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Research Article

A Kind of Complete Moment Convergence for Sums of Independent and Nonidentically Distributed Random Variables

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Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and nonidentically distributed random variables. We obtain a new kind of complete moment convergence for their sums under the Lyapunov condition. Moreover, our result extends and improves the corresponding result of the independent and identically distributed (i.i.d.) cases.

1. Introduction and Main Result

Let $\{X, X_n, n \geq 1\}$ be a sequence of random variables, and $S_n = \sum_{k=1}^{n} X_k$. If for every $\varepsilon > 0$, $\sum_{n=1}^{+\infty} P\{|X_n| > \varepsilon\} < \infty$, then $\{X, X_n, n \geq 1\}$ is said to converge to 0 completely.

Hsu and Robbins [1] proved that if $\{X, X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $EX_1 = \mu$, and $EX_1^2 < \infty$, then $S_n/n \to \mu$ completely.

Erdos [2, 3] proved that if $\{X, X_n, n \geq 1\}$ is a sequence of i.i.d. random variables, then for every $\varepsilon > 0$, $\sum_{n=1}^{+\infty} P\{|S_n|/n > \varepsilon\} < \infty$ holds if and only if $EX_1 = \mu$ and $EX_1^2 < \infty$.

Obviously the sum tends to infinity as $\varepsilon \downarrow 0$, and it is necessary to study the convergence rate in which this occurs; Heyde [4] proved that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{+\infty} P\left(\left|S_n\right| \geq \varepsilon n\right) = EX^2, \tag{1}$$

when $EX = 0$ and $EX^2 < \infty$. This research direction is known as the precise asymptotics. For analogous results in more general case, we refer the reader to [5–14] and the references therein.

Recently, Liu and Lin [15] have introduced a new kind of complete moment convergence and obtained the following result.

Theorem A (see [15]). *Suppose that $\{X, X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables. Then*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} E\left[S_n^2 I\left\{\left|S_n\right| \geq \varepsilon n\right\}\right] = 2\sigma^2 \tag{2}$$

holds if and only if $EX_k = 0$, $E[X_k^2] = \sigma^2$, and $E[X_k^2 \log^+ |X_k|] < \infty$, where $k \in Z$ and $k \geq 1$.

However, the condition of identical distribution is very strong and rather difficult to verify in some real cases. The following theorem gives a sufficient condition of complete moment convergence for independent nonidentically distributed random variables.

Theorem 1. *Let $\{X, X_n, n \geq 1\}$ be independent random variables such that $EX_n = 0$ and $E[X_n^2] = \sigma_n^2 < \infty$, $n \in N$. Assume that there exists a constant c such that $|X_k| \leq cB_n$, a.s., where $B_n^2 = \sum_{k=1}^{n} \sigma_k^2$. Moreover, one also assumes that the following Lyapunov condition [16, page 298] is satisfied:*

$$\lim_{n \to +\infty} B_n^{-2-\delta} \sum_{j=1}^{n} E\left|X_j\right|^{2+\delta} = 0, \tag{3}$$

where $0 < \delta \leq 1$. *Then, one has*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=1}^{+\infty} \frac{1}{n} E\left[\frac{S_n^2}{B_n^2} I_{\{|S_n|/B_n \geq \varepsilon\sqrt{n}\}}\right] = 2. \tag{4}$$

Remark 2. Suppose that $\{X, X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $E[|X_n^{2+\delta}|] < \infty$, where $0 < \delta \leq 1$ is a constant. It is easy to verify the Lyapunov condition (3) in real applications, and so the Lyapunov condition (3) is much weaker than the identically distributed condition. Moreover, the Lyapunov condition (3) constrains the growth rate of moment.

Many sequences of independent random variables satisfy Lyapunov's condition; here we give some examples.

Example 1. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent random variables satisfying $EX_k = \mu_k$, $\operatorname{Var} X_k = \sigma_k^2$, $k \geq 1$, and $B_n^2 \to \infty (n \to \infty)$. Suppose that X_n, $n \geq 1$ are uniformly bounded; that is, there exists a constant M such that $|X_n| \leq M$ for all $n \geq 1$, and then we have

$$\sum_{k=1}^{n} E\left|X_k - \mu_k\right|^{2+1} \leq 2M \sum_{k=1}^{n} E\left|X_k - \mu_k\right|^2 = 2MB_n^2,$$
$$\lim_{n\to+\infty} B_n^{-2-1} \sum_{j=1}^{n} E\left|X_j\right|^{2+1} \leq \lim_{n\to+\infty} \frac{2MB_n^2}{B_n^{2+1}} = 2M \lim_{n\to+\infty} \frac{1}{B_n} = 0, \tag{5}$$

which verifies that $\{X, X_n, n \geq 1\}$ satisfies the Lyapunov condition (3).

Example 2. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent random variables, which satisfies $P(X_n = 1) = p_n$, $P(X_n = 0) = 1 - p_n$, and

$$p_n = \begin{cases} \frac{1}{2}, & n = 2k,\ k \in N, \\ \frac{1}{3}, & n = 2k+1,\ k \in N. \end{cases} \tag{6}$$

By Example 1, we know that $\{X, X_n, n \geq 1\}$ satisfies the Lyapunov condition (3).

Remark 3. Suppose that $\{X, X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables such that $EX_n = 0$, $E[X_n^2] = \sigma^2$ for all $n \geq 1$, $E[|X_n^{2+\delta}|] < \infty$, where $\delta > 0$ is a constant. Then, from Remark 2, we know that it satisfies Lyapunov's condition. Therefore, by Theorem 1, we have

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} E\left[S_n^2 I\left\{\left|S_n\right| \geq \varepsilon n\right\}\right] = 2\sigma^2. \tag{7}$$

Obviously, this case is the result of Liu and Lin [15]. Therefore, our condition of Theorem 1 is different from the conditions of Theorem A, and our result partly extends and improves those given in Liu and Lin [15].

2. Proof of Theorem 1

In this section, we will prove Theorem 1. We first present the following two lemmas, which play a key role in the proof of Theorem 1.

Lemma 4 (see [17]). *Suppose that* $\{X, X_n, n \geq 1\}$ *are independent random variables with* $EX_n = 0$ *and* $E[X_n^2] = \sigma_n^2 < \infty$, *where* $n \in N$. *Let* $B_n^2 = \sum_{j=1}^{n} \sigma_j^2$, $F_n(x) = P((\sum_{j=1}^{n} X_j)/B_n \leq x)$, *and* $\Delta_n(x) = |F_n(x) - \Phi(x)|$, *where* $\Phi(x)$ *is the standard normal distribution function. If* $E|X_j|^{2+\delta} < \infty$, $j = 1, 2, \ldots, n$, *for some* $0 \leq \delta \leq 1$, *then for every* x,

$$\Delta_n(x) \leq AB_n^{-2-\delta} \sum_{j=1}^{n} E\left|X_j\right|^{2+\delta} \left(1 + |x|^{2+\delta}\right)^{-1} \tag{8}$$

holds.

Lemma 5 (see page 73 of [18]). *Under the conditions of Lemma 4, if* $|X_j| \leq cB_n$ *a.s.,* $j = 1, 2, \ldots, n$, *where* $c > 0$, *then for every* $x > 0$,

$$P\left(\frac{\left|S_n\right|}{B_n} \geq x\right) \leq 2\exp\left(-\frac{x}{2c}\sinh\left(\frac{xc}{2}\right)\right). \tag{9}$$

Proof of Theorem 1. Similar to [15], we have

$$\begin{aligned} &\sum_{n=1}^{+\infty} \frac{1}{n} E\left[\frac{S_n^2}{B_n^2} I\left\{\frac{\left|S_n\right|}{B_n} \geq \varepsilon\sqrt{n}\right\}\right] \\ &= \varepsilon^2 \sum_{n=1}^{+\infty} P\left(\frac{\left|S_n\right|}{B_n} \geq \varepsilon\sqrt{n}\right) + \sum_{n=1}^{+\infty} \frac{1}{n} \int_{\varepsilon\sqrt{n}}^{+\infty} 2tP\left(\frac{\left|S_n\right|}{B_n} \geq t\right) dt \\ &:= I_1 + I_2. \end{aligned} \tag{10}$$

To prove Theorem 1, we only need to study I_1 and I_2. We will divide the proof into two steps.

Step 1. We first prove the equality as follows:

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \varepsilon^2 \sum_{n=1}^{+\infty} P\left(\frac{\left|S_n\right|}{B_n} \geq \varepsilon\sqrt{n}\right) = 0. \tag{11}$$

In fact, it follows from Proposition 2.1.1 of [19] that

$$\lim_{\varepsilon\downarrow 0} \varepsilon^2 \sum_{n=1}^{+\infty} P\left(|N| \geq \varepsilon\sqrt{n}\right) = 1. \tag{12}$$

By (12), we obtain

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \varepsilon^2 \sum_{n=1}^{+\infty} P\left(|N| \geq \varepsilon\sqrt{n}\right) = 0. \tag{13}$$

To establish the equality (11), from (13) we only need to prove that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \varepsilon^2 \sum_{n=1}^{+\infty} \left|P\left(\frac{\left|S_n\right|}{B_n} \geq \varepsilon\sqrt{n}\right) - P\left(|N| \geq \varepsilon\sqrt{n}\right)\right| = 0. \tag{14}$$

Obviously, it follows from Lemma 4 that

$$\left|P\left(\frac{|S_n|}{B_n}\geq \varepsilon\sqrt{n}\right)-P\left(|N|\geq \varepsilon\sqrt{n}\right)\right| = 2\Delta_n\left(\varepsilon\sqrt{n}\right) \leq 2AB_n^{-2-\delta}\left(1+\left|\varepsilon\sqrt{n}\right|^{2+\delta}\right)^{-1}\sum_{j=1}^{n}E\left|X_j\right|^{2+\delta}. \quad (15)$$

Combining (3) and (15), we get

$$\lim_{n\to+\infty}\left|P\left(\frac{|S_n|}{B_n}\geq \varepsilon\sqrt{n}\right)-P\left(|N|\geq \varepsilon\sqrt{n}\right)\right| = 0. \quad (16)$$

Since

$$\lim_{\varepsilon\downarrow 0}\sum_{n=1}^{[1/\varepsilon^2]}\varepsilon^2 \leq 1, \quad (17)$$

it follows from Toeplitz's lemma (page 120 of [20]) that

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2\sum_{n=1}^{[1/\varepsilon^2]}\left|P\left(\frac{|S_n|}{B_n}\geq \varepsilon\sqrt{n}\right)-P\left(|N|\geq \varepsilon\sqrt{n}\right)\right| = 0. \quad (18)$$

By (18), we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\varepsilon^2\sum_{n=1}^{[1/\varepsilon^2]}\left|P\left(\frac{|S_n|}{B_n}\geq \varepsilon\sqrt{n}\right)-P\left(|N|\geq \varepsilon\sqrt{n}\right)\right| = 0. \quad (19)$$

On the other hand, it follows from Lemma 5 that

$$\begin{aligned}\varepsilon^2\sum_{n\geq[1/\varepsilon^2]}P\left(\frac{|S_n|}{B_n}\geq \varepsilon\sqrt{n}\right) &\leq 2\varepsilon^2\int_{1/\varepsilon^2}^{+\infty}\exp\left(-\frac{\varepsilon\sqrt{t}}{2c}\sinh\left(\frac{\varepsilon\sqrt{t}c}{2}\right)\right)dt\\ &= 2\varepsilon^2\int_{1}^{+\infty}\exp\left(-\frac{u}{2c}\sinh\left(\frac{uc}{2}\right)\right)\frac{2u}{\varepsilon^2}dt\\ &= 4\int_{1}^{+\infty}u\exp\left(-\frac{u}{2c}\sinh\left(\frac{uc}{2}\right)\right)du.\end{aligned} \quad (20)$$

Noting that

$$\int_{1}^{+\infty}u\exp\left(-\frac{u}{2c}\sinh\left(\frac{uc}{2}\right)\right)du < \infty, \quad (21)$$

the inequality (20) yields

$$\varepsilon^2\sum_{n\geq[1/\varepsilon^2]}P\left(\frac{|S_n|}{B_n}\geq \varepsilon\sqrt{n}\right) < \infty. \quad (22)$$

By (22), we obtain

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\varepsilon^2\sum_{[1/\varepsilon^2]}^{+\infty}\left|P\left(\frac{|S_n|}{B_n}\geq \varepsilon\sqrt{n}\right)-P\left(|N|\geq \varepsilon\sqrt{n}\right)\right| = 0. \quad (23)$$

Combining (19) and (23), we see that the equality (11) is satisfied.

Step 2. Next, we need to prove the following equality:

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=1}^{+\infty}\frac{1}{n}\int_{\varepsilon\sqrt{n}}^{+\infty}2tP\left(\frac{|S_n|}{B_n}\geq t\right)dt = 2. \quad (24)$$

Obviously, it follows from Proposition 3.1 of [15] that

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=1}^{+\infty}\frac{1}{n}\int_{\varepsilon\sqrt{n}}^{+\infty}2tP\left(|N|\geq t\right)dt = 2. \quad (25)$$

To establish (24), from (25) we only need to prove

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=1}^{+\infty}\frac{1}{n}\int_{\varepsilon\sqrt{n}}^{+\infty}2t\left|P\left(\frac{|S_n|}{B_n}\geq t\right)-P\left(|N|\geq t\right)\right|dt = 0. \quad (26)$$

Letting $x = t/\sqrt{n}-\varepsilon$ and $L_n = (A\sum_{j=1}^{n}E|X_j|^{2+\delta})/B_n^{2+\delta}$, we apply Lemma 4 to obtain

$$\begin{aligned}\int_{\varepsilon\sqrt{n}}^{+\infty}2t\left|P\left(\frac{|S_n|}{B_n}\geq t\right)-P\left(|N|\geq t\right)\right|dt &= \int_{0}^{+\infty}2\sqrt{n}(x+\varepsilon)\left|P\left(\frac{|S_n|}{B_n}\geq \sqrt{n}(x+\varepsilon)\right)\right.\\ &\qquad\left.-P\left(|N|\geq \sqrt{n}(x+\varepsilon)\right)\right|\sqrt{n}\,dx \quad (27)\\ &= 2\int_{0}^{+\infty}n(x+\varepsilon)\,2\Delta_n\left(\sqrt{n}(x+\varepsilon)\right)dx\\ &\leq 2\int_{0}^{+\infty}n(x+\varepsilon)L_n\frac{1}{1+\left[\sqrt{n}(x+\varepsilon)\right]^{2+\delta}}dx.\end{aligned}$$

If $n \leq [1/\varepsilon^2]$, then it follows from (3) that

$$\begin{aligned}&\int_{0}^{1/\sqrt{n}}n(x+\varepsilon)L_n\frac{1}{1+\left[\sqrt{n}(x+\varepsilon)\right]^{2+\delta}}dx\\ &\quad\leq \int_{0}^{1/\sqrt{n}}n(x+\varepsilon)L_n\,dx\\ &\quad= \frac{1}{2}nL_n(x+\varepsilon)^2\Big|_{0}^{1/\sqrt{n}} \leq \frac{3}{2}L_n \longrightarrow 0 \quad (n\longrightarrow\infty),\\ &\int_{1/\sqrt{n}}^{1}n(x+\varepsilon)L_n\frac{1}{1+\left[\sqrt{n}(x+\varepsilon)\right]^{2+\delta}}dx\\ &\quad\leq \int_{1/\sqrt{n}}^{1}n^{-\delta/2}(x+\varepsilon)^{-1-\delta}L_n\,dx\end{aligned}$$

$$\le \int_{1/\sqrt{n}}^{1} n^{-\delta/2} x^{-1-\delta} L_n \, dx$$

$$= L_n n^{-\delta/2} \left[-\frac{1}{\delta} x^{-\delta} \right] \Bigg|_{1/\sqrt{n}}^{1} \le \frac{1}{\delta} L_n \longrightarrow 0 \quad (n \longrightarrow \infty),$$

$$\int_1^{+\infty} n(x+\varepsilon) \frac{1}{n^{1+\delta/2}(x+\varepsilon)^{2+\delta}} dx$$

$$= n^{-\delta/2} \int_1^{+\infty} (x+\varepsilon)^{-1-\delta} dx \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{28}$$

Hence, by (27) and (28), we have that for $n \le [1/\varepsilon^2]$, the following holds:

$$\int_{\varepsilon\sqrt{n}}^{+\infty} 2t \left| P\left(\frac{|S_n|}{B_n} \ge t \right) - P(|N| \ge t) \right| dt \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{29}$$

Noting the fact that the weighted average of a sequence that converge to 0 also converges to 0, we have

$$\sum_{n=1}^{[1/\varepsilon^2]} \frac{1}{n} \int_{\varepsilon\sqrt{n}}^{+\infty} 2t \left| P\left(\frac{|S_n|}{B_n} \ge t \right) - P(|N| \ge t) \right| dt \longrightarrow 0, \quad \text{as } \varepsilon \longrightarrow 0, \tag{30}$$

and so

$$\lim_{\varepsilon \downarrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{[1/\varepsilon^2]} \frac{1}{n} \int_{\varepsilon\sqrt{n}}^{+\infty} 2t \left| P\left(\frac{|S_n|}{B_n} \ge t \right) - P(|N| \ge t) \right| dt = 0. \tag{31}$$

If $n \ge [1/\varepsilon^2]$, it follows from (8) and (3) that

$$\int_{\varepsilon\sqrt{n}}^{+\infty} 2t \left| P\left(\frac{|S_n|}{B_n} \ge t \right) - P(|N| \ge t) \right| dt$$

$$\le \int_1^{+\infty} 4tAB_n^{-2-\delta} \left(1 + |t|^{2+\delta}\right)^{-1} \sum_{j=1}^{n} E\left|X_j\right|^{2+\delta} dt \tag{32}$$

$$\le 4L_n \int_1^{+\infty} \frac{1}{t^{1+\delta}} dt \longrightarrow 0 \quad (n \longrightarrow \infty).$$

Obviously, by (32), we get

$$\lim_{\varepsilon \downarrow 0} \frac{1}{-\log \varepsilon} \sum_{n=[1/\varepsilon^2]}^{+\infty} \frac{1}{n} \int_{\varepsilon\sqrt{n}}^{+\infty} 2t \left| P\left(\frac{|S_n|}{B_n} \ge t \right) - P(|N| \ge t) \right| dt = 0. \tag{33}$$

Combining (31) and (33), we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{+\infty} \frac{1}{n} \int_{\varepsilon\sqrt{n}}^{+\infty} 2t \left| P\left(\frac{|S_n|}{B_n} \ge t \right) - P(|N| \ge t) \right| dt = 0, \tag{34}$$

which implies that (24) is satisfied.

Therefore, from (11) and (34), we see that (4) is true. This completes the proof of Theorem 1. □

Conflict of Interests

The authors declare that they have no competing interests.

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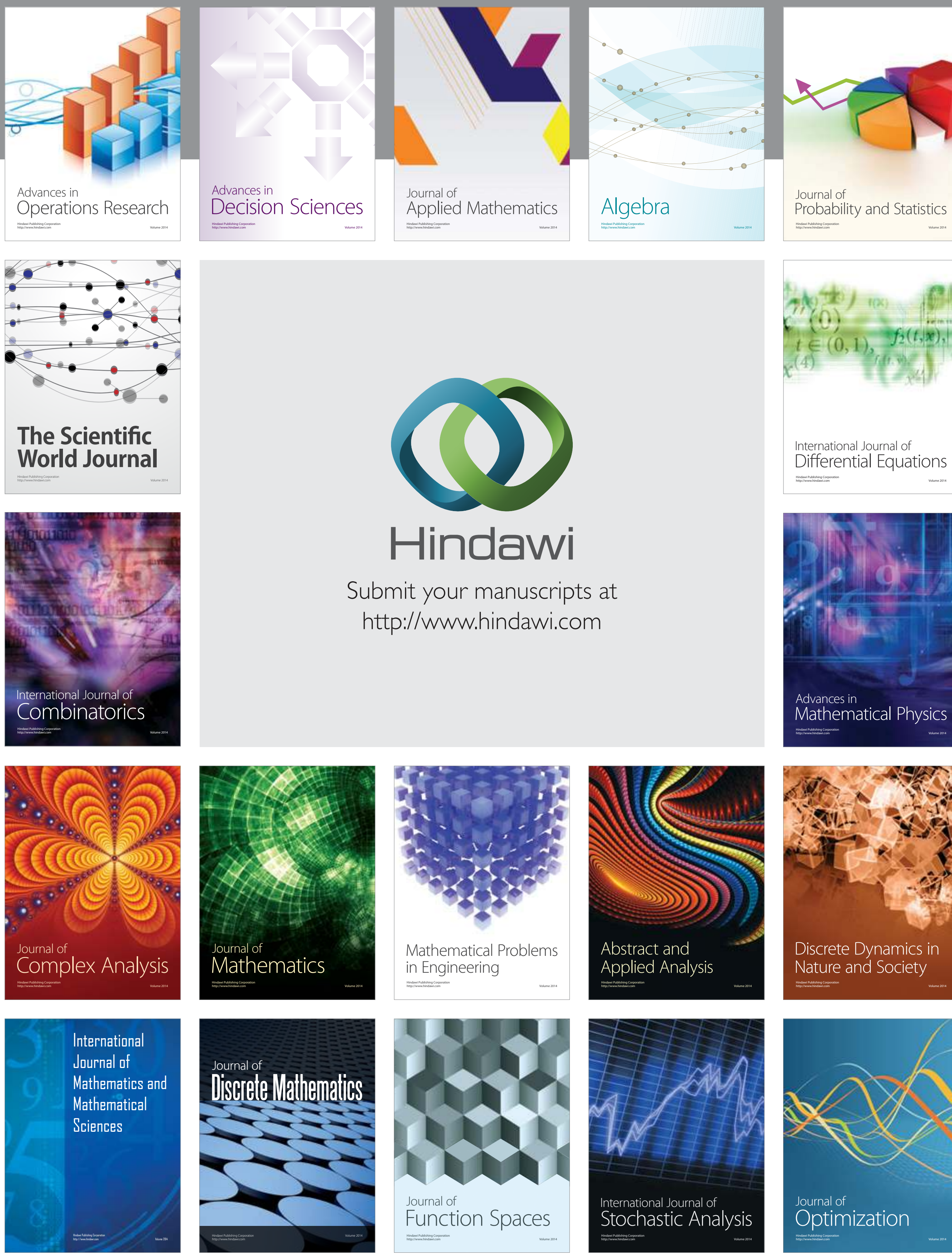
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