Part V. Initial Value Problems

A

Inverse Scattering Method for the Initial Value Problem of the Nonlinear Equation of Evolution

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The inverse spectral problem is outlined exclusively from the viewpoint of applying it to the initial value problem of the nonlinear equation of evolution. Lax's conjecture on the integral of the equation and the spectrum of the associated linear operator is introduced as the basis of the whole discussion. The interrelation between the measure (or weight) function in the completeness relationship of the operator and the corresponding scattering operator is examined. The importance of the discrete spectrum is stressed especially in connection with the soliton. The Gelfand-Levitan equation and Zakharov-Shabat's method of analytic continuation are presented and the time development of the measure function is examined. All discussions are performed on the example of the Kortweg-de Vries equation. An example of the initial value problem for this equation is solved.

§ 1. Introduction

Gardner, Greene, Kruskal and Miura[1] developed an analytical method for solving the initial value problem of the Kortweg-de Vries equation (KdV equation) and Lax[2] extended the method into a form applicable to a wide class of the nonlinear partial differential equation or system. Since then several equations have been investigated by the method; the KdV equation[3], the nonlinear Schrödinger equation,[4,5] the modified KdV equation[6] and the sine-Gordon equation.[7] The method has the distinguished character that it reduces the analysis of the nonlinear partial differential equation into the linear analysis and that by this the concept of the persistent solitary wave —*soliton*— is well established in the sense that this corresponds to discrete eigenvalues of a linear operator. The concept of the soliton and the success of the method combined together attract interest from various fields: applied mathematics, plasma physics, nonlinear optics and so on. In this article the method will be reviewed from the point of studying the initial value problem of the nonlinear equation of evolution.

Consider the linear operator $L=L[u]$ which depends on the function $u(\cdot)$ and acts on some function space. The function $u(\cdot)$ is assumed to develop in time t as specified by $u(\cdot\,, t)$ or by $u(t)$ according to the nonlinear equation of evolution under our consideration:

$$u_t=K[u], \tag{1·1}$$

where K is a nonlinear operator acting on u. In the framework of Lax's theory the spectrum of the operator $L(t)=L[u(t)]$ is invariant with respect to time t, although $L(t)$ evidently varies due to the evolution of $u(t)$ according to Eq. (1·1). We note that Lax requires not only the invariance of the point eigenvalues of $L(t)$ but also the invariance of the whole spectrum. Thus we are led to introduce a unitary operator $T(t)$ with its adjoint $T^*(t)$ such that

$$L(t)=T(t)L(0)T^*(t). \tag{1·2}$$

The evolution of $u(t)$ according to Eq. (1·1) is transferred to the development of $T(t)$. Thus we have decomposed the problem of solving Eq. (1·1) into two parts. The first one is to find an appropriate $L[u]$ and $T[u]$ such that the evolution by Eq. (1·1) is equivalent to the transformation by Eq. (1·2). The second one is to investigate the correspondence between the properties of the operator $L=L[u]$ and the function u when the form of $L[u]$ as a function of u is given explicitly. The latter should be studied in two opposite directions. The one is the direct problem to study the properties of L; properties of the spectrum and the eigenfunctions of L, for given $u(t)$. The other one is the inverse problem to determine the function $u(t)$ from the given properties of the operator $L(t)$ for each t. Analysing the two opposite problems we can replace the study of the initial value problem of the given system, Eq. (1·1), by the study of the equivalent equation (1·2).

We discuss the outline of the direct problem and the inverse problem for u and $L[u]$ in §2. In §§3～5 we examine the detail of the method for the case where the operator L is selfadjoint. We shall present the method for the example of the KdV equation, for which L is the one-dimensional Schrödinger operator on $L_2\,(-\infty,\ \infty)$, and try to cover the essential points which are important, when the method is applied to the other cases. In §3, we discuss the direct problem, and the measure function, the scattering matrix and their interrelation are examined. In §4 two methods for the inverse problem, the Gelfand-Levitan equation[8] and the Zakharov-Shabat method,[4] are described, since they are, among others, frequently used in the initial value problem of the nonlinear equation of evolution. In §5 we show that the evolution due to Eq. (1·2) is fully replaced by the time development of the measure function or the kernel functions of the integral equations in §4 as far as the initial value problem is concerned, and we give also their time development explicitly. Section 6 is devoted to the short illustration of the application of the method to the KdV equation.

We finally mention that Hirota developed another method to obtain the analytical solution of these nonlinear equations. The method has not been developed so far to obtain the theory covering the initial value problem in full extent but has been useful to study the N-soliton problem for the wide class of the nonlinear equations.[9)~13)] It may, therefore, be expected that the method is discussed from various points of view.

§ 2. Invariants of the equation and the formulation of the initial value problem

(1) *Lax's method for determining the invariants*[2)]

In the evolution system of time t such as the system of ordinary differential equations, the hyperbolic system of the partial differential equations and so on, the constant of the motion or the integral of the system plays an important role for the analysis of the solution. As to the nonlinear partial differential equation under our study the forms of various integrals were extensively investigated. For instance in the case of the KdV equation

$$u_t = K[u] = 6uu_x - u_{xxx}, \tag{2·1}$$

there are several works presenting the integrals of the equation.[1),2),14),15)] Gardner, Greene, Kruskal and Miura[1)] found the important fact that, when the solution $u(x, t)$ possesses the characteristic feature of the solitary wave propagating to infinity, the discrete eigenvalue of the one-dimensional Schrödinger operator

$$L[u(x, t)] = -\frac{\partial^2}{\partial x^2} + u(x, t) \quad \text{for} \quad -\infty < x < \infty \tag{2·2}$$

is one of the integrals of Eq. (2·1), i.e., invariant with respect to time t. That is, the eigenvalue E defined by

$$L[u(t)]\psi(t, E) = E\psi(t, E), \tag{2·3}$$

is constant with respect to t, although the corresponding eigenfunction $\psi(t, E)$ changes in time. As described in §1, under Lax's formulation the whole spectrum of $L[u(t)]$ is assumed to be invariant with respect to time evolution, i.e., there is a one-parameter family of unitary operators $T(t)$ such that

$$L(t) = T(t)L(0)T^*(t) \tag{1·2}$$

or

$$\psi(t, E) = T(t)\psi(0, E), \tag{2·4}$$

for every E belonging to the spectrum of the operator $L[u]$. In the case of the KdV equation Lax constructed the unitary operator $T(t)$ from the generator A, where iA is a selfadjoint operator, in the following way:

$$T_t(t)=A(t)T(t) \tag{2·5}$$

with

$$A(t)=A[u(t)]=-4D^3+3Du+3uD, \tag{2·6}$$
$$D\equiv\frac{\partial}{\partial x}.$$

Equation (1·2) is equivalent to the KdV equation (2·1) for L and T given by Eqs. (2·2), (2·4), (2·5) and (2·6). This is seen by differentiation of Eq. (1·2) with respect to t:

$$\begin{aligned}u_t=L_t(t)&=T_t(t)L(0)T^*(t)+T(t)L(0)T_t^*(t)\\&=A(t)T(t)L(0)T^*(t)-T(t)L(0)T^*(t)A(t)\\&=A(t)L(t)-L(t)A(t)\equiv[A(t),L(t)].\end{aligned}$$

The elementary calculation shows that $[A,L]=K[u]=6uu_x-u_{xxx}$ giving the KdV equation (2·1).

Lax formulated the method in the following way.[2)]

Theorem 2.1. Suppose that L is a selfadjoint operator in a Hilbert space H and that it depends on u in the following fashion:

$$L[u]=L_0+M[u],$$

where L_0 is selfadjoint and is independent of u and M depends linearly on u. Suppose that there exists a symmetric operator $iA=iA[u]$ such that $[A[u],\ L[U]]=M[K[u]]$. Then the eigenvalues of $L[u]$ are integrals of $u_t=K[u]$.

In the above formulation, the evolution property of the system $u_t=K[u]$ is transferred to that of $L[u(t)]$ and $A[u(t)]$ and we note that by means of this transformation the time invariants and the time evolution of the system are separated; the whole spectrum of the selfadjoint operator L is invariant whereas the operator L evolves by the unitary transformation $T(t)$ generated by $A(t)$. On this line of thought we are led to consider the more general case where L and/or iA are not restricted to symmetric operators. Suppose that the spectral problem of a linear operator $L[u]$ is solved and that each eigenvalue of $L[u]$ is the integral of the nonlinear system of evolution

$$u_t=K[u]. \tag{2·7}$$

That is, in the equation

$$L[u(t)]\psi(t,E)=E\psi(t,E), \tag{2·8}$$

each complex E belonging to the spectrum of L is independent of t. On the other hand $L[u(t)]$ and also $\psi(t, E)$ vary with time t due to the evolution of $u(t)$ according to Eq. (2·7). Put

$$\psi(t, E)=T(t)\psi(0, E), \tag{2·9}$$

and assume that one-parameter family of the operators $T(t)$ has always its inverse $T^{-1}(t)$ and has its generator $A(t)$ such that

$$\begin{aligned} &T_t(t)=A(t)T(t), \\ &(T^{-1}(t))_t=-T^{-1}(t)A(t), \end{aligned} \tag{2·10}$$

where the second equation is derived from the first. From Eqs. (2·8) and (2·9) we have

$$L(t)T(t)\psi(0, E)=ET(t)\psi(0, E)$$

or

$$T^{-1}(t)L(t)T(t)=L(0).$$

Differentiation of the both sides of the last equation with respect to t gives

$$(T^{-1})_t LT+T^{-1}L_t T+T^{-1}LT_t=0$$

or by means of Eqs. (2·10) we have

$$L_t=AL-LA. \tag{2·11}$$

We see that the derivation of Eq. (2·11) is quite similar to the way used by Lax to obtain Theorem 2.1. In both cases the invariance of the spectrum of the operator L is important. We, thus, obtain a generalization of the preceding Theorem.

Theorem 2.2. Let L and L_0 are linear operators acting in a function space F. Suppose that the spectral problems of L and L_0 are solved and that L depends on u in the following fashion:

$$L[u]=L_0+M[u],$$

where L_0 is independent of u and M depends linearly on u. Suppose that there exists an operator $A=A[u]$ such that $[A, L]=M[K[u]]$. Then the eigenvalue of $L[u]$ is the integral of $u_t=K[u]$, where K is a nonlinear operator on u.

We now present several examples for Theorems 2.1 and 2.2: Theorem 2.1 is applied to the examples i), ii) and v); Theorem 2.2 is applied to the examples iii), iv) and v) (cf. Ref. 16)).

i) KdV equation

$$u_t = 6uu_x - u_{xxx}.$$

We put

$$L = -D^2 + u \quad \text{and} \quad A = -4D^3 + 3(Du + uD),$$

where iA is symmetric and L is the one-dimensional Schrödinger operator and, if $u(x)$ belongs to some class of real-valued function, is selfadjoint in $L_2(-\infty, \infty)$.

ii) Nonlinear Schrödinger equation

$$iu_t + u_{xx} + \kappa |u|^2 u = 0.$$

We put

$$L = i\begin{pmatrix} 1+\mu & 0 \\ 0 & 1-\mu \end{pmatrix} D + \begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix}$$

and

$$iA = -\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D^2 + \begin{pmatrix} |u|^2/(1+\mu) & i\bar{u}_x \\ -iu_x & -|u|^2/(1-\mu) \end{pmatrix}$$

with $\kappa = 2/(1-\mu^2)$. (μ: real)

L and iA are symmetric in $-\infty < x < \infty$ with some boundary condition and for some class of complex-valued function $u(x)$.

iii) Modified KdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0.$$

We put

$$L = L_0 + M[u]$$

with

$$L_0 = i\sigma_1 D, \qquad M[u] = iu\sigma_2$$

$$\text{and} \quad A = -4D^3 - 3(Du^2 + u^2 D) - 3i\sigma_3(Du_x + u_x D).$$

where

$$\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We note that L is not symmetric due to the nonsymmetric perturbation $iu\sigma_2$ to the symmetric operator $i\sigma_1 D$. Also iA is not symmetric due to the presence of the term proportional to $i\sigma_3$.

iv) Sine-Gordon equation

$$u_{TT} - u_{XX} = \sin u \quad \text{or} \quad u_{tx} = \sin u,$$

where $x=(T+X)/2$, $t=(T-X)/2$. In this case we consider the evolution of u_x. We put

$$L=L_0+M[u],\quad L_0=2i\sigma_3 D,\quad M[u]=i\sigma_2 u_x$$

and

$$A=(i/2)(\sigma_3\cos u+\sigma_2\sin u)L^{-1}.$$

L is not symmetric due to the nonsymmetric perturbation $i\sigma_2 u_x$ to the symmetric operator $2i\sigma_3 D$. Also iA is not symmetric.

v) Interaction of three waves

$$\begin{aligned}u_{1t}+v_1u_{1x}&=-\beta_1u_2u_3,\\ u_{2t}+v_2u_{2x}&=\beta_2u_3u_1,\\ u_{3t}+v_3u_{3x}&=-\beta_3u_1u_2,\end{aligned}$$

with real constants v_1, v_2, v_3 and positive constants β_1, β_2, β_3. For real $u_j(j=1,2,3)$ we put

a) $$L=i\rho D+\sum_{j=1}^{3}a_j\tau_ju_j,\qquad iA=iD+\sum_{j=1}^{3}b_j\tau_ju_j,$$

with symmetric matrices satisfying the commutation relations:

$$[\tau_j,\tau_k]=i\tau_l,\qquad (j,k,l)=(1,2,3)\quad\text{in the cyclic order}$$

and

$$[\rho,\tau_j]=0\quad\text{for}\quad j=1,2,3.$$

Then for any v_j, β_j $(j=1,2,3)$ we can find a real set (a_j, b_j) which yields the given system of evolution by Theorem 2.1. L and iA are symmetric.

b) For any set (v_j,β_j) we can find a set (a_j,b_j), which yields the given system by Theorem 2.2., with pure imaginary a_1, a_2, b_1, b_2 and real a_3, b_3. Evidently L and iA are not symmetric.

The examples i), ii) and iii) are frequently discussed in plasma physics and studied also in this volume. The system v) was investigated from the view point that this describes the propagation of three wave packets with amplitudes u_j and group velocities v_j $(j=1,2,3)$. This system, when $v_2=v_3$, is usually analyzed in more simple form, i.e., the sine-Gordon equation of the example iv) initially presented in nonlinear optics. The latter is derived by a transformation of the dependent variables and gives the particular class of the solution of the example v).

(2) *Formulation of the initial value problem*

By means of Lax's method described in (1), the initial value problem of Eq. (2·7) is reducible to the following series of mappings in terms of L and

A given explicitly as functions of u.

a) Eigenvalue problem for operator $L[u_0]$: $u_0 \longmapsto L[u_0]$. For given initial data $u(0)=u_0$ we determine the spectrum $\{E\}$ and the set of corresponding eigenfunctions $\{\psi(0, E)\}$. We call this mapping the direct problem. This corresponds to imposing the initial value $u(0)=u_0$ to Eq. (2·7).

b) Evolution due to the one-parameter family of operators

$$T(t):\ \psi(0, E) \longmapsto T(t)\psi(0, E)=\psi(t, E).$$

In this mapping the generating operator $A[u(t)]$ plays an important role. This evolution by $T(t)$ corresponds to that of $u(t)$ by means of Eq. (2·7).

c) Inverse problem of the mapping a): $L[u(t)] \longmapsto u(t)$. This final mapping is to be performed to fix the function $u(t)$ at each time t. The more precise formulation of the direct and inverse problem is given by Kay and Moses in the case of the selfadjoint operator L as follows:[17)~19)]

Direct problem. One is given:

a) the selfadjoint operator L,

b) the boundary condition on the eigenfunctions of L.

One seeks:

a) the eigenfunctions,

b) the measure functions (or equivalently weight functions) associated with the eigenfunctions which appear in the completeness relationship.

Inverse problem. One is given:

a) the boundary condition on the eigenfunctions of L,

b) the measure functions associated with these eigenfunctions.

One seeks:

a) the eigenfunctions of L,

b) the operator L.

Since in some cases the spectral measure functions associated with the eigenfunctions satisfying certain boundary conditions are connected with the scattering operator in a simple way we can consider the direct problem as the direct scattering problem, in which one determines the scattering operator from the given L, and similarly the inverse problem as the inverse scattering problem, in which one determines the operator L from the given scattering operator. In the case of the non-selfadjoint operator of some class the analogous consideration may be applicable.

In our present problem the discrete spectrum of the operator L plays an important role as well as the continuous one and we present some remarks concerning on this point before proceeding to study the problem in the following sections.

i) Soliton and the point and the continuous spectrum of L.

In Theorems 2.1 and 2.2 the operator L is the sum of L_0 and $M[u]$, the former is independ of u and in all cases of the examples i)~v) they are symmetric operators consisting of differential operators. With analogy to the quantum mechanics we may call L_0 the free part of L since L reduces to L_0 when the potential u vanishes. When the function $u(x, t)$ is the vector-valued function defined on the n-dimensional Euclidian space R_x^n for each t, $\psi(x, t, E)$ also should be a vector-valued function defined on R_x^n, the class of the function $\psi(x, t, E)$ as well as the dimension as a vector-valued function being not necessarily same as those of u. Since, in each of the examples in §2-(1), L_0 is the differential operator acting on R_x^1, it has continuous spectrum extending to infinity and has no point spectrum. Thus we have a problem how does the spectrum of L_0 vary under the perturbation by $M[u]$, in particular, what is the condition for $M[u]$ to yield the point spectrum of $L[u]$ in addition to the continuous one. This mathematical problem has been investigated extensively, especially, for the selfadjoint $L[u]$. The result should be found in literatures[20),21)] and is, expressed quite roughly, that the continuous spectrum of L contains that of L_0 if u is small in some functional space, for example $u \in L_2(R^1) \cap L_1(R^1)$ in the case of the one-dimensional Schrödinger operator.[22)] It is also well known in that example that, for instance, for the potential-well with its depth $v(v>0)$ and its width a, we have one point eigenvalue for $0 < va^2 \leq \pi^2$, two point eigenvalues for $\pi^2 < va^2 \leq 4\pi^2$, and so on.[23)] The occurrence of the point spectrum in the case, in which the domain of definition of $u(x)$ extends to infinity, yields quite interesting feature to the solutions of Eq. (2·7). In each of the examples i)~iv), each point spectrum of $L[u]$ is shown to correspond to one solitary wave, soliton, in the following sense; if Eq. (2·7) has a steady solution called one soliton solution which propagates with constant speed without changing its main feature and without decay of its amplitude, then L has only one point spectrum; if asymptotically in far future and past $(t \to \pm\infty)$ the solution becomes the linear superposition of N solitons, $u = \sum_{j=0}^{N} u_j$, where each u_j propagates according to Eq. (2·7) as one soliton, then $L[u]$ has N eigenvalues, each of which is equal to one of the eigenvalues of $L[u_j]$. As to the solution corresponding to the continuous spectrum of L the analysis has not been performed as fully as in the case of the point spectrum. In most cases (the examples i), ii) and iv)) it has been shown that the solutions around the solitons, except the contribution due to the solitons, vanishes algebraically in t as $t \to \infty$. The case v) has not been solved by Lax's method directly, but it contains the example iv) as a special case and we may expect it to be solved analogously giving solitons for the point spectrum of L, if we could find appropriate L and A of Theorem 2.1 or 2.2.[*)]

*) See Note added in proof.

ii) Periodic solutions in R_x^1.

Consider the solution in a finite part of R_x^1, for instance, $0 \leqq x \leqq d$ in the examples given in (1) with periodic boundary condition, then the spectrum of $L[u]$ as well as of L_0 is discrete. The clear contrast in the case of R_x^1 between solitons and the other part of the solution now disappears. The problem has been not much studied hitherto by analytical method whereas the numerical analysis in the case of the KdV equation established the remarkable feature of the recurrence of the solution for some periodic initial value.[24)]

§ 3. Direct spectral problem

(1) *Eigenfunction and the measure function*

The most important part of the direct problem in the spectral theory of operators, concerning our present problem, is, as described in §2-(2), to determine the eigenfunctions and the measure (or weight) functions of the operator L for given boundary conditions. In this subsection we illustrate this problem for an example of a selfadjoint operator —the one-dimensional Schrödinger operator in the interval $-\infty < x < \infty$— and, in the succeeding subsections, examine the relation between the measure function and the scattering operator. The presentation is given, with necessary modifications, along the way given by Faddeyev for the same operator in the interval $0 \leqq x < \infty$ with boundary condition that the solution should vanish at $x=0$.[25)]

We consider properties of solutions of the equation

$$-y'' + u(x)y = s^2 y, \qquad s = \sigma + i\tau \qquad \text{for} \quad -\infty < x < \infty, \tag{3·1}$$

with a potential $u(x)$ satisfying the condition

$$\int_{-\infty}^{\infty} (1+|x|)\,|u(x)|\,dx = C < \infty, \tag{3·2}$$

where we have put $E = s^2$ in (2·3) with $0 \leqq \arg E \leqq 2\pi$ for complex E. The solutions $f(x, s)$ and $g(x, s)$ are determined by the conditions:

$$\lim_{x\to\infty} e^{-isx} f(x, s) = 1, \tag{3·3}$$

$$\lim_{x\to-\infty} e^{isx} g(x, s) = 1. \tag{3·4}$$

Equation (3·1) with the boundary condition (3·3) or (3·4) is equivalent to the following integral equation, respectively:

$$f(x, s) = e^{isx} - \int_x^{\infty} \frac{\sin s(x-y)}{s} u(y) f(y, s)\,dy \tag{3·5}$$

or

$$g(x, s)=e^{-isx}+\int_{-\infty}^{x} \frac{\sin s(x-y)}{s} u(y)g(y, s)dy. \tag{3·6}$$

Then the following lemmas are proved:

Lemma 3.1. For each x, $f(x, s)$, $g(x, s)$, $f'(x, s)$ and $g'(x, s)$ are analytic in s in the half-plane $\tau>0$ and continuous down to the real axis. Moreover, the inequalities fold for $\tau\geqq 0$:

$$|f(x, s)|\leqq Ke^{-\tau x}, \tag{3·7}$$

$$|g(x, s)|\leqq Ke^{\tau x}. \tag{3·8}$$

The proof is given by the method of the successive approximation with the first approximation $e^{\pm isx}$. From Lemma 3.1 we obtain immediately:

Lemma 3.2. $f(x, s)$ and $g(x, s)$ satisfy the following inequalities for $\tau\geqq 0$:

$$|f(x, s)-e^{isx}|\leqq K\frac{e^{-\tau x}}{|s|}\int_{x}^{\infty}|u(y)|dy, \tag{3·9}$$

$$|g(x, s)-e^{-isx}|\leqq K\frac{e^{\tau x}}{|s|}\int_{-\infty}^{x}|u(y)|dy, \tag{3·10}$$

$$|f(x, s)-e^{isx}|\leqq Ke^{-\tau x}\int_{x}^{\infty}|yu(y)|dy, \tag{3·11}$$

$$|g(x, s)-e^{-isx}|\leqq Ke^{\tau x}\int_{-\infty}^{x}|yu(y)|dy, \tag{3·12}$$

$$|f'(x, s)-ise^{isx}|\leqq Ke^{-\tau x}\int_{x}^{\infty}|u(y)|dy, \tag{3·13}$$

$$|g'(x, s)+ise^{-isx}|\leqq Ke^{\tau x}\int_{-\infty}^{x}|u(y)|dy. \tag{3·14}$$

Analogously to Lemmas 3.1 and 3.2, we have:

Lemma 3.3. For any x, $f(x, s)$, $f'(x, s)$, $g(x, s)$ and $g'(x, s)$ are continuously differentiable with respect to s down to the real axis $\tau=0$ with the possible exception of the point $s=0$. The estimates

$$|\dot{f}(x, s)-ixe^{isx}|\leqq\frac{K}{|s|}e^{-\tau x}, \tag{3·15}$$

$$|\dot{g}(x, s)+ixe^{-isx}|\leqq\frac{K}{|s|}e^{\tau x}, \tag{3·16}$$

$$|\dot{f}'(x, s)+xse^{isx}|\leqq Ke^{-\tau x}, \tag{3·17}$$

$$|\dot{g}'(x, s)+xse^{-isx}|\leqq Ke^{\tau x} \tag{3·18}$$

hold for $\tau\geqq 0$. Here, the dot denotes differentiation with respect to s.

We note that Lemmas 3.1, 3.2, and 3.3 are obtainable also from Theorem 4.1 and the expression (4·2) for $g(x, s)$ and from the corresponding considerations for $f(x, s)$. We write k for s whenever s is real. The solutions $f(x, k)$ and $f(x, -k)$ for $k\neq 0$ are linearly independent and the Wronskian does not vanish:

$$W[f(x, k), f(x, -k)] \equiv f(x, k)f'(x, -k) - f'(x, k)f(x, -k) \\ = -2ik. \tag{3·19}$$

Similarly $g(x, k)$ and $g(x, -k)$ for $k\neq 0$ are linearly independent and

$$W[g(x, k),\ g(x, -k)] = 2ik. \tag{3·20}$$

The following relations are easily derived:

$$f(x, k) = b(k)g(x, k) + a(k)g(x, -k), \tag{3·21}$$

$$g(x, k) = -b(-k)f(x, k) + a(k)f(x, -k), \tag{3·22}$$

$$\left.\begin{aligned} &f(x, -k) = \bar{f}(x, k), \quad g(x, -k) = \bar{g}(x, k), \\ &a(-k) = \bar{a}(k), \quad b(-k) = \bar{b}(k), \end{aligned}\right\} \tag{3·23}$$

$$|a(k)|^2 = 1 + |b(k)|^2, \tag{3·24}$$

where $\bar{f}$ denotes the complex conjugate of f. In addition, we have

$$\left.\begin{aligned} &a(k) = (i/2k)W[f(x, k),\ g(x, k)], \\ &b(k) = (-i/2k)W[f(x, k),\ g(x, -k)]. \end{aligned}\right\} \tag{3·25}$$

By Lemma 3.1 $a(k)$ may be continued analytically into the half-plane of s, $\tau>0$, and the function is denoted by $a(s)$. We note that the analyticity of $b(s)$ is not confirmed at present (see §5). From Eq. (3·1) with $y=f(x, s)$ and $\bar{f}(x, s)$, we have an identity:

$$-[f'(x, s)\bar{f}(x, s) - f(x, s)\bar{f}'(x, s)]_{x=\alpha}^{\beta} = 4i\sigma\tau\int_{\alpha}^{\beta}|f(x, s)|^2 dx. \tag{3·26}$$

If $a(s_n)=0$, then the solution $f(x, s_n)$ and $g(x, s_n)$ are linearly dependent as seen from Eq. (3·25) with s_n in place of k, and the left-hand side of Eq. (3·26) tends to zero as $\alpha\to-\infty$, $\beta\to\infty$ due to Lemma 3.2 and the relation $f(x, s_n) = \text{const}\times g(x, s_n)$. This shows that $a(s)$ can vanish only for $\sigma=0$ or $\tau=0$. The latter possibility, however, is excluded by Eq. (3·24) valid for $\tau=0$. Hence Eq. (3·1) has no positive discrete spectrum. There remains the possibility that $a(s)=0$ for $\sigma=0$, $\tau>0$, which corresponds to the negative discrete spectrum. From the estimates (3·9), (3·10), (3·13), (3·14) and the condition (3·2) we have

$$a(s)=1+O(1) \quad \text{for large} \quad |s|, \tag{3·27}$$

and this shows that $a(s)$ can have only a finite number of zeros for $s=s_n=i\kappa_n(n=1, \cdots, m)$ on the positive imaginary axis. It is easily seen that $f_n(x)=f(x, s_n)$ or $g_n(x)=g(x, s_n)$ is considered as a real-valued function after a multiplication by an appropriate complex factor. The differential operator $-D^2+u(x)$ on the left-hand side of Eq. (3·1) together with the boundary condition defines a selfadjoint operator in $L_2(-\infty, \infty)$. This operator can be obtained by the extension of the symmetric operator, which is defined by the left-hand side of Eq. (3·1) and acts on the dense set of functions in L_2, e.g., twice-continuously differentiable functions vanishing identically outside some finite interval. We shall denote this operator by L.

Consider the kernel

$$\left.\begin{aligned} &R_E(x, y)=g(x, E^{1/2})f(y, E^{1/2})/(-2iE^{1/2}a(E^{1/2})), \\ &\qquad\qquad\qquad\qquad x<y, \\ &R_E(x, y)=R_E(y, x), \qquad x>y, \\ &\qquad\qquad 0\leqq\arg E^{1/2}\leqq\pi, \end{aligned}\right\} \tag{3·28}$$

which is defined for all complex E with the exception of a finite number of points on the negative real axis corresponding to the zeros of $a(E^{1/2})$. Then, it is seen that $R_E(x, y)$ is a solution of the equation:

$$\left(-\frac{\partial^2}{\partial x^2}+u(x)\right)R_E(x, y)-ER_E(x, y)=\delta(x-y), \tag{3·29}$$

and satisfies the boundary condition for each y:

$$R_E(x, y)e^{-iE^{1/2}x} \longrightarrow \text{const} \quad \text{as} \quad x\longrightarrow\infty, \tag{3·30}$$

$$R_E(x, y)e^{iE^{1/2}x} \longrightarrow \text{const} \quad \text{as} \quad x\longrightarrow-\infty. \tag{3·31}$$

Due to the estimates (3·7) and (3·8), we have

$$\begin{aligned} &|R_E(x, y)|\leqq Ke^{-\tau|x-y|}, \\ &\qquad \tau=\operatorname{Im} E^{1/2}>0 \end{aligned} \tag{3·32}$$

for complex E and, hence, the integral operator $R_E(x, y)$ defines a bounded operator R_E in $L_2(-\infty, \infty)$. Since Eq. (3·29) is equivalent to the operator equation

$$(L-EI)R_E=I, \tag{3·33}$$

the operator $R_E=(L-EI)^{-1}$ is the resolvent operator belonging to the selfadjoint operator L. The singularities of R_E in the complex E-plane consist of a cut along the positive real axis and a set of poles $E_n=-\kappa_n^2(n=1, \cdots, m)$ on the negative real axis which correspond to the continuous and dsicrete spectrum, respectively. The jump of the resolvent across the cut and the residues at the poles determine the spectral resolution of the operator L.

We now present the measure function for the boundary condition (3·4).

Theorem 3.1. The functions $\mathcal{g}(x, k, \pm 1)$, $k>0$, $\mathcal{g}_n(x)$ $(n=1, \cdots, m)$ form a complete orthogonal system, where $\mathcal{g}(x, k, \pm 1)=\mathcal{g}(x, \pm k)$ and $\mathcal{g}_n(x)=\mathcal{g}(x, i\kappa_n)$. The completeness relation is given by

$$\sum_{n=1}^{m} C_n \mathcal{g}_n(x)\mathcal{g}_n(y)+\frac{1}{2\pi}\int_0^\infty dk G^T(x, k)W(k)\bar{G}(y, k)=\delta(x-y), \tag{3·34}$$

where

$$G(x, k)=\begin{pmatrix} \mathcal{g}(x, k, -1) \\ \mathcal{g}(x, k, +1) \end{pmatrix}, \quad G^T(x, k)=(\mathcal{g}(x, k, -1), \mathcal{g}(x, k, +1)),$$

$$W(k)=\begin{pmatrix} 1 & \bar{r}(k) \\ r(k) & 1 \end{pmatrix}, \tag{3·35}$$

$$r(k)=b(k)/a(k), \quad r(-k)=\bar{r}(k), \tag{3·36}$$

$$(C_n)^{-1}=i(\mathcal{g}_n(x)/f_n(x))\dot{a}(i\kappa_n)=\int_{-\infty}^{\infty} \mathcal{g}_n^2(x)dx \tag{3·37}$$

and

$$\dot{a}(i\kappa_n)\neq 0, \text{ i.e., } a(s) \text{ has only simple zeros.}$$

Remark. $r(k)$ has the meaning of the reflection coefficient that it gives the amplitude of the plane wave which is reflected toward the left when an incident wave of unit amplitude moves toward the right, where amplitudes are defined at $x\to-\infty$. This is confirmed by Eq. (3·21) and the boundary condition (3·4).

Proof of Theorem 3.1. Equation (3·29) is first obtainable as an operation on a twice-continuously differentiable function $w(x)$ vanishing for large $|x|$. Then, $w_1(y)=-w''(y)+u(y)w(y)$ is continuous and vanishes for large $|x|$ and we have, from Eq. (3·29) multiplied on $w_1(y)$ and integrated with respect to y,

$$\begin{aligned}\frac{1}{E}\int_{-\infty}^{\infty} R_E(x, y)w_1(y)dy&=\frac{1}{E}\int_{-\infty}^{\infty} R_E(x, y)\{-w''(y)+u(y)w(y)\}dy\\&=\frac{1}{E}w(x)+\int_{-\infty}^{\infty} R_E(x, y)w(y)dy.\end{aligned} \tag{3·38}$$

If we integrate both sides of Eq. (3·38) with respect to E along a large circle, $|E|=r$, then the left-hand side vanishes as $r\to\infty$ due to the estimates (3·9), (3·10), (3·27), which hold for $\tau\geqq 0$, i.e., for $0\leqq \arg E^{1/2}\leqq\pi$, and we have

$$w(x)=-\lim_{r\to\infty}\frac{1}{2\pi i}\int_{|E|=r} dE\left[\int_{-\infty}^{\infty} R_E(x, y)w(y)dy\right].$$

We apply Cauchy's integral theorem by deforming the path $|E|=r$ into a sum of m small circles around m poles of R_E and two lines just above and below the positive real axis. Then we obtain

$$w(x)=-\sum_{n=1}^{m}\text{Residue}\left\{\int_{-\infty}^{\infty}R_E(x,y)w(y)dy\right\}\Big|_{E=-\kappa_n^2}$$
$$+\frac{1}{2\pi i}\int_0^{\infty}dE\left[\int_{-\infty}^{\infty}\{R_{E+i0}(x,y)-R_{E-i0}(x,y)\}\,w(y)dy\right],$$

where

$$\text{Residue}\left\{\int_{-\infty}^{\infty}R_E(x,y)w(y)dy\right\}\Big|_{E=-\kappa_n^2}=i(\dot{a}(i\kappa_n))^{-1}f_n(x)\int_{-\infty}^{\infty}g_n(y)w(y)dy,$$

$$\frac{1}{2\pi i}\int_0^{\infty}dE\left[\int_{-\infty}^{\infty}\left\{R_{E+i0}-R_{E-i0}\right\}w(y)dy\right]$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{a(k)}\left[\int_{-\infty}^{x}f(x,k)g(y,k)w(y)dy+\int_x^{\infty}g(x,k)f(y,k)w(y)dy\right]$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dy[r(k)g(x,k)\bar{g}(y,-k)+g(x,k)\bar{g}(y,k)]w(y).$$

Then the expression (3·35) is easily derived. To arrive Eq. (3·37) we proceed as follows. If $a(s)$ has simple zero for $s=i\kappa_p$, then the function $f(x,s)g(y,s)/(sa(s))$ has simple pole at $s=i\kappa_p$, since, otherwise, we have the vanishing numerator, $f(x,i\kappa_p)g(y,i\kappa_p)\equiv 0$. The proof that $a(s)$ has only simple zeros is described below. Thus we have completed the proof of Eq. (3·34) when it operates on $w(y)$. Since the set of $w(x)$ is dense in $L_2(-\infty,\infty)$ and the operators on the both sides are bounded, Eqs. (3·34)~(3·37) are valid for all elements in $L_2\ (-\infty,\infty)$, giving the explicit form of the completeness relation in terms of $r(k)$ and C_n. We give another proof of Eq. (3·37) for the case where $\kappa_m>0$ for all $n=1,\cdots,m$ as follows:
From Eq. (3·1) for $f(x,s)$ and the equation

$$-\dot{g}''+u\dot{g}=2sg+s^2\dot{g},$$

it is easy to derive the identity

$$f'\dot{g}-\dot{g}'f\Big|_{x=\alpha}^{\beta}=2s\int_\alpha^\beta fg\,dx.$$

Taking limits $\alpha\to-\infty$, $\beta\to\infty$ for $s=s_p=i\kappa_p$, we have

$$f'\dot{g}-\dot{g}'f\Big|_{x\to\infty}=2s\int_{-\infty}^{\infty}fg\,dx, \qquad (3\cdot39)$$

since $f, f', \dot{g}, \dot{g}' \to 0$ as $x \to -\infty$ by Lemma 3.3 and by $f(x, s_p) = \text{const.} \times g(x, s_p)$. On the other hand from Eq. (3·25) given for s in place of k we have, by similar consideration as above,

$$-2i\frac{d}{ds}(sa(s)) = \dot{f}g' + f\dot{g}' - \dot{f}'g - f'\dot{g}\Big|_{x\to\infty}$$

$$= fg' - f'g\Big|_{x\to\infty} \quad \text{for} \quad s = s_p. \tag{3·40}$$

From Eqs. (3·39) and (3·40) we obtain

$$is_p\dot{a}(s_p) = s_p\int f(x, s_p)g(x, s_p)dx = s_p\frac{f(x, s_p)}{g(x, s_p)}\int_{-\infty}^{\infty} g_p^2 dx.$$

Hence, if $s_p \neq 0$ we have Eq. (3·37). $\dot{a}(s_p) \neq 0$ is evident. In the case, where $s_p = 0$ for some p, the discussion becomes more complicated but there occurs no essential difficulties and we omit to describe this exceptional case.

(2) *Relation between the measure function and the scattering operator*

The relation between the measure function and the scattering operator is presented in various works:[17)~19),25)~27)] A consice summary mainly from the work of Kay and Moses will be presented here to the extent necessary for the application to our nonlinear equation of evolution given in §2-(1). Although we considered in Theorem 2.2 the general case where L is not always selfadjoint, we examine in this subsection the case of Theorem 2.1, where L and L_0 are selfadjoint in a Hilbert space H and, moreover, for simplicity, the free part L_0 of L has only a continuous (absolutely continuous) spectrum. We note that the one-dimensional Schrödinger operator in the preceding subsection is contained in the framework considered below.

In many cases, e.g., in all examples of §2, the continuous parts of the spectrum of L and L_0 degenerate and it will be convenient to introduce a set of commuting operators, collectively denoted by A_0, each of which commutes with L_0, such that A_0 and L_0 form a complete set of commuting operators in H. For instance in the case treated in §3-(1), $L_0 = -D^2$ with eigenvalue $E = k^2 (-\infty < k < \infty)$ and A_0 is chosen to be the operator whose eigenvalue a gives the sign of the momentum for each E. For the eigenfunction $g(E, a)$ of L_0 and A_0 we have

$$L_0 g_0(E, a) = E g_0(E, a). \tag{3·41}$$

The normalization and the completeness relation are given, respectively as follows:

$$(g_0(E, a),\ g_0(F, b)) = \delta(E-F)\delta(a, b), \tag{3·42}$$

where the inner product and Kronecker's symbol are expressed as usual and

$$\iint (u, \varphi_0(E, a))dEda(\varphi_0(E, a), v) = (u, v) \quad \text{for} \quad u, v \in H. \tag{3·43}$$

Similar consideration holds also for L and A, where A is a set of commuting operators and together with L form a complete set of commuting operators. We denote the eigenfunction of L and A with eigenvalues E and a of L and A, respectively, by $\varphi(E, a)$ with relation

$$L\varphi(E, a) = E\varphi(E, a). \tag{3·44}$$

The spectrum of L has been extensively investigated and it has been revealed in the scattering theory that under various class of perturbation to L_0 the nonsingular (absolutely continuous) spectrum of L is stable and the part of L corresponding to that spectrum is unitarily equivalent to L_0, although it is not clear whether the singular spectrum consists only of discrete eigenvalues or not[20]. On the other hand, in our one-dimensional Schrödinger operator of §3-(1), the continuous spectrum of L is same as the whole spectrum of L_0 including the degeneracy and the singular part consists of a finite number of discrete eigenvalues, which are simple and negative. In this and succeeding subsections we assume that the wave operators in the scattering theory exist and the nonsingular part of L is unitarily equivalent to L_0, which has only a positive and (absolutely) continuous spectrum, and that the singular spectrum of L consists of negative discrete eigenvalues $\{E_n\}$, $n=1, \cdots, m$, m being finite or infinite. We introduce an m-dimensional vector space V^m with its complete orthonormal set $\{\varphi_0(E_n, a)\}$, $\|\varphi_0(E_n, a)\|=1$, $n=1, \cdots, m$, V^m being a Hilbert space when m is infinite. We note that $\varphi_0(E_n, a)$ corresponds to an eigenfunction $\varphi(E_n, a)$ of L with eigenvalue E_n, a of L and A, respectively. If E_n is the simple eigenvalue, then a is not necessary. Construct a Hilbert space $\hat{H}$ as the direct sum of H and V^m:

$$\hat{H} = H \oplus V^m. \tag{3·45}$$

Define projection operators $\hat{P}$ and P in $\hat{H}$ and H, respectively:

$$\hat{P}\hat{H} = H, \qquad \hat{P}V^m = 0, \tag{3·46}$$

$$PH = H_c, \qquad PH_d = 0 \quad \text{with} \quad H = H_c \oplus H_d, \tag{3·47}$$

where H_d is the subspace of H which is sustained by the total set of the eigenfunctions belonging to the discrete spectrum of L, i.e., by $\{\varphi(E_n, a), n=1, \cdots, m.\}$. Also define an extension of L_0 given by Eq. (3·41) to $\hat{L}_0$ in $\hat{H}$ in the following way:

$$\hat{L}_0\varphi_0(E_n, a) = E_n\varphi_0(E_n, a), \; n=1, \cdots, m. \tag{3·48}$$

Then we see that $\hat{L}_0$ defined in $\hat{H}$ has the completely same spectrum as L in H. Further we introduce the transformation operator $\hat{U}$ from $\hat{H}$ to H defined by

$$\mathcal{G}(E, a) = \hat{U}\mathcal{G}_0(E, a), \tag{3·49}$$

for each E belonging to the spectrum of $\hat{L}_0$ and L, then we have

$$L\hat{U}\mathcal{G}_0(E, a) = \hat{U}\hat{L}_0\mathcal{G}_0(E, a)$$

or

$$L\hat{U} = \hat{U}\hat{L}_0. \tag{3·50}$$

Define a contraction U of $\hat{U}$:

$$U = P\hat{U}\hat{P}, \tag{3·51}$$

then U is a transformation operator from H to PH. We can construct the inverse $\hat{U}^{-1}$ and the adjoint $\hat{U}^*$ of $\hat{U}$, as operators from H to $\hat{H}$, and also the inverse U^{-1} and the adjoint U^* of U, as operators from PH to $\hat{P}\hat{H}=H$. It was shown that all possible solutions of Eq. (3·50) satisfy the integral equation in $\hat{H}$:

$$\hat{U} = \hat{N} + \mathrm{P_r}\int \frac{1}{E - L_0} M\hat{U}\delta(E - \hat{L}_0)dE, \tag{3·52}$$

where $M = L - L_0$ and $\hat{N}$ is an arbitrary operator commuting with $\hat{L}_0$. The symbol $\mathrm{P_r}$ means "principal value" in the integration and the range of integration is the range of the spectrum of $\hat{L}_0$. The operator $\delta(E-\hat{L}_0)$ is defined by

$$(f_1, \delta(E - \hat{L}_0)f_2) = \int (f_1, \mathcal{G}_0(E, a))da(\mathcal{G}_0(E, a), f_2) \quad \text{for } f_1, f_2 \in \hat{H}, \tag{3·53}$$

and has the properties:

$$\hat{L}_0\delta(E - \hat{L}_0) = \delta(E - \hat{L}_0)\hat{L}_0 = E\delta(E - \hat{L}_0),$$
$$\int \delta(E - \hat{L}_0)dE = I. \tag{3·54}$$

Let B be any operator commuting with $\hat{L}_0$, then we can define matrix $B(E, a, b)$ for each E:

$$(\mathcal{G}_0(E, a), B\mathcal{G}_0(F, b)) = B(E, a, b)\delta(E - F). \tag{3·55}$$

Since $\hat{N}$ is not unique, the operator U or equivalently the set of eigenfunctions $\{\mathcal{G}(E, a)\}$ is not unique. We saw this in §3-(1), for the one-dimensional Schrödinger operator, where we have obtained two complete sets of eigenfunctions $\{f(x, \pm k), f_n(x)\}$ and $\{g(x, \pm k), g_n(x)\}$ corresponding to different boundary conditions (3·3) and (3·4), respectively. We can construct also the other sets of eigenfunctions corresponding to the other boundary conditions.

We now introduce two particular transformation operators, $U_\pm$ in H satisfying the relation (3·50) in H, which are called the wave operators and play an important role in the scattering theory. $U_\pm$ can be defined, by our assumption, in the following way:

$$U_\pm f = \lim_{s\to\pm\infty} e^{iLs} e^{-iL_0 s} f, \tag{3·56}$$

where $L=L_0+M$ and the limit should be taken in the sense of strong topology in the Hilbert space H. Although f in Eq. (3·56) should be taken from some dense set in H, $U_\pm$ is shown to be definable by extension for all elements in H, and under our assumption we have the relations:

$$U_\pm^* U_\pm = I, \qquad \text{in} \quad H, \tag{3·57}$$

$$U_\pm U_\pm^* P = P, \tag{3·58}$$

$U_\pm$ transforming H to PH. $U_\pm$ is also expressible in the following form:

$$U_\pm = I + \int \gamma_\pm(E-L_0) M U_\pm \delta(E-L_0) dE, \tag{3·59}$$

where

$$\gamma_\pm(x) = \pm i\pi\delta(x) + \mathrm{P_r}\frac{1}{x} = \lim_{\varepsilon\to+0}\frac{1}{x\mp i\varepsilon} \tag{3·60}$$

and $N=\hat{N}\hat{P}$ of Eq. (3·52) is chosen as

$$N = I \pm i\pi \int \delta(E-L_0) M U_\pm \delta(E-L_0) dE. \tag{3·61}$$

The scattering operator S is defined as

$$S = U_+^* U_-. \tag{3·62}$$

Since in our case S is unitary in H:

$$SS^* = S^* S = I, \tag{3·63}$$

the scattering problem is solved completely.

In order to obtain the relation between S and the measure function W introduced in Theorem 3.1, we define according to Kay and Moses[17] the operators $V_\pm$ connecting the operators U and $U_\pm$ in the following way:

$$U = U_\pm V_\pm. \tag{3·64}$$

We note that $V_\pm$ can be expressible also as

$$V_\pm = U - \int \gamma_\pm(E-L_0) M U \delta(E-L_0) dE \quad \text{in} \quad H. \tag{3·65}$$

It is easily seen from Eq. (3·64) that $V_\pm$ commute with L_0 and have their inverses defined in a dense set of elements in H. We have from Eq. (3·64) also the relation

$$U_+ V_+ = U_- V_-$$

or

$$V_+ = U_+^* U_- V_- = S V_-,$$

and, hence

$$S = V_+ V_+^{-1} \quad \text{in} \quad H. \tag{3·66}$$

We have from Eqs. (3·57) and (3·64)

$$I = U_\pm^* U_\pm = (V_\pm^*)^{-1} U^* U V_\pm^{-1}$$

or

$$W U^* U = U^* U W = I \quad \text{in} \quad H, \tag{3·67}$$

and

$$P = U_\pm U_\pm^* P = U V_\pm^{-1} (V_\pm^*)^{-1} U^*$$

or

$$U W \mathrm{U}^* = P \quad \text{in} \quad H, \tag{3·68}$$

where

$$W = V_\pm^{-1} (V_\pm^*)^{-1} \tag{3·69}$$

is commutable with L_0 and positive definite by Eq. (3·69). Expressing W in the matrix form $W(E, a, b)$ given by Eq. (3·55) applied in H

$$(\varphi_0(E, a), W\varphi_0(f, b)) = W(E, a, b)\delta(E-F), \tag{3·70}$$

we have for any $f \in PH$, by Eqs. (3·68), (3·43), (3·70), (3·49) and (3·51)

$$\begin{aligned} f = Pf &= UWU^* f \\ &= \iiint U\varphi_0(E, a) dE da W(E, a, b) db (U\varphi_0(E, b), f) \\ &= \int dE \iint \varphi(E, a) da W(E, a, b) db (\varphi(E, b), f). \end{aligned} \tag{3·71}$$

We see that $W(E, a, b)$ plays the role of the weight function in the expansion of $f \in PH$ in terms of the eigenfunctions $U\varphi_0(E, a) = \varphi(E, a)$ for E belonging to the continuous spectrum of L_0 and L. It is easily seen that $W(E, a, b)$ corresponds to $W(k)$ of Eq. (3·35). Thus Eq. (3·68) gives the completeness relation in PH of the eigenfunctions $U\varphi_0(E, a)$ and we see easily that Eq. (3·67) gives the orthogonality relation of $U\varphi_0(E, a)$. Obviously Eqs. (3·67) and (3·68) are the generalizations of Eqs. (3·57) and (3·58), respectively, which are given for $U_{\pm}\varphi_0(E, a)$ with simple form, $W(E, a, b)=\delta(a, b)$.

It was the object in the inverse problem to determine whether the scattering operator alone yields the unique potential M or not. The problem was resolved with negative answer for the cases where the potential has bound states corresponding to the discrete eigenvalues and it has been known that we should add the data about the bound states to reconstruct L completely from the scattering operator. (The literatures are seen in Ref. 25).) In order to treat the inverse problem we are thus led to consider the extensions of $U_{\pm}$, $V_{\pm}$ in the way already done for the general U and to comprise the data on the bound states, which are absent in the method of the wave operators in the scattering theory, e.g., in Eq. (3·62). Define $\hat{V}_{\pm}$ and $\hat{U}_{\pm}$ as follows:

$$\hat{V}^*_{\pm}\varphi_0(E_n, a)=\hat{V}_{\pm}\varphi_0(E_n, a)=\{C(E_n, a)\}^{-1/2}\varphi(E_n, a), \tag{3·72}$$

$$\hat{U}_{\pm}\varphi_0(E_n, a)=\varphi(E_n, a), \qquad n=1, \cdots, m, \tag{3·73}$$

then we can define the extension of W in $\hat{H}$:

$$\hat{W}\varphi_0(E_n, a)=\hat{V}^{-1}_{\pm}(\hat{V}^*_{\pm})^{-1}\varphi_0(E_n, a)=C(E_n, a)\varphi_0(E_n, a), \qquad n=1, \cdots, m,$$

$$W_d=\hat{W}(I-\hat{P}). \tag{3·74}$$

We have from Eqs. (3·72), (3·73) and (3·74) the orthogonality relation:

$$\hat{U}^*_{\pm}\hat{U}_{\pm}=I \qquad \text{in } \hat{H}, \tag{3·75}$$

$$\hat{W}\hat{U}^*\hat{U}=\hat{U}^*\hat{U}\hat{W}=I \qquad \text{in } \hat{H}, \tag{3·76}$$

and the completeness relation:

$$\hat{U}_{\pm}\hat{U}^*_{\pm}=I \qquad \text{in } H, \tag{3·77}$$

$$\hat{U}\hat{W}\hat{U}^*=I \qquad \text{in } H. \tag{3·78}$$

We remark finally that Eq. (3·74) corresponds to C_n given by Eqs. (3·34) and (3·37) and that C_n has a close connection to W through the analytic continuation of the eigenfunction belonging to the continuous spectrum as was given in Theorem 3.1, and hence C_n cannot be given arbitrarily, when the potential is given, and vice versa. We return to this point in §4.

(3) *Measure function and the reflection coefficient for the one-dimensional Schrödinger operator*

We obtained in §3-(2) the general relation which holds between the measure function W and the scattering operator S via the operators $V_{\pm}$. Since W, S and $V_{\pm}$ are commutable with L_0, we can express them in the matrix form by Eq. (3·55), applied in H in place of $\hat{H}$, as follows:

$$(\varphi_0(E, a), S\varphi_0(F, b))=S(E, a, b)\delta(E-F) \tag{3·79}$$

and

$$(\varphi_0(E, a), V_{\pm}\varphi_0(F, b))=V_{\pm}(E, a, b)\delta(E-F), \tag{3·80}$$

$W(E, a, b)$ being given by Eq. (3·70).

There are examples for which $V_{\pm}$ are eliminated and $W(E, a, b)$ is expressible directly in terms of $S(E, a, b)$. The case of the one-dimensional Schrödinger operator in $0\leqq x<\infty$ with the vanishing boundary condition at $x=0$ is the simplest example and is discussed extensively by Faddeyev.[25] Here we present, following Kay and Moses,[19] the case of the same operator but in the different interval, $-\infty<x<\infty$. Also in this case the problem is rather simple, since the operator A or A_0 has only two eigenvalues ± 1 corresponding to the directions of the momentum.

Consider Eq. (3·1) under the restriction (3·2) and represent $\varphi_{\pm}(E, a)=U_{\pm}\varphi_0(E, a)$ and $\varphi_0(E, a)$ in the x-representation as $\varphi_{\pm}(x, k, a)$ and $\varphi_0(x, k, a)$, respectively, and put

$$\varphi_0(x, k, a)=\frac{1}{2}(\pi k)^{-1/2}e^{-ikax}, \tag{3·81}$$

where $E^{1/2}=k\geqq 0$, $a=\pm 1$. Then, from Eq. (3·59), we have

$$\varphi_{\pm}(x, k, a)=\varphi_0(x, k, a)\pm\frac{i}{2k}\int_{-\infty}^{\infty}dx' e^{\mp ik|x-x'|}u(x')\varphi_{\pm}(x', k, a). \tag{3·82}$$

Then from the formula

$$S=I-2\pi i\int\delta(E-L_0)MU_{-}\delta(E-L_0)dE, \tag{3·83}$$

which is obtained by the standard method in the scattering theory[17] we have

$$\begin{aligned} S(E, a, b)&=\delta_{ab}-2\pi i(\varphi_0(E, a), M\varphi_{-}(E, b)) \\ &=\delta_{ab}-i\sqrt{\frac{\pi}{k}}\int_{-\infty}^{\infty}dx e^{ikax}u(x)\varphi_{-}(x, k, b). \end{aligned} \tag{3·84}$$

Comparing Eq. (3·82) with Eq. (3·84) we have

$$\lim_{x\to\pm\infty}\varphi_{-}(x, k, a)=\varphi_0(x, k, a)+[S(E, \mp 1, a)-\delta_{\mp 1, a}]\varphi_0(x, k, \mp 1). \tag{3·85}$$

We now consider the other set of eigenfunctions $\{\varphi_1(x, k, \pm 1)=U\varphi_0(x, k, \pm 1)\}$ which is transformed from φ_0 by U and obeys the boundary condition (3·4). The set $\{\varphi_1(x, k, \pm 1), k>0\}$ corresponds to the set $\{\varphi(x, k), -\infty<k<\infty\}$

defined in §3-(1) and is identical to the set $\{G(x, k)\}$ of Theorem 3.1, if we put $\mathscr{g}_1(x, k, \pm 1) = \mathscr{g}(x, k, \pm 1)$. (Note the condition (3·4) and the sign of k.) From the relation (3·64), $U = U_- V_-$, or

$$\mathscr{g}(x, k, a) = \sum_{b=\pm 1} \mathscr{g}_-(x, k, b) V_-(E, b, a), \tag{3·86}$$

we shall determine the matrix element $V_-(E, b, a)$ defined by Eq. (3·80). For this purpose, we derive from the boundary condition (3·4) applied to Eq. (3·86) and by Eq. (3·85) the following relation:

$$\begin{aligned}\lim_{x\to-\infty} \mathscr{g}(x, k, a) = \mathscr{g}_0(x, k, a) &= \sum_b \lim_{x\to-\infty} \mathscr{g}_-(x, k, b) V_-(E, b, a) \\ &= \sum_b \mathscr{g}_0(x, k, b) V_-(E, b, a) \\ &\quad + \mathscr{g}_0(x, k, 1) \sum_b [S(E, 1, b) - \delta_{1,b}] V_-(E, b, a).\end{aligned}$$

This relation yields, after renaming of the suffices,

$$V_-(E, a, b) + \delta_{a,1} \sum_c [S(E, 1, c) - \delta_{1,c}] V_-(E, c, b) = \delta_{ab}$$

or

$$V_-(E, a, b) = \begin{pmatrix} 1 & 0 \\ -\dfrac{S(E, 1, -1)}{S(E, 1, 1)} & \dfrac{1}{S(E, 1, 1)} \end{pmatrix}. \tag{3·87}$$

Then by Eq. (3·69) and the unitarity of S, Eq. (3·63), we have

$$W(E, a, b) = \begin{pmatrix} 1 & \bar{S}(E, 1, -1) \\ S(E, 1, -1) & 1 \end{pmatrix} \tag{3·88}$$

which corresponds to $W(k)$ of Eq. (3·35) except the difference of the variables E and k.

§ 4. Inverse problem

We have obtained the measure function $\hat{W} = W + W_d$ appearing in the completeness relation for the eigenfunctions $\mathscr{g}(x, k)$, $\mathscr{g}_n(x)$ defined by Eq. (3·1) with the boundary condition Eq. (3·4). The resolvent kernel $R_E(x, y)$ is regular for each x, y with respect to E except on the spectrum on the real axis and determines $\hat{W}$ in the following way:

W_d: in terms of residues of R_E at the poles $E = -\kappa_n^2$, $n = 1, \cdots, m$ ($m < \infty$).

W : in terms of the boundary value of R_E on the both sides of the positive real axis.

We see from Eq. (3·28) that $R_E(x, y)$ is defined in terms of the function

$f(x, s)\mathcal{g}(y, s)/(sa(s))$ which is analytic in the upper half of s-plane, $(s^2=E)$, except at the poles corresponding to the discrete eigenvalues. In this way W and W_d are connected through the analytic continuations of f, $\mathcal{g}$ and a. We see that this property plays an important role in the following two method which have been employed frequently in the inverse problem for the study of the nonlinear system of evolution.

(1) *Gelfand-Levitan equation*

In §3-(2) we defined the transformation operator U and its extension $\hat{U}$:

$$L\hat{U}=\hat{U}\hat{L}_0 \qquad \text{in} \quad \hat{H}.$$

As discussed there, $\hat{U}$ depends on the boundary condition for the eigenfunction. When $\hat{U}$ and $\hat{U}^{-1}$ is determined, then M is given by

$$M=L-L_0=\hat{U}\hat{L}_0\hat{U}^{-1}-L_0 \qquad \text{in} \quad H. \tag{4·1}$$

We illustrate the method for determining M for the case of the one-dimensional Schrödinger equation in $-\infty<x<\infty$ studied in §3. The procedure is similar to that given by Agranovich and Marchenko[26] except some modifications necessary for the present case. The operators $U=I+K$ and $\hat{U}^{-1}=I+K_0$ are given in the following equations in terms of the integral kernels $K(x, y)$ and $K_0(x, y)$ representing K and K_0, respectively, and yield the transformations between the eigenfunctions of L and L_0:

$$\mathcal{g}(x, s)=e^{-isx}+\int_{-\infty}^{x} K(x, y)e^{-isy}dy, \tag{4·2}$$

$$e^{-isx}=\mathcal{g}(x, s)+\int_{-\infty}^{x} K_0(x, y)\mathcal{g}(y, s)dy. \tag{4·3}$$

Equation (4·2) is easily seen to be equivalent to Eq. (3·1) with the boundary condition (3·4). Thus the integral kernel $\delta(x-y)+K(x, y)$ appearing on the right-hand side of Eq. (4·2) is a representation of the operator $U=I+K$ corresponding to the boundary condition (3·4) and K is related to the potential M by Eq. (4·1). Moreover, the direct relation of $K(x, y)$ to the potential is given in the following Theorem 4.1. On the other hand the kernel $K(x, y)$ is determined as the solution of the Gelfand-Levitan equation in terms of the set $\{\kappa_n, C_n\ (n=1, \cdots, m), r(k)=b(k)/a(k)\}$, i.e., the measure function $\hat{W}$. In this sense $K(x, y)$ is an important function in the inverse problem. We add that Eq. (4·3) is only a modification of Eq. (3·6) and is given in an s dependent form:

$$K_0(x, y)=-\frac{\sin s(x-y)}{s}, \qquad x\geq y,$$
$$=0, \qquad x<y. \tag{4·4}$$

Theorem 4.1. If the restriction (3·2) holds for the potential $u(x)$, then Eq. (4·2) is equivalent to Eq. (3·1) with boundary condition (3·4) and $K(x, y)$ satisfies the integral equation:

$$K(x,y)=\frac{1}{2}\int_{-\infty}^{(x+y)/2}u(t)dt+\frac{1}{2}\int_{(x+y)/2}^{x}u(t)dt\int_{y+t-x}^{y-t+x}K(t,v)dv$$
$$+\frac{1}{2}\int_{-\infty}^{(x+y)/2}u(t)dt\int_{y+t-x}^{t}K(t,v)dv, \tag{4·5}$$

where the kernel $K(x, y)$ is shown to satisfy the inequality

$$|K(x,y)|\leqq\frac{1}{2}\exp[\sigma_1(x)]\sigma\left(\frac{x+y}{2}\right), \tag{4·6}$$

with

$$\begin{aligned}\sigma(x)&=\int_{-\infty}^{x}|u(y)|dy<\infty,\\ \sigma_1(x)&=\int_{-\infty}^{x}|yu(y)|dy<\infty.\end{aligned} \tag{4·7}$$

We see also that Eq. (4·2) holds for $\tau\geqq 0$. Finally we have the relation

$$K(x,x)=\frac{1}{2}\int_{-\infty}^{x}u(y)dy. \tag{4·8}$$

Proof. First we consider the integral equation (3·6), which is equivalent to Eq. (3·1) with the boundary condition (3·4). Substitution of the expression (4·2) for $\varphi(x, s)$ into Eq. (3·6) yields

$$\int_{-\infty}^{x}K(x,y)e^{-isy}dy=\int_{-\infty}^{x}\frac{\sin s(x-t)}{s}u(t)e^{-ist}dt$$
$$+\int_{-\infty}^{x}u(t)dt\int_{-\infty}^{t}\frac{\sin s(x-t)}{s}K(t,v)e^{-isv}dv=J_1+J_2. \tag{4·9}$$

We now insert the relations

$$\frac{\sin s(x-t)}{s}e^{-ist}=\frac{1}{2}\int_{2t-x}^{x}e^{-isy}dy,$$
$$\frac{\sin s(x-t)}{s}e^{-isv}=\frac{1}{2}\int_{v+t-x}^{v+x-t}e^{-isy}dy,$$

into the integrals on the right-hand side of Eq. (4·9) and after the interchange of the order of integration we obtain

$$J_1=\frac{1}{2}\int_{-\infty}^{x}\left\{e^{-isy}\int_{-\infty}^{(x+y)/2}u(t)dt\right\}dy,$$

$$J_2=\frac{1}{2}\int_{-\infty}^{x} e^{-isy}\left\{\int_{(x+y)/2}^{x} u(t)dt\int_{y+t-x}^{y-t+x} K(t, v)dv\right.$$
$$\left.+\int_{-\infty}^{(x+y)/2} u(t)dt\int_{y+t-x}^{t} K(t, v)dv\right\}dy.$$

The change of the order of integration is justified by the estimates (4·6) and (4·7), the former being verified below, but we omit to describe its detail. Then by taking the Fourier transforms on the both sides of Eq. (4·9) we obtain the integral equation (4·5). To solve Eq. (4·5) we apply the method of successive approximations, that is, we start from the first approximation

$$K^{(0)}(x, y)=\frac{1}{2}\int_{-\infty}^{(x+y)/2} u(t)dt,$$

and employing the recursion formula

$$K^{(m)}(x, y)=\frac{1}{2}\int_{(x+y)/2}^{x} u(t)dt\int_{y+t-x}^{y-t+x} K^{(m-1)}(t, v)dv$$
$$+\frac{1}{2}\int_{-\infty}^{(x+y)/2} u(t)dt\int_{y+t-x}^{t} K^{(m-1)}(t, v)dv,$$

we have the solution

$$K(x, y)=\sum_{m=0}^{\infty} K^{(m)}(x, y).$$

By the mathematical induction we have the estimates

$$|K^{(m)}(x, y)|\leqq\frac{1}{2}\sigma\left(\frac{x+y}{2}\right)\frac{\sigma_1^m(x)}{m!},$$

and (4·6) and these confirm the convergence of the above series and we also have the solution of Eq. (4·2). Equation (4·8) follows from Eq. (4·5).

We can now derive the Gelfand-Levitan equation from the relation (3·21) rewritten in the form:

$$\frac{1}{a(k)}f(x, k)=\frac{b(k)}{a(k)}g(x, k)+g(x, -k). \tag{4·10}$$

The Fourier transform of Eq. (4·10) yields, putting $r=b/a$,

$$\int_{-\infty}^{\infty}\frac{1}{a(k)}f(x, k)e^{-iky}dk$$
$$=\int_{-\infty}^{\infty} r(k)g(x, k)e^{-iky}dk+\int_{-\infty}^{\infty} g(x, -k)e^{-iky}dk. \tag{4·11}$$

In the integral on the left-hand side we can close the integration contour into the upper half of s plane, but under the restriction $x>y$, since this is permited by the estimates (3·9) and (3·11). We then obtain by Eq. (3·37) of Theorem 3.1, Lemma 3.1 and Theorem 4.1

$$\int_{-\infty}^{\infty}\frac{1}{a(k)}f(x,k)e^{-iky}dk=2\pi i\sum_{n=1}^{m}\frac{f(x,i\kappa_n)}{\dot{a}(i\kappa_n)}e^{\kappa_n y}$$
$$=-2\pi\sum_{n=1}^{m}C_n e^{\kappa_n y}g(x,i\kappa_n)$$
$$=-2\pi\sum_{n=1}^{m}C_n\left\{e^{\kappa_n(x+y)}+\int_{-\infty}^{x}K(x,v)e^{\kappa_n(y+v)}dv\right\}$$
$$\text{for}\quad x>y.$$

On the other hand the right-hand side is estimated again by Theorem 4.1:

$$\int_{-\infty}^{\infty}r(k)g(x,k)e^{-iky}dk+\int_{-\infty}^{\infty}g(x,-k)e^{-iky}dk$$
$$=\int_{-\infty}^{\infty}r(k)e^{-ik(x+y)}dk+\int_{-\infty}^{\infty}r(k)dk\int_{-\infty}^{x}K(x,v)e^{-ik(y+v)}dv$$
$$+\int_{-\infty}^{\infty}e^{ik(x-y)}dk+\int_{-\infty}^{\infty}dk\int_{-\infty}^{x}K(x,v)e^{-ik(y-v)}dv$$
$$=2\pi F_c(x+y)+2\pi\int_{-\infty}^{x}K(x,v)F_c(x+v)dv+2\pi K(x,y)\qquad\text{for}\quad x>y.$$

Hence we have obtained the Gelfand-Levitan equation

$$K(x,y)=F(x+y)+\int_{-\infty}^{x}K(x,u)F(y+u)du=0\qquad\text{for}\quad x\geqslant y,\tag{4·12}$$

where the validity for $x=y$ is due to the continuity of F[26] and

$$F(z)=F_c(z)+\sum_{n=1}^{m}C_n e^{\kappa_n z},\tag{4·13}$$

$$F_c(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}r(k)e^{-ikz}dk.\tag{4·14}$$

The conditions such that Eq. (4·12) determines by Eq. (4·8) the potential $u(x)$ satisfying the condition (3·2) were studied by Faddeyev[28),29)] and the following conditions on the kernel F_c are derived:

$$r(k)=O(1/|k|)\quad\text{as}\quad |k|\longrightarrow\infty,\qquad |r(k)|\leqq 1\tag{4·15}$$

and

$$\int_{-\infty}^{\infty}(1+|x|)|B'(x)|dx<\infty,\tag{4·16}$$

where

$$B(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}b(k)e^{-ikx}dk.$$

We mention that the method by Gelfand and Levitan is applied also to other examples given in §2.

(2) *Zakharov-Shabat's method*

To solve the inverse problem in the nonlinear Schrödinger equation given in §2-(1) Zakharov and Shabat developed a somewhat different method[4] which is applicable also to the other examples. In this method the connection, which exists between $W(k)$ and W_d through the analytic continuation as described in the beginning of this section, is fully employed.

Consider two sets of eigenfunctions of Eq. (3·1), $f(x, k)$ and $g(x, k)$ corresponding to the boundary conditions (3·3) and (3·4), respectively. These are related in terms of $a(k)$ and $b(k)$ in the following way:

$$f(x, k)=b(k)g(x, k)+a(k)g(x, -k), \tag{3·21}$$

where, for simplicity, we omit to express parameter t in f, g, a, b explicitly, as in the preceding subsection. We introduce a function of s for each x:

$$\Phi(x, s)=\begin{cases} (f(x, s)/a(s))e^{-isx}, & \tau>0, \quad (4·17) \\ \overline{g(x, \bar{s})}e^{-isx}, & \tau<0, \quad (4·18) \end{cases}$$

which is, by Lemma 3.1, analytic with respect to s in the whole plane except on the real axis and at poles in the upper half plane and has the following properties:

i) $\Phi(x, s)\to 1$ as $|s|\to\infty$; this is seen by Eqs. (3·9), (3·10) and (3·27).

ii) $\Phi(x, s)$ has only simple poles with residue $f_n(x)e^{-is_n x}/a(s_n)$ at $s=s_n=i\kappa_n(\kappa_n>0, n=1, \cdots, m)$.

iii) $\phi(x, k)\equiv\Phi(k+i0)-\Phi(k-i0)=r(k)g(x, k)e^{-ikx}, \quad (4·19)$

where

$$r(k)=b(k)/a(k).$$

This implies that the function $(f/a)e^{-isx}$, which is an analytic function of s, except at poles, in the upper half plane, and the function $\overline{g(x, \bar{s})}e^{-isx}$, which is analytic in the whole lower half plane, are combined into a function $\Phi(x, s)$ with residues given above at simple poles $s=s_n(n=1, \cdots, m)$ and the discontinuity $\phi(x, k)$ on the real axis. Hence $\Phi(x, s)$ is expressible in the following form:

$$\begin{aligned}\Phi(x, s)&=1+\sum_{n=1}^{m}\frac{e^{-is_n x}}{s-s_n}\frac{f(x, s_n)}{\dot{a}(s_n)}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\phi(x, k)}{k-s}dk \\ &=1+\sum_{n=1}^{m}\frac{e^{-is_n x}}{s-s_n}iC_n g_n(x)+\frac{1}{2\pi i}\int_{-\infty}^{\infty}r(k)\frac{g(x, k)}{k-s}e^{-ikx}dk, \end{aligned} \tag{4·20}$$

where C_n, $g_n(x)$ was defined in Theorem 3.1.

We proceed to describe the inverse problem to determine the potential

$u(x)$ in Eq. (3·1) from the measure function C_n, κ_n, $r(k)$. From the definition (4·18), we have

$$\mathcal{G}(x, \bar{s})e^{-isx}=1+i\sum_{n=1}^{m}\frac{e^{\kappa_n x}}{s-i\kappa_n}C_n\mathcal{G}_n(x)+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\phi(x,k)}{k-s}dk \quad \text{for} \quad \tau<0, \tag{4·21}$$

where C_n is a real number and $g_n(x)$ is a real valued-function. Taking the complex conjugates of the both sides of Eq. (4·21) and replacing $\bar{s}$ by s we have

$$\mathcal{G}(x, s)e^{isx}=1-i\sum_{n=1}^{m}\frac{e^{\kappa_n x}}{s+i\kappa_n}C_n\mathcal{G}_n(x)-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\bar{\phi}(x,k)}{k-s}dk, \quad \tau>0, \tag{4·22}$$

where $\mathcal{G}(x, s)$ is analytic with respect to s in the upper half plane. In order to solve the integral equation (4·22) for $\mathcal{G}(x, s)$ for $\tau>0$, we put $s=s_p(p=1, \cdots, m)$ and $s=s+io$, then we have the following $(m+1)$ equations:

$$\mathcal{G}_p(x)e^{-\kappa_p x} =1-\sum_{n=1}^{m}\frac{e^{\kappa_n x}}{\kappa_p+\kappa_n}C_n\mathcal{G}_n(x)-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\bar{r}(k')\frac{\bar{\mathcal{G}}(x,k')e^{ik'x}}{k'-i\kappa_p}dk' \quad (p=1, \cdots, m)$$

and

$$\mathcal{G}(x, k)e^{ikx} =1-i\sum_{n=1}^{m}\frac{e^{\kappa_n x}}{k+i\kappa_n}C_n\mathcal{G}_n(x)-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\bar{r}(k')\frac{\bar{\mathcal{G}}(x,k')e^{ik'x}}{k'-k-io}dk'$$

or by Eq. (3·23)

$$\mathcal{G}_p(x)e^{-\kappa_p x} =1-\sum_{n=1}^{m}\frac{e^{\kappa_n x}}{\kappa_p+\kappa_n}C_n\mathcal{G}_n(x)+\frac{1}{2\pi i}\int_{-\infty}^{\infty}r(k')\frac{\mathcal{G}(x,k')e^{-ik'x}}{k'+i\kappa_p}dk' \quad (p=1, \cdots, m) \tag{4·23}$$

and

$$\mathcal{G}(x, k)e^{ikx} =1-i\sum_{n=1}^{m}\frac{e^{\kappa_n x}}{k+i\kappa_n}C_n\mathcal{G}_n(x)+\frac{1}{2\pi i}\int_{-\infty}^{\infty}r(k')\frac{\mathcal{G}(x,k')e^{-ik'x}}{k+k'+io}dk'. \tag{4·24}$$

This is the coupled integral equations to determine the set $\mathcal{G}_n(x)$ $(n=1, \cdots, m)$ and $\mathcal{G}(x, k)$, and is considered to be equivalent to the original equation (3·1). The boundary condition (3·4) follows by applying the Riemann-Lebesgue lemma in the Fourier analysis to the integrals of Eqs. (4·23) and (4·24). Equations (4·23) and (4·24) contain κ_n, C_n and r in place of the potential $u(x)$ in Eq. (3·1). If we can express the solution $\mathcal{G}(x, k)$ and $\mathcal{G}_n(x)$ of Eqs. (4·23) and (4·24) in terms of κ_n, C_n and r and on the other hand, the solution of Eq. (3·1) in terms of $u(x)$, then by comparing these two sets of solutions we can obtain the relation between $\{\kappa_n, C_n, r\}$ and u. We present the solutions of the system (4·23), (4·24) and of Eq. (3·1) by means of the asymptotic expansion with respect to s. First, using the expression (4·22) we obtain

$$\begin{aligned} g(x,s)e^{isx} = 1 + \frac{1}{s}\left\{-i\sum_{n=1}^{m} e^{\kappa_n x} C_n g_n(x) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} r(k) g(x,k) e^{-ikx} dk\right\} \\ + O(1/|s|^2). \end{aligned} \tag{4·25}$$

On the other hand we have for the solution of Eq. (3·1) the following asymptotic form:

$$\begin{aligned} g(x,s) &= g^{(0)}(x,s) + (1/s) g^{(1)}(x,s) + O(1/|s|^2) \\ &= e^{-isx} + (i/2s)\int_{-\infty}^{x} u(y) dy + O(1/|s|^2). \end{aligned} \tag{4·26}$$

From the expressions (4·25) and (4·26) we obtain finally

$$u(x) = -2\frac{d}{dx}\left\{\sum_{n=1}^{m} e^{\kappa_n x} C_n g_n(x) + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k) g(x,k) e^{-ikx} dx\right\}. \tag{4·27}$$

Since g_n and g are obtainable from Eqs. (4·23) and (4·24), the relation (4·27) determines the potential $u(x)$.

§ 5. Evolution of the measure function

By means of Theorem 2.1 or 2.2, the evolution of the system (2·7) is transformed to the evolution of the set of eigenfunctions $\psi(E)$ of the operator $L[u]$:

$$\psi(E) = \psi(0,E) \longmapsto T(t)\psi(0,E) = \psi(t,E). \tag{5·1}$$

This is expressible also in the differential form:

$$\psi_t(t,E) = A(t)\psi(t,E) \tag{5·2}$$

or

$$T_t = AT. \tag{5·3}$$

Here, $A(t) = A[u(t)]$ depends on $u(t)$ and hence $T(t)$ depends on $u(t')$, $0 \leqq t' \leqq t$, i.e., on the whole history of $u(t')$ during this time interval, which is the very solution of Eq. (2·7) to be determined in our study.

Gardner, Greene, Kruskal and Miura,[1] however, found an important fact that the time variation of the measure function is determined by the generator $A[u]$ but without employing the whole function form of $u(t)$ explicitly. In this section we shall examine this problem in detail and present the time dependence of the measure function for the example of Schrödinger operator studied in §§3 and 4. Throughout this section we assume that the potential $u(x, t)$ satisfies the condition (3·2) at each t during the time interval under our consideration. We define the coefficients $a(k)$ and $b(k)$ by Eq. (3·25) in terms of the solutions of Eq. (3·1) with the boundary conditions (3·3) and (3·4) at certain time, say $t=0$. Then for arbitrary time $t\geqq 0$ the both sides of Eq. (3·21) evolve due to $T(t)$ and Eq. (3·21) is deformed to the following relation:

$$\begin{aligned} Tf(x, 0, k)=f(x, t, k)&=b(k)Tg(x, 0, k)+a(k)Tg(x, 0, -k) \\ &=b(k)g(x, t, k)+a(k)g(x, t, -k), \end{aligned} \tag{5·4}$$

where $f(x, t, k)$ and $g(x, t, k)$ obey the following equations:

$$f_t=A(t)f, \tag{5·5}$$

$$g_t=A(t)g. \tag{5·6}$$

Since the eigenvalue k^2 is invariant with respect to the evolution due to $T(t)$, $f(x, t, k)$ and $g(x, t, k)$ are still solutions of Eq. (3·1), i.e., eigenfunctions of $L[u(t)]$ but do not necessarily satisfy the boundary conditions (3·3) and (3·4), respectively. In order to examine the time variation of f and g at the boundary $|x|\to\infty$ we use the fact that under the condition

$$u(x, t) \longrightarrow 0 \quad \text{as} \quad |x| \longrightarrow \infty, \tag{5·7}$$

the following relation holds:

$$A(t)\varphi(x)=A[u(x, t)]\varphi(x) \longrightarrow -4D^3\varphi(x), \tag{5·8}$$
$$\text{at each point } x \text{ with } |x|\geqq X \text{ as } X\to\infty,$$

where $\varphi(x)$ is an arbitrary function such that $D^3\varphi(x)$ exists. Now let us consider, at each t, the eigenfunctions $f(x, t, k)$ and $g(x, t, k)$ of $L(t)=L[u(t)]$, which satisfy the following boundary conditions:

$$f(x, t, k) \longrightarrow \exp[4ik^3t+ikx] \quad \text{as} \quad x\to\infty, \tag{5·9}$$

$$g(x, t, k) \longrightarrow \exp[-4ik^3t-ikx] \quad \text{as} \quad x\to-\infty. \tag{5·10}$$

We note that the conditions (5·9) and (5·10) differ from the conditions (3·3) and (3·4) by the time dependent factors $\exp[4ik^3t]$ for $x\to\infty$ and $\exp[-4ik^3t]$ for $x\to-\infty$, respectively. We see easily that the function $h(x, t)=\exp[4ik^3t+ikx]$, $(-\infty<x<\infty)$, satisfies the equation $h_t=-4D^3h$, $-\infty<x<\infty$, and by (5·8), $\exp[4ik^3t]$ gives the correct time variation of the solution of Eq. (5·5) if it has the x dependence of the form e^{ikx} as $x\to\infty$. In other words, the solution f of Eq. (5·5), which satisfies the boundary condition (3·3) at $t=0$, satisfies the time dependent boundary condition (5·9) for each $t\geqq 0$. The analogous reasoning shows that the solution g of Eq. (5·6), which satisfies the boundary condition (3·4) at $t=0$, satisfies the boundary condition (5·10) for $t\geqq 0$. We define $a(k, t)$ and $b(k, t)$ by

$$\hat{f}(x, t, k)=b(k, t)\hat{g}(x, t, k)+a(k, t)\hat{g}(x, t, -k), \tag{5·11}$$

in terms of the eigenfunctions of $L[u]$, $\hat{f}$ and $\hat{g}$, which satisfy the time independent boundary conditions:

$$\lim_{x\to\infty}\hat{f}(x, t, k)e^{-ikx}=1, \tag{5·12}$$

$$\lim_{x\to-\infty}\hat{g}(x, t, k)e^{ikx}=1. \tag{5·13}$$

We note that $\hat{f}$ and $\hat{g}$ do not satisfy the equations of the type (5·5) and (5·6). It is easily seen that $\hat{f}$ and $f(\hat{g}$ and $g)$ differ only by a factor $\exp[\pm 4ik^3t]$ and that Eqs. (5·11), (5·12) and (5·13) are satisfied by taking as follows:

$$a(k, t)=a(k), \tag{5·14}$$

$$b(k, t)=b(k)\exp[-8ik^3t], \tag{5·15}$$

$$\hat{f}(x, t, k)=f(x, t, k)\exp[-4ik^3t], \tag{5·16}$$

$$\hat{g}(x, t, k)=g(x, t, k)\exp[4ik^3t]. \tag{5·17}$$

From Eqs. (5·14) and (5·15) we obtain the formula

$$r(k, t)\equiv\frac{b(k, t)}{a(k, t)}=\frac{b(k)}{a(k)}\exp[-8ik^3t]=r(k)\exp[-8ik^3t], \tag{5·18}$$

which determines the time development of $W(k)$ given by Eq. (3·35).

Finally we study the time variation of the measure function for the discrete spectrum defined in §3 by

$$C_n=-i(f_n(x)/g_n(x))(\dot{a}(i\kappa_n))^{-1}, \quad n=1, \cdots, m, \tag{3·37}$$

where the factor f_n/g_n is independent of x. We insert here a remark that this factor cannot always be given as $b(s_n)$, from the formal relation

$$f(x, s_n)=b(s_n)g(x, s_n)+a(s_n)g(x, -s_n), \tag{5·19}$$

by simply putting $a(s_n)=0$. If this is allowed, then we have

$$b(s_n, t)=b(s_n, 0)\exp[-8\kappa_n^3t],$$

from Eq. (5·15) but we see that Eq. (5·19) is only a formal analytic continuation of Eq. (3·21) and $b(s)$ and $g(x, -s)$ are not yet defined as analytic continuations to the upper half plane $(\tau>0)$ of $b(k)$ and $g(x, k)$, respectively. The analytic continuation is shown to be possible if we impose further conditions on the potential $u(x, t)$, for instance:

$$u(x,t)=O(\exp[-|x|^{1+\varepsilon}]) \quad \text{as} \quad |x|\to\infty. \ (\varepsilon>0) \tag{5·20}$$

We examine, however, the problem by the analogous method as used to derive Eqs. (5·14)~(5·17).

Since $f_n(x)$ and $g_n(x)$ are the solutions of Eq. (3·1) for $s^2=-\kappa_n^2$ with the boundary conditions given at $t=0$:

$$f_n(x) \longrightarrow \exp[-\kappa_n x] \quad \text{as} \quad x\to\infty \tag{5·21}$$

and

$$g_n(x) \longrightarrow \exp[\kappa_n x] \quad \text{as} \quad x\to-\infty, \tag{5·22}$$

we have by the analogous argument as in the case $s=k$,

$$T(t)f_n(x)=f_n(x,t) \longrightarrow \exp[4\kappa_n^3 t-\kappa_n x] \quad \text{as} \quad x\to\infty, \tag{5·23}$$

$$T(t)g_n(x)=g_n(x,t) \longrightarrow \exp[-4\kappa_n^3 t+\kappa_n x] \quad \text{as} \quad x\to-\infty. \tag{5·24}$$

Also, $f_n(x,t)$ and $g_n(x,t)$ are solutions of Eq. (3·1) for $s^2=-\kappa_n^2$ and since each eigenvalue is simple by Theorem 3.1, there holds the relation

$$f_n(x,t)=\alpha(t)g_n(x,t), \tag{5·25}$$

where $\alpha(t)$ is independent of x. Further, by differentiating the both sides of Eq. (5·25)

$$f_{nt}=\alpha_t g_n+\alpha g_{nt}$$

or

$$A(t)f_n=\alpha_t g_n+\alpha A(t)g_n=\alpha_t g_n+A(t)f_n,$$

we have

$$\alpha(t)=\alpha. \qquad \text{(time independent)} \tag{5·26}$$

For the definition of C_n, we should use $\hat{f}_n(x,t)$ and $\hat{g}_n(x,t)$, which are the solutions of Eq. (3·1) for $s^2=-\kappa_n^2$ with the asymptotic forms same as the relations (5·21) or (5·22):

$$\hat{f}_n(x,t) \longrightarrow \exp[-\kappa_n x] \quad \text{as} \quad x\to\infty \tag{5·27}$$

and

$$\hat{g}_n(x,t) \longrightarrow \exp[\kappa_n x] \quad \text{as} \quad x\to-\infty, \tag{5·28}$$

where $\hat{f}_n$ and $\hat{g}_n$ are obtainable by the analytic continuations of $\hat{f}(x,t,k)$ and $\hat{g}(x,t,k)$ given before for $s=k$ under the boundary conditions (5·12) and (5·13), respectively. By comparing (5·23) and (5·24) with (5·27) and (5·28), we have the relations:

$$\hat{f}_n(x,t)=\exp[-4\kappa_n^3 t]f_n(x,t) \tag{5·29}$$

and

$$\hat{g}_n(x,t)=\exp[4\kappa_n^3 t]g_n(x,t). \tag{5·30}$$

Then the definition (3·37) applied for f_n and g_n at time t in place of f_n and g_n, respectively, yields

$$\begin{aligned}C_n(t)&=-i(\hat{f}_n(x,t)/\hat{g}_n(x,t))(\dot{a}(i\kappa_n,t))^{-1}\\&=-i(f_n(x,t)/g_n(x,t))\exp[-8\kappa_n^3 t](\dot{a}(i\kappa_n))^{-1}\\&=-i(f_n(x)/g_n(x))\exp[-8\kappa_n^3 t](\dot{a}(i\kappa_n))^{-1}\\&=C(0)\exp[-8\kappa_n^3 t],\qquad n=1,\cdots,m.\end{aligned} \tag{5·31}$$

Here we used the relations (5·25), (5·26) and the fact that $a(i\kappa_n,t)$ is determined from $a(s,t)$ and does not depend on time, since $a(s,t)$ is independent of time as the analytic continuation of $a(k)$ given by Eq. (5·14).

We finished the study of the time variation of the measure function and finally remark that it is given in the very simple form, which contains the time variation, due to $T(t)$, of the eigenfunctions of $L(t)$ at $x\to\pm\infty$ but does not contain their full time variation and that this fact enable us to apply Lax's method to the initial value problem of the KdV equation. The application to the other nonlinear equations of evolution might be expected and, in fact, it has been applied to examples given in §2-(1) with favourable results.

§ 6. Examples and remarks

(1) *KdV equation*

Since we established the method for solving the initial value problem of the KdV equation by reducing it to the linear integral equations discussed in §§4 and 5, we now consider the following simple example of the initial value for Eq. (2·1):

$$\begin{aligned}u(x,0)=u_0(x)&=-v,\ (v>0)\quad \text{for}\quad 0\leqq x\leqq l,\\&=0\qquad\qquad\qquad \text{otherwise}.\end{aligned} \tag{6·1}$$

Then the Schrödinger equation (3·1) is to be solved for the squarewell potential (6·1). The solution f or g for the boundary condition (3·3) or (3·4), respectively, is easily derived. For instance f of Eq. (3·21) takes the following form:

$$\begin{aligned}f(x,k)&=e^{ikx}&&\text{for}\quad x>l,\\&=c(k)e^{ihx}+d(k)e^{-ihx}&&\text{for}\quad 0\leqq x\leqq l,\\&=b(k)e^{-ikx}+a(k)e^{ikx}&&\text{for}\quad x<0\end{aligned} \tag{6·2}$$

with

$$h=\sqrt{k^2+v}\,,$$

where $a(k)$ and $b(k)$ give the Wronskians in Eqs. (3·25), from which we determine $\mathcal{G}$ by Eq. (3·22). The reflection coefficient or the measure function given for the eigenfunctions $\mathcal{G}(x, k)$ in Theorem 3.1 is calculated:

$$r(k)=b(k)/a(k)=v\frac{1-e^{2ihl}}{(h+k)^2-(h-k)^2e^{2ihl}} \quad \text{for} \quad k>0,$$

$$r(-k)=\bar{r}(k). \tag{6·3}$$

The analyticity of $r(s)$ for $\tau>0$ is expected, since $u_0(x)$ of Eq. (6·1) satisfies the condition (5·20). In fact, the expression (6·3) shows that $r(k)$ is analytically continuable into the upper half of s plane except at a finite number of poles $s=s_n=i\kappa_n$ $(n=1, \cdots, m)$, which are connected to the roots $y=y_{2p-1}$ and y_{2p} of the following equation:

$$2y/(\sqrt{v}\ l)=\begin{cases} |\cos y|, & (p-1)\pi<y<(p-1/2)\pi, \\ |\sin y|, & (p-1/2)\pi<y<p\pi, \end{cases}$$

$$p=1, 2, \cdots. \tag{6·4}$$

We have

$$\kappa_n=\frac{2}{l}\sqrt{vl^2/4-y_n^2}\,,$$

$$C_n=-i\frac{b(i\kappa_n)}{\dot{a}(i\kappa_n)}=\frac{\kappa_n\sqrt{v-\kappa_n^2}}{v(1+\kappa_n l/2)}\,,$$

$$n=1, 2, \cdots, m, \tag{6·5}$$

where m is an integer such that

$$\sqrt{v}\ l/\pi\leqq m<\sqrt{v}\ l/\pi+1.$$

The kernel (4·13) of the Gelfand-Levitan equation (4·12) is given for $t>0$ by Eqs. (5·18) and (5·31):

$$F(z)=\sum_{n=1}^{m} C_n(t)e^{\kappa_n z}+\frac{1}{2\pi}\int_{-\infty}^{\infty} r(k, t)e^{-ikz}dk \tag{6·6}$$

with

$$C_n(t)=C_n \exp[-8\kappa_n^3 t],$$

$$r(k, t)=r(k)\exp[-8ik^3 t].$$

It is shown that the kernel given by Eqs. (6·6) satisfies the conditions (4·15) and (4·16). The solution for the initial value $u_0(x)$ in the region $x\to\infty$, $t\to\infty$ is expressible as the superposition of m solitons:

$$u(x,t)\cong\sum_{n=1}^{m} u_n(x, t),$$

$$u_n(x, t)=a_n \operatorname{sech}^2[\kappa_n\{(x-x_n)-\lambda_n t)\}]. \tag{6·7}$$

Here $u_n(x, t)$ represents a single soliton solution of Eq. (2·1) with

$$a_n = -2\kappa_n^2 \ (<0),$$
$$\lambda_n = 4\kappa_n^2 \ (>0),$$
$$x_n = -1/(2\kappa_n)\cdot\log\left(\frac{C_n}{2\kappa_n}\right) = 1/(2\kappa_n)\log\frac{2v(1+\kappa_n l/2)}{\sqrt{v-\kappa_n^2}},$$

and this propagates toward the right with velocity proportional to its amplitude. For large $\sqrt{v}\,l/\pi \cong m$ and small n/m we have

$$|a_n| \cong 2v(1-n^2/m^2), \qquad n=1, 2, \cdots, \tag{6·8}$$

indicating that the amplitudes and the velocities of the solitons for $n \ll m$, depend linearly on the potential depth v but not on its width l. On the other hand the "magnitude" of the potential, $\sqrt{v}\,l/\pi$ determines the total number m of solitons.

(2) *Modified KdV equation and the sine-Gordon equation*

Since L is now not symmetric, we should study the non-selfadjoint perturbation to the symmetric operator L_0 in order to solve the spectral problem of L. The theory is not so much developed as for the selfadjoint case. There is, however, a reason to expect that the perturbation is applicable giving some informations on the relation between the spectra of L and L_0 and that the analogous discussion as in §§3∼5 is possible, since L_0 are differential operators in both examples and the perturbing part M may be relatively small under appropriate restriction on u as in the case of the KdV equation.

The Gelfand-Levitan equation for the modified KdV equation was given by Wadati[6] and its solution is connected to the solution of the modified KdV equation by a simple formula. Also the time dependence of the kernel of the Gelfand-Levitan equation is derived.[6] The analogous investigation was made for the sine-Gordon equation by Ablowitz, Kaup, Newell and Segur.[7] Owing to these works the initial value problem for both equations is now becoming manageable in a similar way as in the case of the KdV equation. The result of their analyses shows that L of each equation can have complex discrete eigenvalues giving solitons, which have more complicated structures than those in the KdV equation.

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Note added in proof:

V. E. Zakharov and S. V. Manakov investigated the interaction of three waves by the method of inverse scattering. Their system of equations contains, as special cases, the example v) of §2-(1) in the present article. See ZETF Pis. Redak. **18**, No. 7 (5 Octobar 1973), 413. The work was informed to the author after the manuscript was received by the editor.