq S \in H(x) \cap H(z_j)$. Note that $\nu^*_S > \max\{\nu''_S, \nu''_{Sx}, \nu''_{Sz_j}\}$. Applying the definitions, it suffices to show that for all but at most $\epsilon_{k^3D,3}|F_{xz_j}(T_j-1)(v)|$ vertices $y \in F_{xz_j}(T_j-1)(v) \setminus E_{z_j}(T_j-1)$, $F_{Sx}(T_j)(v)$ is $\epsilon_{\nu^*_S,1}$-regular with $d_S(F(T_j)(v)) = (1 \pm \epsilon_{\nu^*_S,1}) d_{Sx}(F(T_j)) d_S(F(T_j)) \alpha_S$. We claim that

$$F_{Sx}(T_j)(v) = F(T_j-1)^{H+z_j}(vy^c)_S.$$

To see this we apply Definition 6.8, which tells us that for $P \in F(T_j-1)_S$, $v_0 \in F(T_j-1)_x$ and $y_0 \in F(T_j-1)_{z_j}$ we have $Pv_0y_0^c \in F(T_j-1)^{H+z_j}_{Sxz_j^c}$ exactly when $Pv_0 \in F(T_j-1)_{Sx}$ and $(Pv_0)_{S'}y_0 \in F(T_j-1)_{S'z_j}$ for all $S' \subseteq Sx$, $S' \in H(z_j)$. Therefore $F(T_j-1)^{H+z_j}(vy^c)_S$ consists of all $P \in F(T_j-1)_S$ such that $Pv \in F(T_j-1)_{Sx}$ and $(Pv)_{S'} \in F(T_j-1)_{S'z_j}(y)$ for all $S' \subseteq Sx$, $S' \in H(z_j)$. By Definition 3.7 this is equivalent to $Pv \in F(T_j)_{Sx}$, i.e. $P \in F_{Sx}(T_j)(v)$ as claimed. (We do not need to delete y as S and x are in $H(z_j)$.)

By Definition 6.8, $F(T_j-1)^{H+z_j}_{Sxz_j^c}$ is $F(T_j-1)^c_{Sxz_j}$ if $Sxz_j \in H$ or consists of all Py^c with $P \in F(T_j-1)_{Sx}$, $y \in F(T_j-1)_{z_j}$ and $P_{S'}y \in F(T_j-1)_{S'z_j}$ for $S' \subseteq Sx$ with $S' \in H(z_j)$. Thus $F(T_j-1)^{H+z_j}_{Sxz_j^c}(v)$ is $F(T_j-1)^c_{Sxz_j}(v)$ if $Sxz_j \in H$ or $F(T_j-1)[F(T_j-1)_{Sxz_j^{\le}}(v)]^c_{Sz_j}$ if $Sxz_j \notin H$. Either way we can see that is $\epsilon_{\nu''_{Sz_j},2}$-regular: in the first case we write $F(T_j-1)_{Sxz_j}(v) = F(T_j-1)^{Sz_j*v}_{Sz_j}$ and use $A_{3,j-1}$; in the second case we use Lemma 6.19 and the fact that $F(T_j-1)^c_{Sz_j}$ is a copy of $F(T_j-1)_{Sz_j}$, which is $\epsilon_{\nu''_{Sz_j},1}$-regular by Lemma 6.22. Also by $A_{3,j-1}$, $d_{Sz_j^c}(F(T_j-1)^{H+z_j}(v))$ is $(1 \pm \epsilon_{\nu''_{Sz_j},2}) d_{Sz_j}(F(T_j-1)) d_{Sxz_j}(F(T_j-1))$ if $Sxz_j \in H$ or $(1 \pm \epsilon_{\nu''_{Sz_j},2}) d_{Sz_j}(F(T_j-1))$ if $Sxz_j \notin H$. Similarly, $F(T_j-1)^{H+z_j}_{Sx}(v)$ is $\epsilon_{\nu''_S,2}$-regular with $d_S(F(T_j-1)^{H+z_j}(v)) = (1 \pm \epsilon_{\nu''_S,2}) d_S(F(T_j-1)) d_{Sx}(F(T_j-1)) \alpha_S$.

Since $x \in H(z_j)$ we have $F(T_j-1)^{H+z_j}(v)_{z_j^c} = F_{xz_j}(T_j-1)^c(v)$. By Lemma 6.16, for all but at most $\sum_{S' \subseteq Sxz_j} \epsilon_{\nu''_{S'},3} |F_{xz_j}(T_j-1)(v)|$ vertices $y \in F_{xz_j}(T_j-1)(v)$, writing $\eta = \epsilon_{\nu''_S,3} + \epsilon_{\nu''_{Sz_j},3}$, $F_{Sx}(T_j)(v) = F(T_j-1)^{H+z_j}(v)(y^c)_S$ is η-regular with

$$\begin{aligned}
d_S(F(T_j)(v)) &= d_S(F(T_j-1)^{H+z_j}(vy^c)) \\
&= (1 \pm \eta) d_S(F(T_j-1)^{H+z_j}(v)) d_{Sz_j}(F(T_j-1)^{H+z_j}(v)). \\
&= (1 \pm 3\eta) d_S(F(T_j-1)) d_{Sx}(F(T_j-1)) \alpha_S \cdot d_{Sz_j}(F(T_j-1)) d_{Sxz_j}(F(T_j-1))^{1_{Sxz_j \in H}}.
\end{aligned}$$

This gives the required regularity property for $F_{Sx}(T_j)(v)$. Next, $(*_{6.2})$ gives

$$\begin{aligned}
&d_S(F(T_j)) = (1 \pm \epsilon_{\nu^*_S,0}) d_S(F(T_j-1)) d_{Sz_j}(F(T_j-1)) \text{ and} \\
&d_{Sx}(F(T_j)) = (1 \pm \epsilon_{\nu^*_{Sx},0}) d_{Sx}(F(T_j-1)) d_{Sxz_j}(F(T_j-1))^{1_{Sxz_j \in H}},
\end{aligned}$$

so since $\nu_S^* > \max\{\nu_S'', \nu_{Sx}'', \nu_{Sz_j}''\}$ we have $d_S(F(T_j)(v)) = (1 \pm \epsilon_{\nu_S^*,1}) d_{Sx}(F(T_j)) d_S(F(T_j)) \alpha_S$. This proves Claim C.

It remains to prove Claim D. Suppose that $A_{1,j-1}$, $A_{2,j-1}$ and $A_{3,j-1}$ hold, $E \in U(z_j)$, $Z \subseteq E$ and $Z' = Z \cup z_j$. As before we write $\overline{E} = E^{T_j-1}$, $\nu = \nu''_{\overline{E}}(T_j - 1)$, $\nu^* = \nu''_{E^{T_j}}(T_j)$, $B_{z_j} = E_{z_j}(T_j - 1) \cup E^v_{z_j}(T_j - 1)$. Since $E \in U(z_j)$ we have $\nu^* > \nu$. We are required to prove that $|D^{Z'*v}_{z_j,E}(T_j - 1)| < \theta_{\nu^*} |F_{xz_j}(T_j - 1)(v)|$. The proof of Case D.1 when $z_j \in E$ is exactly as before, so we just consider the case $z_j \notin E$.

Suppose that $z_j \notin E$. Then $E^{T_j} = E^{T_j-1} = \overline{E}$. By Lemma 6.10 we have

$$F(T_j)_{\overline{E}} = F(T_j - 1)^{H+z_j}(y^c)_{\overline{E}} \setminus y = \big(F(T_j - 1)^{H+z_j}_{\overline{E}z_j^c} \setminus y\big)(y^c).$$

We claim that

$$F(T_j)^{Z*v}_{\overline{E}^{\le}} = F(T_j - 1)^{H+z_j}(y^c)^{J+v}(v^c)_{\overline{E}^{\le}} \setminus y = F(T_j - 1)^{J'+v}(v^c)^{H+z_j}(y^c)_{\overline{E}^{\le}} \setminus y. \quad (\dagger_{6.28})$$

For the first equality we use $(+_{6.28})$ to get $F(T_j)^{Z*v}_{\overline{E}^{\le}} = F(T_j)^{J+v}(v^c)_{\overline{E}^{\le}}$ and substitute $F(T_j) = F(T_j - 1)^{H+z_j}(y^c) \setminus y$ from Lemma 6.10. For the second equality we apply Definition 6.8 as follows. Suppose $S \subseteq \overline{E}$. We have $P \in F(T_j - 1)^{H+z_j}(y^c)^{J+v}(v^c)_S$ exactly when $P \in F(T_j - 1)^{H+z_j}(y^c)_S$ and $P_{S'}v \in F(T_j - 1)^{H+z_j}(y^c)_{S'x}$ for all $S' \subseteq S$ with $S'x \in J$. Equivalently, $P \in F(T_j - 1)_S$, $P_U y \in F(T_j - 1)_{Uz_j}$ for $U \subseteq S$ with $U \in H(z_j)$, $P_{S'}v \in F(T_j - 1)_{S'x}$ and $(Pv)_{S''}y \in F(T_j - 1)_{S''z_j}$ for $S' \subseteq S$ with $S'x \in J$ and $S'' \subseteq S'x$ with $S'' \in H(z_j)$. Note that it is equivalent to assume $x \in S''$, as otherwise the S'' condition is covered by the U condition. Writing $W = S'' \setminus x$ and using $J = \{A \subseteq Zx : A \in H\}$ we have

$P \in F(T_j - 1)^{H+z_j}(y^c)^{J+v}(v^c)_S$ if and only if $P \in F(T_j - 1)_S$, $P_U y \in F(T_j - 1)_{Uz_j}$ for $U \subseteq S$ with $U \in H(z_j)$, $P_{S'}v \in F(T_j - 1)_{S'x}$ for $S' \subseteq S \cap Z$, $S' \in H(x)$, and $P_W vy \in F(T_j - 1)_{Wxz_j}$ for $W \subseteq S \cap Z$ with $W \in H(xz_j)$.

On the other hand, we have $P \in F(T_j - 1)^{J'+v}(v^c)^{H+z_j}(y^c)_S$ exactly when $P \in F(T_j - 1)^{J'+v}(v^c)_S$ and $P_U y \in F(T_j - 1)^{J'+v}(v^c)_{Uz_j}$ for all $U \subseteq S$ with $U \in H(z_j)$. Equivalently, $P \in F(T_j - 1)_S$, $P_{S'}v \in F(T_j - 1)_{S'x}$ for $S' \subseteq S$ with $S'x \in J'$, $P_U y \in F(T_j - 1)_{Uz_j}$ and $(Py)_{U'}v \in F(T_j - 1)_{U'x}$ for $U \subseteq S$ with $U \in H(z_j)$ and $U' \subseteq Uz_j$ with $U'x \in J'$. Note that it is equivalent to assume $z_j \in U'$, as otherwise the U' condition is covered by the S' condition. Writing $W = U' \setminus z_j$ and using $J' = \{A \subseteq Z'x : A \in H\}$ we have

$P \in F(T_j - 1)^{J'+v}(v^c)^{H+z_j}(y^c)_S$ if and only if $P \in F(T_j - 1)_S$, $P_{S'}v \in F(T_j - 1)_{S'x}$ for $S' \subseteq S \cap Z' = S \cap Z$ with $S' \in H(x)$, $P_U y \in F(T_j - 1)_{Uz_j}$ for $U \subseteq S$ with $U \in H(z_j)$, and $P_W yv \in F(T_j - 1)_{Wz_jx}$ for $W \subseteq S \cap Z$ with $W \in H(xz_j)$.

This proves $(\dagger_{6.28})$. Now, since $M_{\overline{E},E}(T_j) = M_{\overline{E},E}(T_j - 1) \cap F_{\overline{E}}(T_j)$ we have

$$D^{Z'*v}_{z_j,E}(T_j - 1) = \left\{ y \in F_{z_j}(T_j - 1) : \frac{\big|M_{\overline{E},E}(T_j - 1) \cap \big(F(T_j - 1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c} \setminus y\big)(y^c)\big|}{\big|\big(F(T_j - 1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c} \setminus y\big)(y^c)\big|} > \theta_{\nu^*} \right\}.$$

Writing B'_{z_j} for the set of vertices $y \in F(T_j - 1)^{J'+v}(v^c)^{H+z_j}_{z_j} = F(T_j - 1)_{xz_j}(v)$ for which we do not have

$$\big|F(T_j - 1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c}(y^c)\big| = (1 \pm \epsilon_*)\big|F(T_j - 1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c}\big| / |F(T_j - 1)_{xz_j}(v)|,$$

we have $|B'_{z_j}| < \epsilon_* |F(T_j - 1)_{xz_j}(v)|$ by Lemma 6.20. Here we use the fact that the 'double plus' complex is $\epsilon_{k^3 D,3}$-regular; the proof of this is similar to that of Lemma 6.23(i): $F(T_j-1)^{J'+v}$ is $\epsilon_{k^3 D,1}$-regular, $F(T_j-1)^{J'+v}(v^c)$ is $\epsilon_{k^3 D,2}$-regular by Lemma 6.16, $F(T_j-1)^{J'+v}(v^c)^{H+z_j}$ is $\epsilon_{k^3 D,3}$-regular. Then

$$\begin{aligned}
\Sigma &:= \sum_{y \in D^{Z'*v}_{z_j,E}(T_j-1)} \left| M_{\overline{E},E}(T_j-1) \cap \left(F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c} \setminus y \right)(y^c) \right| \\
&> \theta_{v*} \sum_{y \in D^{Z'*v}_{z_j,E}(T_j-1) \setminus B'_{z_j}} \left| \left(F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c} \setminus y \right)(y^c) \right| \\
&> (1-2\epsilon_*)\theta_{v*} \left(\left| D^{Z'*v}_{z_j,E}(T_j-1) \right| - \epsilon_* |F(T_j-1)_{xz_j}(v)| \right) \frac{\left| F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c} \right|}{|F(T_j-1)_{xz_j}(v)|} \\
&= (1-2\epsilon_*)\theta_{v*} \left(\left| D^{Z'*v}_{z_j,E}(T_j-1) \right| / |F(T_j-1)_{xz_j}(v)| - \epsilon_* \right) \left| F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c} \right|.
\end{aligned}$$

We also have

$$\Sigma \le \sum_{y \in F_{xz_j}(T_j-1)(v)} \left| M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c}(y^c) \right|.$$

This counts all pairs (y, P) with $P \in M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}$, $y \in F_{xz_j}(T_j-1)(v)$ and $Py^c \in F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c}$, so we can rewrite it as

$$\Sigma \le \sum_{P \in M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}} \left| F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c}(P) \right|.$$

For all but at most $\epsilon_* |F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}|$ sets $P \in F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}$, we have

$$\left| F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c}(P) \right| = (1 \pm \epsilon_*) \frac{\left| F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c} \right|}{\left| F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}} \right|}.$$

(Recall that the 'double plus' complex is $\epsilon_{k^3 D,3}$-regular and use Lemma 6.20.) Therefore

$$\begin{aligned}
\Sigma &\le \left| M_{\overline{E},E}(T_j-1) \cap F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}} \right| \cdot (1+\epsilon_*) \frac{\left| F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c} \right|}{\left| F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}} \right|} \\
&\quad + \epsilon_* \left| F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}} \right| |F(T_j-1)_{xz_j}(v)|.
\end{aligned}$$

Combining this with the lower bound on Σ we have

$$(1-2\epsilon_*)\theta_{v*}\left(\left|D^{Z'*v}_{z_j,E}(T_j-1)\right|/|F(T_j-1)_{xz_j}(v)|-\epsilon_*\right)$$
$$\leq (1+\epsilon_*)\frac{\left|M_{\overline{E},E}(T_j-1)\cap F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}\right|}{\left|F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}\right|}$$
$$+\epsilon_*\frac{\left|F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}\right||F(T_j-1)_{xz_j}(v)|}{\left|F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c}\right|}.$$

Since $z_j\notin\overline{E}$, Definition 6.8 and equation ($\dagger_{6.28}$) give $F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}} = F(T_j-1)^{J'+v}(v^c)_{\overline{E}} = F(T_j-1)^{J+v}(v^c)_{\overline{E}} = F(T_j-1)^{Z*v}_{\overline{E}}$, so

$$\left|M_{\overline{E},E}(T_j-1)\cap F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}\right| < \theta'_v\left|F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}\right|$$

by $A_{2,j-1}$. We also have

$$\frac{\left|F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}}\right||F(T_j-1)_{xz_j}(v)|}{\left|F(T_j-1)^{J'+v}(v^c)^{H+z_j}_{\overline{E}z_j^c}\right|}\leq d_u^{-2^k}\ll\epsilon_*^{-1},$$

similarly to Lemma 6.23(v) (the statement is only for $F(t)$, but the estimate for the densities is valid for any $\epsilon_{k^3D,3}$-regular complex). Therefore

$$\frac{\left|D^{Z'*v}_{z_j,E}(T_j-1)\right|}{|F_{xz_j}(T_j-1)(v)|}<\frac{(1+\epsilon_*)\theta'_v+\sqrt{\epsilon_*}}{(1-2\epsilon_*)\theta_{v*}}+\epsilon_*<\theta_{v*}.$$

■

The analysis for the conclusion of the algorithm is very similar to that for 3-graphs, with the usual modifications to absolute constants to account for their dependence on k. The only important difference is to take account of restricted positions. Lemma 4.22 (the 'main lemma') holds, provided that we assume that the set Y is disjoint from the set X_* of vertices with restricted positions. We applied Lemma 4.22 in the proof of Theorem 4.23 to show that it is very unlikely that the iteration phase aborts with failure. This required an estimate for the probability that a given set $Y\subseteq X_i$ of size $\delta_Q|X_i|$ is contained in $Q(T)$ (the vertices that have ever been queued). Since $|X_*\cap X_i|\leq c|X_i|$ and $c\ll\delta_Q$, we can apply the same argument to $Y\setminus X_*$, which has size at least $\frac{1}{2}\delta_Q|X_i|$. The remainder of the proof of Theorem 4.23 is checking Hall's condition for the sets $\{A'_z : z\in S\}$, where $S\subseteq X_i(T)\subseteq B$. No changes are required here, as we chose the buffer B to be disjoint from X_*. This completes the proof of Theorem 6.12.

6.5. Obtaining Super-Regularity and Robust Universality

We conclude with some lemmas that will be useful when applying the blow-up lemma. We start with the analogue of Lemma 5.9, showing that one can delete a small number of vertices to enforce super-regularity.

Lemma 6.29. *Suppose* $0 < \epsilon_0 \ll \epsilon \ll \epsilon' \ll d_a \ll \theta \ll 1/D_R, d, 1/k$, *we have a multi-k-complex* R *on* $[r]$ *with maximum degree at most* D_R, *and* (G,M) *is an* R*-indexed marked complex on* $V = \cup_{i\in R} V_i$ *such that when defined* G_S *is* ϵ_0*-regular,* $|M_S| \le \theta|G_S|$, $d_S(G) \ge d_a$ *when* $|S| \ge 2$ *and* $d_S(G) \ge d$ *when* $|S| = k$. *Then we can delete at most* $2\theta^{1/3}|G_i|$ *vertices from each* G_i, $i \in R$ *to obtain an* $(\epsilon, \epsilon', d_a/2, 2\sqrt{\theta}, d/2)$*-super-regular marked complex* $(G^\sharp, M^\sharp)$.

Proof. The proof is similar to that of Lemma 5.9, so we will just sketch the necessary modifications. Similarly to before, for any i, S such that $i \in S$, $|S| = k$ and G_S is defined we let $Y_{i,S\setminus i}$ be the set of vertices $v \in G_i$ for which $|M_S(v)| > \theta|G_S(v)|$. We also let $Z_{i,S\setminus i}$ be the set of vertices $v \in G_i$ such that we do not have $|G_S(v)| = (1 \pm \epsilon)|G_S|/|G_i|$ and $G_{S^\le}(v)$ is ϵ-regular with $d_{S'\setminus i}(G_{S^\le}(v)) = (1 \pm \epsilon)d_{S'\setminus i}(G)d_{S'}(G)$ for $i \subsetneq S' \subseteq S$. As before we have $|Z_{i,S\setminus i}| < \epsilon|G_i|$ and $|Y_{i,S\setminus i}| < 2\sqrt{\theta}$. Next, consider any k-tuple S containing at least one neighbour of i in R such that G_S is defined, and any subcomplex I of $Si^\le$ such that $G_{S'}$ is defined for all $S' \in I$. We let $Y^I_{i,S}$ be the set of vertices $v \in G_i$ for which $|(M \cap G^{I_v})_S| > \sqrt{\theta}|G^{I_v}_S|$. Note that we only need to consider S containing at least one neighbour of i in R, as otherwise we have $G^{I_v}_S = G_S$, and $|M_S| \le \theta|G_S|$ by assumption. We let $Z_{i,S}$ be the union of all $Z_{i,S'}$ with $i \notin S' \subseteq S$, $|S'| = k-1$. Recall from Lemma 6.11 that $G^{I_v}_S = G^{I+i}(v^c)_S$. If $v \notin Z_{i,S}$ then $G^{I+i}_{Si^c}$ is $\sqrt{\epsilon}$-regular by regular restriction and $|G^{I+i}(v^c)_S| = (1 \pm \epsilon')|G^{I+i}_{Si^c}|/|G_i|$ by Lemma 6.20. Then $\Sigma = \sum_{v\in Y^I_{i,S}} |(M \cap G^{I_v})_S| = \sum_{v\in Y^I_{i,S}} |(M \cap G^{I+i}(v^c))_S|$ satisfies

$$\Sigma > \sqrt{\theta} \sum_{v\in Y^I_{i,S}\setminus Z_{i,S}} |(M \cap G^{I+i}(v^c))_S| > \sqrt{\theta}\big(\big|Y^I_{i,S}\big| - k\epsilon|G_i|\big)(1-\epsilon')\big|G^{I+i}_{Si^c}\big|/|G_i|.$$

We also have $\Sigma \le \sum_{v\in G_i} |(M \cap G^{I+i}(v^c))_S|$, which counts all pairs (v,P) with $P \in M_S$, $v \in G_i$ and $Pv^c \in G^{I+i}$. By Lemma 6.20 we have $|G^{I+i}(P)_{i^c}| = (1 \pm \epsilon')|G^{I+i}_{Si^c}|/|G^{I+i}_S|$ for all but at most $\epsilon'|G^{I+i}_S|$ sets $P \in G^{I+i}_S$. Since $G^{I+i}_S = G_S$ we have

$$\Sigma \le \sum_{P\in M_S} |G^{I+i}(P)_{i^c}| \le |M_S|(1+\epsilon')\big|G^{I+i}_{Si^c}\big|/|G_S| + \epsilon'|G_S||G_i|.$$

Combining this with the lower bound on Σ and using $|M_S| \le \theta|G_S|$ we obtain $|Y^I_{i,S}|/|G_i| < \frac{(1+\epsilon')\theta+\sqrt{\epsilon'}}{(1-\epsilon')\sqrt{\theta}} + k\epsilon < 2\sqrt{\theta}$. Let Y_i be the union of all such sets $Y_{i,S\setminus i}$ and $Y^I_{i,S}$. Since $\theta \ll 1/D_R$ we have $|Y_i| < \theta^{1/3}|G_i|$ as in Step 1 of Lemma 5.9. We define $Z'_{i,j}$, Z_i and obtain $|Z_i| < \sqrt{\epsilon}|G_i|$ as in Step 2 of Lemma 5.9. Now we delete $Y_i \cup Z_i$ from G_i for every $i \in R$; as in Step 3 of Lemma 5.9 this gives an $(\epsilon, \epsilon', d_a/2, 2\sqrt{\theta}, d/2)$-super-regular marked complex $(G^\sharp, M^\sharp)$. ∎

The next lemma is analogous to Lemma 5.11; we omit its very similar proof.

Lemma 6.30 (Super-regular restriction). *Suppose* $0 < \epsilon \ll \epsilon' \ll \epsilon'' \ll d_a \ll \theta \ll d, d', 1/k$, *we have a multi-k-complex* R, *and* (G,M) *is a* $(\epsilon, \epsilon', d_a, \theta, d)$*-super-regular marked* R*-indexed complex on* $V = \cup_{i\in R} V_i$ *with* $G_i = V_i$ *for* $i \in R$. *Suppose also that we have* $V'_i \subseteq V_i$ *for* $i \in R$, *write* $V' = \cup_{i\in R} V'_i$, $G' = G[V']$, $M' = M[V']$, *and that* $|V'_i| \ge d'|V_i|$ *and* $|G'_S(v) \cap V'_i| \ge d'|G_S(v)|$ *whenever* $S \in R$ *with* $|S| = 2$ *lies over* $i, j \in R$ *and* $v \in G_j$. *Then* (G', M') *is* $(\epsilon', \epsilon'', d_a/2, \sqrt{\theta}, d/2)$*-super-regular.*

More generally, the same proof shows that super-regularity is preserved on restriction to a dense regular subcomplex Γ, provided that the singleton parts of Γ have large intersection with every vertex neighbourhood. More precisely, suppose (G, M) is as in Lemma 6.30 and Γ is an ϵ'-regular subcomplex of G with $|\Gamma_S| \geq d'|G_S|$ when defined and $|G_S(v) \cap \Gamma_i| \geq d'|G_S(v)|$ whenever $S \in R$ with $|S| = 2$ lies over $i, j \in R$, $v \in G_j$ and Γ_i is defined. Then (G', M') is $(\epsilon', \epsilon'', d'd_a, \sqrt{\theta}, d'd)$-super-regular. Next we will reformulate the blow-up lemma in a more convenient 'black box' form. The following definition of 'robustly universal' is more general than that used for 3-graphs, in that it allows for restricted positions.

Definition 6.31 (Robustly universal). *Suppose R is a multi-k-complex R and J is an R-indexed complex on $Y = \cup_{i\in R} Y_i$ with $J_i = Y_i$ for $i \in R$. We say that J is* $(c^\sharp, c)$-robustly D-universal *if whenever*

(i) *$Y_i' \subseteq Y_i$ with $|Y_i'| \geq c^\sharp |Y_i|$ such that $Y' = \cup_{i\in R} Y_i'$, $J' = J[Y']$ satisfy $|J_S'(v)| \geq c^\sharp |J_S(v)|$ whenever $|S| = k$, J_S is defined, $i \in S$, $v \in J_i'$,*
(ii) *H' is an R-indexed complex on $X' = \cup_{i\in R} X_i'$ of maximum degree at most D with $|X_i'| = |Y_i'|$ for $i \in R$,*
(iii) *$X_* \subseteq X'$ with $|X_* \cap X_i'| \leq c|X_i'|$ for all $i \in R$, and $\Gamma_x \subseteq Y_x'$ with $|\Gamma_x| \geq c^\sharp |Y_x'|$ for $x \in X_*$,*

then there is a bijection $\phi : X' \to V'$ with $\phi(X_i') = V_i'$ for $i \in R$ such that $\phi(S) \in J_S$ for $S \in H'$ and $\phi(x) \in \Gamma_x$ for $x \in X_*$.

More generally, one can allow restrictions to regular subcomplexes in both conditions (i) and (iii) of Definition 6.31, but for simplicity we will not formulate the definition here. As before, one can delete a small number of vertices from a regular complex with a small number of marked k-tuples to obtain a robustly universal complex. As for Theorem 5.13, the proof is immediate from Lemma 6.30, Definition 6.31 and Theorem 6.12.

Theorem 6.32. *Suppose $0 < 1/n \ll 1/n_R \ll \epsilon \ll c \ll d^\sharp \ll d_a \ll \theta \ll c^\sharp, d, 1/k, 1/D_R, 1/D$, we have a multi-$k$-complex R on $[r]$ with maximum degree at most D_R and $|R| \leq n_R$, G is an ϵ-regular R-indexed complex on $V = \cup_{i\in R} V_i$ with $n \leq |V_i| = |G_i| \leq Cn$ for $i \in R$, $d_S(G) \geq d_a$ when $|S| \geq 2$ and $d_S(G) \geq d$ when $|S| = k$, and $M \subseteq G_=$ with $|M_S| \leq \theta |G_S|$ when defined. Then we can delete at most $2\theta^{1/3}|G_i|$ vertices from G_i for $i \in R$ to obtain $G^\sharp$ and $M^\sharp$ so that*

(i) *$d(G_S^\sharp) > d^\sharp$ and $|G_S^\sharp(v)| > d^\sharp |G_S^\sharp|/|G_i^\sharp|$ whenever $|S| = k$, $G_S^\sharp$ is defined, $i \in S$, $v \in G_i^\sharp$, and*
(ii) *$G^\sharp \setminus M^\sharp$ is $(c^\sharp, c)$-robustly D-universal.*

Finally, we mention that one can allow much smaller densities in the restricted positions, provided that one makes an additional assumption to control the marking edges. We can replace c' by d_a in condition (v) of Theorem 6.12, provided that we add the following additional assumptions:

(v.1) $|M_S(v)| \leq \theta |\Gamma_S(v)|$ when $|S| = k$, Γ_S is defined, $i \in S$ and $v \in \Gamma_i$,
(v.2) $|(M \cap G[\Gamma]^{I_v})_S| \leq \theta |G[\Gamma]_S^{I_v}|$ for any submulticomplex I of R, when $|S| = k$, $v \in G_i$ and $S \cap VN_R(i) \neq \emptyset$.

Note that these conditions ensure that the marked edges are controlled in $G[\Gamma]$ exactly as in conditions (ii) and (iii) of super-regularity, so the proof goes through as before. In this general form there is no simplification to be gained by reformulating the statement in a black box form. We suppressed this refined form in the statement of Theorem 6.12 to avoid overburdening the reader with technicalities, but we note that it may be needed in some applications. Indeed, one may well have to generate restricted positions using neighbourhood complexes in G, and then c' will be of the order of the densities in G.

6.6. Concluding Remarks

The theory of regularity and super-regularity for hypergraphs is considerably more involved than that for graphs. As explained in Section 3, these technicalities cannot be avoided, but the black box reformulation in Lemma 6.32 should make the hypergraph blow-up lemma more convenient for future applications. The graph blow-up lemma has had many applications in modern graph theory, so it is natural to look for hypergraph generalisations of these results. However, many such applications build on basic results for graphs for which the hypergraph analogue is unknown. For example, in our application in Section 5 we only needed a matching, and were able to rely on Kahn's matching theorem, which is already quite a difficult result. Thus one may expect it will take longer for the hypergraph blow-up lemma to achieve its full potential.

Another question for future research is to obtain an algorithmic version of our theorem, along the lines of the algorithmic graph blow-up lemma in [27]. In applications this could be combined with an algorithmic version of hypergraph regularity given by [6]. A rather different direction of research would be along the lines of the 'infinitary' versions of hypergraph regularity theory, whether probabilistic [1, 45], analytic [34], model theoretic [7] or algebraic [10, 38]. It is natural to ask whether the blow-up lemma has an interpretation in any of these frameworks.

Further refinements could include estimating the number of embeddings, rather than just proving the existence of a single embedding as in this paper. Here it may be helpful to note that one can combine Lemma 6.28 with martingale estimates to show that with high probability there will be at least $\frac{1}{2}p|B \cap X_v|$ vertices $x \in B \cap X_v$ such that $\phi(H(x)) \subseteq (G \setminus M)(v)$. We also note that small improvements to the tail decay of our martingales may be obtained from the Optional Sampling Theorem (see e.g. [14] p. 462). One could also try to obtain (nearly) perfect edge-decompositions of super-regular complexes into copies of a given bounded degree hypergraph. For example, one could ask for hypergraph generalisations of a result of Frieze and Krivelevich [11] that one can cover almost all edges of an ϵ-regular graph by edge-disjoint Hamilton cycles.[8]

ACKNOWLEDGMENTS

I thank Daniela Kühn, Deryk Osthus, Richard Mycroft and Mathias Schacht for drawing my attention to a significant oversight in an earlier version of this paper. That version contained a weaker version of the blow-up lemma, which assumed a dense environment that cannot be guaranteed in many applications. In particular it was not adequate for the application that I give in this paper. I also thank the anonymous referees, Daniela Kühn, Deryk Osthus and Richard Mycroft for carefully reading and commenting extensively on various versions of this paper, and Mathias Schacht for explaining [41, 42] to me.

[8]Note added during revisions: Frieze and Krivelevich now have a preprint on this topic.

REFERENCES

[1] T. Austin, On exchangeable random variables and the statistics of large graphs and hypergraphs, Probab Surveys 5 (2008), 80–145.

[2] K. Azuma, Weighted sums of certain dependent random variables, Tôhoku Math J 19 (1967), 357–367.

[3] D. Conlon, J. Fox, and B. Sudakov, Ramsey numbers of sparse hypergraphs, Random Struct Algorithm 35 (2009), 1–14.

[4] O. Cooley, N. Fountoulakis, D. Kühn, and D. Osthus, Embeddings and Ramsey numbers of sparse k-uniform hypergraphs, Combinatorica 29 (2009), 263–297.

[5] B. Csaba, On the Bollobás - Eldridge conjecture for bipartite graphs, Combin Probab Comput 16 (2007), 661–691.

[6] Y. Dementieva, P. E. Haxell, B. Nagle, and V. Rödl, On characterising hypergraph regularity, Random Struct Algorithm 21 (2002), 293–335.

[7] G. Elek and B. Szegedy, Limits of hypergraphs, removal and regularity lemmas, A Non-standard Approach, arXiv:0705.2179, 2007.

[8] P. Erdős, On extremal problems of graphs and generalised graphs, Israel J Math 2 (1964), 183–190.

[9] P. Frankl and V. Rödl, Extremal problems on set systems, Random Struct Algorithm 20 (2002), 131–164.

[10] M. Freedman, L. Lovász, and A. Schrijver, Reflection positivity, rank connectivity, and homomorphism of graphs, J AMS 20 (2007), 37–51.

[11] A. Frieze and M. Krivelevich, On packing Hamilton cycles in epsilon-regular graphs, J Combin Theory Ser B 94 (2005), 159–172.

[12] W. T. Gowers, Quasirandomness, counting and regularity for 3-uniform hypergraphs, Combin Probab Comput 15 (2006), 143–184.

[13] W. T. Gowers, Hypergraph regularity and the multidimensional Szemerédi theorem, Ann Math 166 (2007), 897–946.

[14] G. R. Grimmett and D. R. Stirzaker, Probability and random processes, 2nd edition, Oxford University Press, Oxford, 1992.

[15] H. Hán and M. Schacht, Dirac-type results for loose Hamilton cycles in uniform hypergraphs, J Combin Theory Ser B 100 (2010), 332–346.

[16] Y. Ishigami, A simple regularisation of hypergraphs, arXiv:math.CO/0612838, 2006.

[17] J. Kahn, A linear programming perspective on the Frankl-Rödl-Pippenger theorem, Random Struct Algorithm 8 (1996), 149–157.

[18] P. Keevash, A hypergraph regularity method for generalised Turán problems, Random Struct Algorithm 34 (2009), 123–164.

[19] P. Keevash, D. Kühn, R. Mycroft, and D. Osthus, Loose Hamilton cycles in hypergraphs, Disc Math (in press).

[20] P. Keevash, D. Kühn, and D. Osthus, An exact minimum degree condition for Hamilton cycles in oriented graphs, J Lond Math Soc 79 (2009), 144–166.

[21] D. Kühn, R. Mycroft, and D. Osthus, Hamilton ℓ-cycles in uniform hypergraphs, J Combin Theory Ser A 117 (2010), 910–927.

[22] Y. Kohayakawa, V. Rödl, and J. Skokan, Hypergraphs, quasi-randomness, and conditions for regularity, J Combin Theory Ser A 97 (2002), 307–352.

[23] J. Komlós, Tiling Turán theorems, Combinatorica 20 (2000), 203–218.

[24] J. Komlós, G. N. Sárközy, and E. Szemerédi, Blow-up lemma, Combinatorica 17 (1997), 109–123.

[25] J. Komlós, G. N. Sárközy, and E. Szemerédi, Proof of the Seymour conjecture for large graphs, Ann Comb 2 (1998), 43–60.

[26] J. Komlós, G. N. Sárközy, and E. Szemerédi, Proof of the Alon-Yuster conjecture, Combinatorics (Prague, 1998), Discrete Math 235 (2001), 255–269.

[27] J. Komlós, G. N. Sárközy, and E. Szemerédi, An algorithmic version of the blow-up lemma, Random Struct Algorithm 12 (1998), 297–312.

[28] J. Komlós and M. Simonovits, Szemerédi's regularity lemma and its applications in graph theory, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), 295–352, Bolyai Soc. Math. Stud., 2, János Bolyai Math. Soc., Budapest, 1996.

[29] T. Kövari, V. Sós, and P. Turán, On a problem of K. Zarankiewicz, Colloquium Math 3 (1954), 50–57.

[30] D. Kühn and D. Osthus, Loose Hamilton cycles in 3-uniform hypergraphs of large minimum degree, J Combin Theory Ser B 96 (2006), 767–821.

[31] D. Kühn and D. Osthus, Packings in dense regular graphs, Combin Probab Comput 14 (2005), 325–337.

[32] D. Kühn and D. Osthus, The minimum degree threshold for perfect graph packings, Combinatorica 29 (2009), 65–107.

[33] D. Kühn and D. Osthus, Embedding large subgraphs into dense graphs, Surveys in Combinatorics, Cambridge University Press, 2009, pp. 137–167.

[34] L. Lovász and B. Szegedy, Szemerédi's lemma for the analyst, J Geom Func Anal 17 (2007), 252–270.

[35] B. Nagle, S. Olsen, V. Rödl, and M. Schacht, On the Ramsey number of sparse 3-graphs, Graphs Combin 24 (2008), 205–228.

[36] B. Nagle and V. Rödl, Regularity properties for triple systems, Random Struct Algorithm 23 (2003), 264–332.

[37] B. Nagle, V. Rödl, and M. Schacht, The counting lemma for regular k-uniform hypergraphs, Random Struct Algorithm 28 (2006), 113–179.

[38] A. Razborov, Flag algebras, J Symbolic Logic 72 (2007), 1239–1282.

[39] V. Rödl and A. Ruciński, Dirac-type questions for hypergraphs – a survey (or more problems for Endre to solve), An Irregular Mind (Szemerédi is 70), Bolyai Soc Math Stud 21 (2010).

[40] V. Rödl, A. Ruciński, and E. Szemerédi, An approximate Dirac-type theorem for k-uniform hypergraphs, Combinatorica 28 (2008), 229–260.

[41] V. Rödl and M. Schacht, Regular partitions of hypergraphs: Regularity lemmas, Combin Probab Comput 16 (2007), 833–885.

[42] V. Rödl and M. Schacht, Regular partitions of hypergraphs: counting lemmas, Combin Probab Comput 16 (2007), 887–901.

[43] V. Rödl and J. Skokan, Regularity lemma for uniform hypergraphs, Random Struct Algorithm 25 (2004), 1–42.

[44] E. Szemerédi, Regular partitions of graphs, Problèmes combinatoires et théorie des graphes, Colloq Int CNRS 260 (1978), 399–401.

[45] T. Tao, A variant of the hypergraph removal lemma, J Combin Theory Ser A 113 (2006), 1257–1280.

[46] T. Tao, A correspondence principle between (hyper)graph theory and probability theory, and the (hyper)graph removal lemma, J Anal Math 103 (2007), 1–45.

INDEX OF NOTATION AND TERMINOLOGY