Логические исследования
2020. Т. 26. № 1. С. 110–123
УДК 510.64

Logical Investigations
2020, Vol. 26, No. 1, pp. 110–123
DOI: 10.21146/2074-1472-2020-26-1-110-123

Janusz Ciuciura

A lattice of the paracomplete calculi

Janusz Ciuciura
University of Łódź,
Faculty of Philosophy and History,
Institute of Philosophy,
Lindleya 3/5, 90–131 Łódź, Poland.
E-mail: janusz.ciuciura@uni.lodz.pl

Abstract: Paracomplete logic is intended to cope with the problem of vagueness, or uncertain and incomplete data. It deals with the situation when some propositions and their negations are allowed to be simultaneously false, which is obviously impossible in the classical and many non–classical propositional logics. In paracomplete logic, such classical laws as tertium non datur or consequentia mirabilis are not generally accepted. This implies that the logic is defined negatively.

In this paper, we introduce a family of the paracomplete calculi that will be defined in a Hilbert-style formalization. We propose the so-called bi–valuational semantics and prove the key metatheorems for the calculi. We also discuss a generalization of the paracomplete calculus QD^1 to the hierarchy of related calculi.

Keywords: paracomplete logic, paracompleteness, the law of exluded middle, tertium non datur, consequentia mirabilis, weakly–intuitionistic logic, literal–paracomplete logic, paraconsistent logic

For citation: Ciuciura J. "A lattice of the paracomplete calculi", *Logicheskie Issledovaniya / Logical Investigations*, 2020, Vol. 26, No. 1, pp. 110–123. DOI: 10.21146/2074-1472-2020-26-1-110-123

1. Introduction

Let *var* denote a (non-empty) denumerable set of all propositional variables. The set of formulas $\mathcal{F}$ is inductively defined as follows:

$$\varphi ::= p \mid \neg\alpha \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \alpha \to \alpha,$$

where $p \in var$, $\alpha \in \mathcal{F}$ and the symbols $\neg$, $\vee$, $\wedge$, $\to$ denote negation, disjunction, conjunction and implication, respectively. The connective of equivalence, $\alpha \leftrightarrow \beta$, is treated as an abbreviation for $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

Paracomplete logic can be defined in various ways, for instance,

Definition 1. A logic $\langle \mathcal{L}, \vdash \rangle$ is said to be paracomplete if, and only if
(1) $\{\beta \to \alpha, \neg\beta \to \alpha\} \nvdash \alpha$, for some $\alpha, \beta \in \mathcal{F}$; or
(2) $\emptyset \nvdash \alpha \vee \neg\alpha$, for some $\alpha \in \mathcal{F}$; or
(3) $\emptyset \nvdash (\neg\alpha \to \alpha) \to \alpha$, for some $\alpha \in \mathcal{F}$; or
(4) $\emptyset \nvdash (\alpha \to \neg\alpha) \to \neg\alpha$, for some $\alpha \in \mathcal{F}$.[1]

It is noticeable that paracomplete logic is specified negatively: any logic is paracomplete if it meets at least one of the criteria listed above. The definitions may seem too general at first sight; in particular, they may suggest some logics which have nothing in common with *paracompleteness*. Suffice it to note that Łukasiewicz's three–valued logic meets the four requirements. It is not by accident, however, that the example has been cited here. From philosophical perspective, paracomplete calculi are expected to cope with the problem of vagueness,[2] or uncertain and incomplete information.[3] Seen from this viewpoint, Łukasiewicz's logic is a good example of how to interpret uncertainty in relation to the issue of determinism or fatalism, whereas paracomplete calculi – with regard to the dynamic character of information or knowledge. Metaphorically speaking, in paracomplete logic, the dilemma of 'Tomorrow's sea fight' has been reduced to 'Today's communication'.

The paracomplete calculi are expected to deal with the situation when some propositions and their negations are allowed to be simultaneously false, which is impossible in the classical and many non–classical propositional logics. The calculi are also viewed as being *dual* to their paraconsistent counterparts, in a sense that "(...) a logic is paraconsistent if it can be the underlying logic of theories containing contradictory theorems which are both true. (...) a logical system is paracomplete if it can function as the underlying logic of theories in which there are (closed) formulas such that these formulas and their negations are simultaneously false" [Loparić, da Costa, 1984, p. 119].

In what follows, we will consider axiomatic propositional calculi in a Hilbert-style formalization with the sole rule of inference (MP): $\alpha \to \beta,\ \alpha\ /\ \beta$. Such a calculus $\mathcal{C}$, identified with the triple $\langle \mathcal{F}, Ax_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$, is determined by its set of axioms $Ax_{\mathcal{C}}$ which is included in $\mathcal{F}$. We will require for each paracomplete calculus that it contains all axiom schemas of the positive fragment of Classical Propositional Calculus ($\mathcal{CPC}^{+}$, for short), that is, all instances of the following schemas:

[1] *Cit. per* [Petrukhin, 2018, pp. 425–426]. Some interesting examples of the paracomplete calculi are given in [Batens et all, 1999; Bolotov et all, 2018; Ciuciura, 2015; Karpenko, Tomova, 2017; Loparić, da Costa, 1984; Popov, 2002; Sette, Carnielli, 1995].

[2] See [Arruda, Alves, 1979; Arruda, Alves, 1979] and [Beall, 2017, Section 4.1], for details.

[3] See [Bolotov et all, 2018], for details.

(*A*1) $\alpha \to (\beta \to \alpha)$
(*A*2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
(*A*3) $((\alpha \to \beta) \to \alpha) \to \alpha$
(*A*4) $(\alpha \wedge \beta) \to \alpha$
(*A*5) $(\alpha \wedge \beta) \to \beta$
(*A*6) $\alpha \to (\beta \to (\alpha \wedge \beta))$
(*A*7) $\alpha \to (\alpha \vee \beta)$
(*A*8) $\beta \to (\alpha \vee \beta)$
(*A*9) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$,

admits the rule (MP), and fulfils all the criteria listed in *Definition 1.* To put it more accurately:

Definition 2. A calculus $\langle \mathcal{F}, Ax_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$ is said to be paracomplete if, and only if it contains $\mathcal{CPC}^+$, admits (MP) and cumulatively meets the conditions:
(1) $\{\beta \to \alpha, \neg\beta \to \alpha\} \nvdash \alpha$, for some $\alpha, \beta \in \mathcal{F}$
(2) $\emptyset \nvdash \alpha \vee \neg\alpha$, for some $\alpha \in \mathcal{F}$
(3) $\emptyset \nvdash (\neg\alpha \to \alpha) \to \alpha$, for some $\alpha \in \mathcal{F}$
(4) $\emptyset \nvdash (\alpha \to \neg\alpha) \to \neg\alpha$, for some $\alpha \in \mathcal{F}$.

Observe that many non-classical logics, esp. Intuitionistic and Łukasiewicz's three–valued logic, do not come within the scope of *paracompleteness*.

Definition 3. For $\mathcal{C}$, any $\alpha \in \mathcal{F}$ and any $\Gamma \subseteq \mathcal{F}$, we say that α *is provable from* Γ *within* $\mathcal{C}$ (in symbols: $\Gamma \vdash_{\mathcal{C}} \alpha$) iff there is a finite sequence of formulas, $\beta_1, \beta_2, \ldots, \beta_n$ such that $\beta_n = \alpha$ and for each $i \leqslant n$, either $\beta_i \in \mathit{\Gamma}$, or $\beta_i \in Ax_{\mathcal{C}}$, or for some $j, k \leqslant i$ we have $\beta_k = \beta_j \to \beta_i$. A formula α is a thesis of $\mathcal{C}$ iff α is provable from $\emptyset$ within $\mathcal{C}$ (in symbols: $\emptyset \vdash_{\mathcal{C}} \alpha$).

Definition 4. Let $\mathcal{T}(\mathcal{C})$ be the set of all theses of $\mathcal{C}$. For any calculi $\mathcal{C}$ and $\mathcal{C}\star$ in $\mathcal{F}$, we say that $\mathcal{C}$ is an extension of $\mathcal{C}\star$ if, and only if $\mathcal{T}(\mathcal{C}\star) \subseteq \mathcal{T}(\mathcal{C})$. We say that $\mathcal{C}\star$ is a *proper subsystem* of $\mathcal{C}$ (in symbols: $\mathcal{C}\star \sqsubset \mathcal{C}$) if, and only if $\mathcal{T}(\mathcal{C}\star) \subseteq \mathcal{T}(\mathcal{C})$ and $\mathcal{T}(\mathcal{C}) \nsubseteq \mathcal{T}(\mathcal{C}\star)$.

Let us recall a few well–known facts about $\mathcal{C}$, where $\mathcal{C} = \mathcal{CPC}^+ + $ (MP).

Theorem 1. *Deduction theorem holds for* $\mathcal{C}$*.*

Proof. This follows from the fact that $\mathcal{C}$ includes (*A*1) and (*A*2), and the sole rule of inference in $\mathcal{C}$ is (MP). ■

Lemma 1. *Let* $\Gamma, \Delta \subseteq \mathcal{F}$ *and* α, β, $\gamma \in \mathcal{F}$.

(1) If $\alpha \in \Gamma$, *then* $\Gamma \vdash_{\mathcal{C}} \alpha$
(2) If $\Gamma \subseteq \Delta$ *and* $\Gamma \vdash_{\mathcal{C}} \alpha$, *then* $\Delta \vdash_{\mathcal{C}} \alpha$
(3) $\Gamma \vdash_{\mathcal{C}} \alpha$ *iff for some finite* $\Delta \subseteq \Gamma$, $\Delta \vdash_{\mathcal{C}} \alpha$
(4) If $\Delta \vdash_{\mathcal{C}} \alpha$ *and, for every* $\beta \in \Delta$ *it is true that* $\Gamma \vdash_{\mathcal{C}} \beta$, *then* $\Gamma \vdash_{\mathcal{C}} \alpha$
(5) If $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \gamma$ *and* $\Gamma \cup \{\beta\} \vdash_{\mathcal{C}} \gamma$, *then* $\Gamma \cup \{\alpha \vee \beta\} \vdash_{\mathcal{C}} \gamma$
(6) If $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta$ *and* $\Delta \vdash_{\mathcal{C}} \alpha$, *then* $\Gamma \cup \Delta \vdash_{\mathcal{C}} \beta$
(in particular, if $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta$ *and* $\emptyset \vdash_{\mathcal{C}} \alpha$, *then* $\Gamma \vdash_{\mathcal{C}} \beta$*)*

Proof. We refer the interested reader to [Wójcicki, 1988] and [Pogorzelski, Wojtylak, 2008] for details. ■

Remark 1. The relation $\vdash_{\mathcal{C}}$ is a finitary consequence relation satisfying Tarskian properties (reflexivity, monotonicity, transitivity).

2. Paracomplete calculi. Axioms

The *basic* paracomplete calculus discussed in this section is $CLaN$. $CLaN$, as introduced in [Batens et all, 1999], is defined by (MP), CPC^{+} and the law of explosion (DS): $\alpha \to (\neg\alpha \to \beta)$. In the succeeding paragraphs, we consider some extensions of $CLaN$. They are obtained from $CLaN$ by adding to it at least one of the schemas:

(ExM^2) $\alpha \vee \neg\alpha \vee \neg\neg\alpha$
($NN^{\star}$) $\alpha \to \neg\neg\alpha$.

As a result, we obtain three such extensions, namely,

$D_{min} = CLaN + (NN^{\star})$
$Q^1 = CLaN + (ExM^2)$
$QD^1 = CLaN + (ExM^2) + (NN^{\star})$.

The calculus Q^1 was introduced in [Ciuciura, 2019]; D_{min} was briefly discussed in [Carnielli, Marcos, 1999]; QD^1 seems to be pretty new. Notice that the calculi (incl. QD^1) are proper subsystems of I^1. The propositional calculus I^1 was originally defined by (MP), ($A1$), ($A2$),

($I1$) $(\neg\neg\alpha \to \neg\beta) \to ((\neg\neg\alpha \to \beta) \to \neg\alpha)$
($I2$) $\neg\neg(\alpha \to \beta) \to (\alpha \to \beta)$.[4]

The connectives of $\neg$ and $\to$ are taken as primitives. Conjunction, disjunction and equivalence are useful abbreviations. They can be introduced *via* the definitions:

[4] [Sette, Carnielli, 1995, pp. 182–183].

$\alpha \wedge \beta =_{df} \neg(((\alpha \to \alpha) \to \alpha) \to \neg((\beta \to \beta) \to \beta))$
$\alpha \vee \beta =_{df} (\neg(\beta \to \beta) \to \beta) \to ((\alpha \to \alpha) \to \alpha)$
$\alpha \leftrightarrow \beta =_{df} (\alpha \to \beta) \wedge (\beta \to \alpha)$.[5]

It is noteworthy that I^1 gave an impulse for further research and several alternative axiomatizations for the calculus were proposed. In [*Ciuciura*, 2015], for instance, the *consequentia mirabilis* (*cf. Defition 1, (3)*) plays the key role; in [Fernández, Coniglio, 2003], the role is taken by the *tertium non datur* (*cf. Defition 1, (2)*) which suggests that the connective of disjunction (and conjunction) formally appears in formulas. Indeed, Fernández–Coniglio's axiomatization consists of $(A1)$, $(A2)$, $(A4)$–$(A9)$, $(NN^{\star})$ and

(nC) $\neg(\alpha \wedge \neg\alpha)$
$(NI^{\star})$ $(\alpha \vee \neg\alpha) \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$
$(ExM^{\neg})$ $\neg\alpha \vee \neg\neg\alpha$
$(ExM^{\ddagger})$ $(\alpha \ddagger \beta) \vee \neg(\alpha \ddagger \beta)$, where $\ddagger \in \{\wedge, \vee, \to\}$.

The sole rule of inference is (MP).

We prove now that $CLaN, D_{min}, Q^1$ and QD^1 meet the criteria mentioned in *Definition 2*; let $\mathcal{C} \in \{CLaN, D_{min}, Q^1, QD^1\}$, for the sake of brevity.

Remark 2. (1) The formulas
(ExM) $p \vee \neg p$
$(CM1)$ $(p \to \neg p) \to \neg p$
$(CM2)$ $(\neg p \to p) \to p$
(NN) $\neg\neg p \to p$
are not provable in $\mathcal{C}$.

(2) Neither (a) $\{\beta \to \alpha, \neg\beta \to \alpha\} \vdash_{\mathcal{C}} \alpha$, nor (b) $\{\neg(\alpha \to \beta)\} \vdash_{\mathcal{C}} \neg\beta$, nor (c) $\{\alpha \to \neg\beta, \alpha \to \beta\} \vdash_{\mathcal{C}} \neg\alpha$ hold, for any $\alpha, \beta \in \mathcal{F}$.

Proof. Apply the matrix $\mathcal{M}_I = \langle \{1, 2, 0\}, \{1\}, \neg, \wedge, \vee, \to \rangle$, where $\{1, 2, 0\}$ is the set of logical values, 1 is the designated truth value in $\mathcal{M}_I$ and the connectives $\neg, \wedge, \vee, \to$ are defined in the same way as it is done in [Sette, Carnielli, 1995] (see pp. 190, 199), that is,

$\to$	1	2	0
1	1	0	0
2	1	1	1
0	1	1	1

	$\neg$
1	0
2	0
0	1

[5] See *Ibid.*, p. 199.; see also [Karpenko, Tomova, 2017, p. 14].

$\wedge$	1	2	0
1	1	0	0
2	0	0	0
0	0	0	0

$\vee$	1	2	0
1	1	1	1
2	1	0	0
0	1	0	0

Observe that $(A1)$–$(A9)$, (DS), (ExM^2), $(NN^\star)$ are valid in $\mathcal{M}_I$ and (MP) preserves validity. To demonstrate that (ExM), $(CM1)$, $(CM2)$ and (NN) are unprovable in $\mathcal{C}$, it suffices to assign 2 to p in the formulas $p \vee \neg p$, $(p \to \neg p) \to \neg p$, $(\neg p \to p) \to p$ and $\neg\neg p \to p$, respectively. This shows that the claim *(1)* holds. For *(2)*, assign 2 to α and β in *(a)*; 1 to α and 2 to β in *(b)*; 2 to α and 0 to β in *(c)*. ■

Remark 3. The calculus QD^1 can be defined, in a Hilbert-style formalization, by the axiom schemas of CPC^+, (DS), $(ExM^\neg)$ $\neg\alpha \vee \neg\neg\alpha$ and (MP).

Proof. We need to show that *(1)* $(ExM^\neg)$ is a thesis of QD^1, and *(2)* (ExM^2) and $(NN^\star)$ are provable in $QD^1_\star$, where $QD^1_\star$ is defined by CPC^+, (DS), $(ExM^\neg)$ and (MP). *(1)*: This can be easily done by means of (ExM^2), $(NN^\star)$, the thesis of CPC^+ $(\alpha \vee \beta \vee \gamma) \to ((\alpha \to \gamma) \to (\beta \vee \gamma))$ and (MP). *(2)*: Assume that α (by the deduction theorem). Then, we obtain $\neg\alpha \to \neg\neg\alpha$ by (DS), the assumption and (MP). Notice that $\emptyset \vdash_{QD^1_\star} (\neg\alpha \to \neg\neg\alpha) \to \neg\neg\alpha$ by $(ExM^\neg)$, the thesis of CPC^+ $(\alpha \vee \beta) \to ((\alpha \to \beta) \to \beta)$ and (MP). If $\neg\alpha \to \neg\neg\alpha$ and $(\neg\alpha \to \neg\neg\alpha) \to \neg\neg\alpha$, then $\neg\neg\alpha$, and finally $\emptyset \vdash_{QD^1_\star} \alpha \to \neg\neg\alpha$ by the deduction theorem. To prove that (ExM^2) is a thesis of $QD^1_\star$, it suffices to apply (MP) to $(A8)$ and $(ExM^\neg)$. ■

Remark 4. $CLaN \sqsubset Q^1 \sqsubset QD^1$ and $CLaN \sqsubset D_{min} \sqsubset QD^1$.

Proof. It is clear that Q^1 and D_{min} are the extensions of $CLaN$. A proof that $CLaN$ is a proper subsystem of Q^1 immediately follows from the classical truth tables for implication, conjunction and disjunction plus the following one for negation:

	$\neg$
1	0
0	0

The designated value is 1. As expected, $(A1)$–$(A9)$, (DS) are valid under the interpretation and (MP) preserves validity. Now, assign 0 to p in $p \vee \neg p \vee \neg\neg p$ to demonstrate that there is a thesis of Q^1 which is unprovable in $CLaN$.

A proof that $CLaN$ is a proper subsystem of D_{min} basically follows from the fact that $p \to \neg\neg p$ is not provable in $CLaN$. This can be shown by modifying the matrix $\mathcal{M}_I$ appropriately, that is, by replacing the truth table for negation with the so-called rotary negation:

	$\neg$
1	2
2	0
0	1

and assigning 1 to p in $p \to \neg\neg p$. Let $\mathcal{M}_3$ denote the resulting matrix, henceforth.

It is obvious that QD^1 is an extension of Q^1 and D_{min}. Now, we show that the formula $\neg p \vee \neg\neg p$ is provable neither in Q^1 nor D_{min}. Case Q^1: apply $\mathcal{M}_3$ and assign 1 to p in $\neg p \vee \neg\neg p$. Case D_{min}: consider the matrix $\mathcal{M}_{3\star} = \langle \{1,2,0\}, \{1\}, \neg, \wedge, \vee, \to \rangle$, where the connectives $\wedge, \vee, \to$ are defined in the same way as in $\mathcal{M}_I$, but the truth table for negation is as follows:

	$\neg$
1	0
2	2
0	1

The axiom schemas of D_{min} are valid in the matrix and (MP) preserves validity; to *falsify* $\neg p \vee \neg\neg p$, it is enough to assign 2 to p. ■

Remark 5. (1) $D_{min} \not\sqsubset Q^1$
(2) $Q^1 \not\sqsubset D_{min}$.

Proof. *(1)*: Apply the matrix $\mathcal{M}_{3\star}$ and assign 2 to p in $p \vee \neg p \vee \neg\neg p$, to show that the formula $p \vee \neg p \vee \neg\neg p$ is unprovable in D_{min}. *(2)*: Use the matrix $\mathcal{M}_3$ and assign 1 to p in $p \to \neg\neg p$, to demonstrate that $p \to \neg\neg p$ is unprovable in Q^1. ■

Remark 6. $QD^1 \sqsubset I^1 \sqsubset CPC$, where CPC denotes the classical propositional calculus.

Proof. It is known that $I^1 \sqsubset CPC$.[6] All we have to do is to prove that $QD^1 \sqsubset I^1$. Since (MP) is the sole rule of inference of both calculi and each axiom schema of QD^1 is provable in I^1, then I^1 is an extension of QD^1. Now, we prove that (nC^p) $\neg(p \wedge \neg p)$ is not a thesis of QD^1 (*cf.* Fernández–Coniglio's axiomatization of I^1). For this purpose, consider the matrix $\mathcal{M}_{3\star\star} = \langle \{1,2,0\}, \{1\}, \neg, \wedge, \vee, \to \rangle$, where 1 is the only designated value in $\mathcal{M}_{3\star\star}$, the connectives of negation and implication are specified in the same way as in $\mathcal{M}_I$, but conjunction and disjunction are defined as follows:

[6] See [Sette, Carnielli, 1995; Karpenko, Tomova, 2017; Ciuciura, 2015], for details.

$\wedge$	1	2	0
1	1	2	2
2	2	2	2
0	2	2	0

$\vee$	1	2	0
1	1	1	1
2	1	2	2
0	1	2	0

Each axiom schema of QD^1 is valid in $\mathcal{M}_{3\star\star}$ and the rule of detachment preserves validity. To show that (nC^p) is unprovable in QD^1, it is enough to assign 2 to p in $\neg(p \wedge \neg p)$. ■

Since $CLaN$, D_{min}, Q^1 and QD^1 are proper subsystems of I^1, I^1 is the *strongest* calculus among the paracomplete calculi that have been discussed so far. Moreover, I^1 is *maximal* in the sense that if we enrich the calculus with any classical tautology, which is not valid in I^1, the resulting calculus collapses into CPC. It means that there is no *structural* proper subsystem of CPC stronger than I^1. But 'Is $CLaN$ the *weakest* paracomplete calculus?', or: 'Is there a proper subsystem of $CLaN$ admitting CPC^+ and (MP)?' Some results supporting a positive answer were suggested in *Section 7* of [Nowak, 1998]. The requested calculus, denoted as $\vdash_{Cl1}$, is defined by CPC^+, (MP) and $(DS^\star)$ $\alpha \to (\neg\alpha \to \neg\beta)$.

Remark 7. $CPC^+ \sqsubset \vdash_{Cl1} \sqsubset CLaN$.

Proof. It is obvious that $CPC^+ \sqsubset \vdash_{Cl1}$. For $\vdash_{Cl1} \sqsubset CLaN$, note that $(DS^\star)$ is an instance of (DS). Thus all the axiom schemas of $\vdash_{Cl1}$ are theses of $CLaN$. To show that $p \to (\neg p \to q)$ is unprovable in $\vdash_{Cl1}$, apply the classical truth tables for implication, conjunction and disjunction plus the following one for negation (1 is the designated value):

	$\neg$
1	1
0	1

■

Let us summarize that the lattice relationships between the calculi can be represented by the structure of Figure 1.

3. Paracomplete calculi. Semantics

A Kripke-type semantics for $\vdash_{Cl1}$ was given in [Nowak, 1998, p. 98]; a valuation semantics for $CLaN$ was introduced in [Batens et all, 1999, p. 32]; and a three-valued semantics for I^1 was proposed in [Sette, Carnielli, 1995, p. 190]; an alternative semantics for I^1 was discussed in [Fernández, Coniglio, 2003]. In this section, we propose a bi-valuational semantics for the calculi Q^1 and QD^1; let $\mathcal{C}\star \in \{Q^1, QD^1\}$, for the sake of brevity.

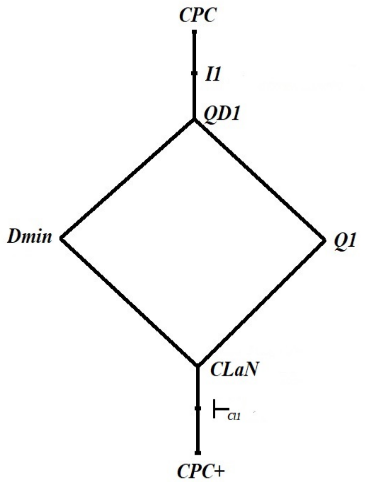

Fig. 1. A lattice of the paracomplete calculi.

Definition 5. A $\mathcal{C}\star$-valuation is any function $v : \mathcal{F} \longrightarrow \{1, 0\}$ that satisfies, for any $\alpha, \beta \in \mathcal{F}$, the following conditions:

($\vee$) $v(\alpha \vee \beta)=1$ iff $v(\alpha)=1$ or $v(\beta)=1$

($\wedge$) $v(\alpha \wedge \beta)=1$ iff $v(\alpha)=1$ and $v(\beta)=1$

($\rightarrow$) $v(\alpha \rightarrow \beta)=1$ iff $v(\alpha)=0$ or $v(\beta)=1$

($\neg$) if $v(\neg\alpha)=1$, then $v(\alpha)=0$,

and additionally,

($\neg\neg$) if $v(\neg\neg\alpha)=0$, then ($v(\alpha)=1$ or $v(\neg\alpha)=1$), for $\mathcal{C}\star = Q^1$

($\neg\neg$) if $v(\neg\neg\alpha)=0$, then $v(\neg\alpha)=1$, for $\mathcal{C}\star = QD^1$.

Definition 6. A formula α is a $\mathcal{C}\star$-tautology if, and only if for every $\mathcal{C}\star$-valuation v, $v(\alpha) = 1$. For any $\alpha \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$, α is a semantic consequence of Γ ($\Gamma \models_{C\star} \alpha$, in symbols) iff for any $\mathcal{C}\star$-valuation v: if $v(\beta) = 1$ for any $\beta \in \Gamma$, then $v(\alpha) = 1$.

The proof of soundness can be obtained in the standard way, by induction on the length of a derivation in $C\star$.

Theorem 2. *For every $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, we have if $\Gamma \vdash_{C\star} \alpha$, then $\Gamma \models_{C\star} \alpha$.*

For the proof of completeness, we apply the method which is based on the notion of maximal non-trivial sets of formulas. We use the technique proposed in [Carnielli, Coniglio, 2016, Section 2.2]. Before going further, let us recall some important definitions and results. Let $\mathcal{C} = \langle \mathcal{F}, Ax_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$ be a calculus (satisfying Tarskian properties) and $\Delta \subseteq \mathcal{F}$.

Definition 7. We say that Δ is a closed theory of $\mathcal{C}$ if, and only if for any $\beta \in \mathcal{F}$: $\Delta \vdash_{\mathcal{C}} \beta$ *iff* $\beta \in \Delta$. We say that Δ is maximal non-trivial with respect to $\alpha \in \mathcal{F}$ in $\mathcal{C}$, if, and only if (i) $\Delta \nvdash_{\mathcal{C}} \alpha$, and (ii) for every $\beta \in \mathcal{F}$, if $\beta \notin \Delta$ then $\Delta \cup \{\beta\} \vdash_{\mathcal{C}} \alpha$.

Lemma 2 ([Carnielli, Coniglio, 2016], Lemma 2.2.5). *Every maximal non-trivial set with respect to some formula is a closed theory.*

Observe that the lemma holds for $\mathcal{C}\star$. Moreover, we have:

Lemma 3. *For any maximal non-trivial set Δ with respect to α in $\mathcal{C}\star$ the mapping $v : \mathcal{F} \longrightarrow \{1, 0\}$ defined, for any $\delta \in \mathcal{F}$, as ($\star$): $v(\delta) = 1$ if and only if $\delta \in \Delta$, is a $\mathcal{C}\star$-valuation.*

Proof. We only prove the clauses for negation. The rest of the proof is similar to that of *Theorem 2.2.7* in [Carnielli, Coniglio, 2016].

Assume, for a contradiction, that $v(\neg\beta) = 1$ and $v(\beta) = 1$. Thus we have $\neg\beta \in \Delta$ and $\beta \in \Delta$ by ($\star$). This implies, by *Lemma 1(1)*, that $\Delta \vdash_{\mathcal{C}\star} \neg\beta$ and $\Delta \vdash_{\mathcal{C}\star} \beta$. But, if $\Delta \vdash_{\mathcal{C}\star} \neg\beta$ and $\Delta \vdash_{\mathcal{C}\star} \beta$, then $\Delta \vdash_{\mathcal{C}\star} \{\neg\beta, \beta\}$. Since $\emptyset \vdash_{\mathcal{C}\star} \beta \to (\neg\beta \to \gamma)$, thus $\{\beta, \neg\beta\} \vdash_{\mathcal{C}\star} \gamma$, by the deduction theorem. The relation $\vdash_{\mathcal{C}\star}$ is transitive, so $\Delta \vdash_{\mathcal{C}\star} \gamma$. Notice that Δ is a closed theory, so $\alpha \in \Delta$. But $\alpha \notin \Delta$ (by the main assumption). This yields a contradiction.

If $\mathcal{C}\star = Q^1$, we need to show that the mapping v satisfies the following clause: if $v(\neg\neg\beta) = 0$ then ($v(\beta) = 1$ or $v(\neg\beta) = 1$), for any $\beta \in \mathcal{F}$. Assume, for a contradiction, that $v(\neg\neg\beta) = 0$ and $v(\neg\beta) = v(\beta) = 0$. Thus we have $\neg\neg\beta \notin \Delta$, $\neg\beta \notin \Delta$ and $\beta \notin \Delta$ by ($\star$). Since Δ is a maximal non-trivial set with respect to α, $\Delta \cup \{\beta\} \vdash_{Q^1} \alpha$, $\Delta \cup \{\neg\beta\} \vdash_{Q^1} \alpha$ and $\Delta \cup \{\neg\neg\beta\} \vdash_{Q^1} \alpha$. Consequently, $\Delta \cup \{\beta \vee \neg\beta \vee \neg\neg\beta\} \vdash_{Q^1} \alpha$, by *Lemma 1 (5)*. Note that $\emptyset \vdash_{Q^1} \beta \vee \neg\beta \vee \neg\neg\beta$, so $\Delta \vdash_{Q^1} \alpha$, by *Lemma 1 (6)*. Since Δ is a closed theory, then $\alpha \in \Delta$. But $\alpha \notin \Delta$. This yields a contradiction.

If $\mathcal{C}\star = QD^1$, we have to prove that the mapping v satisfies the clause: if $v(\neg\neg\beta) = 0$ then $v(\neg\beta) = 1$, for any $\beta \in \mathcal{F}$. Assume, for a contradiction, that $v(\neg\neg\beta) = 0$ and $v(\neg\beta) = 0$. Then we have $\neg\neg\beta \notin \Delta$ and $\neg\beta \notin \Delta$ by ($\star$). Since Δ is a maximal non-trivial set with respect to α, then $\Delta \cup \{\neg\neg\beta\} \vdash_{QD^1} \alpha$ and $\Delta \cup \{\neg\beta\} \vdash_{QD^1} \alpha$. Consequently, $\Delta \cup \{\neg\beta \vee \neg\neg\beta\} \vdash_{QD^1} \alpha$, by *Lemma 1 (6)*. It is known that $\emptyset \vdash_{QD^1} \neg\beta \vee \neg\neg\beta$, so $\Delta \vdash_{QD^1} \alpha$, by *Lemma 1 (6)*. Since Δ is a closed theory, then $\alpha \in \Delta$. But $\alpha \notin \Delta$. This yields a contradiction. ■

Note that the so-called Lindenbaum–Łoś' theorem holds, for any finitary calculus $\mathcal{C} = \langle \mathcal{F}, Ax_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$.

Lemma 4 ([Pogorzelski, Wojtylak, 2008], Theorem 3.31; [Carnielli, Coniglio, 2016], Theorem 2.2.6). *For any $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$ such that $\Gamma \nvdash_{\mathcal{C}} \alpha$, there is a maximal non-trivial set Δ with respect to α in $\mathcal{C}$ such that $\Gamma \subseteq \Delta$.*

Thus, the completeness of $\mathcal{C}\star$ follows:

Theorem 3. *For all* $\Gamma \subseteq \mathcal{F}$ *and* $\alpha \in \mathcal{F}$*: if* $\Gamma \models_{\mathcal{C}\star} \alpha$*, then* $\Gamma \vdash_{\mathcal{C}\star} \alpha$*.*

Proof. Assume that $\Gamma \nvdash_{\mathcal{C}\star} \alpha$ and Δ be a maximal non-trivial set with respect to α in $\mathcal{C}\star$ such that $\Gamma \subseteq \Delta$. Then $\alpha \notin \Delta$. Because *Lemma 3* holds, there is a valuation v such that $v(\alpha) = 0$ and $v(\beta) = 1$, for any $\beta \in \Gamma$. Hence $\Gamma \not\models_{\mathcal{C}\star} \alpha$. ■

4. A hierarchy of the paracomplete calculi

The $\vdash_{Cl1}$, $CLaN$, D_{min}, Q^1, QD^1 and I^1 are not the only paracomplete calculi that satisfy the criteria specified in *Definition 2*. In fact, there are infinitely many such calculi, for example, Q^1, Q^2, ..., Q^n; or QD^1, QD^2,... QD^n. The hierarchy of Q^n–calculi, $n \in \mathbb{N}$, was considered in [Ciuciura, 2019]. In the subsequent paragraphs we will discuss the hierarchy of QD^n–calculi. The hierarchy is obtained by replacing $(ExM^{\neg})$ with a more general schema, that is,

$(ExM^{\neg n})$ $\neg^n\alpha \vee \neg^{n+1}\alpha$,

where $n \in \mathbb{N}$ and $\neg^n\alpha$ is an abbreviation for $\overbrace{\neg\neg...\neg}^{n}\alpha$. To put it more precisely, for each $n \in \mathbb{N}$, let QD^n be obtained from CPC^+ (and (MP)) by adding to it the axiom schemas:

(DS) $\alpha \to (\neg\alpha \to \beta)$
$(ExM^{\neg n})$ $\neg^n\alpha \vee \neg^{n+1}\alpha$.[7]

For each $n \in \mathbb{N}$, the semantics for QD^n results from replacing the evaluation condition for $(\neg\neg)$ with a more general one, *i.e.*

$(\neg^{n+1})$ if $v(\neg^{n+1}\alpha)$=0, then $v(\neg^n\alpha)$=1.

The semantic clauses for $(\vee)$, $(\wedge)$, $(\to)$ and $(\neg)$ remain unchanged, *i.e.*

Definition 8. A QD^n-valuation is any function $v : \mathcal{F} \longrightarrow \{1, 0\}$ that satisfies, for any $\alpha, \beta \in \mathcal{F}$, the conditions:
$(\vee)$ $v(\alpha \vee \beta)$=1 iff $v(\alpha)$=1 or $v(\beta)$=1
$(\wedge)$ $v(\alpha \wedge \beta)$=1 iff $v(\alpha) = 1$ and $v(\beta)$=1

[7]If $n = 0$, then $QD^0 = CPC$. Some other examples of the hierarchies are known in the logical literature. For instance, the hierarchy of I^n–calculi is proposed in [Sette, Carnielli, 1995] and [Fernández, Coniglio, 2003]. There are also interesting hierarchies in a Newton da Costa-style presentation, *e.g.* da Costa and Marconi's hierarchy of paracomplete calculi P_n, see [da Costa, Marconi, 1986]; or Arruda–Alves' logic of vagueness, see [Arruda, Alves, 1979] and [Arruda, Alves, 1979].

($\to$) $v(\alpha \to \beta)$=1 iff $v(\alpha)$=0 or $v(\beta)$=1
($\neg$) if $v(\neg\alpha)$=1, then $v(\alpha)$=0,
($\neg^{n+1}$) if $v(\neg^{n+1}\alpha)$=0, then $v(\neg^{n}\alpha)$=1, where $n \in \mathbb{N}$.

The definition of QD^n-tautology (and semantic consequence $\models_{QD^n}$) is analogous to that of *Definition 6.*

Theorem 4. *For every* $\Gamma \subseteq \mathcal{F}$ *and* $\alpha \in \mathcal{F}$, $\Gamma \vdash_{QD^n} \alpha$ *iff* $\Gamma \models_{QD^n} \alpha$, $n \in \mathbb{N}$.

Proof. Proceed analogously to the proof of *Theorems 2 and 3.* ■

At the end of this section, we state a few simple facts about the QD^n–calculi.

Remark 8. If $n > 1$, then the formula $p \to \neg\neg p$ is not provable in QD^n.

Proof. This follows from the completeness of QD^n–calculi. ■

Remark 9. If $n > 1$, then
(1) $D_{min} \not\subseteq QD^n$
(2) $QD^n \not\subseteq D_{min}$.

Proof. *(1)*: Although $(ExM^{\neg n})$ is an axiom schema of QD^n, the formula $\neg^n p \vee \neg^{n+1} p$ is not provable in D_{min} (it is enough to apply the semantics and completeness theorem for D_{min}, *cf.* [Carnielli, Marcos, 1999], *Proposition 6.2*). *(2)*: This is a consequence of *Remark 8* and the fact that $(NN^\star)$ is an axiom schema of D_{min}. ■

Remark 10. For any r, $m \in \mathbb{N}$ such that $r > m$, we have $QD^r \sqsubset QD^m$.

Proof. The proof follows from the completeness of QD^n–calculi. ■

Remark 11. Enriching the set of axiom schemas of any QD^n-calculus ($n \in \mathbb{N}$) with the formula (NN) $\neg\neg\alpha \to \alpha$, results in obtaining the axiom system of CPC.

Proof. This follows from the fact that the axiom schemas $(ExM^{\neg n})$ and (NN) are equivalent to (ExM) $\alpha \vee \neg\alpha$ in CPC. ■

Remark 12. Enriching the set of axiom schemas of any QD^n-calculus (for $n > 1$) with the formula $\neg\neg\neg\alpha \to \neg\alpha$, results in obtaining the calculus QD^1.

Proof. Notice that *(1)* $\neg\neg\neg\alpha \to \neg\alpha$ is a thesis of QD^1, and *(2)* $\neg\neg\neg p \to \neg p$ is not provable in any QD^n-calculus that is weaker than QD^1. Now it suffices to show that $(ExM^{\neg n})$, where $n > 1$, and $\neg\neg\neg\alpha \to \neg\alpha$ are equivalent to $(ExM^{\neg})$ in QD^1. ■

Acknowledgements. I am very grateful to anonymous reviewers for their helpful comments on an earlier draft of this paper.

References

Arruda, Alves, 1979 – Arruda, A.I., Alves, E.H. "Some remarks on the logic of vagueness", *Bulletin of the Section of Logic*, 1979, Vol. 8, No. 3, pp. 133–138.

Arruda, Alves, 1979 – Arruda A.I., Alves, E.H. "A semantical study of some systems of vagueness logic", *Bulletin of the Section of Logic*, 1979, Vol. 8, No. 3, pp. 139–144.

Batens et all, 1999 – Batens, D., De Clercq, K., Kurtonina, N. "Embedding and interpolation for some paralogics. The propositional case", *Reports on Mathematical Logic*, Vol. 33, pp. 29–44.

Beall, 2017 – Beall, Jc. "There is no logical negation: true, false, both and neither", *Australasian Journal of Logic*, 2017, Vol. 14, No. 1, pp. 1–29.

Bolotov et all, 2018 – Bolotov, A., Kozhemiachenko, D., Shangin, V. "Paracomplete Logic Kl – Natural Deduction, its Automation, Complexity and Applications", *Journal of Applied Logics*, 2018, Vol. 5, No. 1, pp. 221–261.

Carnielli, Coniglio, 2016 – Carnielli, W., Coniglio, M.E., *Paraconsistent Logic: Consistency, Contradiction and Negation.* Logic, Epistemology, and the Unity of Science. Vol. 40. Springer International Publishing, 2016.

Carnielli, Marcos, 1999 – Carnielli, W., Marcos, J., "Limits for paraconsistent calculi", *Notre Dame Journal of Formal Logi*, 1999, Vol. 40, No. 3, pp. 375–390.

Ciuciura, 2015 – Ciuciura, J., "A Weakly-Intuitionistic Logic I1", *Logical Investigations*, 2015, Vol. 21, No. 2, pp. 53—60.

Ciuciura, 2019 – Ciuciura, J. "Paraconsistency and Paracompleteness", *Logical Investigations*, 2019, Vol. 25, No. 2, pp. 46–60.

da Costa, Marconi, 1986 – da Costa, N.C.A., Marconi, D., "A note on paracomplete logic", in: *Atti della Accademia Nazionale dei Lincei*, Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, 1986, Vol. 80, No. 7–12, pp. 504–509.

Fernández, Coniglio, 2003 – Fernández, V.L., Coniglio, M.E., "Combining Valuations with Society Semantics", *Journal of Applied Non-Classical Logics*, 2003, Vol. 13, No. 1, pp. 21–46.

Karpenko, Tomova, 2017 – Karpenko, A., Tomova, N., "Bochvar's three-valued logic and literal paralogics: Their lattice and functional equivalence", *Logic and Logical Philosophy*, 2017, Vol. 26, No. 2, pp. 207–235.

Loparić, da Costa, 1984 – Loparić, A., da Costa, N.C.A., "Paraconsistency, Paracompleteness, and Valuations", *Logique et Analyse*, 1984, Vol. 27, No. 106, pp. 119–131.

Nowak, 1998 – Nowak, M. "Kripke semantics for some paraconsistent logics", *Logica Trianguli*, 1998, Vol. 2, pp. 87–101.

Petrukhin, 2018 – Petrukhin, Y. "Generalized Correspondence Analysis for Three-Valued Logics", *Logica Universalis*, 2018, Vol. 12, No. 3–4, pp. 423–460.

Pogorzelski, Wojtylak, 2008 – Pogorzelski, W.A., Wojtylak, P., *Completeness Theory for Propositional Logics*, Studies in Universal Logic, Birkhäuser Basel, 2008.

Popov, 2002 – Popov, V.M. "On a three-valued paracomplete logic", *Logical Investigations*, 2002, Vol. 9, pp. 175–178. (In Russian)

Sette, Carnielli, 1995 – Sette, A.M., Carnielli, W.A. "Maximal weakly-intuitionistic logics", *Studia Logica*, 1995, Vol. 55, No. 1, pp. 181–203.

Wójcicki, 1988 – Wójcicki, R. *Theory of Logical Calculi. Basic Theory of Consequence Operations*, Synthese Library, Vol. 199, Springer Netherlands, 1988.