A KLT-inspired Node Centrality for Identifying Influential Neighborhoods in Graphs

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***Abstract*—We present principal component centrality (PCC) as a measure of centrality that is more general and encompasses eigenvector centrality (EVC). We explain some of the difficulties in applying EVC to graphs and networks that contain more than just one neighborhood of nodes with high influence. We demonstrate the shortcomings of traditional EVC and contrast it against PCC. PCC's ranking procedure is based on spectral analysis of the network's graph adjacency matrix and identification of its most significant eigenvectors.**

I. Introduction

Centrality [3], [4], [6], [14], [20] is a measure to assess the criticality of a node's position. Node centrality is a measure of a node's importance by virtue of its criticality to the control/ ability to disrupt the flow of commodity in a network. Over the years several different meanings of centrality have emerged based on the context. Among the many centrality measures, eigenvalue centrality (EVC) is arguably the most successful tool for detecting the most influential node(s) within a social graph. Thus, EVC has been a highly popular centrality measure in the social sciences ([16], [23], [3], [13], [11], [12], [24], [22], [5], [4], [18], [14], [25]), where it is often referred to simply as centrality. As we demonstrate later in this paper, a key shortcoming of EVC is its focus on (virtually) only a single community of nodes clustered into a single neighborhood. In other words, EVC has the tendency of identifying a set of influential nodes that are all within the same region of a graph. Meanwhile, when dealing with massive networks/graphs, it is not necessarily always the case that there is only a single community of influential nodes; rather, there may be multiple communities. EVC by its very nature focuses on the largest community, to the detriment of other, smaller but perhaps still significant communities.

In order to identify influential neighborhoods, there is a need to associate such neighborhoods with some form of an objective measure of centrality that can be evaluated and searched for. To that end, one can think of a centrality plane that is overlaid over the underlying graph under consideration. This centrality plane may contain multiple centrality score maxima, each of which is centered on an influential neighborhood. Nodes that have centrality score higher than other nodes are located under a centrality peak and are more central than any of their neighbors. We use the term hubs to refer to nodes forming centrality maxima. Figure 1 illustrates this concept. Thus, these hubs form the kernel of influential neighborhoods in networks. We will show that EVC has a tendency to be too narrowly focused on a dominating neighborhood. To this end, we introduce a new measure of centrality that we call principal component centrality (PCC) that gradually widens the focus of EVC in a controlled manner. More importantly, PCC provides a general framework for transforming graphs into a spectral space analogous to popular signal transforms that operate on random signals. In essence, PCC is a general transform of graphs that can provide vital insight into the centrality and related characteristics of such graphs. Similar to the Karhunen Loève transform (KLT) of a signal, the proposed PCC of a graph gives a form of compact representation that identifies *influential nodes* and, more importantly, *influential neighborhoods*. Hence, PCC provides an elegant *graph transform* framework.

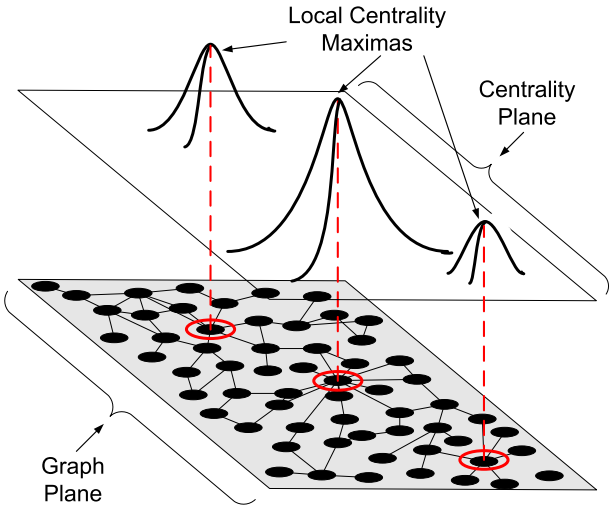

Fig. 1. This figure shows a graph on the lower plane, overlayed with another plane of the interpolated surface plot of node centrality scores. The centrality planes typically exhibit a number of peaks or local maxima.

The rest of this chapter is organized as follows. Section II gives a brief review of EVC accompanied by a critique of its application to graph topologies found in wireless networks

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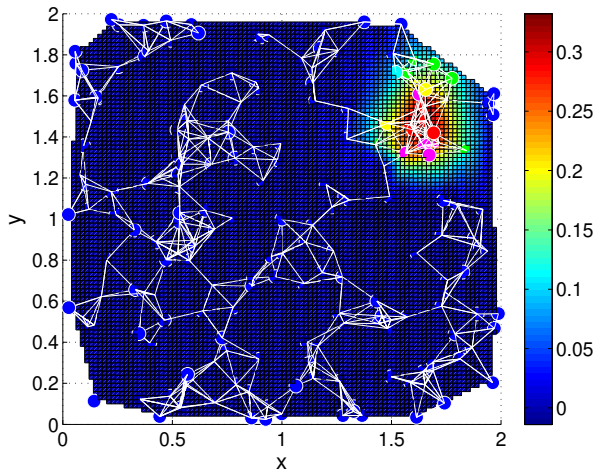

Fig. 2. A spatial graph of 200 nodes. Node colors are indicative of the range in which their EVC falls.

that motivated the development of a new node centrality. Section III introduces PCC, a node centrality measure that is inspired by the KLT and principal component analysis (PCA). In particular, we demonstrate EVC's shortcoming by using both EVC and PCC to compute node centralities in a network small enough to allow meaningful illustration. We also develop the equivalence of an inverse PCC transform that attempts to reconstruct a representation of the original graph from its influential neighborhoods. Section IV describes in detail the advantages, mathematical interpretation, visualization and the effect of varying number of features of PCC. Section V concludes the chapter.

II. Background

Let $\mathbf{A}$ denote the adjacency matrix of a graph $G(V, E)$ consisting of the set of nodes $V = \{v_1, v_2, v_3, \ldots, v_N\}$ of size N and set of undirected edges E. When a link is present between two nodes v_i and v_j both $A_{i,j}$ and $A_{j,i}$ are set equal to 1 and set to 0 otherwise. Let $\Gamma(v_i)$ denote the neighborhood of v_i, the set of nodes v_i is connected to directly. EVC is a relative score recursively defined as a function of the number and strength of connections to its neighbors and as well as those neighbors' centralities. Let $x(i)$ be the EVC score of a node v_i. Then,

$$\begin{aligned} x(i) &= \frac{1}{\lambda} \sum_{j \in \Gamma(v_i)} x(j) \\ &= \frac{1}{\lambda} \sum_{j=1}^{N} A_{i,j} x(j) \end{aligned} \tag{1}$$

Here λ is a constant. Equation 1 can be rewritten in vector form equation 2 where $\mathbf{x} = [x(1), x(2), x(3), \ldots, x(N)]'$ is the vector of EVC scores of all nodes.

$$\begin{aligned} \mathbf{x} &= \frac{1}{\lambda} \mathbf{A}\mathbf{x} \\ \lambda \mathbf{x} &= \mathbf{A}\mathbf{x} \end{aligned} \tag{2}$$

This is the well known eigenvector equation where this centrality takes its name from. λ is an eigenvalue and $\mathbf{x}$ is the corresponding eigenvector of matrix $\mathbf{A}$. Obviously several eigenvalue/eigenvector pairs exist for an adjacency matrix $\mathbf{A}$. The EVC of nodes are defined on the basis of the Perron eigenvalue λ_A (the Perron eigenvalue is the largest of all eigenvalues of $\mathbf{A}$ and is also called the principal eigenvalue). If λ is any other eigenvalue of $\mathbf{A}$ then $\lambda_A > |\lambda|$. The eigenvector $\mathbf{x} = [x(1), x(2), \ldots, x(N)]'$ corresponding to the Perron eigenvalue is the Perron eigenvector or principal eigenvector. Thus the EVC of a node v_i is the corresponding element $x(i)$ of the Perron eigenvector $\mathbf{x}$. Note that when the adjacency matrix $\mathbf{A}$ is symmetric all elements of the principal eigenvector $\mathbf{x}$ are positive.

In computing a node's EVC it takes into consideration its neighbors's EVC scores. Because of its recursive definition, EVC is suited to measure nodes' power to influence other nodes in the network both directly and indirectly through its neighbors. Connections to neighbors that are in turn well connected themselves are rated higher than connections to neighbors that are weakly connected. Like closeness and betweenness, the EVC of a node provides a network-wide perspective. At the same time it can take advantage of distributed methods of computing eigenvectors/eigenvalues of a matrix but does not have to bear the overhead of excess network traffic. Sankaralingam [21], Kohlschütter [17] and Canright, Engø-Monsen and Jelasity [7], Bischof [2], Bai [1] and Tisseur [22] proposed some parallel algorithms for computing eigenvectors and eigenvalues of adjacency matrices.

EVC has been used extensively to great effect in the study and analysis of a wide variety of networks that are shown to exhibit *small-world* and *scale-free* properties. In [8] Canright and Engø-Monsen correlated EVC with the instantaneous rate of spread of contagion on a Gnutella network peer-to-peer graph, a social network of students in Oslo, a collaboration graph of researchers at Telenor R&D and a snapshot of a collaboration graph of the Santa Fe Institute. In [19] Newman analyzed the use of EVC in a lexical network of co-occuring words in Reuters newswire stories. In [9] Carreras et al. used EVC to study the spread of epidemics in mobile networks.

Now consider the graph in figure 2. It consists of 200 nodes and is typical of wireless (sensor) networks such as the ones described by Gupta and Kumar in [15]. Its nodes are assigned one of six colors from the adjacent color palette. Each of the six colors represents one of six bins of a histogram spanning, in uniform step sizes, the range from the smallest to the largest EVCs. As the legend accompanying figure 2 shows, blue represents the lowest EVCs and red the highest. We make the following observations:

1) EVCs are tightly clustered around a very small region

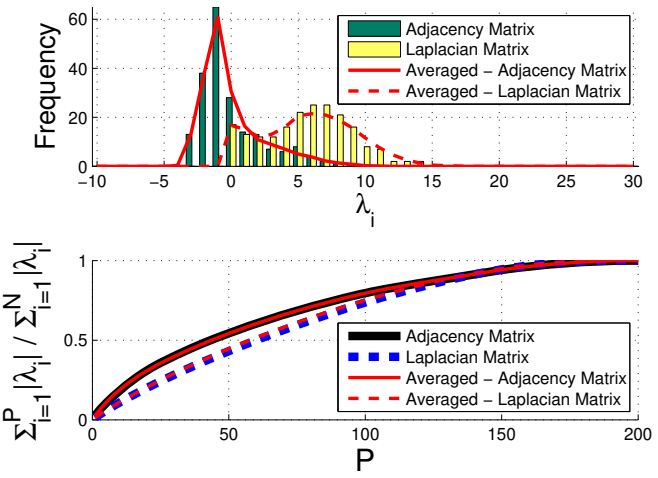

Fig. 3. [Top] Histogram of eigenvalues of adjacency matrix and Laplacian matrix $\mathbf{A}$ of network in figure 2; [Bottom] Cumulative sum of the sequence of eigenvalues of adjacency matrix and Laplacian matrix of network in figure 2 when sorted in descending order of magnitudes. In both figures the lines plotted in red color are averages of 50 networks generated randomly with the same parameters.

with respect to the total size of the network and drops off sharply as one moves away from the node of peak EVC.

2) EVC is unable to provide much centrality information for the vast majority of nodes in the network.
3) The position of the peak EVC node appears somewhat 'arbitrary' because a visual inspection shows that almost equally significant clusters of nodes can be visually spotted in other locations in the graph. Counter to intuition, the high EVC cluster is connected to the rest of the network by a single link.

III. Principal Component Centrality

The EVC of a node is recursively defined as a measure of centrality that is proportional to the number of neighbors of a node and their respective EVCs. As equation 2 shows, the mathematical expression for the vector of node EVCs is equivalent to the principal eigenvector. Our motivation for PCC as a new measure of node centrality may be understood by looking at EVC through the lens of the KLT. When the KLT is derived from an $N \times N$ covariance matrix of N random variables, the principal eigenvector is the most dominant feature vector, i.e. the direction in N-dimensional hyperspace along which the spread of data points is maximized. Similarly, the second eigenvector (corresponding to the second largest eigenvalue) is representative of the second most significant feature of the data set. It may also be thought of as the most significant feature after the data points are collapsed along the direction of the principal eigenvector. When the covariance matrix is computed empirically from a set of data points, the eigendecomposition is the well known PCA [11]. Since we are operating on the adjacency matrix derived from graph data we call the node centrality proposed in this research PCC. In a covariance matrix, a non-zero entry with a 'large' magnitude at positions (i, j) and (j, i) is representative of a strong relationship between the i-th and j-th random variables. A non-zero entry in the adjacency matrix representing a link from one node to another is, in a broad sense, also an indication of a 'relationship' between the two nodes. Based on this understanding we draw an analogy between graph adjacency matrix and covariance matrix.

EVC is the node centrality most often used in the study of social networks and other networks with small-world properties. While EVC assigns centrality to nodes according to the strength of the most dominant feature of the data set, PCC takes into consideration additional, subsequent features. We define the PCC of a node in a graph as the Euclidean distance/ℓ^2 norm of a node from the origin in the P-dimensional eigenspace formed by the P most significant eigenvectors. For a graph consisting of a single connected component, the N eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_N| = 0$ correspond to the normalized eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$. The eigenvector/eigenvalue pairs are indexed in order of descending magnitude of eigenvalues. When $P = 1$, PCC equals a scaled version of EVC. Unlike other measures of centrality, the parameter P in PCC can be used as a tuning parameter to adjust the number of eigenvectors included in the PCC. The question of selection of an appropriate value of P will be addressed in subsequent subsection IV-D. Let $\mathbf{X}$ denote the $N \times N$ matrix of concatenated eigenvectors $\mathbf{X} = [\mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_N]$ and let $\Lambda = [\lambda_1 \lambda_2 \ldots \lambda_N]'$ be the vector of eigenvalues. Furthermore, if $P < N$ and if matrix $\mathbf{X}$ has dimensions $N \times N$, then $\mathbf{X}_{N \times P}$ will denote the submatrix of $\mathbf{X}$ consisting of the first N rows and first P columns. Then PCC can be expressed in matrix form as:

$$\mathbf{C}_P = \sqrt{((\mathbf{A}\mathbf{X}_{N\times P}) \odot (\mathbf{A}\mathbf{X}_{N\times P}))\, \mathbf{1}_{P\times 1}} \tag{3}$$

The '$\odot$' operator is the Hadamard (or entrywise product or Schur product) operator. Equation 3 can also be written in terms of the eigenvalue and eigenvector matrices Λ and $\mathbf{X}$, of the adjacency matrix $\mathbf{A}$:

$$\mathbf{C}_P = \sqrt{(\mathbf{X}_{N\times P} \odot \mathbf{X}_{N\times P})\, (\Lambda_{P\times 1} \odot \Lambda_{P\times 1})}. \tag{4}$$

It is important to note a major difference between a traditional "signal transform" under KLT as compared with the proposed PCC "graph transform". First, recall that, under KLT, a transform matrix $\mathbf{T}$ is derived from a covariance matrix $\mathbf{C}$; and then the eigenvector-based transform $\mathbf{T}$ is applied on any realization of the random signal that has covariance $\mathbf{C}$. Meanwhile, under the proposed PCC, the adjacency matrix $\mathbf{A}$ plays a dual role: at one hand, it plays the role of the covariance matrix of the KLT; and on the other hand, one can think of $\mathbf{A}$ as being the "signal" that is represented compactly by the PCC vector $\mathbf{C}_P$. Effectively, the adjacency matrix $\mathbf{A}$ represents the graph (i.e., "signal") that we are interested in analyzing; and at the same time $\mathbf{A}$ is used to derive the eigendecomposition; and hence, we have the dual role for $\mathbf{A}$.

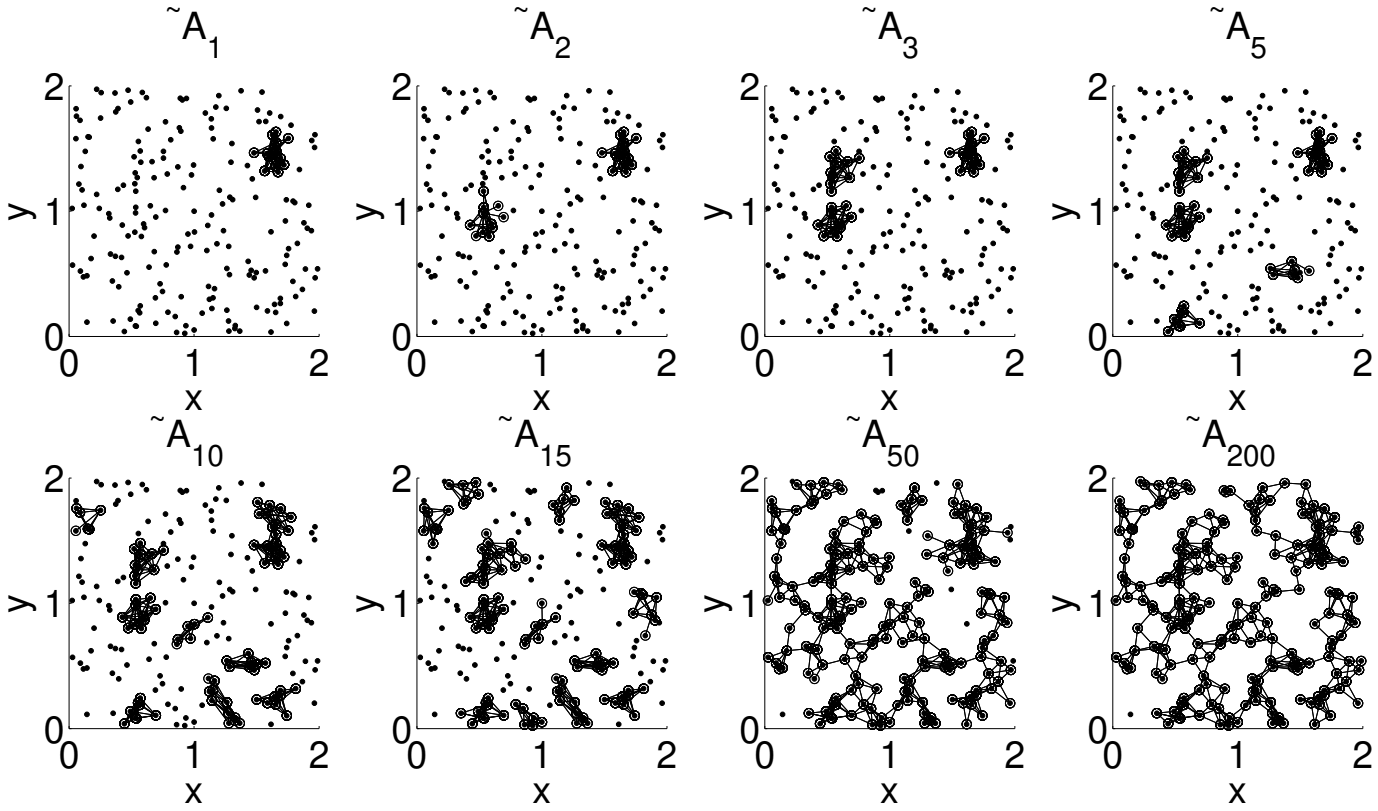

Fig. 4. Reconstructed topologies of the graph from figure 2 using only the first 1, 2, 3, 5, 10, 15, 50 and all 200 eigenvectors.

Later, we will develop the equivalence of an inverse PCC, and we will see this dual role of the adjacency matrix $\mathbf{A}$ again.

IV. Evaluation

A. Interpretation of Eigenvalues

The definition of PCC is based on the graph adjacency matrix $\mathbf{A}$. For a matrix $\mathbf{A}$ of size $N \times N$ its eigenvectors $\mathbf{x}_i$ for $1 \leq i \leq N$ are interpreted as N-dimensional features (feature vectors) of the set of N-dimensional data points represented by their covariance (adjacency) matrix $\mathbf{A}$. The magnitude of an eigenvalue corresponding to an eigenvector provides a measure of the importance and prominence of the feature represented by it. The eigenvalue λ_i is the power of the corresponding feature $\mathbf{x}_i$ in $\mathbf{A}$.

An alternative representation of a graph's topology is the graph Laplacian matrix which is frequently used in spectral graph theory [10]. The graph Laplacian can be obtained from the adjacency matrix by setting the diagonal entries of the adjacency matrix to $A_{i,i} = -\sum_{j=1;i\neq j}^{N} A_{i,j}$, i.e. a diagonal entry in a Laplacian matrix is the negative of the sum of all off-diagonal entries in the same row in the adjacency matrix. This definition applies equally to weighted and unweighted graphs. The graph Laplacian is always positive-semidefinite which means all of its eigenvalues are non-negative with at least one eigenvalue equal to 0. The adjacency matrix, however, does not guarantee positive semidefiniteness and typically has several negative eigenvalues. This is the reason the ordering of features is based on magnitudes of eigenvalues. The bar chart at the top of figure 3 plots histograms of eigenvalues for both adjacency and Laplacian matrices of the network in figure 2. But why then, did we not use the Laplacian matrix in the first place? The reason is that the eigendecomposition of the adjacency matrix yields greater energy compaction than that of the Laplacian. The middle plot in figure 3 shows the normalized, cumulative function of the sorted sequence of eigenvalue powers. The line for the eigenvalue derived from the adjacency matrix rises faster than that of the Laplacian matrix. The adjacency matrix' curve indicates that 25%, 50% and 75% of total power is captured by the first 15 (7.5%), 44 (22%) and 89 (44.5%) features, respectively. In contrast, the Laplacian matrix' eigendecomposition shows that the same power levels are contained in its first 26 (13%), 61 (30.5%) and 103 (51.5%) features, respectively. Thus eigendecomposition of the adjacency matrix of graphs offers more energy compaction, i.e. a set of features of the adjacency matrix captures more energy than the same number of features of the corresponding Laplacian matrix.

B. Interpretation of Eigenvectors

EVC interprets the elements of the Perron-eigenvector $\mathbf{x}_1$ of adjacency matrix $\mathbf{A}$ as measures of corresponding nodes' centralities in the network topology (see section **??**). Research on scale-free network topologies has demonstrated EVC's

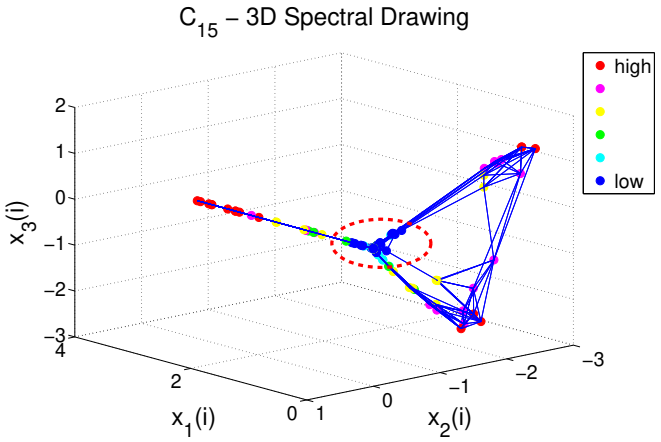

Fig. 5. Spectral drawing of graph in three dimensions using entries of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ for the three coordinate axes. Nodes are colored according to their $\mathbf{C}_{15}$ PCC.

usefulness. However, when applied to large spatial graphs of uniformly, randomly deployed nodes such as the one in figure 1, EVC fails to assign significant scores to a large fraction of nodes. For a broader understanding that encompasses all eigenvectors we revert to the interpretation of eigenvectors as features. One way of understanding PCC is in terms of PCA [11], where PCC takes part of its name from. PCA finds the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_N$ and eigenvalues of G's adjacency matrix $\mathbf{A}$. Every eigenvector represents a feature of the adjacency matrix. To understand how these feature vectors are to be interpreted in graphical terms, refer to equation 5 which uses eigenvectors and eigenvalues to reconstruct an approximation $\tilde{\mathbf{A}}_P$ of the adjacency matrix $\mathbf{A}$. Reconstruction can be performed to varying degrees of accuracy depending on P, the number of features/ eigenvectors used. If we set $P = N$ in equation 5 (all eigenvectors/eigenvalues are used), the adjacency matrix can be reconstructed without losses (see He [16]). Here, $\widehat{\Lambda}$ denotes the diagonal matrix of eigenvalues sorted in descending order of magnitude on the diagonal (from upper left corner to lower right corner).

$$\tilde{\mathbf{A}}_P = \mathbf{X}_{N\times P}\widehat{\Lambda}_{P\times N}\mathbf{X}^T_{N\times N} \tag{5}$$

where $\widehat{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$. To illustrate, consider the unweighted, undirected graph $G(V, E)$ shown in figure 2 with adjacency matrix $\mathbf{A}$. $\mathbf{A}$'s entries are either 0 or 1. However, this is not necessarily true for $\tilde{\mathbf{A}}_P$, the version of the matrix reconstructed using the P most significant eigenvectors. The entries in $\tilde{\mathbf{A}}_P$ will very likely contain a lot of fractions. Therefore, before viewing the recovered topology in the reconstructed adjacency matrix $\tilde{\mathbf{A}}_P$ its entries have to be thresholded. Prior to plotting the topology, we rounded values less than 0.5 down to 0 and round values larger than or equal to 0.5 up to 1. Figure 4 plots the adjacency matrix reconstructed from the most significant 1, 2, 3, 5, 10, 15, 50 and all 200 feature vectors. The plot for $\tilde{\mathbf{A}}_1$ shows that the recovered topology information is highly localized to the vicinity of nodes with the highest EVC. The plot using $\tilde{\mathbf{A}}_2$ adds another highly connected but still very localized cluster to the network. Adding more feature vectors extends the set of connected nodes in various parts of the network. As more eigenvectors are added to the computation of PCC it has the effect of increasing the resolution of centrality scores in nodes lying in less well connected regions of the network.

C. Graphical Interpretation of PCC

In this section we evaluate the usefulness of the PCC scores assigned to nodes of a network. Recall that a node's PCC is its ℓ^2 norm in P-dimensional eigenspace. Perceptional limitations restrict us from redrawing the graph in any eigenspace with more than 3 dimensions. Figure 5 is a drawing of the graph in figure 2 in the 3-dimensional eigenspace formed by the 3 most significant eigenvectors of the adjacency matrix $\mathbf{A}$. Nodes are colored according to their $\mathbf{C}_{15}$ PCC scores, derived from the 15 most significant eigenvectors, divided into 6 equally sized intervals between the lowest and highest PCC score. Based on the interpretation of PCC we expect nodes with higher (red) PCC scores to be located farther away from the origin at $(0, 0, 0)$ than nodes with lower (blue) PCC scores. From figure 5 we can see that this is clearly the case. For clarification, the cluster of low-PCC nodes around the origin $(0, 0, 0)$ is marked with a red, dashed oval.

D. Effect of Number of Features on PCC

In this section we study the effect varying the number of eigenvectors P has on PCC. For an illustrated example we revert to the randomly generated network topology of 200 nodes in figure 2. We compute PCC while varying P from 1 through 2, 3, 5, 10, 15, 50 and 200. Figures 6a, 6b, 6c, 6d, 6e, 6f, 6g and 6h re-plot the network with nodes colored to indicate their PCC scores. The bin size for all histograms is set to 0.25. Recall that since PCC score at $P = 1$ are a scaled versions of EVC, the figure 6a represents the baseline case of EVC. In figure 6a, EVC identifies a small cluster in the upper right corner as the nodes most central to the network. Note that ironically this cluster is separable from the larger graph by the removal of merely one link! On the other hand, clusters of nodes in the larger, better connected part of the graph are assigned EVC on the low end of the scale. As P is increased from figure 6b through 6h, more clusters of high PCC nodes pop up. As expected, the accompanying histograms below each graph plot show that this has the effect of increasing the variance of PCC scores. Adding successively more features/eigenvectors will have the obvious effect of increasing the sum total of node PCC scores, i.e. $\mathbf{1}_{1\times N}\mathbf{C}_m > \mathbf{1}_{1\times N}\mathbf{C}_n$ when $m > n$. However, it is unclear how much PCC's scores change as P is varied from 1 through N. In [7] Canright *et al.* use the phase difference between

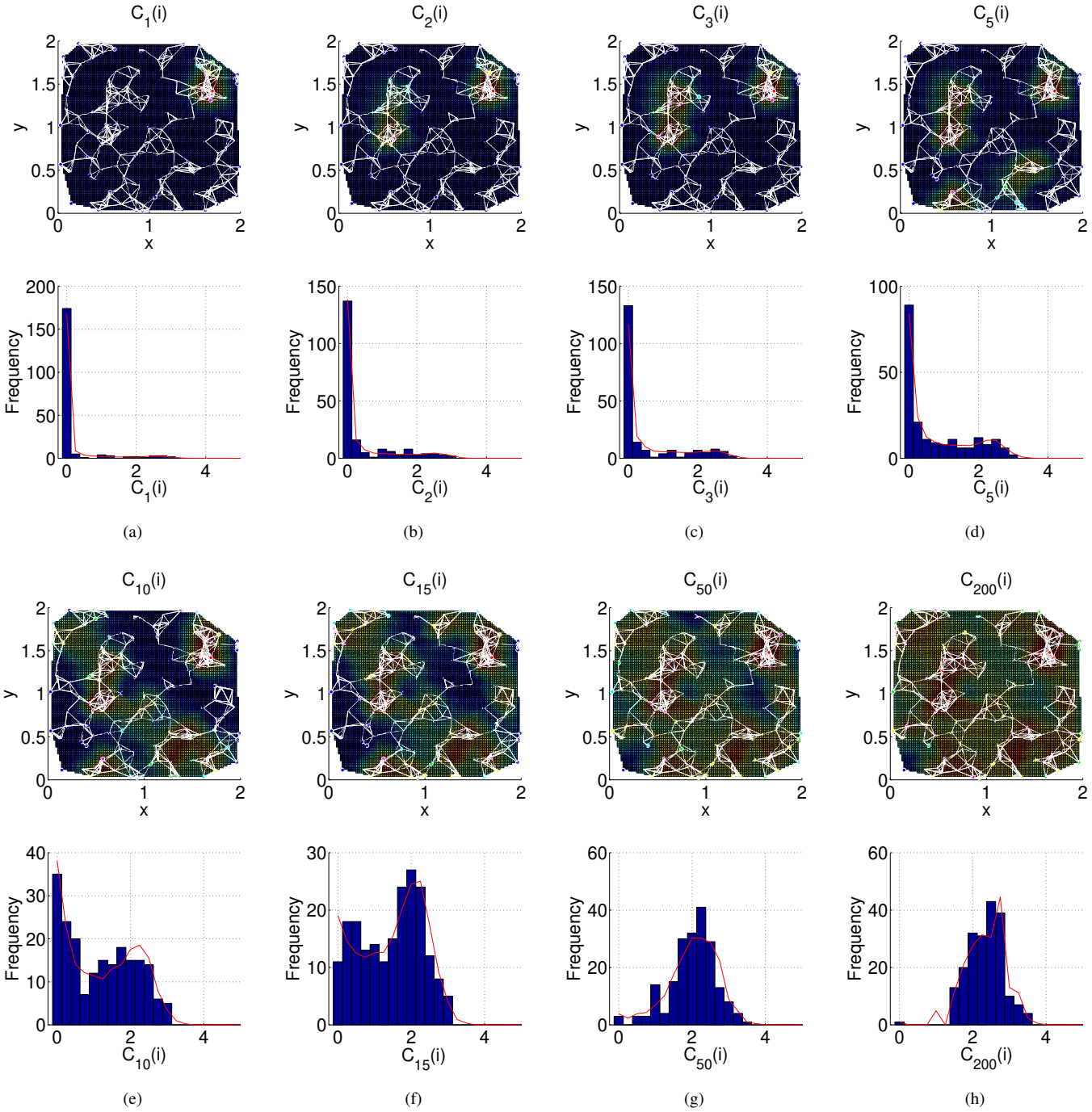

Fig. 6. PCC of nodes in network of figure 2 when computed using first (a) 1, (b) 2, (c) 3, (d) 5, (e) 10, (f) 15, (g) 50 and (h) 200 eigenvectors. The histograms accompanying each graph plot show the distribution of PCC of their nodes. The lineplot in the histogram represents the average PCC histograms of 50 randomly generated networks with the same parameters as the network in figure 2.

eigenvectors computed in successive iterations as a stopping criteria for their fully distributed method for computing the principal eigenvector. We use the phase angle between PCC vectors and EVC to study the effect of adding more features. We compute the phase angle $\phi(n)$ of a PCC vector using n features with the EVC vector as,

$$\phi(P) = \arccos\left(\frac{\mathbf{C}_P}{|\mathbf{C}_P|} \cdot \frac{\mathbf{C}_E}{|\mathbf{C}_E|}\right). \tag{6}$$

Here, '$\cdot$' denotes the inner product operator. The relationship of the phase angle with the number of features used in PCC

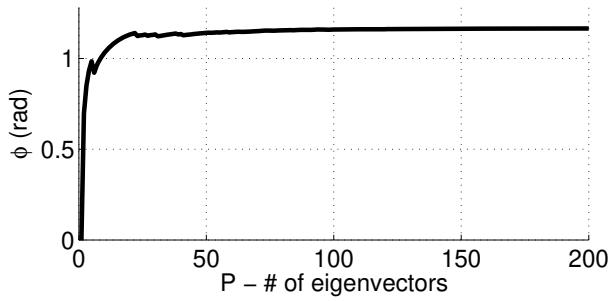

Fig. 7. Plot of phase angles ϕ (in radians) of PCC vectors with the EVC vector for the graph in figure 6.

for the network under consideration is plotted in figure 7. Initially, the function of phase angle ϕ rises sharply and then levels off almost completely at 22 features. This means that, in this example, the relative PCCs of nodes cease to change with the addition of more features beyond the first 22 features. The phase angle plot may be used for determining how many features are sufficient for the computation of PCC of a network.

V. Conclusions

We reviewed previously defined measures of centrality and pointed out their shortcomings in general and EVC in particular. We introduced PCC, a new measure of node centrality. PCC is based on PCA and the KLT which takes the view of treating a graphs adjacency matrix as a covariance matrix. PCC interprets a node's centrality as its ℓ^2 norm from the origin in the eigenspace formed by the P most significant feature vectors (eigenvectors) of the adjacency matrix. Unlike EVC, PCC allows the addition of more features for the computation of node centralities. We explore two criteria for the selection of the number of features to use for PCC; a) The relative contribution of each feature's power (eigenvalue) to the total power of adjacency matrix and b) Incremental changes in the phase angle of the PCC with P features and the EVC as P is increased. We also provide a visual interpretation of significant eigenvectors of an adjacency matrix. The use of the adjacency matrix is compared with that of the Laplacian and it is shown that eigendecomposition of the adjacency matrix yields significantly higher degree of energy compaction than does the Laplacian at the same number of features. We also investigated the effect of adding successive eigenvectors and the information they contain by looking at reconstructions of the original graph's topology using a subset of features.

In the future we intend to extend the definition of PCC so it can be applied to both directed and undirected graphs. Furthermore, we propose to formulate a distributed method for computing PCC along the lines of Canright's method [7] for computing EVC in peer-to-peer systems.

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