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A Kerr Metric Solution in New General Relativity

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We give an exact solution of the vacuum gravitational field equation in new general relativity. The solution gives the Kerr metric and the parallel vector fields are axially symmetric. A parameter h in the expression of the metric is related to the angular momentum of the rotating source, when the spin density $S_{ij}{}^{\mu}$ of the gravitational source satisfies the condition $\partial_\mu S_{ij}{}^{\mu}=0$. In the Kerr metric space-time, we cannot discriminate new general relativity from general relativity, so far as scalar, the Dirac and the Yang-Mills fields and macroscopic bodies are used as probes. *The space-time given by the solution does not have singularities at all, although it has an "effective singularity".* Two kinds of Schwarzschild metric solutions, one is our solution with $h=0$ and the other is a solution given by Hayashi and Shirafuji, are physically equivalent with each other. Nevertheless, these are markedly different from each other with regard to the asymptotic behavior of the torsion tensor for $r\to\infty$ and the space-time singularities.

§ 1. Introduction

New general relativity (N. G. R.) is a gravitational theory formulated by Hayashi and Nakano[1)] and by Hayashi and Shirafuji.[2)] A feature of this theory is the absolute parallelism the notion of which is first introduced by Einstein;[3)] thus the space-time is the Weitzenböck space-time, which is characterized by vanishing curvature tensor and by the metricity condition. The theory is invariant under *global* Lorentz transformations, but it is not under general *local* ones.*)

Fundamental fields of gravitation are the parallel vector fields $\boldsymbol{b}_k=b_k{}^{\mu}\partial/\partial x^{\mu}$ which give a vierbein system to this theory. The components of the metric tensor $\boldsymbol{g}=g_{\mu\nu}dx^{\mu}\otimes dx^{\nu}$ are related to the dual components $b^k{}_{\mu}$ of the parallel vector fields through

$$g_{\mu\nu}=\eta_{kl}b^k{}_{\mu}b^l{}_{\nu} \tag{1·1}$$

with $(\eta_{kl})\overset{\text{def}}{=}\text{diag}(-,+,+,+)$. The affine connection coefficients $\Gamma^{*\lambda}_{\mu\nu}$ are given by

$$\Gamma^{*\lambda}_{\mu\nu}=b_k{}^{\lambda}\partial_\nu b^k{}_{\mu}\,, \tag{1·2}$$

as a result of the absolute parallelism.[2)] The Christoffel symbol

$$\begin{Bmatrix}\rho\\ \mu\ \ \nu\end{Bmatrix}=\frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu}) \tag{1·3}$$

does *not* represent the affine connection of the Weitzenböck space-time. But it is important in the description of the motions of macroscopic test bodies.[2)]

The gravitational Lagrangian in this theory consists of quadratic terms of the torsion tensor

*) In Ref. 4), an extended new general relativity, which has additionally the *local internal* translational invariance, has been proposed. So far as vacuum gravitational field equation is concerned, there is no difference between the extended theory and N. G. R.

$$T^{\lambda}{}_{\mu\nu}=b_k{}^{\lambda}(\partial_\nu b^k{}_\mu-\partial_\mu b^k{}_\nu)\,. \tag{1·4}$$

The following Lagrangian employed in this paper,*) in particular,

$$L_G=-\frac{1}{3\kappa}(t^{\mu\nu\lambda}t_{\mu\nu\lambda}-v^\mu v_\mu)+\xi a^\mu a_\mu \tag{1·5}$$

is quite favorable experimentally.[2] Here κ is the Einstein gravitational constant and ξ is a real constant parameter and $t_{\mu\nu\lambda}$, v_μ and a_μ are the irreducible components of the torsion tensor $T^{\lambda}{}_{\mu\nu}$:

$$t_{\mu\nu\lambda}\overset{\text{def}}{=}\frac{1}{2}(T_{\mu\nu\lambda}+T_{\nu\mu\lambda})+\frac{1}{6}(g_{\lambda\mu}v_\nu+g_{\lambda\nu}v_\mu)-\frac{1}{3}g_{\mu\nu}v_\lambda\,, \tag{1·6}$$

$$v_\mu\overset{\text{def}}{=}T^{\lambda}{}_{\lambda\mu}\,, \tag{1·7}$$

$$a_\mu\overset{\text{def}}{=}\frac{1}{6}\varepsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma}\,. \tag{1·8}$$

Here, $\varepsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor, normalized to $\varepsilon_{0123}=-\sqrt{-g}$.

Fukui and Hayashi[5] has pointed out that axially symmetric and stationary solutions of N. G. R. with the gravitational Lagrangian L_G are different from those of general relativity (G. R.), but no explicit solution has been given there. Also, no one has discussed the question whether the Kerr metric is allowed or not in N. G. R.

The main purpose of this paper is to give explicitly an axially symmetric exact solution of N. G. R., which gives the Kerr metric. In § 2, the solution is given, and the physical meanings of the parameters a and h appearing in the solution are discussed in § 3. In § 4, the singularities and "effective singularities"[6] of the space-time given by the solution are examined. In § 5, two kinds of Schwarzschild metric solutions, one is our solution with $h=0$ and the other is a solution given in Ref. 2), are examined from the points of the asymptotic behavior and of the space-time singularities. Finally in § 6, we give a summary.

§ 2. A Kerr metric solution

The gravitational field equations are given by[2),**)]

$$G^{\mu\nu}(\{\,\})+K^{\mu\nu}=\kappa T^{(\mu\nu)}\,, \tag{2·1}$$

$$b_i{}^\mu b_j{}^\nu\partial_\rho(\sqrt{-g}\,J^{ij\rho})=\lambda\sqrt{-g}\,T^{[\mu\nu]}\,, \tag{2·2}$$

where $G^{\mu\nu}(\{\,\})$ is the Einstein tensor:

$$G_{\mu\nu}(\{\,\})=R_{\mu\nu}(\{\,\})-\frac{1}{2}g_{\mu\nu}R(\{\,\})\,, \tag{2·3}$$

*) We will use the natural unit $\hbar=c=1$.

**) $T_{(\mu\nu)}\overset{\text{def}}{=}\frac{1}{2}(T_{\mu\nu}+T_{\nu\mu})$, $T_{[\mu\nu]}\overset{\text{def}}{=}\frac{1}{2}(T_{\mu\nu}-T_{\nu\mu})$.

$$R^{\rho}{}_{\sigma\mu\nu}(\{\,\})\overset{\text{def}}{=}\partial_{\mu}\begin{Bmatrix}\rho\\ \sigma\ \nu\end{Bmatrix}-\partial_{\nu}\begin{Bmatrix}\rho\\ \sigma\ \mu\end{Bmatrix}+\begin{Bmatrix}\rho\\ \tau\ \mu\end{Bmatrix}\begin{Bmatrix}\tau\\ \sigma\ \nu\end{Bmatrix}-\begin{Bmatrix}\rho\\ \tau\ \nu\end{Bmatrix}\begin{Bmatrix}\tau\\ \sigma\ \mu\end{Bmatrix}, \tag{2·4}$$

$$R_{\mu\nu}(\{\,\})\overset{\text{def}}{=}R^{\rho}{}_{\mu\rho\nu}(\{\,\}), \tag{2·5}$$

$$R(\{\,\})\overset{\text{def}}{=}g^{\mu\nu}R_{\mu\nu}(\{\,\}), \tag{2·6}$$

and $T^{\mu\nu}$ is the energy-momentum tensor of a source field with Lagrangian L_M:

$$\sqrt{-g}\,T^{\mu\nu}\overset{\text{def}}{=}b^{i\mu}\frac{\delta(\sqrt{-g}\,L_M)}{\delta b^{i}{}_{\nu}}. \tag{2·7}$$

The tensors $K^{\mu\nu}$ and $J^{ij\mu}$ are defined by

$$K^{\mu\nu}\overset{\text{def}}{=}\frac{\kappa}{\lambda}\left[\frac{1}{2}\{\varepsilon^{\mu\rho\sigma\lambda}(T^{\nu}{}_{\rho\sigma}-T_{\rho\sigma}{}^{\nu})+\varepsilon^{\nu\rho\sigma\lambda}(T^{\mu}{}_{\rho\sigma}-T_{\rho\sigma}{}^{\mu})\}a_{\lambda}-\frac{3}{2}a^{\mu}a^{\nu}-\frac{3}{4}g^{\mu\nu}a^{\lambda}a_{\lambda}\right], \tag{2·8}$$

$$J^{ij\mu}\overset{\text{def}}{=}-\frac{3}{2}b^{i}{}_{\rho}b^{j}{}_{\sigma}\varepsilon^{\rho\sigma\mu\nu}a_{\nu}, \tag{2·9}$$

respectively. Here, λ is a parameter defined by

$$\frac{1}{\lambda}\overset{\text{def}}{=}\frac{4}{9}\xi+\frac{1}{3\kappa}. \tag{2·10}$$

In this section, we consider the case of the vacuum gravitational field:

$$T^{(\mu\nu)}=T^{[\mu\nu]}=0. \tag{2·11}$$

We will seek a solution having the following form:

$$b^{k}{}_{\mu}=\delta^{k}{}_{\mu}+\frac{a}{2}l^{k}l_{\mu}, \tag{2·12}$$

where a is an arbitrary constant parameter, and l_μ and l^k are quantities satisfying the conditions:

$$\eta^{\mu\nu}l_{\mu}l_{\nu}=0, \tag{2·13}$$

$$l^{k}=\delta^{k}{}_{\mu}\eta^{\mu\nu}l_{\nu}. \tag{2·14}$$

For l_μ and l^k, we will use the convention that Greek and Latin indices are raised and lowered by $(\eta^{\mu\nu})$, $(\eta_{\mu\nu})$, (η^{kl}) and (η_{kl}) and are converted into each other by $(\delta^{k}{}_{\mu})$ and $(\delta_{k}{}^{\mu})$. Substituting Eq. (2·12) into Eq. (1·4), we get

$$T_{\lambda\mu\nu}=a\partial_{[\nu}(l_{\mu]}l_{\lambda}). \tag{2·15}$$

The axial vector part (1·8) of the torsion tensor (2·15) vanishes identically:

$$a_{\mu}=0, \tag{2·16}$$

and hence field equations (2·1) and (2·2) are reduced to the Einstein equation:

$$G^{\mu\nu}(\{\,\})=0. \tag{2·17}$$

An axially symmetric and stationary solution of Eq. (2·17) is well known in G. R.[7),8)]

Equations (2·12) and (2·17) lead to the following:*)

(i) $l^\lambda \partial_\lambda l_\mu$ is parallel to l_μ:

$$l^\lambda \partial_\lambda l_\mu = -A l_\mu \tag{2·18}$$

with A being a function.

(ii) For a stationary space-time, if we express (l_μ) as

$$(l_\mu) = l_0(1, \lambda_1, \lambda_2, \lambda_3)\,, \tag{2·19}$$

where l_0 and λ_a $(a=1,2,3)$**) are x^0-independent functions, and λ_a satisfy

$$\lambda_a\lambda_a \overset{\text{def}}{=} \sum_{a=1}^{3} \lambda_a\lambda_a = 1\,, \tag{2·20}$$

then

$$\partial_b\lambda_a = \alpha(\delta_{ab} - \lambda_a\lambda_b) + \beta\varepsilon_{abc}\lambda_c\,, \tag{2·21}$$

$$\lambda_a\partial_a\alpha = \beta^2 - \alpha^2\,, \tag{2·22}$$

$$\lambda_a\partial_a\beta = -2\alpha\beta\,, \tag{2·23}$$

$$\alpha \overset{\text{def}}{=} \mathrm{Re}(\gamma)\,, \quad \beta \overset{\text{def}}{=} \mathrm{Im}(\gamma)\,. \tag{2·24}$$

Here γ is a complex valued function which satisfies the equations:

$$\triangle\gamma \overset{\text{def}}{=} \partial_a\partial_a\gamma = 0\,, \tag{2·25}$$

$$\partial_a\left(\frac{1}{\gamma}\right)\cdot\partial_a\left(\frac{1}{\gamma}\right) = 1\,. \tag{2·26}$$

(iii) The stationary solution of Eq. (2·17) is given by

$$(l_0)^2 = \mathrm{Re}(\gamma) = \alpha\,, \tag{2·27}$$

$$\lambda_a = \frac{(\beta^2-\alpha^2)\partial_a\alpha - 2\alpha\beta\partial_a\beta - \varepsilon_{abc}\partial_b\alpha\cdot\partial_c\beta}{(\beta^2-\alpha^2)^2 + (\partial_b\beta\cdot\partial_b\beta)}\,. \tag{2·28}$$

We take γ as

$$\gamma = \frac{1}{\sqrt{r^2 - h^2 + 2ihz}}\,, \tag{2·29}$$

where $r \overset{\text{def}}{=} \sqrt{x^a x^a}$, $z \overset{\text{def}}{=} x^3$ and h is a real constant parameter, and we make the coordinate transformation from (x^μ) to $(t, \rho, \theta, \varphi)$ defined by

*) These results are quoted from Ref. 8).

**) In our convention, the initial part of the Latin alphabets, $a, b, c, \cdots$, refers to as 1, 2 and 3.

$$\left\{\begin{array}{l} x^0 \stackrel{\text{def}}{=} t+\frac{a}{2}\ln|\rho^2+h^2-a\rho|+\frac{a^2}{2}B\,, \\ x^1 \stackrel{\text{def}}{=} (\rho\cos\Phi+h\sin\Phi)\sin\theta\,, \\ x^2 \stackrel{\text{def}}{=} (\rho\sin\Phi-h\cos\Phi)\sin\theta\,, \\ x^3 \stackrel{\text{def}}{=} \rho\cos\theta\,, \\ B \stackrel{\text{def}}{=} \int^{\rho}\frac{d\rho}{\rho^2+h^2-a\rho}\,, \quad \Phi \stackrel{\text{def}}{=} \varphi-hB\,. \end{array}\right. \tag{2·30}$$

Then the metric has the expression:

$$ds^2=-\left(1-\frac{a\rho}{\Sigma}\right)dt^2+\frac{\Sigma}{\Delta}d\rho^2+\Sigma d\theta^2$$
$$+\left\{(\rho^2+h^2)\sin^2\theta+\frac{a\rho h^2\sin^4\theta}{\Sigma}\right\}d\varphi^2+2\frac{a\rho h\sin^2\theta}{\Sigma}dtd\varphi \tag{2·31}$$

with

$$\Sigma \stackrel{\text{def}}{=} \rho^2+h^2\cos^2\theta\,, \tag{2·32}$$

$$\Delta \stackrel{\text{def}}{=} \rho^2+h^2-a\rho\,. \tag{2·33}$$

Equation (2·31) gives the Kerr metric written in the Boyer-Lindquist coordinates. The parallel vector fields*) $b^k{}_\mu$ are expressed as**)

$$\left\{\begin{array}{ll} b^0{}_t=1-\dfrac{a\rho}{2\Sigma}\,, & b^0{}_\rho=\dfrac{a\rho}{2\Delta}\,, \\ b^0{}_\theta=0\,, & b^0{}_\varphi=-\dfrac{a\rho h\sin^2\theta}{2\Sigma}\,, \\ b^1{}_t=\dfrac{a\rho\sin\theta\cos\Phi}{2\Sigma}\,, & b^1{}_\rho=\dfrac{\rho\sin\theta}{\Delta}\left(X-\dfrac{a}{2}\cos\Phi\right), \\ b^1{}_\theta=X\cos\theta\,, & b^1{}_\varphi=-Y\sin\theta+\dfrac{a\rho h\sin^3\theta\cos\Phi}{2\Sigma}\,, \\ b^2{}_t=\dfrac{a\rho\sin\theta\sin\Phi}{2\Sigma}\,, & b^2{}_\rho=\dfrac{\rho\sin\theta}{\Delta}\left(Y-\dfrac{a}{2}\sin\Phi\right), \\ b^2{}_\theta=Y\cos\theta\,, & b^2{}_\varphi=X\sin\theta+\dfrac{a\rho h\sin^3\theta\sin\Phi}{2\Sigma}\,, \\ b^3{}_t=\dfrac{a\rho\cos\theta}{2\Sigma}\,, & b^3{}_\rho=\left(1+\dfrac{a\rho}{2\Delta}\right)\cos\theta\,, \\ b^3{}_\theta=-\rho\sin\theta\,, & b^3{}_\varphi=\dfrac{a\rho h\sin^2\theta\cos\theta}{2\Sigma}\,, \end{array}\right. \tag{2·34}$$

*) In what follows, "the dual components $b^k{}_\mu$ of the parallel vector fields" are simply called "the parallel vector fields $b^k{}_\mu$".

**) The solution (2·34) is different from solutions discussed in Ref. 5), which can be seen by comparing Eq. (2·34) with Eq. (7) of Ref. 5). This can be seen also by noting that the field equation (12) in Ref. 5) is not reduced, unless the function ω appearing therein is constant, to the Einstein equation.

which is obtainable by the use of Eqs. (2·12), (2·19), (2·24), (2·27)∼(2·29) and coordinate transformations (2·30). Here, X and Y are defined by

$$X \overset{\text{def}}{=} \rho\cos\Phi + h\sin\Phi\,, \quad Y \overset{\text{def}}{=} \rho\sin\Phi - h\cos\Phi\,. \tag{2·35}$$

Thus, a vacuum solution which gives the Kerr metric has been given. The parallel vector fields $\boldsymbol{b}^k = b^k{}_\mu dx^\mu$ given by Eq. (2·34) are axially symmetric in the sense that they are form invariant under the transformation,[5]

$$\begin{aligned} &\varphi \to \varphi + \delta\varphi\,, \quad \boldsymbol{b}^0 \to \boldsymbol{b}^0\,, &&\boldsymbol{b}^1 \to \boldsymbol{b}^1\cos\delta\varphi - \boldsymbol{b}^2\sin\delta\varphi\,, \\ &\boldsymbol{b}^2 \to \boldsymbol{b}^1\sin\delta\varphi + \boldsymbol{b}^2\cos\delta\varphi\,, &&\boldsymbol{b}^3 \to \boldsymbol{b}^3\,. \end{aligned} \tag{2·36}$$

General relativity has a solution which gives the Kerr metric:

$$e^k{}_\mu = \begin{array}{cc} & \to \mu \\ \downarrow k & \begin{pmatrix} \sqrt{1-\dfrac{a\rho}{\Sigma}} & 0 & 0 & -\dfrac{a\rho h\sin^2\theta}{\Sigma - a\rho}\sqrt{1-\dfrac{a\rho}{\Sigma}} \\ 0 & \sqrt{\dfrac{\Sigma}{\Delta}} & 0 & 0 \\ 0 & 0 & \sqrt{\Sigma} & 0 \\ 0 & 0 & 0 & \sqrt{\dfrac{\Delta\Sigma}{\Sigma - a\rho}}\sin\theta \end{pmatrix} \end{array}. \tag{2·37}$$

The fields $e^k{}_\mu$ are related to the parallel vector fields $b^k{}_\mu$ of Eq. (2·34) through a *local* Lorentz transformation, but $e^k{}_\mu$ do *not* give a solution in N. G. R., so far as $h \neq 0$. This is not unnatural, because N. G. R. is not required to be *local* Lorentz invariant.

We mention here a method of finding a vacuum solution of N. G. R. from a vacuum solution $e^k{}_\mu$ of G. R. Under a local Lorentz transformation

$$b^k{}_\mu \overset{\text{def}}{=} \Lambda^k{}_l e^l{}_\mu\,, \tag{2·38}$$

the axial vector part of the torsion tensor defined by Eq. (1·8) transforms as

$$a_\mu{}^{(b)} = a_\mu{}^{(e)} + \frac{1}{3}\varepsilon_\mu{}^{\nu\rho\sigma} e^i{}_\nu e^j{}_\rho \eta_{kl} \Lambda^k{}_i \partial_\sigma \Lambda^l{}_j\,, \tag{2·39}$$

where $a_\mu{}^{(b)}$ and $a_\mu{}^{(e)}$ denote the axial vector part of the torsion tensor made of $b^k{}_\mu$ and $e^k{}_\mu$, respectively. If we can get

$$a_\mu{}^{(b)} = 0 \tag{2·40}$$

by choosing $\Lambda^k{}_l$ appropriately, then $(b^k{}_\mu)$ is a vacuum solution of N. G. R., because $K^{\mu\nu}$ and $J^{ij\mu}$ defined by Eqs. (2·8) and (2·9) both vanish and $b^k{}_\mu$ satisfy the gravitational field equations (2·1) and (2·2) in vacuum. The Kerr metric solution (2·34) gives an example of this statement.

§ 3. Physical meaning of a and of h

To clarify the physical meaning of the parameters a and h in the solution (2·34), we now consider the weak-field approximation:

$$b^k{}_\nu = \delta^k{}_\nu + a^k{}_\nu\,, \qquad |a^k{}_\nu| \ll 1\,. \tag{3·1}$$

The field $a_{\mu\nu} \overset{\text{def}}{=} \delta^k{}_\mu \eta_{kl} a^l{}_\nu$ can be expressed as*)

$$a_{\mu\nu} = \frac{1}{2} h_{\mu\nu} + A_{\mu\nu} \tag{3·2}$$

with $h_{\mu\nu} = h_{\nu\mu}$ and $A_{\mu\nu} = -A_{\nu\mu}$. The gravitational field equations (2·1) and (2·2) now take the following forms:

$$\Box \bar{h}_{\mu\nu} = -2\kappa T_{(\mu\nu)}\,, \tag{3·3}$$

$$\Box A_{\mu\nu} = -\lambda T_{[\mu\nu]}\,, \tag{3·4}$$

under the gauge conditions[2)]

$$\partial_\nu \bar{h}^{\mu\nu} = 0\,, \tag{3·5}$$

$$\partial_\nu A^{\mu\nu} = 0\,, \tag{3·6}$$

where $\bar{h}_{\mu\nu}$ and $\Box$ are defined by

$$\bar{h}_{\mu\nu} \overset{\text{def}}{=} h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(\eta^{\rho\sigma} h_{\rho\sigma})\,, \tag{3·7}$$

$$\Box \overset{\text{def}}{=} \eta^{\mu\nu}\partial_\mu \partial_\nu\,, \tag{3·8}$$

respectively.

The compatibility of Eqs. (2·12) and (3·1) in the weak-field situations requires that $A_{\mu\nu}$ vanishes identically:

$$A_{\mu\nu} = 0\,. \tag{3·9}$$

The tensor $T_{[\mu\nu]}$ is related to the spin density

$$\boldsymbol{S}_{kl}{}^{\mu} \overset{\text{def}}{=} i \frac{\partial(\sqrt{-g}\,L_M)}{\partial \psi_{,\mu}} S_{kl}\psi \tag{3·10}$$

of the source field ψ, through

$$\sqrt{-g}\, T_{[\mu\nu]} = \frac{1}{2} b^k{}_\mu b^l{}_\nu \partial_\lambda \boldsymbol{S}_{kl}{}^{\lambda}\,, \tag{3·11}$$

as a result of global Lorentz invariance of L_M and the field equation of ψ. Here, S_{kl} is the representation of the Lie algebra of the Lorentz group to which ψ belongs.

*) In this section, we will use the convention that Greek and Latin indices are raised and lowered by $(\eta^{\mu\nu})$, $(\eta_{\mu\nu})$, (η^{kl}) and (η_{kl}) and are converted into each other by $(\delta^k{}_\mu)$ and $(\delta_k{}^\mu)$.

Therefore, we get*)

$$\partial_\mu S_{kl}{}^\mu = 0 \tag{3·12}$$

by the use of Eqs. (3·4), (3·9) and (3·11). We will use Eq. (3·12) instead of Eq. (3·9) as the compatibility condition. The condition (3·12) is satisfied when effects due to the intrinsic spin of constituent fundamental particles can be ignored for the gravitational source, as are the cases for usual macroscopic objects such as planets and stars.

Under the condition (3·12), the physical meaning of a and h are given by

$$a = \frac{\kappa M}{4\pi}, \tag{3·13}$$

$$h = -\frac{J}{M}, \tag{3·14}$$

where M is the gravitational mass of a central gravitating body and J represents the angular momentum of the rotating source. The relations (3·13) and (3·14) are obtained by comparing the metric for the solution (2·34) with Lense and Thirring's metric:[8),9)]

$$ds^2 = -\left(1 - \frac{\kappa M}{4\pi r}\right)dt^2 + \left(1 + \frac{\kappa M}{4\pi r}\right)dx^a dx^a - \frac{\kappa J}{2\pi r}\sin^2\theta dt d\varphi\,. \tag{3·15}$$

The Kerr metric space-time in N. G. R. given by Eq. (2·34) cannot be discriminated from the Kerr space-time in G. R., so far as scalar, the Dirac and the Yang-Mills fields and macroscopic bodies are used as probes because of the following reasons:

(i) The field equations and measurable quantities for these fields in the space-time given by Eq. (2·34) agree with those in the Kerr space-time in G. R.[2)]

(ii) In macroscopic gravitational phenomena, only the metric tensor plays significant roles.[2)]

§ 4. Singularities

In N. G. R., by singularity of the space-time, we mean[6)] the singularity of the scalar concomitants of the torsion and curvature tensors, and by "effective singularity" is meant[6)] the singularity of the scalar concomitants of the Riemann-Christoffel curvature tensor.**)

The space-time given by the solution (2·34) *does not have singularities at all,* because we have

$$t^{\mu\nu\lambda} t_{\mu\nu\lambda} = v^\mu v_\mu = a^\mu a_\mu = 0\,, \tag{4·1}$$

and the curvature tensor is vanishing. Equation (4·1) is obtainable from Eqs. (1·6),

*) Conversely, Eq. (3·9) can be derived from Eq. (3·12) by noting that we are dealing with a static solution with the condition: $A_{\mu\nu} \to 0$ $(r \to \infty)$.

**) The Riemann-Christoffel curvature tensor is not the curvature tensor of the Weitzenböck space-time, but in N. G. R., macroscopic test bodies "feel" effectively this curvature.

(1·7), (2·13), (2·15), (2·16) and (2·18).

From Eq. (2·17), it is evident that

$$R^{\mu\nu}(\{\ \})R_{\mu\nu}(\{\ \})=R(\{\ \})=0\,, \tag{4·2}$$

and hence the "effective singularities" are given by those of $R^{\rho\sigma\mu\nu}(\{\ \})R_{\rho\sigma\mu\nu}(\{\ \})$. By the use of Eqs. (1·1), (1·3), (2·4), (2·13), (2·18)~(2·23), we obtain

$$R^{\rho\sigma\mu\nu}(\{\ \})R_{\rho\sigma\mu\nu}(\{\ \})=12\alpha^2(\alpha^2-\beta^2)\{(\alpha^2-\beta^2)^2-12\alpha^2\beta^2\}\,. \tag{4·3}$$

Substituting Eq. (2·24) with Eq. (2·29) into Eq. (4·3) and using the coordinate system $(t, \rho, \theta, \varphi)$, we get

$$R^{\rho\sigma\mu\nu}(\{\ \})R_{\rho\sigma\mu\nu}(\{\ \})=\frac{12a^2}{\Sigma^6}\Lambda(\Lambda^2-12\rho^2h^2\cos^2\theta)\,, \tag{4·4}$$

where

$$\Lambda\overset{\text{def}}{=}\rho^2-h^2\cos^2\theta\,. \tag{4·5}$$

Thus, we find that there is an "effective singularity" at $(\rho, \theta)=(0, \pi/2)$ when $h\neq0$ and at $\rho=0$ when $h=0$. This "effective singularity" agrees with the singularity of the Kerr space-time in G. R.

§ 5. The special case with $h=0$

We consider the special case with $h=0$. In this case, the parallel vector fields $b^k{}_\mu$ given by Eq. (2·34) and the vierbein fields $e^k{}_\mu$ given by Eq. (2·37) are reduced to

$$b^k{}_\mu=\begin{array}{c} \rightarrow\mu \\ \downarrow k \end{array}\begin{pmatrix} 1-\dfrac{a}{2r} & \dfrac{a}{2(r-a)} & 0 & 0 \\ \dfrac{a}{2r}\sin\theta\cos\varphi & \dfrac{2r-a}{2(r-a)}\sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \dfrac{a}{2r}\sin\theta\sin\varphi & \dfrac{2r-a}{2(r-a)}\sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \dfrac{a}{2r}\cos\theta & \dfrac{2r-a}{2(r-a)}\cos\theta & -r\sin\theta & 0 \end{pmatrix} \tag{5·1}$$

and

$$e^k{}_\mu = \begin{array}{c|c} & \rightarrow \mu \\ \hline \begin{matrix}\downarrow \\ k\end{matrix} & \begin{pmatrix} \sqrt{1-\dfrac{a}{r}} & 0 & 0 & 0 \\ 0 & \sqrt{\dfrac{r}{r-a}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix} \end{array}, \tag{5·2}$$

respectively. The fields $b^k{}_\mu$ and $e^k{}_\mu$ give the Schwarzschild metric:

$$ds^2 = -\left(1-\frac{a}{r}\right)dt^2 + \left(1-\frac{a}{r}\right)^{-1} dr^2 + r^2\{d\theta^2 + \sin^2\theta d\varphi^2\}\,. \tag{5·3}$$

A set of vierbein fields (5·2) is a solution of N. G. R., as has been mentioned in p. 3540 of Ref. 2). Therefore, *not only the parallel vector fields* $b^k{}_\mu$ *but also the vierbein fields* $e^k{}_\mu$ *are solutions in* N. G. R.

The parallel vector fields of Eq. (5·1) are related to the vierbein fields of Eq. (5·2) through the *local* Lorentz transformations:

$$b^k{}_\mu = \Lambda^k{}_l e^l{}_\mu\,, \tag{5·4}$$

where

$$\Lambda^k{}_l = \begin{array}{c|c} & \rightarrow l \\ \hline \begin{matrix}\downarrow \\ k\end{matrix} & \begin{pmatrix} \dfrac{2r-a}{2\sqrt{r(r-a)}} & \dfrac{a}{2\sqrt{r(r-a)}} & 0 & 0 \\ \dfrac{a\sin\theta\cos\varphi}{2\sqrt{r(r-a)}} & \dfrac{(2r-a)\sin\theta\cos\varphi}{2\sqrt{r(r-a)}} & \cos\theta\cos\varphi & -\sin\varphi \\ \dfrac{a\sin\theta\sin\varphi}{2\sqrt{r(r-a)}} & \dfrac{(2r-a)\sin\theta\sin\varphi}{2\sqrt{r(r-a)}} & \cos\theta\sin\varphi & \cos\varphi \\ \dfrac{a\cos\theta}{2\sqrt{r(r-a)}} & \dfrac{(2r-a)\cos\theta}{2\sqrt{r(r-a)}} & -\sin\theta & 0 \end{pmatrix} \end{array}, \tag{5·5}$$

which leaves the axial vector part of torsion tensor invariant.*) Hence, these two solutions are physically equivalent with each other as known by applying discussions of Ref. 10).

Nevertheless, they are markedly different from each other in the following respects:

(i) For the torsion tensors $T^{(b)}_{\lambda\mu\nu}$ and $T^{(e)}_{\lambda\mu\nu}$ corresponding to the solutions (5·1) and (5·2), respectively, we have

$$T^{(b)}_{00\alpha} = -\frac{a}{2r^2}\frac{x^\alpha}{r}, \quad \alpha = 1, 2, 3$$

*) The axial vector parts of the torsion tensors for the solutions (5·1) and (5·2) both vanish.

$$T^{(b)}_{\alpha 0\beta} = \frac{a}{2r^2}\left\{\delta^{\alpha\beta} - \frac{2r-a}{r-a}\frac{x^\alpha x^\beta}{r^2}\right\}, \quad \alpha, \beta = 1, 2, 3\,,$$

$$T^{(b)}_{\alpha\beta\gamma} = -\frac{a\delta^{\alpha[\beta}}{r(r-a)}\frac{x^{\gamma]}}{r}, \quad \alpha, \beta, \gamma = 1, 2, 3\,,$$

(the other independent components vanish), (5·6)

and

$$T^{(e)}_{00\alpha} = -\frac{a}{2r^2}\frac{x^\alpha}{r}, \quad \alpha = 1, 2, 3$$

$$T^{(e)}_{112} = \frac{y}{r^2-z^2}, \qquad T^{(e)}_{221} = \frac{x}{r^2-z^2},$$

$$T^{(e)}_{113} = \frac{z}{r^2-z^2}\left(\frac{x}{r}\right)^2, \qquad T^{(e)}_{223} = \frac{z}{r^2-z^2}\left(\frac{y}{r}\right)^2,$$

$$T^{(e)}_{331} = \frac{x}{r^2}, \qquad T^{(e)}_{332} = \frac{y}{r^2},$$

$$T^{(e)}_{123} = T^{(e)}_{213} = \frac{xyz}{r^2(r^2-z^2)},$$

(the other independent components vanish), (5·7)

where

$$x \overset{\text{def}}{=} x^1 = r\sin\theta\cos\varphi\,, \quad y \overset{\text{def}}{=} x^2 = r\sin\theta\sin\varphi\,, \quad z \overset{\text{def}}{=} x^3 = r\cos\theta\,. \qquad (5\cdot 8)$$

Equations (5·6) and (5·7) show that $T^{(b)}_{\lambda\mu\nu}$ approaches zero as $r\to\infty$ faster than $T^{(e)}_{\lambda\mu\nu}$ does.

(ii) From the discussion in § 4, it is clear that the space-time given by Eq. (5·1) does not have singularities at all. While, the space-time given by Eq. (5·2) is singular at $r=0$, $r=a$ and $x=y=0$, as is seen from the following:

$$t^{(e)\lambda\mu\nu}t^{(e)}_{\lambda\mu\nu} = \frac{(2r-3a)^2}{4r^3(r-a)} + \frac{z^2}{r^4(r^2-z^2)}, \qquad (5\cdot 9)$$

$$v^{(e)\mu}v^{(e)}{}_{\mu} = \frac{(4r-3a)^2}{4r^3(r-a)} + \frac{z^2}{r^4(r^2-z^2)}, \qquad (5\cdot 10)$$

where $t^{(e)}_{\lambda\mu\nu}$ and $v^{(e)}_\mu$ are the irreducible components of $T^{(e)}_{\lambda\mu\nu}$.

§ 6. Summary

The results of the preceding sections can be summarized as follows:

⟨1⟩ An exact solution (2·34) which gives the Kerr metric has been given. It is axially symmetric, but is different from solutions discussed in Ref. 5).
⟨2⟩ Any vacuum solution of G. R., such that a_μ can be made to vanish by a choice of gauge, can be transformed into a vacuum solution of N. G. R. The solution (2·34)

gives an example of this statement.

⟨3⟩ When the gravitational source satisfies the condition (3·12), the parameter h is related to the angular momentum of the rotating source through Eq. (3·14).

⟨4⟩ There appears no difference between our solution (2·34) and the Kerr solution in G. R., so far as scalar, the Dirac and the Yang-Mills fields and the macroscopic test bodies are concerned.

⟨5⟩ The space-time given by the solution (2·34) does not have singularities at all. But, "effective singularity" exists at $(\rho, \theta)=(0, \pi/2)$ when $h\neq 0$ and at $\rho=0$ when $h=0$.

⟨6⟩ Two kinds of the solution, (5·1) and (5·2), both of which give the Schwarzschild metric, are physically equivalent with each other. Nevertheless, these are markedly different from each other with regard to the asymptotic behavior of the torsion tensors for $r\to\infty$ and the space-time singularities.

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