Progress of Theoretical Physics, Vol. 69, No. 1, January 1983

A Jacobi Equation on the Coset Manifold $SO(2N)/U(N)$ and the Quasi-Particle RPA Equation

Seiya NISHIYAMA

Department of Physics, Kochi University, Kochi 780

(Received March 19, 1982)

We study the variational properties of the action functional appeared in the time-dependent Hartree-Bogoliubov (TDHB) theory. Through the calculus of the second variation on the coset manifold $SO(2N)/U(N)$, the Jacobi equation is obtained. Assuming the periodic Jacobi field, we derive in a natural way the equation for the quasi-particle random phase approximation (RPA) describing the collective excitation around certain static HB fields. The present method of obtaining the $SO(2N)$ RPA may be useful to get the $SO(2N+1)$ RPA.

§ 1. Introduction

For the purpose of constructing a theory suitable for the description of collective motions with large amplitudes in soft nuclei, in the previous paper[1] (referred to as I), we have proposed a quantized time-dependent Hartree-Bogoliubov (TDHB) theory of many fermion systems with pair correlations. In order to take account of the pair correlations, we were forced to enlarge the symmetry of the Lie group from the $U(N)$ (the unitary group of N-dimension) to the $SO(2N)$ (the special orthogonal group of $2N$-dimension). Here N denotes the number of the single particle state of fermions. Our theory was obtained not by using *a priori* quantized method[2),3)] but by using the path integrals on the coset space $SO(2N)/U(N)$.[4),5)] Kleinert proposed some years ago a functional integral approach to a theory of collective excitations in the systems given above. He introduced, in a different way from our method, external sources represented in terms of the Grassmann variables adopting a simple schematic model.[6)]

As is well known, the path integral formalism provides the natural connection between the classical problem and its quantized version. It is particularly useful for the semiclassical treatment of quantum systems. Recently Kuratsuji and Mizobuchi investigated the semiclassical analysis of a spin system taking account of effects arising from the second variation of the action functional around the classical path. They obtained a closed form of the semiclassical propagator and derived a semiclassical quantization condition.[7)] On the other hand, Dewitt-Morette et al. pointed out that there is a remarkable analogy between the role played by classical paths in the study of quantum systems and the role played by equilibrium points in the study of classical dynamical systems.[8] In our case, the classical path $G(t)$ with the end points (G_0, t_0) and (G_1, t_1) can be considered as an "equilibrium point" in the limit $\hbar=0$. $G(t)$ satisfies the variational principle $\delta S[G(t)]=0$ and the qualitative features of the quantum system whose classical limit is $G(t)$ are determined by the second variation of the action functional $\delta^2 S[G(t)]$. Then it becomes necessary to study the small deviations from "equilibrium point" or the Jacobi equation in the corresponding classical system.

The conventional approach to collective excitations starts with the random phase approximation (RPA). The quasi-particle $SO(2N)$ RPA treats the collective states of many fermion systems with pair correlations. Its derivation is usually made by e.g. the well-known linearization method. Though quitely self-evident in a sense, we will stress it can be also given through the above-mentioned Jacobi equation. The $SO(2N)$ RPA, however, is applicable only to even fermion systems. We know no extension of the $SO(2N)$ RPA to the $SO(2N+1)$ one for odd systems to include both paired and unpaired modes. A way of solving an unknown problem of constructing the $SO(2N+1)$ RPA may be found since embedding the $SO(2N+1)$ group into the $SO(2N+2)$ one[4),5)] we can obtain the corresponding Jacobi equation. Then it becomes very meaningful to reproduce the $SO(2N)$ RPA from the Jacobi equation on the coset manifold $SO(2N)/U(N)$ in this paper.

In the present paper, we first aim at studying the variational properties of the action functional S and obtaining the Jacobi equation through the calculus of the second variation on the coset manifold $SO(2N)/U(N)$. Our second purpose is to show how the following treatment is possible: Assuming the periodic Jacobi fields, we derive the equation for the quasi-particle RPA describing the collective excitation around certain static HB fields. In § 2 from the calculus of the second variation of S, we will give the *exact* Jacobi equation on the coset manifold $SO(2N)/U(N)$. In § 3 assuming the form of the Jacobi fields to be the simple periodic form, we will derive the quasi-particle RPA equation in a natural way. Finally we will add some remarks. In Appendices, we give the detailed calculus of variations needed for our discussion.

The notations used in this paper are the same as those in I.

§ 2. The second variation and the Jacobi equation on the coset manifold $SO(2N)/U(N)$

As was shown in I, the first variation of the action functional S leads to the following classical TDHB equation:

$$\left.\begin{aligned} \dot{q} &= -\frac{1}{2}\frac{i}{\hbar}(1-M^*)^{-1}\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}(1-M)^{-1}\,, \\ \dot{q}^* &= \frac{1}{2}\frac{i}{\hbar}(1-M)^{-1}\frac{\partial\langle H\rangle_{G,G}}{\partial q}(1-M^*)^{-1}\,, \end{aligned}\right\} \qquad (2\cdot 1)$$

in which the classical path satisfies the end point conditions $q(t_0)=q_0$, $q(t_1)=q_1$ together with their complex conjugate (the Dirichlet data).[9] The above equation is easily rewritten in the conventional type of the TDHB equation as seen in Appendix I.

Let us introduce the path variation from the classical path, $\xi(t)=(\xi_{\alpha\beta}(t))$, which is a function on $[t_1, t_0]$ with $\xi^T(t)=-\xi(t)$ and is vanishing on the boundary, $\xi(t_0)=\xi(t_1)=0$. The second variation of the action functional S is readily found as follows:

$$\left.\begin{aligned}
&\delta^2 S=\int_{t_0}^{t_1} 2\Omega dt\,,\\
&2\Omega\equiv(\xi,\xi^*,\dot{\xi},\dot{\xi}^*)\begin{pmatrix}
\dfrac{\partial^2 L}{\partial q\partial q} & \dfrac{\partial^2 L}{\partial q\partial q^*} & \dfrac{\partial^2 L}{\partial q\partial \dot{q}} & \dfrac{\partial^2 L}{\partial q\partial \dot{q}^*}\\
\dfrac{\partial^2 L}{\partial q^*\partial q} & \dfrac{\partial^2 L}{\partial q^*\partial q^*} & \dfrac{\partial^2 L}{\partial q^*\partial \dot{q}} & \dfrac{\partial^2 L}{\partial q^*\partial \dot{q}^*}\\
\dfrac{\partial^2 L}{\partial \dot{q}\partial q} & \dfrac{\partial^2 L}{\partial \dot{q}\partial q^*} & \dfrac{\partial^2 L}{\partial \dot{q}\partial \dot{q}} & \dfrac{\partial^2 L}{\partial \dot{q}\partial \dot{q}^*}\\
\dfrac{\partial^2 L}{\partial \dot{q}^*\partial q} & \dfrac{\partial^2 L}{\partial \dot{q}^*\partial q^*} & \dfrac{\partial^2 L}{\partial \dot{q}^*\partial \dot{q}} & \dfrac{\partial^2 L}{\partial \dot{q}^*\partial \dot{q}^*}
\end{pmatrix}\begin{pmatrix}\xi\\ \xi^*\\ \dot{\xi}\\ \dot{\xi}^*\end{pmatrix},
\end{aligned}\right\} \tag{2·2}$$

where $\partial^2 L/\partial q_{\alpha\beta}\partial q_{\gamma\delta}$, etc., are evaluated along the classical path. It is seen that the second variation (2·2) is itself a functional with respect to the path variations $\xi(t)$. The functional $\Omega=\Omega(\xi,\xi^*,\dot{\xi},\dot{\xi}^*)$ defined in the above is the so-called secondary Lagrangian.

A necessary condition for the classical path $q(t)$ to be a minimum of S is that the above second variation $\delta^2 S$ should be nonnegative for all admissible functions $\xi(t)$. As is clear from the structure of Eq. (2·2), $\delta^2 S$ is defined in the quadratic functional form of the functions $\xi(t)$. Then to assure the conditions $\delta^2 S\geqq 0$ with the Dirichlet data $\xi(t_0)=\xi(t_1)=0$ together with their complex conjugate, we must set up the following well-known conditions as necessary ones:[10]

(I) *Legendre's condition*

A matrix $P(t)$ is nonnegative*) on $[t_0, t_1]$ where the matrix $P(t)$ is defined as

*) If the matrix $P(t)$ defined by Eq. (2·3) on page 103 is positive definite, then the conditions I and II become necessary and sufficient conditions for $\delta^2 S>0$.[10]

$$P(t)=\begin{pmatrix} \dfrac{\partial^2 L}{\partial\dot q\partial\dot q} & \dfrac{\partial^2 L}{\partial\dot q\partial\dot q^*} \\ \dfrac{\partial^2 L}{\partial\dot q^*\partial\dot q} & \dfrac{\partial^2 L}{\partial\dot q^*\partial\dot q^*} \end{pmatrix}, \qquad P^T(t)=P(t). \tag{2·3}$$

(II) *Jacobi's condition*

The path variations $\xi(t)$ are subjected to the following Euler-Lagrange equation for the secondary Lagrangian Ω with the initial condition $\xi(t_0)=0$ together with their complex conjugate:

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial\Omega}{\partial\dot\xi}\right)-\frac{\partial\Omega}{\partial\xi}=0\,, \\ &\frac{d}{dt}\left(\frac{\partial\Omega}{\partial\dot\xi^*}\right)-\frac{\partial\Omega}{\partial\xi^*}=0\,, \end{aligned}\right\} \tag{2·4}$$

which is called the Jacobi equation.*) There exists no *conjugate point* **) to $q_0(=q(t_0))$ on $[t_0, t_1]$.

The explicit expression on the coset manifold $SO(2N)/U(N)$ of the above Jacobi equation is given in the following form:

$$\begin{aligned} &-\frac{\partial^2 L}{\partial\dot q^*_{\alpha\beta}\partial\dot q^*_{\gamma\delta}}\ddot\xi^*_{\gamma\delta}+\left\{\frac{\partial^2 L}{\partial q^*_{\alpha\beta}\partial\dot q^*_{\gamma\delta}}-\frac{\partial^2 L}{\partial\dot q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}-\frac{d}{dt}\frac{\partial^2 L}{\partial\dot q^*_{\alpha\beta}\partial\dot q^*_{\gamma\delta}}\right\}\dot\xi^*_{\gamma\delta} \\ &\qquad+\left\{\frac{\partial^2 L}{\partial q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}-\frac{d}{dt}\frac{\partial^2 L}{\partial\dot q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}\right\}\xi^*_{\gamma\delta} \\ &\qquad-\frac{\partial^2 L}{\partial\dot q^*_{\alpha\beta}\partial\dot q_{\gamma\delta}}\ddot\xi_{\gamma\delta}+\left\{\frac{\partial^2 L}{\partial q^*_{\alpha\beta}\partial\dot q_{\gamma\delta}}-\frac{\partial^2 L}{\partial\dot q^*_{\alpha\beta}\partial q_{\gamma\delta}}-\frac{d}{dt}\frac{\partial^2 L}{\partial\dot q^*_{\alpha\beta}\partial\dot q_{\gamma\delta}}\right\}\dot\xi_{\gamma\delta} \\ &\qquad+\left\{\frac{\partial^2 L}{\partial q^*_{\alpha\beta}\partial q_{\gamma\delta}}-\frac{d}{dt}\frac{\partial^2 L}{\partial\dot q^*_{\alpha\beta}\partial q_{\gamma\delta}}\right\}\xi_{\gamma\delta} \\ &=0\,. \end{aligned} \tag{2·5}$$

Since our "Lagrangian" L is linear in $\dot q$ and $\dot q^*$, then we have

$$\frac{\partial^2 L}{\partial\dot q^*\partial\dot q^*}=\frac{\partial^2 L}{\partial\dot q^*\partial\dot q}=0\,, \tag{2·6}$$

and their complex conjugate which satisfies the so-called *weak Legendre's condition.*

---

*) Note that, though we use the same symbols, the path variations $\xi(t)$ in Eq. (2·4) are not identical with the admissible functions $\xi(t)$ which must satisfy $\xi(t_0)=\xi(t_1)=0$. A nonzero solution $\xi(t)$ of the Jacobi equation does not always satisfy $\xi(t_1)=0$ and is normalized through $\dot\xi(t_0)=c$ (c being certain constant numbers).[10]

**) Two points q_0 and q_1 are said to be conjugate along a stationary path $q(t)$, if there exists a nonzero solution $\xi(t)$ of the Jacobi equation along $q(t)$ vanishing at t_0 and t_1.

With the aid of the property $\partial^2 L/\partial\dot{q}^*\partial q=\partial^2 L/\partial q\partial\dot{q}^*$, etc., the equations given in Appendix II and the antisymmetry of ξ, the Jacobi equation is rewritten in the following form:

$$\begin{aligned}&\left\{\frac{\partial^2 L}{\partial q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}-\left(q\frac{\partial\langle H\rangle_{G,G}}{\partial q}q-\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}\right)_{\alpha\gamma}K_{\delta\beta}\right.\\&\qquad\left.-K_{\alpha\gamma}\left(q\frac{\partial\langle H\rangle_{G,G}}{\partial q}q-\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}\right)_{\delta\beta}\right\}\xi^*_{\gamma\delta}\\&\qquad+4i\hbar\{(1-M^*)_{\alpha\gamma}(1-M)_{\delta\beta}\}\dot{\xi}_{\gamma\delta}\\&+\left\{\frac{\partial^2 L}{\partial q^*_{\alpha\beta}\partial q_{\gamma\delta}}-\left(q\frac{\partial\langle H\rangle_{G,G}}{\partial q}-\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}q^*\right)_{\alpha\gamma}(1-M)_{\delta\beta}\right.\\&\qquad\left.-(1-M^*)_{\alpha\gamma}\left(\frac{\partial\langle H\rangle_{G,G}}{\partial q}q-q^*\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}\right)_{\delta\beta}\right\}\xi_{\gamma\delta}=0\,. \qquad (2\cdot7)\end{aligned}$$

Note that in the above equation the components $\dot{\xi}^*_{\gamma\delta}$ automatically vanish.

Further substituting the explicit expressions of the second derivatives $\partial^2 L/\partial q^*\partial q^*$ and $\partial^2 L/\partial q^*\partial q$ calculated in Appendix III and using the antisymmetry of ξ, the above equation is transformed into the following form:

$$\begin{aligned}&\frac{1}{2}\left\{K_{\alpha\gamma}\frac{\partial\langle H\rangle_{G,G}}{\partial q^*_{\delta\beta}}+\frac{\partial\langle H\rangle_{G,G}}{\partial q^*_{\alpha\gamma}}K_{\delta\beta}-\frac{1}{2}\frac{\partial^2\langle H\rangle_{G,G}}{\partial q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}\right\}\xi^*_{\gamma\delta}\\&+i\hbar\{(1-M^*)_{\alpha\gamma}(1-M)_{\delta\beta}\}\dot{\xi}_{\gamma\delta}\\&+\frac{1}{2}\left\{(1-M^*)_{\alpha\gamma}\left(q^*\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}\right)_{\delta\beta}+\left(\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}q^*\right)_{\alpha\gamma}(1-M)_{\delta\beta}\right.\\&\left.-\frac{1}{2}\frac{\partial^2\langle H\rangle_{G,G}}{\partial q^*_{\alpha\beta}\partial q_{\gamma\delta}}\right\}\xi_{\gamma\delta}=0\,. \qquad (2\cdot8)\end{aligned}$$

Now multiplying $(1-M^*)^{-1}$ and $(1-M)^{-1}$ on the left and right hand sides, respectively and using the relations $q=(1-M^*)^{-1}K=K(1-M)^{-1}$, we obtain

$$\begin{aligned}&\frac{1}{2}\left[q_{\alpha\gamma}\left\{\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}(1-M)^{-1}\right\}_{\delta\beta}+\left\{(1-M^*)^{-1}\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}\right\}_{\alpha\gamma}q_{\delta\beta}\right.\\&\left.-\frac{1}{2}(1-M^*)^{-1}{}_{\alpha\alpha'}\frac{\partial^2\langle H\rangle_{G,G}}{\partial q^*_{\alpha'\beta'}\partial q^*_{\gamma\delta}}(1-M)^{-1}{}_{\beta'\beta}\right]\xi^*_{\gamma\delta}+i\hbar\dot{\xi}_{\alpha\beta}\\&+\frac{1}{2}\left[\delta_{\alpha\gamma}\left\{q^*\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}(1-M)^{-1}\right\}_{\delta\beta}+\left\{(1-M^*)^{-1}\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}q^*\right\}_{\alpha\gamma}\delta_{\delta\beta}\right.\\&\left.-\frac{1}{2}(1-M^*)^{-1}{}_{\alpha\alpha'}\frac{\partial^2\langle H\rangle_{G,G}}{\partial q^*_{\alpha'\beta'}\partial q_{\gamma\delta}}(1-M)^{-1}{}_{\beta'\beta}\right]\xi_{\gamma\delta}=0\,. \qquad (2\cdot9)\end{aligned}$$

Finally substituting the explicit forms of the Hessians into the above, we can get the following simple equation:

$$i\hbar\dot{\xi}_{\alpha\beta}-A_{\alpha\beta\gamma\delta}(t)\xi_{\gamma\delta}-B_{\alpha\beta\gamma\delta}(t)\xi^{*}_{\gamma\delta}=0\,, \tag{2·10a}$$

$$\begin{aligned}A_{\alpha\beta\gamma\delta}(t)\equiv&(F+qD^{*})_{\alpha\gamma}\delta_{\delta\beta}+\delta_{\alpha\gamma}(F^{*}+D^{*}q)_{\delta\beta}\\&-[\alpha\gamma'|\delta''\gamma'']q_{\gamma'\beta}(1-M^{*})_{\gamma''\gamma}K^{*}_{\delta\delta''}\\&-[\beta\gamma'|\delta''\gamma'']q_{\alpha\gamma'}(1-M^{*})_{\gamma''\gamma}K^{*}_{\delta\delta''}\\&+\frac{1}{2}[\gamma''\gamma'|\delta''\delta']q_{\alpha\gamma'}q_{\delta'\beta}K^{*}_{\gamma''\gamma}K^{*}_{\delta\delta''}\\&+\frac{1}{2}[\alpha\gamma'|\beta\delta'](1-M^{*})_{\gamma'\gamma}(1-M)_{\delta\delta'}\,,\end{aligned} \tag{2·10b}$$

$$\begin{aligned}B_{\alpha\beta\gamma\delta}(t)\equiv&-[\alpha\gamma'|\gamma''\delta'']q_{\gamma'\beta}(1-M)_{\gamma''\gamma}K_{\delta\delta''}\\&-[\beta\gamma'|\gamma''\delta'']q_{\alpha\gamma'}(1-M)_{\gamma''\gamma}K_{\delta\delta''}\\&+\frac{1}{2}[\gamma''\gamma'|\delta''\delta']q_{\alpha\gamma'}q_{\delta'\beta}(1-M)_{\gamma''\gamma}(1-M^{*})_{\delta\delta''}\\&+\frac{1}{2}[\alpha\gamma'|\beta\delta']K_{\gamma'\gamma}K_{\delta\delta'}\,,\end{aligned} \tag{2·10c}$$

where we have used the antisymmetry of ξ. Up to the present stage, all the expressions are *exact*. Then the *exact* solution of the above equation becomes the Jacobi fields on the coset manifold $SO(2N)/U(N)$. Kuratsuji and Mizobuchi constructed the explicit form of the Jacobi fields in the case of a spin system and evaluated the value of the reduced propagator by using them.[7] It will be an interesting and future problem to construct explicitly the Jacobi fields on the coset manifold $SO(2N)/U(N)$.

§ 3. Derivation of the quasi-particle RPA equation

In the preceding section, we have obtained the Jacobi equation through the calculus of the second variation on the coset manifold $SO(2N)/U(N)$. We are now at a position to derive the quasi-particle RPA equation from the Jacobi fields. For this aim, we assume the form of the Jacobi fields to be the following simple periodic form:

$$\left.\begin{aligned}&\xi_{\alpha\beta}(t)=\sum_{n}(\psi^{n}_{\alpha\beta}e^{-i\omega nt/\hbar}+\phi^{n*}_{\alpha\beta}e^{i\omega nt/\hbar}),\\&\psi^{n}_{\alpha\beta}=-\psi^{n}_{\beta\alpha}\,,\qquad \phi^{n}_{\alpha\beta}=-\phi^{n}_{\beta\alpha}\,.\end{aligned}\right\} \tag{3·1}$$

To include no conjugate point to $q_0(=q(t_0))$, the period $T(=2\pi/\omega)$ of the above periodic Jacobi fields is supposed to obey the following conditions: It is much longer than the period of the intrinsic motion in the stationary TDHB solution and also it is almost the same as that of the periodic solution of the TDHB equation. Substituting this periodic Jacobi field into Eq. (2·10a) and demanding the coefficients of $e^{i\omega_n t/\hbar}$ and $e^{-i\omega_n t/\hbar}$ to vanish, respectively, in each index n, we can get

$$\left.\begin{aligned}\omega_n\psi^n_{\alpha\beta}&=A_{\alpha\beta\gamma\delta}\psi^n_{\gamma\delta}+B_{\alpha\beta\gamma\delta}\phi^n_{\gamma\delta}\,,\\ \omega_n\phi^n_{\alpha\beta}&=-B^*_{\alpha\beta\gamma\delta}\psi^n_{\gamma\delta}-A^*_{\alpha\beta\gamma\delta}\phi^n_{\gamma\delta}\,,\end{aligned}\right\}\tag{3·2}$$

where the matrices $A_{\alpha\beta\gamma\delta}$ and $B_{\alpha\beta\gamma\delta}$ should be time-independent. Next we will transform the above equation into the one represented in the quasi-particle frame. To this end, it is very useful to bring to mind the group theoretical Tamm-Dancoff method developed by Fukutome.[11]

Following Fukutome, let G', G and $\tilde{G}$ be the $SO(2N)$ matrices having the same form as that in I and suppose that the matrices G and $\tilde{G}$ denote the stationary matrix and the fluctuating one, respectively. Then, they satisfy $G'=\tilde{G}G$. Further introducing the matrix variables $q'(=b'a'^{-1})$, $q(=ba^{-1})$ and $\tilde{q}\,(=\tilde{b}\,\tilde{a}^{-1})$ they are shown to be governed by the following relation:[11]

$$q'=q+\mathcal{P}(1-e^*\mathcal{P})^{-1}\,,\tag{3·3a}$$

$$\mathcal{P}=a^{T-1}\tilde{q}a^{-1}\,,\tag{3·3b}$$

$$e=-q(1-q^*q)^{-1}\,.\tag{3·3c}$$

Up to this time, in the calculus of variations, the differential $\partial/\partial q$ has been understood as not the covariant differential but the ordinary one. This means that in the second term of the above equation (3·3a), we take out only the first order term in $\mathcal{P}$. So, this $\mathcal{P}$ corresponds to the path variation ξ used in the previous sections within our first order approximation. Due to this fact, from Eq. (3·3b) the path variation represented in the quasi-particle frame is expressed as

$$\tilde{q}=a^T\xi a=\tilde{\xi}\,.\tag{3·4}$$

Now we set up the time-dependence of the stationary TDHB solutions as

$$\left.\begin{aligned}a&=a^{(0)}e^{i\varepsilon^{(0)}t/\hbar}\,,\\ b&=b^{(0)}e^{i\varepsilon^{(0)}t/\hbar}\,,\end{aligned}\right\}\tag{3·5}$$

where $\varepsilon^{(0)}=(\delta_{ij}\varepsilon^{(0)}_i)$ is an N-dimensional diagonal matrix. Substituting the above solution into the well-known equation of TDHB, the TDHB equation leads to the following equation for the time-independent static matrices $a^{(0)}$ and $b^{(0)}$:

$$\begin{pmatrix} F^{(0)} & D^{(0)} \\ -D^{(0)*} & -F^{(0)*} \end{pmatrix}\begin{pmatrix} b^{(0)} \\ a^{(0)} \end{pmatrix}_i = -\varepsilon_i^{(0)}\begin{pmatrix} b^{(0)} \\ a^{(0)} \end{pmatrix}_i, \quad \text{(not summed for } i\text{)} \tag{3·6}$$

$$\left.\begin{aligned} F_{\alpha\beta}^{(0)} &= h_{\alpha\beta} + [\alpha\beta|\gamma\delta](b^{(0)*}b^{(0)T})_{\gamma\delta}, \\ D_{\alpha\beta}^{(0)} &= \frac{1}{2}[\alpha\gamma|\beta\delta](a^{(0)*}b^{(0)T})_{\delta\gamma}, \end{aligned}\right\} \tag{3·7}$$

where we have used the relations $M = b^* b^T$ and $K = -a^* b^T$. The above equations (3·6) and (3·7) are nothing but the usual HB eigenvalue equation.

According to the transformation (3·4) of the path variation to a quasi-particle frame, with the use of the static HB amplitude $a^{(0)}$, Eq. (3·2) is transformed into the following form represented in the quasi-particle frame:

$$\left.\begin{aligned} \omega_n \psi_{ij}^{(0)n} &= A_{iji'j'}^{(0)} \psi_{i'j'}^{(0)n} + B_{iji'j'}^{(0)} \phi_{i'j'}^{(0)n}, \\ \omega_n \phi_{ij}^{(0)n} &= -B_{iji'j'}^{(0)*} \psi_{i'j'}^{(0)n} - A_{iji'j'}^{(0)*} \phi_{i'j'}^{(0)n}, \end{aligned}\right\} \tag{3·8a}$$

$$\left.\begin{aligned} \psi_{ij}^{(0)n} &\equiv a^{(0)\alpha}{}_i a^{(0)\beta}{}_j \psi_{\alpha\beta}^n, \quad \psi_{ij}^{(0)n} = -\psi_{ji}^{(0)n}, \\ \phi_{ij}^{(0)n} &\equiv a^{(0)\alpha*}{}_i a^{(0)\beta*}{}_j \phi_{\alpha\beta}^n, \quad \phi_{ij}^{(0)n} = -\phi_{ji}^{(0)n}, \end{aligned}\right\} \tag{3·8b}$$

$$\left.\begin{aligned} A_{iji'j'}^{(0)} &\equiv a^{(0)\alpha}{}_i a^{(0)\beta}{}_j A_{\alpha\beta\gamma\delta} (a^{(0)-1})_{i'}{}^{\gamma} (a^{(0)-1})_{j'}{}^{\delta}, \\ B_{iji'j'}^{(0)} &\equiv a^{(0)\alpha}{}_i a^{(0)\beta}{}_j B_{\alpha\beta\gamma\delta} (a^{(0)*-1})_{i'}{}^{\gamma} (a^{(0)*-1})_{j'}{}^{\delta}. \end{aligned}\right\} \tag{3·8c}$$

With the aid of the HB eigenvalue equation, the matrices $A^{(0)}$ and $B^{(0)}$ are calculated to be

$$\left.\begin{aligned} A_{iji'j'}^{(0)} &= (\varepsilon_i^{(0)} + \varepsilon_j^{(0)})\delta_{ii'}\delta_{jj'} + \mathcal{V}_{iji'j'}^{(0)X} - \mathcal{V}_{jii'j'}^{(0)X} \\ &= A_{i'j'ij}^{(0)*}, \\ B_{iji'j'}^{(0)} &= 4\mathcal{V}_{ii'jj'}^{(0)V} - 4\mathcal{V}_{ji'ij'}^{(0)V} - 2\mathcal{V}_{i'j'ij}^{(0)V} - 2\mathcal{V}_{iji'j'}^{(0)V} \\ &= B_{i'j'ij}^{(0)}, \end{aligned}\right\} \tag{3·9}$$

where the matrices $\mathcal{V}^{(0)X}$ and $\mathcal{V}^{(0)V}$ are defined as

$$\left.\begin{aligned} \mathcal{V}_{iji'j'}^{(0)X} &= \frac{1}{4}[\alpha\beta|\gamma\delta]\{a^{(0)\alpha}{}_i a^{(0)\gamma}{}_j a^{(0)\delta*}{}_{j'} a^{(0)\beta*}{}_{i'} \\ &\quad + b^{(0)*}{}_{\alpha j'} b^{(0)*}{}_{\gamma i'} b^{(0)}{}_{\delta i} b^{(0)}{}_{\beta j} \\ &\quad + 4a^{(0)\alpha}{}_i b^{(0)*}{}_{\gamma j'} a^{(0)\delta*}{}_{i'} b^{(0)}{}_{\beta j}\}, \\ \mathcal{V}_{iji'j'}^{(0)V} &= \frac{1}{4}[\alpha\beta|\gamma\delta] a^{(0)\alpha}{}_i a^{(0)\gamma}{}_j b^{(0)}{}_{\delta j'} b^{(0)}{}_{\beta i'}. \end{aligned}\right\} \tag{3·10}$$

In the above calculation, we have used the relations $q=ba^{-1}$, $1-M=aa^{\dagger}$ and $K=-a^{*}b^{T}$ together with Eq. (3·5). In Eq. (3·9), as is clear from the structures and the symmetry properties of the matrices $A^{(0)}$ and $B^{(0)}$, we can immediately see that the matrix $\mathcal{R}$ defined as

$$\mathcal{R}=\begin{pmatrix} A^{(0)} & B^{(0)} \\ -B^{(0)*} & -A^{(0)*} \end{pmatrix}, \quad A^{(0)\dagger}=A^{(0)}, \quad B^{(0)T}=B^{(0)}, \tag{3·11}$$

become quite equivalent to the so-called HB stability matrix. Thus, assuming the periodic Jacobi fields, we obtain at the final stage the matrix equation written in a very compact form as

$$\mathcal{R}\begin{pmatrix} \psi^{(0)} \\ \phi^{(0)} \end{pmatrix}^{n}=\omega_{n}\begin{pmatrix} \psi^{(0)} \\ \phi^{(0)} \end{pmatrix}^{n}. \quad \text{(not summed for } n\text{)} \tag{3·12}$$

This is just identical with the quasi-particle RPA equation to describe the collective excitation around certain static HB fields.[12] In Eq. (3·4), if we replace the path variation $\tilde{\xi}$ represented in the quasi-particle frame by the RPA bosons X_n,

$$\tilde{\xi}_{ij}=\sum_{n}(\psi_{ij}^{(0)n}X_n+\phi_{ij}^{(0)n*}X_n{}^{\dagger}), \tag{3·13}$$

the RPA orthogonality conditions are easily derived.

§ 4. Concluding remarks

In the present paper, we first have studied the variational properties of the action functional S and obtained the Jacobi equation through the calculus of the second variation on the coset manifold $SO(2N)/U(N)$. Next, assuming the periodic Jacobi fields, we have been able to derive the quasi-particle RPA equation describing the collective excitation around certain HB fields.

Throughout this paper, the calculus of variation has been carried out by regarding the differential $\partial/\partial q$ as not the covariant differential but the ordinary one. This means that in Eq. (3·3) we take out only the first order term in $\mathcal{P}$ as an approximated path variation ξ. However, for the path variation ξ, if we will hope to take into account effects of the higher order terms in $\mathcal{P}$, we must inevitably use the covariant differential. Then under this treatment, it is expected to get an interesting equation beyond the RPA one.[11]

Very recently we have shown that the extension of the formalism in I to the $SO(2N+1)$ group is possible. Embedding the $SO(2N+1)$ group into the $SO(2N+2)$ group and performing the calculus of the first variation on the coset manifold $SO(2N+2)/U(N+1)$, we got the classical form of the $SO(2N+1)$ TDHB theory which is applicable to both even and odd fermion systems.[13] As was pointed out in the preceding section, we know no extension of the $SO(2N)$ RPA to the $SO(2N+1)$ one for odd fermion systems to include both the paired and unpaired modes. Then we have an interesting problem to obtain the quasi-particle $SO(2N+1)$ RPA equation along the same line as the present work. As the first step, we can easily obtain the Jacobi equation on the coset manifold $SO(2N+2)/U(N+1)$ in a manner quite similar to the present one. If it is possible to solve the static $SO(2N+1)$ HB equation, we will get the $SO(2N+1)$ RPA equation which leads to a new information of collective excitations of the unpaired modes coupled to those of the paired ones. The detailed discussion will be given elsewhere.

Acknowledgements

The author expresses his sincere thanks to Professor H. Fukutome and Professor M. Yamamura and Dr. A. Kuriyama and Dr. M. Iwasaki for their valuable discussions and comments. He is also thankful to them for continual encouragement and interest.

Appendix I

Throughout this paper, the following formulae are very available:

$$\left.\begin{array}{l} M^{\dagger}=M \qquad K^{T}=-K\,, \\ q(1-M)=(1-M^{*})q=K\,, \\ q^{*}K=K^{*}q=-M\,, \end{array}\right\} \tag{AI·1a}$$

$$\left.\begin{array}{l} \dfrac{\partial M_{\gamma\delta}}{\partial q_{\alpha\beta}}=-K^{*}_{\gamma\alpha}(1-M)_{\beta\delta}+K^{*}_{\gamma\beta}(1-M)_{\alpha\delta}\,, \\ \dfrac{\partial M_{\gamma\delta}}{\partial q^{*}_{\alpha\beta}}=-(1-M)_{\gamma\alpha}K_{\beta\delta}+(1-M)_{\gamma\beta}K_{\alpha\delta}\,, \\ \dfrac{\partial K_{\gamma\delta}}{\partial q_{\alpha\beta}}=(1-M^{*})_{\gamma\alpha}(1-M)_{\beta\delta}-(1-M^{*})_{\gamma\beta}(1-M)_{\alpha\delta}\,, \\ \dfrac{\partial K_{\gamma\delta}}{\partial q^{*}_{\alpha\beta}}=K_{\gamma\alpha}K_{\beta\delta}-K_{\gamma\beta}K_{\alpha\delta}\,. \end{array}\right\} \tag{AI·1b}$$

The above formulae are fully utilized to execute the calculus of variations in the following Appendices.

With the use of the formulae (AI·1), we can get the following relation:

$$\frac{\partial\langle H\rangle_{G,G}}{\partial q_{\alpha\beta}}=\frac{\partial\langle H\rangle_{G,G}}{\partial M_{\gamma\delta}}\frac{\partial M_{\gamma\delta}}{\partial q_{\alpha\beta}}+\frac{\partial\langle H\rangle_{G,G}}{\partial M^{*}_{\gamma\delta}}\frac{\partial M^{*}_{\gamma\delta}}{\partial q_{\alpha\beta}}$$
$$+\frac{\partial\langle H\rangle_{G,G}}{\partial K_{\gamma\delta}}\frac{\partial K_{\gamma\delta}}{\partial q_{\alpha\beta}}+\frac{\partial\langle H\rangle_{G,G}}{\partial K^{*}_{\gamma\delta}}\frac{\partial K^{*}_{\gamma\delta}}{\partial q_{\alpha\beta}}$$

$$=2\left\{K^*\frac{\partial\langle H\rangle_{G,G}}{\partial M}(1-M^*)+(1-M)\frac{\partial\langle H\rangle_{G,G}}{\partial M^*}K^*\right.$$
$$\left.+(1-M)\frac{\partial\langle H\rangle_{G,G}}{\partial K}(1-M^*)+K^*\frac{\partial\langle H\rangle_{G,G}}{\partial K^*}K^*\right\}_{\alpha\beta}. \tag{AI·2}$$

Substituting Eq. (AI·2) into the TDHB equation (2·1) and using the relation $q=(1-M^*)^{-1}K=K(1-M)^{-1}$, we obtain

$$\dot{q}=q\frac{\partial\langle H\rangle_{G,G}}{\partial M^*}+\frac{\partial\langle H\rangle_{G,G}}{\partial M}q+\frac{\partial\langle H\rangle_{G,G}}{\partial K^*}+q\frac{\partial\langle H\rangle_{G,G}}{\partial K}q\,. \tag{AI·3}$$

On the other hand, from the definition of q, $\dot{q}$ is given as

$$\dot{q}=\dot{b}a^{-1}-ba^{-1}\dot{a}a^{-1}\,. \tag{AI·4}$$

Further substituting Eq. (AI·4) into the left-hand side of Eq. (AI·3) and right multiplying by a for both sides of the equation, we can get

$$i\hbar\dot{b}-Fb-Da-ba^{-1}(i\hbar\dot{a}+F^*a+D^*b)=0\,, \tag{AI·5}$$

where we have used the well-known forms of Hartree-Bogoliubov matrices F and D.

Now our HB amplitudes a and b satisfy the orthogonality conditions $G^\dagger G=GG^\dagger=1$ in terms of the $SO(2N)$ matrix G defined in I. In order that the TDHB equation (2·1) is compatible with the above orthogonality conditions, we require the relation

$$(i\hbar\dot{a}+D^*b+F^*a)a^\dagger=a(-i\hbar\dot{a}^\dagger-b^\dagger D+a^\dagger F^*) \tag{AI·6}$$

does hold.

Making good use of the indeterminancy of the unitary matrix appeared in a decomposition of the matrix G, we may adopt the conventional type of the TDHB equation which of course satisfies both Eqs. (AI·5) and (AI·6).

Appendix II

With the use of the "Lagrangian" L given in I, we get the relation

$$\left.\begin{aligned}&\frac{\partial L}{\partial q^*}=i\hbar\{(1-M^*)\dot{q}(1-M)-K\dot{q}^*K\}-\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}\,,\\&\frac{\partial L}{\partial\dot{q}^*}=-i\hbar K\,.\end{aligned}\right\} \tag{AII·1}$$

Further using the TDHB equation (2·1), we obtain

$$\left.\begin{aligned}
\frac{\partial L}{\partial q^*}&=\frac{1}{2}\left(q\frac{\partial\langle H\rangle_{G,G}}{\partial q}q-\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}\right),\\
i\hbar\dot{K}&=-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}^*}=-\frac{1}{2}\left(q\frac{\partial\langle H\rangle_{G,G}}{\partial q}q-\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}\right),\\
i\hbar(1\dot{-}M)&=-\frac{1}{2}\left(\frac{\partial\langle H\rangle_{G,G}}{\partial q}q-q^*\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}\right),
\end{aligned}\right\}\qquad(\mathrm{AII}\cdot 2)$$

where we have used the relation $q=(1-M^*)^{-1}K=K(1-M)^{-1}$.

Appendix III

With the help of the formulae (AI·1) and Eq. (AII·2), the second derivative $\partial^2L/\partial q^*\partial q^*$ is calculated to be

$$\begin{aligned}
\frac{\partial^2 L}{\partial q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}&=i\hbar\frac{\partial}{\partial q^*_{\alpha\beta}}\{(1-M^*)\dot{q}(1-M)-K\dot{q}^*K\}_{\gamma\delta}-\frac{\partial^2\langle H\rangle_{G,G}}{\partial q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}\\
&=i\hbar[K_{\alpha\gamma}\{(1-M^*)\dot{q}(1-M)\}_{\delta\beta}-K_{\alpha\delta}\{(1-M^*)\dot{q}(1-M)\}_{\gamma\beta}\\
&\quad+K_{\beta\delta}\{(1-M^*)\dot{q}(1-M)\}_{\gamma\alpha}-K_{\beta\gamma}\{(1-M^*)\dot{q}(1-M)\}_{\delta\alpha}\\
&\quad-K_{\alpha\gamma}(K\dot{q}^*K)_{\delta\beta}+K_{\alpha\delta}(K\dot{q}^*K)_{\gamma\beta}\\
&\quad-K_{\beta\delta}(K\dot{q}^*K)_{\gamma\alpha}+K_{\beta\gamma}(K\dot{q}^*K)_{\delta\alpha}]\\
&\quad-\frac{\partial^2\langle H\rangle_{G,G}}{\partial q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}\,.
\end{aligned}\qquad(\mathrm{AIII}\cdot 1)$$

Further putting the TDHB equation (2·1) into the above, we can get the following expression:

$$\begin{aligned}
\frac{\partial^2 L}{\partial q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}&=\frac{1}{2}\Bigg[K_{\alpha\gamma}\left\{\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}+q\frac{\partial\langle H\rangle_{G,G}}{\partial q}q\right\}_{\delta\beta}\\
&\quad-K_{\alpha\delta}\left\{\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}+q\frac{\partial\langle H\rangle_{G,G}}{\partial q}q\right\}_{\gamma\beta}\\
&\quad+\left\{\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}+q\frac{\partial\langle H\rangle_{G,G}}{\partial q}q\right\}_{\alpha\gamma}K_{\delta\beta}\\
&\quad-\left\{\frac{\partial\langle H\rangle_{G,G}}{\partial q^*}+q\frac{\partial\langle H\rangle_{G,G}}{\partial q}q\right\}_{\alpha\delta}K_{\gamma\beta}\Bigg]\\
&\quad-\frac{\partial^2\langle H\rangle_{G,G}}{\partial q^*_{\alpha\beta}\partial q^*_{\gamma\delta}}\,.
\end{aligned}\qquad(\mathrm{AIII}\cdot 2)$$

Similarly, the second derivative $\partial^2 L/\partial q^* \partial q$ can be calculated easily. Finally the Hessians $\partial^2\langle H\rangle_{G,G}/\partial q^* \partial q^*$, etc., are given by a tedious but straightforward calculation, though we omit to express their explicit forms here.

References

1) S. Nishiyama, Prog. Theor. Phys. **66** (1981), 348.
2) M. Yamamura and S. Nishiyama, Prog. Theor. Phys. **56** (1976), 124.
3) H. Fukutome, M. Yamamura and S. Nishiyama, Prog. Theor. Phys. **57** (1977), 1554.
4) H. Fukutome, Prog. Theor. Phys. **58** (1977), 1692.
5) H. Fukutome, Prog. Theor. Phys. **65** (1981), 809.
6) H. Kleinert, Phys. Letters **69B** (1979), 9.
7) H. Kuratsuji and Y. Mizobuchi, J. Math. Phys. **22** (1981), 757.
H. Kuratsuji and Y. Mizobuchi, Phys. Letters **82A** (1981), 279.
8) C. Dewitt-Morette, A. Maheshwari and B. Nelson, Phys. Reports **50** (1979), 255.
9) H. Kuratsuji and T. Suzuki, Phys. Letters **92B** (1980), 19.
10) I. M. Gelfand and S. V. Formin, *Calculus of Variations* (Prentice-Hall, New Jersey, USA, 1963).
11) H. Fukutome, Prog. Theor. Phys. **60** (1978), 1624.
12) Y. Mizobuchi, S. Nishiyama and M. Yamamura, Prog. Theor. Phys. **57** (1977), 96, 1797.
13) S. Nishiyama, Prog. Theor. Phys. **68** (1982), 680.