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α-INCLUSIONS APPLIED TO GROUP THEORY VIA SOFT SET AND LOGIC

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Abstract. Soft set theory, initiated by Molodtsov, is a tool for modeling various types of uncertainty. In this paper, upper and lower α-inclusions of a soft set are defined. By using these new notions, some analyzes with respect to group theory are made and it is shown that some of the subgroups of a group can be obtained easily with the help of these notions. It is also illustrated that a soft int-group and a soft uni-group can be obtained by its upper α-subgroups and lower α-subgroups, respectively. Furthermore, soft int-group by its family of upper α-subgroups is characterized under a certain equivalence relation. Finally, a new method used to construct a soft int-group with the help of its upper α-subgroups are introduced and an application of this method is given.

1. Introduction

The notion of soft set was introduced in 1999 by Molodtsov [1]. Since its inception, Maji et al. [2] and Ali et al. [3] introduced several operations of soft sets and Sezgin et al. [4] studied on soft set operations in more detail. And the theory of soft set has gone through remarkably rapid strides with a wide-ranging applications especially in decision making [5, 6, 7, 8, 9, 10, 11, 12, 13]. There have been also attempt to softificate various mathematical structures like groups [14], semirings [15], BCK/BCI-algebras [16, 17], p-ideals [18], BCH-algebras [19], rings [20], near-rings [21], mappings [22], soft lattices [23], substructures of rings, fields and modules [24]. Çağman et al. [25] introduced the concept of soft int-group which is based on the inclusion relation and the intersection of sets, and therefore more functional for developing the theory of soft groups. With the motivation of soft int-group, Sezgin et al. [26] moreover defined soft uni-group which is based on the inclusion relation and the union of sets and Muştuoğlu et al. [27] studied its basic properties of soft uni-groups as regards normal soft uni-groups as well.

In this paper, first the notions of upper/lower α-inclusions and proper upper/lower α-inclusions of a soft set are introduced and their basic properties are

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studied. In this connection, the notions of upper α-subgroups of a soft uni-group and lower α-subgroups of a soft uni-group are introduced. It is shown that some of the subgroups of a group can be obtained via a soft int-group and its upper α-subgroups and via a soft uni-group and its lower α-subgroups. And also it is illustrated that under certain conditions, one can obtain a soft int-group with the help of its upper α-subgroups and a soft uni-group with the help of its lower α-subgroups. Since the theory of groups occupies a significant role in mathematics, with many practical applications, these notions are analyzed in more detail and some applications of these notions on group theory especially more concerning with upper α-subgroups of a soft int-group are made. An answer whether the family of upper α-subgroups of a soft int-group determine the soft int-group uniquely or not are tried to be found. Firstly, an example to show that two soft int-groups may have an identical family of upper α-subgroups, but may not be soft equal is given, and it is proved that for the equality, their image sets have to be equal. Also, it is shown that given any chain of subgroups of a group, there always exists a soft int-group whose upper α-subgroups are exactly the members of this chain. Further, a method used for constructing a soft int-group of G with the help of the subgroups of G is defined. Finally, it is shown how an upper α-subgroup of a soft int-group is softificated.

2. Preliminaries

In this section, some basic definition of soft set theory for the sake of completeness are recalled. Throughout this paper, U refers to an initial universe, E is a set of parameters, $A \subseteq E$ and $P(U)$ is the power set of U .

Definition 1. [1] *A soft set f_A over U is a set defined by*

$$f_A : E \to P(U) \text{ such that} f_A(x) = \emptyset \text{ if } x \notin A.$$

Here, f_A is also called approximate function. A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

It is clear to see that a soft set is a parametrized family of subsets of the set U. It is worth noting that the sets $f_A(x)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection. Throughout this paper, $f_A(x) \neq \emptyset$ if $x \in A$, all of the soft sets are over U unless otherwise specified and $Im(f_A)$ denotes the image set of f_A.

Definition 2. [6] *Let f_A, f_B be soft sets over U. Then, f_A is a soft subset of f_B, denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

Definition 3. [6] *Let f_A, f_B be soft sets over U. Then, f_A and f_B are soft equal, denoted by $f_A = f_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$.*

Definition 4. [6] *The complement of the soft set f_A over U is defined by f_A^c, where $f_A^c(\alpha) = U \setminus f_A(\alpha)$ for all $\alpha \in E$.*

Definition 5. [25] *Let G be a group and f_G be a soft set over U. Then, f_G is called a soft int-group of G over U if*

i) $f_G(xy) \supseteq f_G(x) \cap f_G(y)$ *for all* $x, y \in G$,
ii) $f_G(x^{-1}) = f_G(x)$ *for all* $x \in G$.

Definition 6. [26] *Let G be a group and f_G be a soft set over U. Then, f_G is called an soft uni-group of G over U if*

i) $f_G(xy) \subseteq f_G(x) \cup f_G(y)$ *for all* $x, y \in G$,
ii) $f_G(x^{-1}) = f_G(x)$ *for all* $x \in G$.

For the sake of ease, a soft uni-group is designated by SU-group and a soft int-group by SI-group in what follows.

Theorem 7. [25] *A soft set f_G over U is a SI-group over U if and only if $f_G(xy^{-1}) \supseteq f_G(x) \cap f_G(y)$ for all $x, y \in G$.*

Theorem 8. [26] *A soft set f_G over U is a SU-group over U if and only if $f_G(xy^{-1}) \subseteq f_G(x) \cup f_G(y)$ for all $x, y \in G$.*

Theorem 9. [25] *Let f_G be a SI-group over U. Then, $f_G(e) \supseteq f_G(x)$ for all $x \in G$.*

Theorem 10. [26] *Let f_G be a soft set over U. Then, f_G is a SU-group over U if and only if f_G^c is a SI-group over U.*

3. α-inclusions of a soft set

In this section, first upper and proper upper α-inclusions, lower and proper lower α-inclusions of a soft set are introduced and their basic properties are given. From now on, f_A is a soft set over U, α and α_i, where $i \in I$ and I is an arbitrary finite index set, represents the subsets of U.

Definition 11. [25] *Let f_A be a soft set over U. Then, upper α-inclusion of the soft set f_A, denoted by $f_A^{\supseteq\alpha}$, is defined as*

$$f_A^{\supseteq\alpha} = \{x \in A : f_A(x) \supseteq \alpha\}.$$

In particular, proper upper α-inclusion of the soft set f_A, denoted by $f_A^{\supsetneq\alpha}$, is defined as

$$f_A^{\supsetneq\alpha} = \{x \in A : f_A(x) \supsetneq \alpha\},$$

where $f_A(x) \supsetneq \alpha$ means that $f_A(x) \supseteq \alpha$ and $f_A(x) \neq \alpha$.

Definition 12. *Let f_A be a soft set over U. Then, lower α-inclusion of the soft set f_A, denoted by $f_A^{\subseteq\alpha}$, is defined as*

$$f_A^{\subseteq\alpha} = \{x \in A : f_A(x) \subseteq \alpha\}.$$

In particular, proper lower α-inclusion of the soft set f_A, denoted by $f_A^{\subsetneq\alpha}$, is defined as

$$f_A^{\subsetneq\alpha} = \{x \in A : f_A(x) \subsetneq \alpha\},$$

where $f_A(x) \subsetneq \alpha$ means that $f_A(x) \subseteq \alpha$ and $f_A(x) \neq \alpha$.

Note that, $f_A^{\subseteq U} = f_A^{\supseteq\emptyset} = A$ and $f_A^{\subseteq\alpha} \cup f_A^{\supseteq\alpha} \neq A$.
In the following theorem, it is shown that each approximate elements of the soft set can be characterized with the help of lower and upper α-inclusions.

Proposition 13. *Let f_A be a soft set over U and $x \in A$. Then,*

(1) $f_A(x) = \bigcap\{\alpha \subseteq U : x \in f_A^{\subseteq\alpha}\}$.
(2) $f_A(x) = \bigcup\{\alpha \subseteq U : x \in f_A^{\supseteq\alpha}\}$.

Proposition 14. *Let f_A be a soft set over U. Then,*

(1) $f_A^{\supsetneq\alpha} \subseteq f_A^{\supseteq\alpha}$ *and* $f_A^{\subsetneq\alpha} \subseteq f_A^{\subseteq\alpha}$.
(2) $f_A^{\supsetneq\alpha} = f_A^{\supseteq\alpha}$ *and* $f_A^{\subsetneq\alpha} = f_A^{\subseteq\alpha}$ *if and only if $\alpha \notin Im(f_A)$, that is to say, there is no $x \in A$ such that $f_A(x) = \alpha$.*

Proposition 15. *Let f_A be a soft set over U and $A = E$. Then, if $\alpha_1, \alpha_2 \in Im(f_A)$ such that $\alpha_1 \neq \alpha_2$, then $f_A^{\supseteq\alpha_1} \neq f_A^{\supseteq\alpha_2}$ and $f_A^{\subseteq\alpha_1} \neq f_A^{\subseteq\alpha_2}$.*

Proposition 15 gives the idea that different elements of $Im(f_A)$ generates different upper/lower α-inclusions.

Note 16. *It is obvious that if $A \neq E$, then $\emptyset$ is always an element of $Im(f_A)$. Thus, Proposition 15 is not valid if there is an element, say α_k, in $Im(f_A)$ which is contained in all of the elements of the image set and not equal to null set. Because, then $f_A^{\supseteq\emptyset} = f_A^{\supseteq\alpha_k}$, but $\emptyset \neq \alpha_k$.*

Proposition 17. *Let f_A be a soft set over U and $\alpha_1, \alpha_2 \subseteq U$ such that $\alpha_1 \subseteq \alpha_2$. Then,*

(1) $f_A^{\supsetneq\alpha_2} \subseteq f_A^{\supsetneq\alpha_1}$ *and* $f_A^{\supseteq\alpha_2} \subseteq f_A^{\supseteq\alpha_1}$.
(2) $f_A^{\subsetneq\alpha_1} \subseteq f_A^{\subsetneq\alpha_2}$ *and* $f_A^{\subseteq\alpha_2} \subseteq f_A^{\supseteq\alpha_2}$.

In Proposition 17, it is shown that if $\alpha_1 \subseteq \alpha_2$, then whether α_1 and α_2 are elements of $Im(f_A)$ does not become an issue. However in Proposition 18, 19 and 20, it is seen that if $\alpha_1 \subsetneq \alpha_2$, then it has great importance whether α_1 and α_2 are elements of $Im(f_A)$.

Proposition 18. *Let f_A be a soft set over U and $\alpha_1, \alpha_2 \subseteq U$ such that $\alpha_1 \subsetneq \alpha_2$. If $\alpha_1, \alpha_2 \notin Im(f_A)$, then*

(1) $f_A^{\supsetneq\alpha_2} \subseteq f_A^{\supsetneq\alpha_1}$ *and* $f_A^{\supseteq\alpha_2} \subseteq f_A^{\supseteq\alpha_1}$.
(2) $f_A^{\subsetneq\alpha_1} \subseteq f_A^{\subsetneq\alpha_2}$ *and* $f_A^{\subseteq\alpha_1} \subseteq f_A^{\subseteq\alpha_2}$.

Proof. *i)* Let $x \in f_A^{\supsetneq \alpha_2}$, then

$$f_A(x) \supsetneq \alpha_2 \supsetneq \alpha_1 \Rightarrow x \in f_A^{\supsetneq \alpha_1} \Rightarrow f_A^{\supsetneq \alpha_2} \subseteq f_A^{\supsetneq \alpha_1}.$$

$f_A^{\supseteq \alpha_2} \subseteq f_A^{\supseteq \alpha_1}$ follows from $f_A^{\supsetneq \alpha_2} \subseteq f_A^{\supsetneq \alpha_1}$ and Proposition 14.

ii) Let $x \in f_A^{\subsetneq \alpha_1}$, then

$$f_A(x) \subsetneq \alpha_1 \subsetneq \alpha_2 \Rightarrow x \in f_A^{\subsetneq \alpha_2} \Rightarrow f_A^{\subsetneq \alpha_1} \subseteq f_A^{\subsetneq \alpha_2}.$$

$f_A^{\subseteq \alpha_1} \subseteq f_A^{\subseteq \alpha_2}$ follows from $f_A^{\subsetneq \alpha_1} \subseteq f_A^{\subsetneq \alpha_2}$ and Proposition 14. □

Proposition 19. *Let f_A be a soft set over U and $\alpha_1, \alpha_2 \subseteq U$ such that $\alpha_1 \subsetneq \alpha_2$. If $\alpha_1 \notin Im(f_A)$ and $\alpha_2 \in Im(f_A)$, then*

(1) $f_A^{\supsetneq \alpha_2} \subsetneq f_A^{\supsetneq \alpha_1}$ *and* $f_A^{\subsetneq \alpha_1} \subseteq f_A^{\subsetneq \alpha_2}$.
(2) $f_A^{\supseteq \alpha_2} \subseteq f_A^{\supseteq \alpha_1}$ *and* $f_A^{\subseteq \alpha_1} \subseteq f_A^{\subseteq \alpha_2}$

Proposition 20. *Let f_A be a soft set over U and $\alpha_1, \alpha_2 \subseteq U$ such that $\alpha_1 \subsetneq \alpha_2$. If $\alpha_1, \alpha_2 \in Im(f_A)$, then*

(1) $f_A^{\supsetneq \alpha_2} \subsetneq f_A^{\supsetneq \alpha_1}$ *and* $f_A^{\subsetneq \alpha_1} \subsetneq f_A^{\subsetneq \alpha_2}$.
(2) $f_A^{\supseteq \alpha_2} \subsetneq f_A^{\supseteq \alpha_1}$ *and* $f_A^{\subseteq \alpha_1} \subsetneq f_A^{\subseteq \alpha_2}$.

Note that in Proposition 18, 19 and 20, whether the set $Im(f_A)$ is ordered by inclusion or not has an important role, too.

Proposition 21. *Let f_A be a soft set over U and $\alpha' = U \setminus \alpha$. Then,*

$$f_A^{\subseteq \alpha} = (f_A^c)^{\supseteq \alpha'} \text{ and } f_A^{\supseteq \alpha} = (f_A^c)^{\subseteq \alpha'}.$$

Proof. Let f_A be a soft set over U. Then,

$$\begin{aligned} (f_A^c)^{\supseteq \alpha'} &= \{x \in A : f_A^c(x) \supseteq \alpha'\} \\ &= \{x \in A : U \setminus f_A(x) \supseteq \alpha'\} \\ &= \{x \in A : f_A(x) \subseteq \alpha\} \\ &= f_A^{\subseteq \alpha} \end{aligned}$$

The other equality can be shown similarly, hence omitted. □

4. Applications of α-inclusions to group theory

In this section, the notion of upper α-subgroups of a SI-groups and lower α-subgroups of a SU-groups are introduced and these notions are studied with respect to group theory. From now on, G denotes a group with identity e and also G is the set of parameters under consideration unless otherwise specified.

Theorem 22. [25] *Let f_G be a soft set over U and α be a subset of U such that $\emptyset \subseteq \alpha \subseteq f_G(e)$. If f_G is a SI-group over U, then $f_G^{\supseteq \alpha}$ is a subgroup of G.*

Theorem 23. *Let f_G be a soft set over U and α be a subset of U such that $\emptyset \subseteq f_G(e) \subseteq \alpha$. If f_G is a SU-group over U, then $f_G^{\subseteq\alpha}$ is a subgroup of G.*

Proof. Let f_G be a SU-group over U, then f_G^c is a SI-group over U by Theorem 22. It is needed to show that $f_G^{\subseteq\alpha}$ is a subgroup of G. Since $f_G^{\subseteq\alpha} = (f_G^c)^{\supseteq\alpha'}$ by Proposition 21, it is enough to show that $(f_G^c)^{\supseteq\alpha'}$ is a subgroup of G. Since $f_G(e) \subseteq \alpha$, it follows that $e \in f_G^{\subseteq\alpha}$ and

$$\emptyset \neq f_G^{\subseteq\alpha} \subseteq G, \text{ so } \emptyset \neq (f_G^c)^{\supseteq\alpha'} \subseteq G.$$

Now assume that $x, y \in (f_G^c)^{\supseteq\alpha'}$, then

$$f_G^c(x) \supseteq \alpha' \text{ and } f_G^c(y) \supseteq \alpha'.$$

Since f_G^c is a SI-group over U, then

$$f_G^c(xy^{-1}) \supseteq f_G^c(x) \cap f_G^c(y) \supseteq \alpha' \cap \alpha' = \alpha'.$$

This shows that $xy^{-1} \in (f_G^c)^{\supseteq\alpha'}$, which completes the proof. □

From now on, if f_G is a SI-group over U, then it is assumed that $\emptyset \subseteq \alpha \subseteq f_G(e)$ and if f_G is a SU-group over U, then $\emptyset \subseteq f_G(e) \subseteq \alpha$ is satisfied for all $\alpha \subseteq U$.

Definition 24. *If f_G is a SI-group over U, then the subgroups $f_G^{\supseteq\alpha}$ are called upper α-subgroups of f_G. If f_G is a SU-group over U, then the subgroups $f_G^{\subseteq\alpha}$ are called lower α-subgroups of f_G.*

It is known that if G is a finite group, then the number of subgroups of G is finite. Now, consider the upper α-subgroups of a SI-group f_G. It seems that as the subsets of U changes, so does the upper α-subgroups of f_G. But, since each upper α-subgroup of f_G is in fact a subgroup of G, it can be deduced that not all these upper α-subgroups are distinct, that is, some of the subsets of U generates the same upper α-subgroups. Otherwise, the number of the subgroups of G would be equal to $2^{|P(U)|}$, which is not the case.

This idea motivates us to find the same upper α-subgroups of a SI-group and lower α-subgroups of a SU-group while characterizing the number of upper α-subgroups and lower α-subgroups subgroups with respect to the number of subgroups of a group. First let us give an example to clarify what is wanted to be meaned.

Example 25. *Let $G = \mathbb{Z}_4$ be the set of parameters and $U = \mathbb{Z}_4$ be the universal set. IF a soft set over U is constructed by*
$f_G(0) = \mathbb{Z}_4$
$f_G(1) = \{0, 1\}$
$f_G(2) = \{0, 1, 3\}$
$f_G(3) = \{0, 1\}$,
then one can easily show that the soft set f_G is a SI-group over $\mathbb{Z}_4$. Here, the universal set $\mathbb{Z}_4$ has $2^4 = 16$ different improper subsets. However, all the upper

α-subgroups of f_G are:
$f_{\overline{G}}^{\supseteq\mathbb{Z}_4} = f_{\overline{G}}^{\supseteq\{0,2,3\}} = f_{\overline{G}}^{\supseteq\{1,2,3\}} = f_{\overline{G}}^{\supseteq\{0,1,2\}} = f_{\overline{G}}^{\supseteq\{2,3\}} = f_{\overline{G}}^{\supseteq\{1,2\}} = f_{\overline{G}}^{\supseteq\{0,2\}} = f_{\overline{G}}^{\supseteq\{2\}} = \{0\}$,
$f_{\overline{G}}^{\supseteq\{0,1,3\}} = f_{\overline{G}}^{\supseteq\{1,3\}} = f_{\overline{G}}^{\supseteq\{0,3\}} = f_{\overline{G}}^{\supseteq\{3\}} = \{0,2\}$ *and*
$f_{\overline{G}}^{\supseteq\{0,1\}} = f_{\overline{G}}^{\supseteq\{1\}} = f_{\overline{G}}^{\supseteq\{0\}} = f_{\overline{G}}^{\supseteq\emptyset} = \mathbb{Z}_4$.

Not to our surprise, some of the upper α-subgroups coincide with each others, since $\mathbb{Z}_4$ has only three subgroups, that is, $\{0\}$, $\{0,2\}$ and $\mathbb{Z}_4$. Also since $Im(f_G) = \{\{0,1\},\{0,1,3\},\mathbb{Z}_4\}$, it follows from Note 16 and Proposition 15 that $f_{\overline{G}}^{\supseteq\{0,1\}} \neq f_{\overline{G}}^{\supseteq\{0,1,3\}} \neq f_{\overline{G}}^{\supseteq\mathbb{Z}_4}$. These ideas may evoke the following questions:

(1) Does the image set of f_G have a role in determining the number of upper α-subgroups of f_G?
(2) Why are some of the upper α-subgroups of f_G are the same and some of them different? Under which conditions does the upper α-subgroups of f_G coincide with each others?
(3) In Example 25, is it a coincidence that $f_{\overline{G}}^{\supseteq\alpha}$, where $\alpha \in Im(f_G)$ constitutes all the upper α-subgroups of f_G? If so, can we obtain all the subgroups of G by all the upper α-subgroups of f_G?

The same questions can be raised for a SU-groups and its lower α-subgroups. The answers of the first question is found in Theorem 33, the second in Theorem 26 and Theorem 28 and the third in Theorem 33 and Note 34.

Theorem 26. *Let f_G be a SI-group over U. Then, the subgroups $f_{\overline{G}}^{\supseteq\alpha_1}$ and $f_{\overline{G}}^{\supseteq\alpha_2}$ (with $\alpha_1 \subsetneq \alpha_2$ and $\alpha_1,\alpha_2 \notin Im(f_G)$) of f_G are equal if and only if there is no $x \in G$ such that $\alpha_1 \subseteq f_G(x) \subsetneq \alpha_2$.*

Proof. ($\Rightarrow$) Assume that $f_{\overline{G}}^{\supseteq\alpha_1} = f_{\overline{G}}^{\supseteq\alpha_2}$ and there exists $x \in G$ such that $\alpha_1 \subseteq f_G(x) \subsetneq \alpha_2$. It follows that $x \in f_{\overline{G}}^{\supseteq\alpha_1}$ and $x \notin f_{\overline{G}}^{\supseteq\alpha_2}$, implying $f_{\overline{G}}^{\supseteq\alpha_1} \nsubseteq f_{\overline{G}}^{\supseteq\alpha_2}$, which is a contradiction with the hypothesis.
($\Leftarrow$) Let there be no $x \in G$ such that $\alpha_1 \subseteq f_G(x) \subsetneq \alpha_2$. Since $\alpha_1 \subsetneq \alpha_2$, and $\alpha_1,\alpha_2 \notin Im(f_G)$, then $f_{\overline{G}}^{\supseteq\alpha_2} \subseteq f_{\overline{G}}^{\supseteq\alpha_1}$ by Proposition 18 (i). Now, let $x \in f_{\overline{G}}^{\supseteq\alpha_1}$, then $f_G(x) \supseteq \alpha_1$ and $f_G(x) \supseteq \alpha_2$ because $f_G(x)$ can not be a proper subset of α_2. Thus, $x \in f_{\overline{G}}^{\supseteq\alpha_2}$ and $f_{\overline{G}}^{\supseteq\alpha_1} = f_{\overline{G}}^{\supseteq\alpha_2}$. □

In Theorem 26, it is handled the subgroups $f_{\overline{G}}^{\supseteq\alpha_1}$ and $f_{\overline{G}}^{\supseteq\alpha_2}$ of f_G with $\alpha_1 \subsetneq \alpha_2$ and $\alpha_1,\alpha_2 \notin Im(f_G)$ and investigate the condition that makes them equal. We can not make such a characterization for the subgroups $f_{\overline{G}}^{\supseteq\alpha_1}$ and $f_{\overline{G}}^{\supseteq\alpha_2}$ with $\alpha_1 \subsetneq \alpha_2$ and $\alpha_1,\alpha_2 \in Im(f_G)$. Because, if $\alpha_1,\alpha_2 \in Im(f_G)$ such that $\alpha_1 \neq \alpha_2$, then obviously $f_{\overline{G}}^{\supseteq\alpha_1} \neq f_{\overline{G}}^{\supseteq\alpha_2}$.

Example 27. *In Example 25, $\{0,2\} \notin Im(f_G)$ and $\{0,2,3\} \notin Im(f_G)$ also $\{0,2\} \subsetneq \{0,2,3\}$. Since there is no $x \in G$ such that*

$$\{0,2\} \subseteq f_G(x) \subsetneq \{0,2,3\},$$

it follows that $f_G^{\supseteq \{0,2\}} = f_G^{\supseteq \{0,2,3\}}$. *Now,* $\{0\} \notin Im(f_G)$ *and* $\{0,1,2\} \notin Im(f_G)$ *and* $\{0\} \subsetneq \{0,1,2\}$. *However, since there is* $x = 1 \in G$ *such that*

$$\{0\} \subseteq f_G(1) \subsetneq \{0,1,2\},$$

it follows that $f_G^{\supseteq \{0\}} \neq f_G^{\supseteq \{0,1,2\}}$.

Theorem 28. *Let* f_G *be a* SU*-group over* U*. Then, the subgroups* $f_G^{\subseteq \alpha_1}$ *and* $f_G^{\subseteq \alpha_2}$ *(with* $\alpha_1 \subsetneq \alpha_2$ *and* $\alpha_1, \alpha_2 \notin Im(f_G)$*) of* f_G *are equal if and only if there is no* $x \in G$ *such that* $\alpha_1 \subsetneq f_G(x) \subseteq \alpha_2$.

Proof. ($\Rightarrow$) Assume that $f_G^{\subseteq \alpha_1} = f_G^{\subseteq \alpha_2}$ and there exists $x \in G$ such that $\alpha_1 \subsetneq f_G(x) \subseteq \alpha_2$. It follows that $x \in f_G^{\subseteq \alpha_2}$ and $x \notin f_G^{\subseteq \alpha_1}$, implying $f_G^{\subseteq \alpha_2} \nsubseteq f_G^{\subseteq \alpha_1}$, which contradicts the hypothesis.
($\Leftarrow$) Conversely, let there be no $x \in G$ such that $\alpha_1 \subsetneq f_G(x) \subseteq \alpha_2$. Since $\alpha_1 \subsetneq \alpha_2$, and $\alpha_1, \alpha_2 \notin Im(f_G)$, then $f_G^{\subseteq \alpha_1} \subseteq f_G^{\subseteq \alpha_2}$ by Proposition 18 (ii). Now, let $x \in f_G^{\subseteq \alpha_2}$, then $f_G(x) \subseteq \alpha_2$ and $f_G(x) \subseteq \alpha_1$ because α_1 can not be a proper subset of $f_G(x)$. Thus, $x \in f_G^{\subseteq \alpha_1}$ and so $f_G^{\subseteq \alpha_1} = f_G^{\subseteq \alpha_2}$. □

In Theorem 22 and Theorem 23, it is shown that some of the subgroups of a group can be obtained by using a SI-group and its upper α-subgroups and by a SU-group and its lower α-subgroups. With the following two theorems, it is illustrated that under certain conditions, a SI-group and a SU-group can be obtained by their upper α-subgroups and lower α-subgroups, respectively.

Theorem 29. *Let* f_G *be a soft set over* U*,* $f_G^{\supseteq \alpha}$ *be upper* α*-subgroups of* f_G *for each* $\alpha \subseteq U$ *and the set* $Im(f_G)$ *be ordered by inclusion. Then,* f_G *is a* SI*-group over* U*.*

Proof. Let $x, y \in G$ and $f_G(x) = \alpha_1$ and $f_G(y) = \alpha_2$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in f_G^{\supseteq \alpha_1}$ and $y \in f_G^{\supseteq \alpha_2}$. Since $\alpha_1 \subseteq \alpha_2$, it follows by Proposition 17 that $x, y \in f_G^{\supseteq \alpha_1}$ and since $f_G^{\supseteq \alpha}$ is a subgroup of G for all $\alpha \subseteq U$, it follows that $xy^{-1} \in f_G^{\supseteq \alpha_1}$. Hence,

$$f_G(xy^{-1}) \supseteq \alpha_1 = \alpha_1 \cap \alpha_2 = f_G(x) \cap f_G(y).$$

Thus, f_G is a SI-group over U. □

Theorem 30. *Let* f_G *be a soft set over* U*,* $f_G^{\subseteq \alpha}$ *be lower* α*-subgroups of* f_G *for each* $\alpha \subseteq U$ *and the set* $Im(f_G)$ *be ordered by inclusion. Then,* f_G *is a* SU*-group over* U*.*

Proof. Let $x, y \in G$ and $f_G(x) = \alpha_1$ and $f_G(y) = \alpha_2$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in f_G^{\subseteq \alpha_1}$ and $y \in f_G^{\subseteq \alpha_2}$. Since $\alpha_1 \subseteq \alpha_2$, it follows from by Proposition 17 that $x, y \in f_G^{\subseteq \alpha_2}$ and since $f_G^{\subseteq \alpha}$ is a subgroup of G for all $\alpha \subseteq U$, it follows that $xy^{-1} \in f_G^{\subseteq \alpha_2}$. Hence,

$$f_G(xy^{-1}) \subseteq \alpha_2 = \alpha_1 \cup \alpha_2 = f_G(x) \cup f_G(y).$$

Therefore, f_G is a SU-group over U. □

From now on, it will be more concerned with a SI-group and its upper α-subgroups.

Note 31. *If f_G is a SI-group over U, then the set $Im(f_G)$ does not need to be ordered by inclusion as in the case of Example 32. However, it is an immediate result of Definition 24 that f_G attains an infimum on all the upper α-subgroups even $Im(f_G)$ does not form a chain. This significant fact is used in the proof of Theorem 33.*

Example 32. *Assume that $U = S_3$ is the universal set and $G = \mathbb{Z}_6$ is the set of parameters. If a soft set is constructed by*

$$\begin{aligned} f_G(0) &= S_3 \\ f_G(1) &= f_G(5) = \{(12), (13), (132)\} \\ f_G(2) &= f_G(4) = \{(12), (13), (23), (123), (132)\} \\ f_G(3) &= \{(1), (12), (13), (132)\} \end{aligned}$$

then f_G is a SI-group over U, whose image set is not ordered by inclusion. Note that

$$\begin{aligned} f_{\overline{G}}^{\supseteq S_3} &= \{0\} \\ f_{\overline{G}}^{\supseteq\{(1),(12),(13),(132)\}} &= \{0,3\} \\ f_{\overline{G}}^{\supseteq\{(12),(13),(23),(123),(132)\}} &= \{0,2,4\} \\ f_{\overline{G}}^{\supseteq\{(12),(13),(132)\}} &= \mathbb{Z}_6. \end{aligned}$$

Thus, upper α-subgroup $\{0\}$ attains an infimum on 0, $\{0,3\}$ on 3, $\{0,2,4\}$ on 2 and 4, since $f_G(2) = f_G(4)$, and $\mathbb{Z}_6$ on 1 and 5.

Theorem 33. *Let G be a finite group, f_G be a SI-group over U and I be an arbitrary finite index set. Then,*

$$G(f_{\overline{G}}^{\supseteq\alpha_i}) = \{f_{\overline{G}}^{\supseteq\alpha_i} : i \in I, \alpha_i \in Im(f_G)\}$$

contains all the upper α-subgroups of f_G.

Proof. Let $f_{\overline{G}}^{\supseteq\alpha}$ be any upper α-subgroup of f_G. It is wanted to be shown that $f_{\overline{G}}^{\supseteq\alpha} \in G(f_{\overline{G}}^{\supseteq\alpha_i})$. If $\alpha = \alpha_i$ for some $i \in I$, then there is nothing to prove. Assume that $\alpha \neq \alpha_i$ for all $i \in I$. Then, there does not exist $x \in G$ such that $f_G(x) = \alpha$. Let $H = \{x \in G : f_G(x) \supsetneq \alpha\}$. First, let us show that H is a subgroup of G. Since f_G is a SI-group over U, where $\emptyset \subseteq \alpha \subseteq f_G(e)$ and since there does not exist $x \in G$ such that $f_G(x) = \alpha$, it follows that $e \in f_{\overline{G}}^{\supsetneq\alpha}$ and $\emptyset \neq f_{\overline{G}}^{\supsetneq\alpha} \subseteq G$. Now assume that $x, y \in f_{\overline{G}}^{\supsetneq\alpha}$, then $f_G(x) \supsetneq \alpha$ and $f_G(y) \supsetneq \alpha$, thus

$$f_G(xy^{-1}) \supseteq f_G(x) \cap f_G(y) \supseteq \alpha.$$

Since there does not exist $x \in G$ such that $f_G(x) = \alpha$, $f_G(xy^{-1}) \supsetneq \alpha$, so $xy^{-1} \in H$, implying that H is a subgroup of G. Moreover, it is obvious from the definition of H that f_G attains an infimum on H (similar to Note 31). Thus, there exists $h^\star \in H$ such that

$$f_G(h^\star) = Inf\{f_G(h) : h \in H\}.$$

Now, $f_G(h^\star) \in Im(f_G)$ and thus, $f_G(h^\star) = \alpha_{i^*}$ for some $i^* \in I$. Then, we have

$$Inf\{f_G(x) : f_G(x) \supsetneq \alpha\} = \alpha_{i^*}.$$

Clearly, $\alpha_{i^*} \supsetneq \alpha$. Also, since there does not exist $x \in G$ such that $\alpha \subseteq f_G(x) \subsetneq \alpha_{i^*}$, by Proposition 19 and 26, $f_{\overline{G}}^{\supseteq\alpha} = f_{\overline{G}}^{\supseteq\alpha_{i^*}}$, thus $f_{\overline{G}}^{\supseteq\alpha} \in G(f_{\overline{G}}^{\supseteq\alpha_i})$, completing the proof. □

Note 34. *Theorem 33 does not mean that we can obtain all the subgroups of G by constructing a SI-group and then finding its upper α-subgroups $f_{\overline{G}}^{\supseteq\alpha}$, where $\alpha \in Im(f_G)$.*

In Example 25, we could obtain all the subgroups of $\mathbb{Z}_4$ with the help of upper α-subgroups, where $\alpha \in Im(f_G) = \{\{0\}, \{0, 1, 3\}, \{\mathbb{Z}_4\}\}$. But this is not always the case. Because there is not only one SI-group of a group G over U. While constructing a SI-group, we are free, even we can change the number of the elements of $Im(f_G)$. For example, consider Example 25. If we define a soft set h_G over $U = \mathbb{Z}_4$ such that $h_G(0) = \{0, 3\}$, $h_G(1) = h_G(2) = h_G(3) = \{3\}$, then h_G is an SI-group, where G is again $\mathbb{Z}_4$. But, since $Im(h_G) = \{\{3\}, \{0, 3\}\}$, by Theorem 33, the only upper α-subgroups of h_G are $h_{\overline{G}}^{\supseteq\{3\}} = \mathbb{Z}_4$ and $h_{\overline{G}}^{\supseteq\{0,3\}} = \{0\}$. This means that we can not obtain the other subgroup $\{0, 2\}$ of $\mathbb{Z}_4$ by h_G and its upper α-subgroups. Namely, there is not any $\alpha \in Im(h_G)$ such that $h_{\overline{G}}^{\supseteq\alpha} = \{0, 2\}$. However, see the following:

Theorem 35. *Any subgroup H of a group G can be realized as an upper α-subgroup of some SI-group over U.*

Proof. Let f_G be a soft set over U defined by

$$f_G(x) = \begin{cases} \alpha, & \text{if } x \in H \\ \emptyset, & \text{if } x \notin H \end{cases}$$

Then, f_G is a SI-group over U. Let $a, b \in G$.
Case 1: Suppose $a \in H$ and $b \in H$, then $ab \in H$. It follows that $f_G(ab) = \alpha$ and $f_G(a) = f_G(b) = \alpha$. Thus, $f_G(ab) \supseteq f_G(a) \cap f_G(b)$. And also if $a \in H$, then so is a^{-1}, thus $f_G(a) = f_G(a^{-1}) = \alpha$.

Case 2: Now, suppose $a \in H$ and $b \notin H$, then $ab \notin H$. It follows that $f_G(a) = \alpha$ and $f_G(b) = f_G(ab) = \emptyset$. Therefore, $f_G(ab) \supseteq f_G(a) \cap f_G(b)$, furthermore $f_G(a) = f_G(a^{-1})$ if $a \in H$ or $a \notin H$.

Case 3: Now, suppose that $a \notin H$ and $b \notin H$. Then either $ab \in H$ or $ab \notin H$. It is easy to show that in any cases, $f_G(ab) \supseteq f_G(a) \cap f_G(b)$ and $f_G(a) = f_G(a^{-1})$. Hence, f_G is a SI-group over U. Moreover for this SI-group, $f_G^{\supseteq \alpha} = H$. □

Note 36. *It is known that if f_G is a SI-group over U, then $f_G(e) \supseteq f_G(x)$ for all $x \in G$. Let $f_G(e) = \alpha_e$, then it turns out to be an interesting case to investigate the upper α_e-subgroup $f_G^{\supseteq \alpha_e}$ of f_G. Because, if $x \in f_G^{\supseteq \alpha_e}$, then $f_G(x) \supseteq \alpha = f_G(e)$ and it appears that only $e \in f_G^{\supseteq \alpha_e}$. But that is not always the case as seen in the following example.*

Example 37. *Consider the SI-group f_G in Theorem 35. Assume that $H \neq \{e\}$ and $H \neq G$. It is known that f_G is a SI-group over U and $Im(f_G) = \{\emptyset, \alpha\}$. Thus, by Theorem 33, two upper α-subgroups are*

$$f_G^{\supseteq \emptyset} = G \text{ and } f_G^{\supseteq \alpha} = \{x \in G : f_G(x) \supseteq \alpha\} = H.$$

Since $e \in H$, $f_G(e) = \alpha$; but $f_G^{\supseteq \alpha} = H$, which is not equal to "e".

Definition 38. [26] *Let f_G be a SU-group over U. Then e-set of f_G, denoted by G_{f_G}, is defined as*

$$G_{f_G} = \{x \in G : f_G(x) = f_G(e)\}.$$

Theorem 39. *Let f_G be a SI-group over U. If $f_G(e) = \alpha_e$, then $f_G^{\supseteq \alpha_e} = G_{f_G}$.*

Proof. $f_G^{\supseteq \alpha_e} = \{x \in G : f_G(x) \supseteq \alpha_e\} = \{x \in G : f_G(x) = \alpha_e\}$, since $\alpha_e \supseteq f_G(x)$, $\forall x \in G$ by Theorem 9. Writing $\alpha_e = f_G(e)$,

$$f_G^{\supseteq \alpha_e} = \{x \in G : f_G(x) = f_G(e)\} = G_{f_G}.$$

□

In Theorem 33, it is seen that the image set of a SI-group has a significant role in determining the upper α-subgroups, thus from now on the image set of the SI-group is more concerned.

Note 40. *When considering Proposition 20 and Theorem 33 together, it can be deduced that if f_G is a SI-group over U and $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \in Im(f_G)$, satisfying that $\alpha_0 \supsetneq \cdots \supsetneq \alpha_n$, then the family of upper α-subgroups form a chain, which is denoted by $C(f_G)$ as below:*

$$C(f_G) = f_G^{\supseteq \alpha_0} \subsetneq f_G^{\supseteq \alpha_1} \cdots \subsetneq f_G^{\supseteq \alpha_n},$$

where $f_G^{\supseteq \alpha_0} = G_{f_G}$. Moreover since f_G attains an infimum on all of the i- upper α-subgroups and $f_G^{\supseteq \alpha_i}$, where $\alpha_i \in Im(f_G)$ constitutes all the upper α-subgroups, $f_G^{\supseteq \alpha_n}$ which is at the end point of the chain is of course equal to G.

Not to our surprise, only some of the upper α-subgroups of f_G form a chain, since all the subgroups of G, in general, does not form a chain. That is, it makes no sense to hope all the upper α-subgroups form a chain. In this connection, see

Example 32, $\{0,3\} \nsubseteq \{0,2,4\}$ and $\{0,2,4\} \nsubseteq \{0,3\}$. Of course if the members of the $Im(f_G)$ forms a chain, so does the upper α-subgroups of f_G. For further detail, refer to Theorem 41.

Theorem 41. *Let G be a finite group, f_G be a SI-group over U and I be an arbitrary finite index set and*

$$G(f_{\tilde{G}}^{\supseteq \alpha_i}) = \{f_{\tilde{G}}^{\supseteq \alpha_i} : i \in I, \alpha_i \in Im(f_G)\}.$$

Then, we have the followings:

(1) *There exists a unique $i_e \in I$ such that $\alpha_{i_e} \supseteq \alpha_i$, $\forall i \in I$.*
(2) $G_{f_G} = \bigcap_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i} = f_{\tilde{G}}^{\supseteq \alpha_{i_e}}$.
(3) $G = \bigcup_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i}$.
(4) *If the members of $Im(f_G)$ forms a chain, so does $G(f_{\tilde{G}}^{\supseteq \alpha_i})$.*

Proof. i) Since $f_G(e) \in Im(f_G)$, there exists a unique $i_e \in I$ such that $f_G(e) = \alpha_{i_e}$. By Theorem 9, $f_G(e) \supseteq f_G(x)$ for all $x \in G$. It follows that $\alpha_{i_e} \supseteq f_G(x)$, $\forall x \in G$. Thus, $\alpha_{i_e} \supseteq \alpha_i$, $\forall i \in I$.

ii) Since in Theorem 39, it is proved that $G_{f_G} = f_{\tilde{G}}^{\supseteq \alpha_{i_e}}$, where $f_G(e) = \alpha_{i_e}$, it is only shown that $f_{\tilde{G}}^{\supseteq \alpha_{i_e}} = \bigcap_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i}$. Since $\alpha_{i_e} \supseteq \alpha_i$, $\forall i \in I$, $f_{\tilde{G}}^{\supseteq \alpha_{i_e}} \subseteq f_{\tilde{G}}^{\supseteq \alpha_i}$, $\forall i \in I$ by Proposition 17. Thus, $f_{\tilde{G}}^{\supseteq \alpha_{i_e}} \subseteq \bigcap_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i}$. Now, let $x \in \bigcap_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i}$, then $x \in f_{\tilde{G}}^{\supseteq \alpha_i}$, $\forall i \in I$. Since $i_e \in I$, thus $x \in f_{\tilde{G}}^{\supseteq \alpha_{i_e}}$, implying that $\bigcap_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i} = f_{\tilde{G}}^{\supseteq \alpha_{i_e}}$. Thus, the proof is completed.

iii) Since $f_{\tilde{G}}^{\supseteq \alpha_i} \subseteq G$, $\forall i \in I$, $\bigcup_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i} \subseteq G$. Now let $x \in G$. It is obvious that since $f_G(x) \in Im(f_G)$, there exists $i_x \in I$ such that $f_G(x) = \alpha_{i_x}$. Clearly, $x \in f_{\tilde{G}}^{\supseteq \alpha_{i_x}}$ and thus, $x \in \bigcup_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i}$. Therefore, $G \subseteq \bigcup_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i}$, so $G = \bigcup_{i \in I} f_{\tilde{G}}^{\supseteq \alpha_i}$.

iv) Assume that the members of $Im(f_G)$ form a chain under proper inclusion. Then, for any $i, j \in I$, either $\alpha_i \subsetneq \alpha_j$ or $\alpha_j \subsetneq \alpha_i$. It follows by Proposition 20 that $f_{\tilde{G}}^{\supseteq \alpha_i} \subsetneq f_{\tilde{G}}^{\supseteq \alpha_j}$ or $f_{\tilde{G}}^{\supseteq \alpha_j} \subsetneq f_{\tilde{G}}^{\supseteq \alpha_i}$. Of course, if $\alpha_i \subseteq \alpha_j$ or $\alpha_j \subseteq \alpha_i$, it follows that $f_{\tilde{G}}^{\supseteq \alpha_i} \subseteq f_{\tilde{G}}^{\supseteq \alpha_j}$ or $f_{\tilde{G}}^{\supseteq \alpha_j} \subseteq f_{\tilde{G}}^{\supseteq \alpha_i}$ by Proposition 17. □

5. Characterization of SI-groups

In this section, it is tried to find an answer whether the family of upper α-subgroups of a SI-group determine the SI-group uniquely or not. The following example shows that two SI-groups of a G over U may have an identical family of upper α-subgroups, but the SI-groups may not be soft equal.

Example 42. *Let $G = \mathbb{Z}_4$ be the set of parameters and $U = \{1, -1, i, -i\}$ be the universal set. If a soft set over U is constructed by $f_G(0) = \{1, -1, i, -i\}$,*

$f_G(1) = f_G(3) = \{i\}$, $f_G(2) = \{1, i\}$, clearly f_G is a SI-group over U. Here, $Im(f_G) = \{\{i\}, \{1, i\}, \{1, -1, i, -i\}\}$, thus all the upper α-subgroups of f_G are

$$f_G^{\supseteq\{i\}} = \mathbb{Z}_4,\ f_G^{\supseteq\{1,i\}} = \{0, 2\},\ f_G^{\supseteq\{1,-1,i,-i\}} = \{0\}.$$

Now, let us define a soft set over U such that $h_G(0) = \{1, -1, i\}$, $h_G(1) = h_G(3) = \{-1\}$, $h_G(2) = \{-1, i\}$. Obviously, h_G is a SI-group over U, too and the family of upper α-subgroups of h_G are

$$h_G^{\supseteq\{-1\}} = \mathbb{Z}_4,\ h_G^{\supseteq\{-1,i\}} = \{0, 2\},\ h_G^{\supseteq\{1,-1,i\}} = \{0\}.$$

It is seen that two SI-groups f_G and h_G have the same family of upper α-subgroups, however f_G is not soft equal to h_G.

Proposition 43. *Let $\mathcal{S}_G$ be the class of SI-groups of a group G over U. If I define a relation $\mathcal{R}$ on $\mathcal{S}_G$ by $f_G\mathcal{R}h_G$ if and only if f_G and h_G have an identical family of upper α-subgroups, then the relation $\mathcal{R}$ is an equivalence relation.*

In Example 42, it is shown that f_G and h_G may be such that $f_G\mathcal{R}h_G$ but f_G and h_G need not to be soft equal. The equivalence relation defined in Proposition 43 partitions $\mathcal{S}_G$ into equivalence classes. Let $f_G \in \mathcal{S}_G$ and $[f_G]$ denote the equivalence class contained f_G. If the group G is finite, then the number of possible distinct upper α-subgroups are finite, as each upper α-subgroups is a subgroup of G in the usual sense. In Theorem 35, it is shown that any subgroup of a group can be realized as an upper α-subgroup of a SI-group. All these remarks lead us to the conclusion that the number of possible chains of upper α-subgroups is finite. Since each equivalence class can be characterized by its chain of upper α-subgroups, we have with the following that the number of equivalence classes is finite, although $\mathcal{S}_G$ is an infinite family when U is infinite.

Corollary 44. *If G is a finite group, then the number of distinct equivalence classes in $\mathcal{S}_G$ under the definition of equivalence defined in Proposition 43 is finite. Moreover, $\mathcal{S}_G$ can be written as a disjoint union*

$$\mathcal{S}_G = [f_G^1]\dot\cup[f_G^2]\dot\cup\cdots\dot\cup[f_G^k]$$

where $[f_G^i], 1 \le i \le k$ are all distinct equivalence classes. Here, again note that $[f_G^i]$ has an infinite number of SI-groups when U is infinite.

Theorem 45. *Let f_G and h_G be two SI-groups of a finite group G having the identical family of upper α-subgroups and the sets $Im(f_G)$ and $Im(h_G)$ be ordered by inclusion. If $Im(f_G) = \{\alpha_0, \ldots, \alpha_m\}$ and $Im(h_G) = \{\beta_0, \ldots, \beta_n\}$, then*

(1) *$m = n$,*
(2) *$f_G^{\supseteq\alpha_i} = h_G^{\supseteq\beta_i}, 0 \le i \le m$,*
(3) *If $x \in G$ such that $f_G(x) = \alpha_i$, then $h_G(x) = \beta_i, 0 \le i \le m$.*

Proof. i) By Theorem 33 that the only upper α-subgroups of f_G and h_G are the families of $f_G^{\supseteq\alpha_i}$ and $h_G^{\supseteq\beta_i}$, respectively. Since f_G and h_G have the identical family of upper α-subgroups, it follows that $m = n$.

ii) Since $\alpha_0 \supsetneq \cdots \supsetneq \alpha_m$ and $\beta_0 \supsetneq \cdots \supsetneq \beta_n$, by Theorem 33 and Theorem 45 (i) that two chains of upper α-subgroups are

$$f_G^{\supseteq \alpha_0} \subsetneq f_G^{\supseteq \alpha_1} \subsetneq \cdots \subsetneq f_G^{\supseteq \alpha_m} = G, \;\; h_G^{\supseteq \beta_0} \subsetneq h_G^{\supseteq \beta_1} \subsetneq \cdots \subsetneq h_G^{\supseteq \beta_n} = G.$$

Since the two upper α-subgroups are identical, it is obvious that $f_G^{\supseteq \alpha_0} = h_G^{\supseteq \beta_0} = \{e\}$. Let $f_G^{\supseteq \alpha_1} = h_G^{\supseteq \beta_j}$ for some $j > 0$ (since $f_G^{\supseteq \alpha_0} = h_G^{\supseteq \beta_0}$). Suppose that $f_G^{\supseteq \alpha_1} = h_G^{\supseteq \beta_j}$ for some $j > 1 (j \neq 1)$. Again, $h_G^{\supseteq \beta_1} = f_G^{\supseteq \alpha_i}$ for some $\alpha_1 \supsetneq \alpha_i$. It is obvious that $\alpha_i \neq \alpha_1$. Thus,

$$f_G^{\supseteq \alpha_i} = h_G^{\supseteq \beta_1} \subsetneq h_G^{\supseteq \beta_j} (\text{since } \beta_1 \supsetneq \beta_j), \text{ so } f_G^{\supseteq \alpha_i} \subsetneq h_G^{\supseteq \beta_j}.$$

Now

$$h_G^{\supseteq \beta_j} = f_G^{\supseteq \alpha_1} \subseteq f_G^{\supseteq \alpha_i} \text{ (since } \alpha_1 \supsetneq \alpha_i) \text{ so } h_G^{\supseteq \beta_j} \subsetneq f_G^{\supseteq \alpha_i}$$

Note that, $f_G^{\supseteq \alpha_i} \subsetneq h_G^{\supseteq \beta_j}$ and $h_G^{\supseteq \beta_j} \subsetneq f_G^{\supseteq \alpha_i}$ contradicts one another, because the inclusion are both proper inclusion, so I must have $f_G^{\supseteq \alpha_1} \subsetneq h_G^{\supseteq \beta_1}$. The rest of the proof follows by induction on i. Finally it is obtained that $f_G^{\supseteq \alpha_i} = h_G^{\supseteq \beta_i}, 0 \leq i \leq m$.

iii) Let $x \in G$ such that $f_G(x) = \alpha_i$ and $h_G(x) = \beta_j$. By Theorem 45 (ii), $f_G^{\supseteq \alpha_i} = h_G^{\supseteq \beta_i}, 0 \leq i \leq m$. Thus, $x \in h_G^{\supseteq \beta_i}$ implies that $h_G(x) = \beta_j$ such that $\beta_j \supseteq \beta_i$. So, $h_G^{\supseteq \beta_j} \subsetneq h_G^{\supseteq \beta_i}$ by Proposition 17. Similarly, by Theorem 45 (ii), $h_G^{\supseteq \beta_j} = f_G^{\supseteq \alpha_j}$. Therefore, since $x \in h_G^{\supseteq \beta_j}$ (as $h_G(x) = \beta_j$), $x \in f_G^{\supseteq \alpha_j}$ and so, $f_G(x) = \alpha_i \supseteq \alpha_j$. It follows by Proposition 17 that $f_G^{\supseteq \alpha_i} \subsetneq f_G^{\supseteq \alpha_j}$. However, by Theorem 45 (ii), $f_G^{\supseteq \alpha_i} = h_G^{\supseteq \beta_i}$ and $f_G^{\supseteq \alpha_j} = h_G^{\supseteq \beta_j}$. So I have that

$$h_G^{\supseteq \beta_i} = f_G^{\supseteq \alpha_i} \subsetneq f_G^{\supseteq \alpha_j} = h_G^{\supseteq \beta_j},$$

thus $h_G^{\supseteq \beta_i} \subsetneq h_G^{\supseteq \beta_j}$, which contradicts the fact that $h_G^{\supseteq \beta_j} \subsetneq h_G^{\supseteq \beta_i}$ if we do not have $h_G^{\supseteq \beta_j} = h_G^{\supseteq \beta_i}$. We know that $h_G^{\supseteq \beta_j} = h_G^{\supseteq \beta_i}$ if and only if $\beta_j = \beta_i$. Thus, $f_G(x) = \alpha_i$ and $h_G(x) = \beta_j = \beta_i$, completing the proof. □

Theorem 46. *Let f_G and h_G be two SI-groups of a finite group G such that their family of upper α-subgroups are identical and their image sets are both ordered by inclusion. Then,*

$$f_G = h_G \Leftrightarrow Im(f_G) = Im(h_G).$$

Proof. Let $f_G = h_G$, then $Im(f_G) = Im(h_G)$ is obvious. Conversely, suppose that $Im(f_G) = Im(h_G)$. Let $Im(f_G) = \{\alpha_0, \alpha_1, \cdots \alpha_r\}$ and $Im(h_G) = \{\beta_0, \beta_1, \ldots \beta_r\}$, such that $\alpha_0 \supsetneq \alpha_1 \supsetneq \ldots \supsetneq \alpha_r$ and $\beta_0 \supsetneq \beta_1 \supsetneq \cdots \supsetneq \beta_r$. Let

$$\beta_0 \in Im(f_G), \text{ thus } \beta_0 = \alpha_{t_0} \text{ for some } t_0.$$

Let $\alpha_{t_0} \neq \alpha_0$. It follows that $\alpha_{t_0} \not\supseteq \alpha_0$, since α_0 is the maximal element of the chain. Now, let $\beta_1 \in Im(f_G)$ and so $\beta_1 = \alpha_{t_1}$ for some t_1. Since $\beta_0 \supsetneq \beta_1$, it implies that $\alpha_{t_0} \supsetneq \alpha_{t_1}$. Continuing similarly,

$$\alpha_{t_0} \supsetneq \alpha_{t_1} \supsetneq \cdots \supsetneq \alpha_{t_r}, \text{ where } \beta_0 = \alpha_{t_0} \not\supseteq \alpha_0.$$

This means that there does not exist any $\beta_i \in Im(h_G)$ such that $\alpha_0 = \beta_i$. But this contradicts the fact that $Im(f_G) = Im(h_G)$. Hence we must have $\beta_0 = \alpha_0$. Similarly, one can obtain that $\beta_i = \alpha_i$, $0 \leq i \leq r$. Now, let $a_0, a_1, \dots a_r$ be different elements of G such that

$$f_G(a_i) = \alpha_i, 0 \leq i \leq r.$$

By Theorem 45 (iii), $h_G(a_i) = \beta_i$, $0 \leq i \leq r$. Since $\alpha_i = \beta_i$, then $f_G(a_i) = h_G(a_i)$, $\forall a_i \in G$. Hence $f_G = h_G$, completing the proof. □

Since all the subgroups of G, in general, do not form a chain, we can conclude that not all subgroups of G are upper α-subgroups of a given SI-group whose image sets form a chain. Therefore, it turns out to be an interesting problem to find a SI-group whose image sets form a chain and which accommodates as many subgroups of G as possible in the chain of upper α-subgroups of the SI-group. For this characterization, I have the following:

Theorem 47. *Let G be a group, H_i be subgroups of G such that*

$$\{e\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_r = G$$

and α_i be any sets such that $\alpha_0 \supsetneq \alpha_1 \supsetneq \ldots \supsetneq \alpha_r$ for all $i = 1, 2 \ldots r$. If

$$\begin{aligned} f_G(H_0) &= \alpha_0, \\ f_G(H_i \setminus H_{i-1}) &= \alpha_i, (0 \leq i \leq r), \end{aligned}$$

for all $i = 1, 2 \ldots r$, then f_G is a SI-group over U.

Here, note that the length of the arbitrary chain of sets and the subgroups have to be the same.

Proof. Before starting the proof, note that if $h \in H_i$, then $f_G(h) \supseteq \alpha_i$. In fact, $H_{i-1} \subsetneq H_i$. Then, if $h \in H_i$, $h \in H_i \setminus H_j$ or $h \in H_j$, where $j < i$. If $h \in H_i \setminus H_j$, then $f_G(h) = \alpha_i$, if $h \in H_j$ (say $h \in H_j \setminus H_{j-1}$), then $f_G(h) = \alpha_j$, where $j < i$. Since $j < i$, it follows that $\alpha_j \supsetneq \alpha_i$. It means that $f_G(h) \supsetneq \alpha_i$. Let $x, y \in G$. I handle the proof in two cases: **Case 1**: Let $x, y \in H_i$, but not in H_{i-1}, namely, $x, y \in H_i \setminus H_{i-1}$. Then,

$$f_G(x) = f_G(y) = \alpha_i.$$

Since H_i is a subgroup of G, it follows that $xy \in H_i$. Thus,

$$f_G(xy) \supseteq \alpha_i = f_G(x) \cap f_G(y).$$

Moreover, since $x \in H_i$ (and not in H_{i-1}), and H_i is a subgroup of G, $x^{-1} \in H_i$. Thus,

$$f_G(x^{-1}) = \alpha_i = f_G(x).$$

Case 2: Let $x \in H_i \setminus H_{i-1}$ and $y \in H_j \setminus H_{j-1}$ and assume that $i > j$. It implies that $\alpha_j \supsetneq \alpha_i$ and $H_j \subsetneq H_i$. Moreover, I have $f_G(x) = \alpha_i$ and $f_G(y) = \alpha_j$, thus $f_G(x) \cap f_G(y) = \alpha_i$. Also, since $y \in H_j \setminus H_{j-1}$ and $i > j$, thus $y \in H_i$. It follows that $x, y \in H_i$ and $xy \in H_i$ (since H_i is a subgroup of G). This implies that

$$f_G(xy) \supseteq \alpha_i = f_G(x) \cap f_G(y) = f_G(x).$$

Moreover,

$$f_G(x^{-1}) = \alpha_i = f_G(x).$$

Therefore f_G is a SI-group over U. □

Example 48. *Consider the group $G = \mathbb{Z}_8$. In order to construct a SI-group over U, I only need an arbitrary chain of sets and subgroups of $\mathbb{Z}_8$ forming a chain. It is known that these two chains have to be with the same length. Let the chain of subgroups of $\mathbb{Z}_8$ be*

$$\{0\} \subsetneq \{0, 4\} \subsetneq \{0, 2, 4, 6\} \subsetneq \mathbb{Z}_8.$$

Here, $H_0 = \{0\}$, $H_1 = \{0, 4\}$, $H_2 = \{0, 2, 4, 6, 8\}$, $H_3 = \mathbb{Z}_8$. Let any chain of sets whose length is four be the following:

$$\{1, 2, 3, 4, 5\} \supsetneq \{1, 2, 4\} \supsetneq \{1, 2\} \supsetneq \{1\}$$

Here, $\alpha_0 = \{1, 2, 3, 4, 5\}$, $\alpha_1 = \{1, 2, 4\}$, $\alpha_2 = \{1, 2\}$, $\alpha_3 = \{1\}$. Now it is time to construct the SI-group.
Since $H_0 = \{0\}$, then $f_G(0) = \alpha_0 = \{1, 2, 3, 4, 5\}$,
Since $H_1 \setminus H_0 = \{4\}$, then $f_G(4) = \alpha_1 = \{1, 2, 4\}$,
Since $H_2 \setminus H_1 = \{2, 6\}$, then $f_G(2) = f_G(6) = \alpha_2 = \{1, 2\}$,
Since $H_3 \setminus H_2 = \{1, 3, 5, 7\}$, then $f_G(1) = f_G(3) = f_G(5) = f_G(7) = \alpha_3 = \{1\}$.
One can easily show that f_G is a SI-group over U.

Theorem 49. *Let G be a group, H_i be subgroups of G such that*

$$H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_r = G$$

for all $i = 1, 2 \ldots r$. Then, there exists a SI-group of G whose upper α-subgroups are exactly the members of this chain.

Proof. Let us consider the chain of sets $\alpha_0 \supsetneq \alpha_1 \supsetneq \ldots \supsetneq \alpha_r$. If a soft set over U is defined by

$$\begin{aligned} f_G(H_0) &= \alpha_0, \\ f_G(H_i \setminus H_{i-1}) &= \alpha_i, (0 \leq i \leq r) \end{aligned}$$

then f_G is a SI-group over U by Theorem 47. Now, $Im(f_G) = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$. Since α_i is the element of the chain for all $i \in I = \{1, 2 \ldots r\}$, all the upper α-subgroups of f_G are given the chain of subgroups

$$f_G^{\supseteq \alpha_0} \subsetneq f_G^{\supseteq \alpha_1} \subsetneq \cdots \subsetneq f_G^{\supseteq \alpha_r} = G.$$

For the final of the proof, note that $H_i \subseteq f_G^{\supseteq \alpha_i}$. In fact, if $x \in H_i$, then $f_G(x) \supseteq \alpha_i$ and it follows that $x \in f_G^{\supseteq \alpha_i}$. Moreover, if $x \in f_G^{\supseteq \alpha_i}$, then $f_G(x) \supseteq \alpha_i$, so $f_G(x) \in$

$\{\alpha_0, \alpha_1, \ldots \alpha_i\}$. Therefore, $x \in H_s$ for some $s \leq i$. Since $H_s \subsetneq H_i$ for $s \leq i$, $x \in H_i$. It follows that $f_G^{\supseteq\alpha_i} \subseteq H_i$, completing the proof. $\square$

Example 50. *In Example 48, $\alpha_0 = \{1,2,3,4,5\}$, $\alpha_1 = \{1,2,4\}$, $\alpha_2 = \{1,2\}$, $\alpha_3 = \{1\}$, that is, $\alpha_0 \supsetneq \alpha_1 \supsetneq \alpha_2 \supsetneq \alpha_3$, so when f_G is considered, we have*

$$f_G^{\supseteq\alpha_0} = \{0\}, f_G^{\supseteq\alpha_1} = \{0,4\}, f_G^{\supseteq\alpha_2} = \{0,2,4,6,8\} \text{ and } f_G^{\supseteq\alpha_3} = \mathbb{Z}_8.$$

It follows that $f_G^{\supseteq\alpha_3} \supsetneq f_G^{\supseteq\alpha_2} \supsetneq f_G^{\supseteq\alpha_1} \supsetneq f_G^{\supseteq\alpha_0}$. Moreover, since $H_0 = \{0\}$, $H_1 = \{0,4\}$, $H_2 = \{0,2,4,6,8\}$ and $H_3 = \mathbb{Z}_8$, it follows that $H_3 \supsetneq H_2 \supsetneq H_1 \supsetneq H_0$. Here, note that $H_0 = f_G^{\supseteq\alpha_0}, H_1 = f_G^{\supseteq\alpha_1}, H_2 = f_G^{\supseteq\alpha_2}$ and $H_3 = f_G^{\supseteq\alpha_3}$, as required.

Finally, it is time to softificate an upper α-subgroup of a SI-group over U with the following definition.

Definition 51. *Let f_G be a SI-group over U and $f_G^{\supseteq\alpha}$ be an upper α-subgroup of f_G. Softificated $f_G^{\supseteq\alpha}$ is a soft set $f^\star_{f_G^{\supseteq\alpha}}$ defined by,*

$$f^\star_{f_G^{\supseteq\alpha}}(x) = \begin{cases} f_G(x), & \text{if } x \in f_G^{\supseteq\alpha}, \\ \emptyset, & \textit{otherwise} \end{cases}$$

for all $x \in G$. Clearly, $f^\star_{f_G^{\supseteq\alpha}} \tilde{\subseteq} f_G$.

Theorem 52. *Let f_G be a SI-group over U and $f_G^{\supseteq\alpha}$ be an upper α-subgroup of G. Then, $f^\star_{f_G^{\supseteq\alpha}}$ is a SI-group over U.*

Proof. Assume that $x, y \in G$ and $x, y \in f_G^{\supseteq\alpha}$. Since $f_G^{\supseteq\alpha}$ is a subgroup of G, it follows that $xy^{-1} \in f_G^{\supseteq\alpha}$. Then,

$$f^\star_{f_G^{\supseteq\alpha}}(xy^{-1}) = f_G(xy^{-1}) \supseteq f_G(x) \cap f_G(y) = f^\star_{f_G^{\supseteq\alpha}}(x) \cap f^\star_{f_G^{\supseteq\alpha}}(y)$$

Now suppose that $x \in f_G^{\supseteq\alpha}$ and $y \notin f_G^{\supseteq\alpha}$. It follows that $xy^{-1} \notin f_G^{\supseteq\alpha}$, then

$$f^\star_{f_G^{\supseteq\alpha}}(xy^{-1}) = \emptyset \supseteq f_G(x) \cap \emptyset = f^\star_{f_G^{\supseteq\alpha}}(x) \cap f^\star_{f_G^{\supseteq\alpha}}(y)$$

Finally, suppose that $x, y \notin f_G^{\supseteq\alpha}$. It follows that either $xy^{-1} \notin f_G^{\supseteq\alpha}$ or $xy^{-1} \in f_G^{\supseteq\alpha}$. If $xy^{-1} \notin f_G^{\supseteq\alpha}$, then

$$f^\star_{f_G^{\supseteq\alpha}}(xy^{-1}) = \emptyset \supseteq \emptyset \cap \emptyset = f^\star_{f_G^{\supseteq\alpha}}(x) \cap f^\star_{f_G^{\supseteq\alpha}}(y)$$

If $xy^{-1} \in f_G^{\supseteq\alpha}$, then

$$f^\star_{f_G^{\supseteq\alpha}}(xy^{-1}) = f_G(xy^{-1}) \supseteq \emptyset \cap \emptyset = f^\star_{f_G^{\supseteq\alpha}}(x) \cap f^\star_{f_G^{\supseteq\alpha}}(y)$$

Thus, $f^\star_{f_G^{\supseteq\alpha}}$ is a SI-group over U. $\square$

6. Conclusion

In this paper, the notions of upper (proper) α-inclusions and lower (proper) α-inclusions of a soft set, upper α-subgroup of a SI-group and lower α-subgroup of a SU-group are defined and these notions are analyzed with respect to group theory in the mean of subgroups of a group in more detail. An answer to the question whether the family of upper α-subgroup of a SI-group determine the SI-group uniquely or not have bee found. Besides, a method which helps us to construct a SI-group of G with the help of the upper α-subgroups of f_G is introduced. Finally, it is shown how an upper α-subgroup of a SI-group is softificated.

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